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On the impact of zealots in a population of susceptible agents in a best-of-n problem within a heterogeneous network

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Abstract

Both humans and social animals live in groups and are frequently forced to choose between options with different qualities. When there are no leader agents controlling the group decision, consensus can be achieved through repeated interactions among group members. Various studies on collective decision-making illustrate how the dynamics of the opinions are determined by the structure of the social network and the methods that individuals use to share and update their opinion upon a social interaction. In this paper, we are interested in further exploring how cognitive, social, and environmental factors interactively contribute to determining the outcome of a collective best-of-n decision process involving asymmetric options, i.e., different costs and/or benefits for each option. We propose and study a novel model capturing those different factors, i) the cognitive load in processing social information, ii) the number of zealots (i.e., asocial agents who never change their opinion), iii) the option qualities, iv) the social connectivity structure, and v) the degree centrality of the asocial agents (i.e., the number of neighbours). By using the heterogeneous mean-field approach, we study the impact of the above-mentioned factors in the decision dynamics. Our findings indicate that when susceptible agents, i.e., individuals who change their opinion to conform with others, use the voter model as a mechanism to update their opinion, both the number and the degree of connectivity of the zealots can lead the population to converge towards the lowest quality option. Instead, when susceptible agents use methods demanding a larger cognitive cost (e.g., the majority rule), the group is marginally impacted by the presence of zealots. The results of the analytical model are complemented and extended by agent-based simulations. Our analysis also shows that the network topology can modulate the influence of zealots on group dynamics. In fact, in homogeneous networks where all nodes have the same degree, any location of the zealots has similar impact on the group dynamics. Instead, when the network is heterogeneous, our simulations confirm the model predictions showing that placing the zealots in the network hubs (nodes with several neighbours) has a much larger impact than placing them in lower-degree nodes.

1. Introduction

Collective decision-making in humans and other social animals is based on information exchange between individuals and can be considered a complex process. In spite of the fact that the outcomes of these decisions influence the life of all individuals of a community, in some cases the decision process is governed by a limited number of dominant individuals (e.g., political leaders in humans, or socially dominant individuals in primates). In other cases, the decisions are genuinely collective, meaning that they are the results of social interactions between peers. Collective decisions are characterised by the fact that the decision, once made, is not attributable to any individual of the group [?]. For instance, primate groups like chimpanzees and baboons engage in a collective decision-making process to determine their direction after a period of rest [? ? ?]. Similarly, a flock of birds collectively decides when to leave a foraging patch [? ?]; a swarm of honeybees, during reproductive swarming, decides where to build their new nest [? ? ? ?].

Scenarios requiring collective decisions have been studied in different scientific disciplines and with different methods [? ? ?], including computational modelling and simulations [? ? ?] and social network analysis [? ? ?]. In [?], authors provide a descriptive framework for collective decision-making processes based on a costs/benefits analysis for sampling and selecting the available options. In particular, they distinguished between symmetric conditions (i.e., same costs and/or same benefits for each option) and asymmetric conditions (i.e., different costs and/or different benefits for each option). A number of studies have shown that consensus for the best alternative can be achieved even when individuals operate without a leader and make noisy individual estimates of the option's qualities. Indeed, to make collective decisions, it is sufficient that the individuals use the estimated quality of the selected option to modulate the frequency (or persistence) with which they share their opinion. This mechanism has been observed in group-living animals, e.g., ants and honeybees [?], and then employed to design artificial distributed systems, such as robot swarms [? ?]. There are multiple causal factors, interacting in a complex way, that contribute to determining the outcome of collective decision-making processes. Among these causal factors, a significant share of research work has focused on the behaviours that each individual follows to develop and update her opinion. It has been shown that decision-makers with limited perceptual and cognitive resources can follow simple rules to make, as a group, complex collective decisions [? ? ?]. Among these simple rules, there are those based on social feedback, like the voter model [? ?] where each individual simply copies the opinion of a randomly selected individual within her social network, and the majority model [? ?] where each individual selects the option held by the majority of the individuals in her social network.

This study aims to unravel the mutual influence of psychological, social, and environmental factors on the outcome of a collective decision-making process. It focuses specifically on the best-of-n scenario with $n = 2$ options and an asymmetric condition, i.e., the two options have different costs and/or different benefits [? ? ? ?]. The best-of-n problem requires selecting which option, out of n available ones, is the best alternative [?]. Considerable work has been dedicated to the study of the best-of-n problem, in order to unveil opinion dynamics characterising a wide range of phenomena, from political polarisation [?], to the spread of rumours [?], and misinformation [?]. We focus on an asymmetric best-of-2 scenario since we intend to investigate which factors among those that we modelled hinder the group from achieving a consensus on the best option. In particular, we provide an analysis of the combined effects of the following parameters (which we define in detail in the next paragraph) : i) the cognitive load, related to the cost of acquiring information from peers, which in our model is determined by the update strategy, i.e., voter model (small cognitive load) versus majority one (large cognitive load) [1], ii) the number of zealots present in populations of susceptible agents, with zealots being individuals that never change their opinion, while susceptible agents do (see [**? ? ? ?**]). Let us observe that zealots are indistinguishable from susceptible agents during the group interaction and thus they share their option with the same strength a susceptible age does. iii) the options quality, a property of each option, which influences the probability with which each agent communicates her opinion to neighbours ; iv) the social network structure referring to how agents are connected and how information flows between them (see [**?**]) ; and v) the degree centrality of zealot agents, i.e., the higher the degree centrality of zealots the greater the number of direct connections to them (see [**? ?**]).

The parameters mentioned above have been separately studied in previous research works. For example, **?**] investigates the effect of the position and the number of zealots on the consensus formation and time to reach consensus in both random and scale-free networks in a symmetric opinions scenario. The results show that the degree centrality of the zealots has a great impact on the time required for the group to achieve a consensus. Other studies have shown that, in different quality options scenarios, the asymmetry facilitates the formation of consensus towards the option with the higher quality (or lower cost) [**? ? ?**]. There are several works that investigated the dynamics of the voter model [**? ? ? ?**] and the majority model [**? ? ? ? ?**] on networks.

Contrary to previous research works, mostly looking at the influence of specific variables on the decision process, we propose a single modelling framework that allow us to study the combined causal relationship among multiple elements bearing upon the decision dynamics of populations engaged in a best-of-n with $n = 2$ scenario. In particular, we build upon the mathematical model presented in [**?**], which set the basis for studying the effects of cognitive load in a homogeneous population of agents engaged in the best-of-2 decision-making scenario. We first extend the model in [**?**] by building a new analytical model that includes elements such as the number of zealots, their degree centrality, and the social network structure. Moreover, we run numerical simulations with an agent-based model (hereafter, ABM), to corroborate the predictions of the analytical model.

1. Let us observe that another interpretation of this parameter can be done [**?**], namely to be associated to the pooling error arising in processing and integrating the sampled opinions : the higher the number of opinions sampled the higher the pooling error.

By looking at populations with different numbers of zealots committed to the lowest quality option, we show interesting correlations between the cognitive load, the number of zealots in the population and their degree centrality, in differently connected populations. In particular, we identify the conditions under which zealots favouring the lowest-quality option manage to counterbalance the quality difference and drive the population toward a consensus on the lowest-quality option. We also demonstrate the effect of network heterogeneity, in particular its sparsity, on the conditions for achieving consensus towards the low-quality option.

The rest of this work is organised as follows. In Section 2, we describe the agent-based model with social interactions occurring on a scale-free network and in Section 3 we present the results of the numerical simulation of this agent-based model. Section 4 presents the mathematical model defined by an ordinary differential equation (ODE) allowing us to study the evolution of group opinion taking into account multiple parameters such as the cognitive load, the heterogeneity of the population, the option qualities, the connectivity structure, and the degree centrality of the zealot agents. In Sections 5 and 6, we show the numerical results of this analytical model showing the combined effects of the previous parameters. Finally, we present our conclusions in Section 7.

2. The Agent-Based Model

We consider N agents represented by nodes (in the following, node and agent will be indistinguishably used to denote a member or the group) of a scale-free network composed of L undirected edges, representing social interaction between agents. We assume the number of neighbours of a generic node i (i.e., its degree k) follows a power-law distribution [**? ?**], $p_k \sim 1/k^\gamma$, where $\gamma > 2$. Note that the closer γ to 2 the more heterogeneous the degree distribution. Nodes with a very large degree can be observed since $\langle k^2 \rangle$ becomes unbounded as $\gamma \to 2$. When $\gamma \gg 3$ the network becomes sparse, very high degree nodes are very rare and the degree spread is well described by finite variance of the degree distribution. The networks we consider are simple (i.e., at most one edge can connect any two nodes), undirected (i.e., all social interactions are reciprocate) and connected (i.e., starting from any node it is possible to reach any other node by traversing the network using the available links).

We model a best-of-n problem with $n = 2$ options where each node can hold either opinion A or opinion B and we will denote by $n_A(t)$ (resp. $n_B(t)$) the number of nodes with opinion A (resp. B) at time t. In the following we assume that initially half of the nodes hold opinion A and the other half hold opinion B, with opinions randomly assigned to each of the N agents. We classify nodes into susceptible (i.e., representing agents capable of changing their opinions) and zealots (i.e., agents that never change their opinion under any circumstance). Moreover agents with opinion A are susceptible to modify their opinion after social exchanges, while a number Z of agents with opinion B are zealots. Therefore, the population holding opinion B consists of both susceptible and zealots, $n_B(t) = S_B(t) + Z$, with $S_B(t)$ the number of susceptible agents holding opinion B at time t and then, the total population at each time t is expressed as $N = n_A(t) + S_B(t) + Z$. We also associate to each option a quality, $Q_A > 0$ for opinion A, and $Q_B > 0$ for opinion B. The quality defines the strength or the probability with which the option is communicated to the neighbours. Without loss of generality, in the rest of the work, we assume $Q_A = 1$, $Q_B \leq Q_A$ and hence the quality ratio $Q := Q_B/Q_A \leq 1$. In this work, zealots hold the opinion with the lowest quality option (i.e., opinion B), since we are interested in investigating under which circumstances they can hinder the population from reaching a consensus (i.e., a general agreement) on the option with the best quality. Let us stress again that zealots behave as normal agents during the social interaction and thus disseminate their option, i.e. B, with the same strength Q_B as susceptible B agents do. The only difference being the fact that zealots do not modify their option, they can thus be assimilated to "bots" or "asocial agents".

The system evolves asynchronously : at each time step, one agent is randomly selected with uniform probability from the population ; if she is a zealot then nothing happens, she goes back to the group and a new selection is performed. On the other hand, if she is susceptible she interacts with her peers and possibly changes her opinion. More precisely, let us assume the selected agent i holds opinion A (a similar analysis can be performed in the case she holds opinion B) then she modifies her opinion with probability $P_\alpha(n^{\#}_{i,B})$ or she will keep the original one with probability $P_\alpha(n^{\#}_{i,A}) = 1 - P_\alpha(n^{\#}_{i,B})$, where

$$n^{\#}_{i,A} = \frac{Q_A n_{i,A}}{Q_A n_{i,A} + Q_B n_{i,B}} \text{ and } n^{\#}_{i,B} = \frac{Q_B n_{i,B}}{Q_A n_{i,A} + Q_B n_{i,B}} \,, \tag{1}$$

denote the quality-weighted proportion of neighbours holding opinion A (i.e., $n^{\#}_{i,A}$), or B (i.e., $n^{\#}_{i,B}$), $n_{i,A}$ and $n_{i,B}$ being the number of neighbours of agent i holding opinion A and B, respectively. Let us observe that the latter can be rewritten as follows

$$n^{\#}_{i,A} = \frac{n_{i,A}/k_i}{(1-Q)n_{i,A}/k_i + Q} \quad \text{and} \quad n^{\#}_{i,B} = 1 - n^{\#}_{i,A} \quad \forall i\,. \tag{2}$$

where $k_i = n_{i,A} + n_{i,B}$ is the degree of agent i.

Finally the functional form of the probability $P_\alpha(x)$ is given by [**?**]

$$P_\alpha(x) = \begin{cases} \frac{1}{2} - \frac{1}{2}(1-2x)^\alpha & \text{if } 0 \le x \le \frac{1}{2} \\ \frac{1}{2} + \frac{1}{2}(2x-1)^\alpha & \text{if } \frac{1}{2} < x \le 1 \end{cases}, \tag{3}$$

where the non negative parameter α is inversely correlated to the cognitive load. Indeed if $\alpha = 0$ the function becomes a step function (see red curve in Fig. 1), hence the agent updates her opinion by following a majority rule. The cognitive load is thus very large because the agent has to interact with the whole group. On the other hand if $\alpha = 1$ (see green curve in Fig. 1), the function $P_\alpha(x)$ is linear and agents change their opinion by copying the opinion of a randomly selected neighbour, by replicating thus a (weighted) voter model, for which the cognitive load is small. For $\alpha \gg 1$ (see yellow curve in Fig. 1 for the value $\alpha = 100$), the agent randomly chooses with probability 1/2 either of the options. In this case, the cognitive load is almost zero. Therefore, by varying α, the strategy encoded by P_α is capable of generalising among several opinion selection mechanisms existing in the literature.

The proposed model combines the aggregation of agents' opinions with the communication signal strength, determined by the qualities Q_A and Q_B of the options A and B, respectively. Each agent i receives an aggregated signal arising from her peers weighted by the number of agents signalling that opinion and the option quality. Hence, the voting process allows agents to consider not only the number of neighbours with a particular opinion but also the strength of their convictions.

3. Numerical results of the agent-based-model

In this section we present the results of the numerical simulation of the agent-based model with social interactions occurring on a scale-free network.

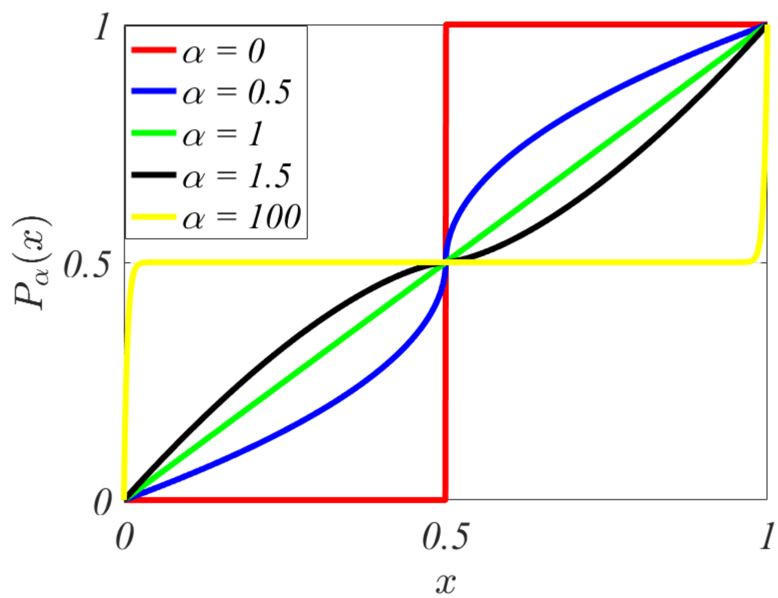

FIGURE 1: The probability function $P_\alpha(x)$, defined in Eq. (3), determines how likely agent i will change their opinion based on the weighted fraction x of neighbours holding opinion A (or B), i.e., $x = n^{\#}_{i,A}$ or $x = n^{\#}_{i,B}$. This function depends on the parameter α that is inversely correlated to the cognitive load required by each agent to perform her action. For $\alpha = 0$, our model corresponds to the weighted majority model, and for $\alpha = 1$, it corresponds to the weighted voter model. In the interval $0 < \alpha < 1$, our model interpolates between these two models. However, for $\alpha \gg 1$, agents experience very small cognitive load, being the decision outcome the result of a random process.

We limit our analysis to studying the combined effects of three factors : i) the number of zealots, ii) their location in the network, and iii) the cognitive load, while we keep constant the network structure and the option quality ratio $Q = Q_B/Q_A = 0.9$.

We consider a system composed of $N = 1000$ agents whose social interactions are described by a scale-free network obtained by using the Barabási-Albert algorithm [**?**] with parameter $m = 8$ (i.e., a network in which the minimum degree is $k_{min} = 8$). The exponent of the power law of such a network is $\gamma = 3$. We consider cases where the zealots, committed to opinion B, are either peripheral agents (i.e., associated with nodes with the smallest degree $k_{\min} = 8$) ; or agents associated to nodes with an intermediate degree ; or with a large degree. The initial distribution of opinions among the agents is random, with half of agents holding opinion A and the other half holding opinion B. The maximum simulation time is set to $T_{max} = 400\,000$. Each experimental plan is repeated 30 times by changing each time the seed of the pseudo-random number generator.

In Fig. 2, we present the results of numerical simulations performed by adopting the strategy of introducing zealots with low degree centrality. In Fig. 2a we illustrate the average final fraction of susceptible agents holding opinion A, computed over $n_{iter} = 30$ independent replicas, in formula :

$$\left\langle \frac{n_A(T_{max})}{N} \right\rangle = \frac{1}{n_{iter}} \sum_{\ell=1}^{n_{iter}} \frac{n_A^{(\ell)}(T_{max})}{N} \,,$$

with $n_A^{(\ell)}(T_{max})$ the number of agents supporting option A at time T_{max} for the ℓ-th simulation. We can observe that in the absence of zealots ($Z = 0$), the system always makes a decision in favour of the option with the larger quality, i.e., A for $\alpha \leq Q_A/Q_B \approx 1.11$ (see the yellow region in Fig. 2a), corresponding to $\langle n_A(T_{max})/N\rangle \approx 1.0$). Conversely for $\alpha \geq Q_A/Q_B$, a decision deadlock may occurs, leading the population to a state where both opinions A and B coexist. In other words, due to a low cognitive load (i.e., $\alpha > 1$), agents fail to coordinate with each other, the population remains polarized and unable to reach a consensus for any alternative, especially when the quality difference between the two options is small. The presence of zealots in the population introduces strong changes in the dynamics depending on the value of the cognitive load. If the latter is maximal, i.e., $\alpha = 0$, despite the presence of zealots, the system reaches a consensus in favour of the high-quality option, i.e., A. This can be observed in Fig. 2b, where we show the distribution of $n_A^{(\ell)}(T_{max})$ for $Z = 128$. All the susceptible agents are committed to A, and only the zealots support option B. By increasing α, i.e., by decreasing the cognitive load, the system fate changes and exhibits a non monotone behaviour. For $\alpha \sim 1$ and still $Z = 128$, the system converges toward a consensus to the option with lower quality, i.e., B (see Fig. 2c). Interestingly enough, if the number of zealots is sufficiently large, i.e., $Z \geq 128$, we can find a large interval of α values, i.e., $\sim$ [0.5,1], for which the whole population converges to a consensus toward the lower quality option B (see dark blue region on the right of Fig. 2a). For larger values of α the mechanism is somehow decreased and agents with opinion A persist in the population and coexist with agents of opinion B (see Fig. 2d for $\alpha = 1.5$ and $Z = 128$). This behaviour is due to the small cognitive load that keeps the population polarized and deadlocked in indecision because of the (almost) random decision. This observation holds true even for a large number of zealots, e.g., $Z = 128$ and $Z = 160$.

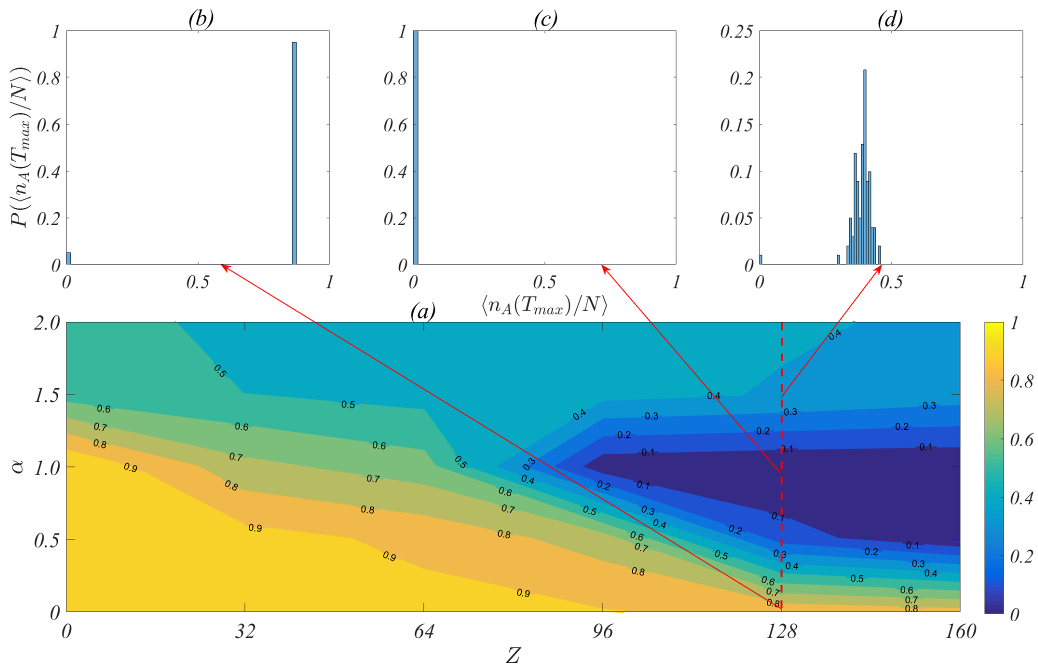

FIGURE 2: Panel a) displays the average final fraction of agents holding opinion A as a function of the number of zealots (Z) committed to opinion B positioned in leaf nodes (i.e., $k_i = 8$) and the cognitive load ($1/\alpha$). The average is computed over 30 independent simulations in a population of $N = 1000$ agents interacting on a Barabási-Albert network with $m = 8$. For a fixed number of zealots, $Z = 128$, and three different values of the parameter α : $\alpha = 0.0$ (panel b), $\alpha = 1.0$ (panel c) and $\alpha = 1.5$ (panel d), we report the distribution of $n_A^{(\ell)}(T_{max})$ for 100 independent simulations. The quality factors are $Q_A = 1$ for opinion A and $Q_B = 0.9$ for opinion B.

To study the combined impact of the number of zealots and their location in the social network, we conduct a further series of simulations to verify whether the position of the zealots within a scale-free network of $N = 1000$ nodes determines some outcome on the population fate. We consider various experimental conditions, defined by different numbers of zealots $Z \in \{0, 1, 4, 8, 16\}$, three values of $\alpha \in \{0.5, 1.0, 1.5\}$ (see respectively Figs. 3a, 3b and 3c), and four different strategies for locating zealots based on the agent degree k.

In the first strategy, zealots are placed on nodes with small degree, i.e., $k_{\min} = 8$. In the second strategy, zealots are placed on nodes with degrees ranging from 15 to 25. The third strategy involved placing zealots on nodes with degrees ranging from 35 to 45. Finally, in the fourth strategy, zealots are placed on nodes with degrees larger than 60. For the last three strategies, zealots are first assigned to nodes with the highest degrees in the considered interval, and then progressively associated to nodes with lower degrees only when higher-degree nodes were no longer available. The quality values and the initial conditions are the same of the ones used to obtain the results shown in Fig. 2. Each experiment is repeated 30 times, by using different random seeds and each simulation lasted for $T_{\max} = 400\,000$ time steps.

The results of these simulations are shown in Fig. 3, where we report the average fraction of agents holding opinion B at $t = T_{max}$ for each considered value of Z and assignment strategy. For $\alpha = 0.5$ (see Fig. 3a) and for $\alpha = 1.0$ (see Fig. 3b), we notice two clear trends : first, for any given positive number of zealots within the population, the higher their degree the higher the final proportion of susceptible agents terminating with opinion B. Second, given an assignment strategy, the higher the number of zealots, the higher the final proportion of susceptible agents terminating with opinion B. The increase in the proportion of susceptible agents committed to opinion B at the end of the simulations appears to be non-linearly correlated with the number of zealots and their location in the social network, i.e., their degree. Let us also emphasise that in the case zealots are placed on lower degree nodes (see Fig. 2) and the cognitive load is set to $1/\alpha = 1.0$, then more than 100 zealots are needed to drive the population toward the lower quality opinion, on the other hand if zealots are placed in higher-degree nodes (see Fig. 3b), then $Z = 16$ zealots are enough to obtain consensus toward B. A similar conclusion holds true also for $\alpha = 0.5$. Let us conclude by observing that in the case $\alpha = 1.5$ (see Fig. 3c) for any number of zealots, and regardless of their degree, the population always splits into two groups of (almost) equal sizes with different opinions.

The results shown in this section indicate that adding to the population zealots with the lower quality option can drive the system to a consensus toward the latter. More importantly, this outcome is generated by a smaller number of zealots if they have a high degree. The goal of the next section is to move a step forward and to provide support to this claim by using the Heterogeneous Mean-Field theory.

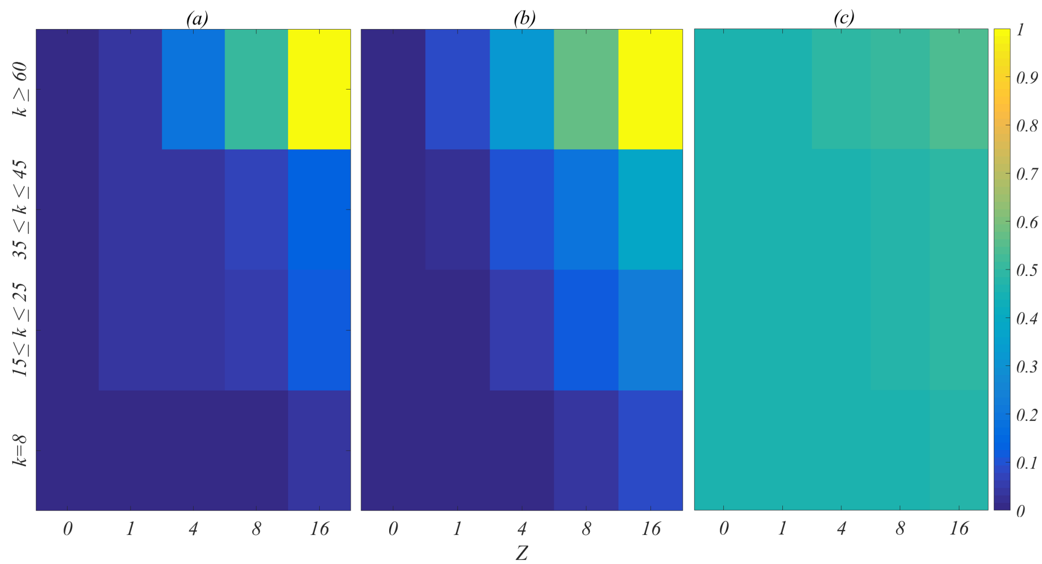

Figure 3: For several values of zealots Z and four assignment strategies of zealots to nodes with given degrees, we report the average final fraction of agents holding opinion B computed from 30 independent simulations in a population of 1000 agents interacting in a Barabási-Albert network with $m = 8$. Panel (a) corresponds to $\alpha = 0.5$, panel (b) to $\alpha = 1.0$, and panel (c) to $\alpha = 1.5$. The option qualities are $Q_A = 1$ for option A and $Q_B = 0.9$ for option B.

4. Mathematical model

We build a mathematical model defined by an ordinary differential equation (ODE) to look at the combined effects of the cognitive load, the ratio of the opinion qualities $Q = Q_B/Q_A$, the fraction of zealots, and the network structure γ, i.e., the exponent of the power law, on the evolution of opinion dynamics in the best-of-2 problem.

To make some analytical progress we rely on the Heterogeneous Mean Field (HMF) assumption [**? ? ?**], namely we hypothesise that nodes with the same degree are dynamically equivalent. Therefore, nodes are grouped into degree classes, more precisely we define A_k (resp. B_k), as the number of nodes with degree k and opinion A (resp. opinion B). To distinguish between susceptible agents with opinion B and zealots, we introduce Z_k to denote the number of zealots with opinion B and degree k, and S_k to count the number of susceptible agents with opinion B and degree k. Let N_k to denote the total number of nodes with degree k, then

$$A_k + Z_k + S_k = N_k \,. \tag{4}$$

By taking into account all the degree classes J, the relation Eq. (4) returns :

$$\sum_{k \in J} (A_k + Z_k + S_k) = N \,.$$

For all k, let us denote by $a_k = A_k/N_k$ the fraction of agents with opinion A and degree k, $b_k = B_k/N_k$ the fraction of susceptible agents with opinion B and degree k, and $\zeta_k = Z_k/N_k$ the fraction of zealots with opinion B and degree k. Thus, Eq. (4) can be rewritten as follows :

$$a_k + b_k + \zeta_k = 1 \,. \tag{5}$$

Let us study the temporal evolution of $a_k(t)$ for a given k ; because of Eq. (5), the function $b_k(t)$ will be determined by $b_k(t) = 1 - a_k(t) - \zeta_k$. The proportion of agents holding opinion A and degree k, increases (resp. decreases) when susceptible agents with opinion B and degree k switch to opinion A (resp. agents with A and degree k change, for opinion B) according to Eqs. (2) and (3). Our goal is to cast the latter two equations into the HMF scheme.

Consider thus a randomly selected focal agent i and assume she has opinion A and a degree k. Let us introduce the excess degree

$$q_k = \frac{(k+1)\, p_{k+1}}{\langle k \rangle} \quad \forall k \geq 0 \,,$$

with p_k the probability that a randomly chosen node has a degree k and $\langle k \rangle = \sum_k k p_k$ the average node degree, and where $\sum_k q_k = 1$. Then following [**?**], let q_{j_1} to represent the probability that an agent labelled i_1, who is connected to the focal agent i, has an excess degree $j_1 \geq 0$. We assume a_{j_1+1} to be the probability that agent i_1 holds opinion A, and $b_{j_1+1} + \zeta_{j_1+1} = 1 - a_{j_1+1}$ is the probability that she holds opinion B. For the k agents connected to the focal agent i, let $q_{j_1}, \ldots, q_{j_k}$ denote the joint probability that each agent, connected via any of the k edges, has excess degrees $j_1, \ldots, j_k$. We hypothesise that $\pi_{k,\ell}$ represents the probability that, among the k connected agents, ℓ agents hold opinion B, while the remaining $k-\ell$ agents hold opinion A. Based on Eq. (1), the weighted proportion of agents with opinion A or opinion B can expressed as :

$$n^{\#}_{i,A} = \frac{k-\ell}{k-\ell+Q\ell} \quad n^{\#}_{i,B} = \frac{Q\ell}{k-\ell+Q\ell} \,.$$

Therefore, the agent i with opinion B can switch to opinion A with probability

$$p_{B\to A} \sim q_{j_1}\dots q_{j_k}\pi_{k,\ell}P_\alpha\left(\frac{k-\ell}{k-\ell+Q\ell}\right),$$

similarly an agent with opinion A will move to B with probability

$$p_{A\to B} \sim q_{j_1}\dots q_{j_k}\pi_{k,\ell}P_\alpha\left(\frac{Q\ell}{k-\ell+Q\ell}\right) = 1-p_{B\to A}.$$

Thus, for any degree k, the variation in the fraction of agents holding opinion A and degree k can be expressed by

$$\begin{aligned}\frac{da_k}{dt} &= (1-a_k-\zeta_k)\sum_{j_1,\dots,j_k} q_{j_1}\dots q_{j_k}\sum_{\ell=0}^{k}\pi_{k,\ell}P_\alpha\left(\frac{k-\ell}{k-\ell+\ell Q}\right)\\ &\quad -a_k\sum_{j_1,\dots,j_k} q_{j_1}\dots q_{j_k}\sum_{\ell=0}^{k}\pi_{k,\ell}\left(1-P_\alpha\left(\frac{k-\ell}{k-\ell+\ell Q}\right)\right). \qquad (6)\end{aligned}$$

This equation captures the dynamics of opinion changes by taking into account the cognitive load, α, the heterogeneity of the population, ζ_k, the quality ratio, Q and the connectivity structure, q_k. More precisely the term $(1-a_k-\zeta_k)$ represents the probability that the focal agent has degree k and holds opinion B, the remaining sum denotes the probability she can switch to opinion A because of the social interaction composed by several contributions. The sum over the indexes $j_1,\dots,j_k$ accounts for all possible combinations of the excess degrees. Given a combination of $j_1,\dots,j_k$, the term $\pi_{k,\ell}$ denotes the probability that ℓ nodes among the k have opinion B, and thus $k-\ell$ have opinion A. The sum $\sum_{\ell=0}^{k}$ encompasses all possibilities, ranging from $\ell=0$ (where all agents have opinion A) to $\ell=k$ (where all agents have opinion B). The second term in the formula express the probability that the focal agent has degree k, holds opinion A and she can change her mind according to the probability captured by the remaining sum.

After some straightforward computations, Eq. (6) can be rewritten in the form

$$\frac{da_k}{dt} = -a_k + (1-\zeta_k)\sum_{j_1,\dots,j_k} q_{j_1}\dots q_{j_k}\sum_{\ell=0}^{k}\pi_{k,\ell}P_\alpha\left(\frac{k-\ell}{k-\ell+\ell Q}\right). \qquad (7)$$

To proceed with the analytical study, let us define the quantity $\langle a\rangle := \sum_k q_k a_{k+1}$. Let us observe that the latter is a sort of weighted average of fraction of nodes with degree k and opinion A; a similar quantity can be used in epidemics model once dealing with the HMF analysis [?]. We can obtain the time evolution of $\langle a\rangle$ by leveraging the independence of probabilities and applying some combinatory, we can show (we refer the interested reader to [?] for more details) :

$$\sum_{j_1,\dots,j_k} q_{j_1}\dots q_{j_k}\pi_{k,\ell} = \binom{k}{\ell}\langle a\rangle^{k-\ell}(1-\langle a\rangle)^{\ell} .$$

Based on this claim, Eq.(7) can be rewritten as

$$\frac{da_k}{dt} = -a_k + (1-\zeta_k)\sum_{\ell=0}^{k-1}\binom{k}{\ell}\langle a\rangle^{k-\ell}(1-\langle a\rangle)^{\ell} P_\alpha\left(\frac{k-\ell}{k-\ell+\ell Q}\right) . \quad (8)$$

By renaming the index $k\to k+1$, multiplying both sides by q_k and then summing over k we obtain

$$\begin{aligned}\frac{d\langle a\rangle}{dt} &= -\langle a\rangle + \sum_k q_k(1-\zeta_{k+1})\sum_{\ell=0}^{k+1}\binom{k+1}{\ell}\langle a\rangle^{k+1-\ell}\times \\ &\quad (1-\langle a\rangle)^{\ell} P_\alpha\left(\frac{k+1-\ell}{k+1-\ell+\ell Q}\right) . \end{aligned} \quad (9)$$

The equilibria of the HMF equation and their stability determine the system fate. The latter are obtained by setting the right-hand side of Eq. (9) equal to zero. Let us thus define the function $f_\alpha(a)$

$$\begin{aligned}f_\alpha(a) :&= -\langle a\rangle + \sum_k q_k(1-\zeta_{k+1})\sum_{\ell=0}^{k+1}\binom{k+1}{\ell}\langle a\rangle^{k+1-\ell}\times \\ &\quad (1-\langle a\rangle)^{\ell} P_\alpha\left(\frac{k+1-\ell}{k+1-\ell+\ell Q}\right) , \end{aligned} \quad (10)$$

hence by denoting $\langle a^*\rangle$ a system equilibrium, we have by definition

$$f_\alpha(\langle a^*\rangle) = 0 .$$

A direct inspection of Eq. (10) allows to demonstrate that $f_\alpha(0)=0$, hence $\langle a^*\rangle = 0$, i.e., the absence of agents holding opinion A represents an equilibrium of the system. Conversely, $f_\alpha(1) = -\sum_k q_k\zeta_{k+1} \neq 0$, signifying that the presence of zealots (with opinion B) prevents the system from converging to a situation where all susceptible agents hold opinion A. Finally, the existence of a nontrivial solution $0 < \langle a^* \rangle < 1$ to the equation $f(\langle a^* \rangle) = 0$ indicates an equilibrium for the coexistence of opinions A and B in the network.

The stability of those equilibria can be determined by evaluating the derivative of the function f_α at those points. This analysis will be detailed in the following section, where we will also explore the effects of the key model parameters.

5. Results of the HMF model

In this section, we present the results obtained for the analytical model described in the previous section. As previously noted, our focus lies on the influence of the cognitive load as measured by $1/\alpha$, the network structure encapsulated into the exponent γ of the power law, and the ratio of zealots present in the population and their degree centrality, more precisely if they sit onto hubs or leaves nodes. To place zealots in hubs, we set $\zeta_k = 1$ for all $k \geq k_M$, for some sufficiently large $k_M > 0$; this accounts to add into the model an average number of zealots equal to $Z_{tot} = \sum_{k \geq k_M} N_k \sim \sum_{k \geq k_M} N c_\gamma / k^\gamma$, where c_γ is a normalisation constant such that $\sum_k p_k = 1$ and N is the total number of nodes in the network. In the scenario where zealots are assumed to be on leaf nodes, for a fair comparison with the prior condition, we maintain the same number of zealots as positioned in the hubs. This is achieved by assuming $\zeta_{k_{min}} = Z_{tot}/N_{k_{min}}$, where $k_{min} > 0$ represents a sufficiently small degree. Specifically :

$$\zeta_{k_{min}} = \frac{Z_{tot}}{N_{k_{min}}} \sim \frac{Z_{tot}}{N p_{k_{min}}} = k_{min}^\gamma \sum_{k \geq k_M} \frac{1}{k^\gamma} \sim \frac{k_M}{\gamma - 1} \left(\frac{k_{min}}{k_M} \right)^\gamma .$$

Let us observe that the above strategy implies that ζ_k can be interpreted as the probability to find a zealot into the class of nodes with degree k; the final number of added zealots will be always finite, indeed in any network realisation, e.g., by using the configuration model, there is a finite number of nodes with degree larger than k_M and thus Z_{tot} is also a finite quantity.

Fig. 4 summarises our main results. The quality of options A and B are set as in the previous section, i.e., $Q_A = 1$ and $Q_B = 0.9$, resulting in $Q = Q_B/Q_A = 0.9$. We then proceed to vary the power law exponent γ

and the location of zealots in the network. Subsequently, we (numerically) determine the zeros of the function f_α for varying values of α within the range of [0,2], by enabling us to derive the system equilibria. Once the latter have been found, we evaluate the derivative of f_α and we determine its sign, if $f_\alpha(\langle a^* \rangle) > 0$ then the equilibrium $\langle a^* \rangle$ is unstable and marked with red points in Fig. 4. On the other hand, if $f_\alpha(\langle a^* \rangle) < 0$ then the equilibrium $\langle a^* \rangle$ is stable and we represent it in green. The two top panels (see Fig. 4a and 4b) refer to the strategy consisting of setting the zealots in the leaves (here $k_{min} = 8$), and the two bottom panels (see Fig. 4c and 4d) refer to the opposite strategy with the zealots in the hubs, $k_M \geq 60$; this strategy is thus the same of the one used for the ABM in Section 3. Moving from left to right we increase γ, passing from $\gamma = 2.5$ (Fig. 4a and 4c), to $\gamma = 3.5$ (Fig. 4b and 4d).

Several conclusions can be drawn from those results. For large enough values of α, the system always sets into a state where opinions A and B coexist, leading to a state of decision deadlock where the population is unable to choose one opinion over the other. The larger is α the closer is the stable equilibrium to 0.5; this behaviour is independent of the strategy used to place the zealots or the network structure determined by the exponent γ. Hence, a small cognitive load prevents the agents from reaching a consensus for their alternative regardless of the position of zealots.

For intermediate values of the cognitive load, e.g., $1/\alpha \sim 1$, placing the zealots into the low-degree nodes allows the system to reach a consensus for the best option (see Fig. 4a and 4b) irrespective from the value of γ, indeed there is a stable equilibrium $\langle a^* \rangle$ very close to 1. This behaviour completely changes once zealots are set into hubs nodes (see Fig. 4c and 4d), indeed for small γ, i.e., a network with a pronounced degree heterogeneity, there exists an interval of $\alpha \in [0.41, 1.10]$ (see Fig. 4c) values for which the unique stable equilibrium is $\langle a^* \rangle = 0$, the HMF predicts thus the system to converge to the opinion with lower quality, B. By increasing γ, i.e., dealing with a more homogeneous network, we recover once again convergence to a large fraction of agents committed to opinion A.

Finally, for very low values of α, i.e., $\alpha \sim 0$, placing zealots in low-degree nodes allows the system to reach a consensus for the best option. Therefore, a stable equilibrium $\langle a^* \rangle \sim 1$ is achieved for any value of γ (see Fig. 4a and 4b). However, when the zealots are placed on hubs in a network with much greater heterogeneity, the stable equilibrium representing a majority of agents with opinion A reduces to values lower than 1. Thus, while most agents adopt

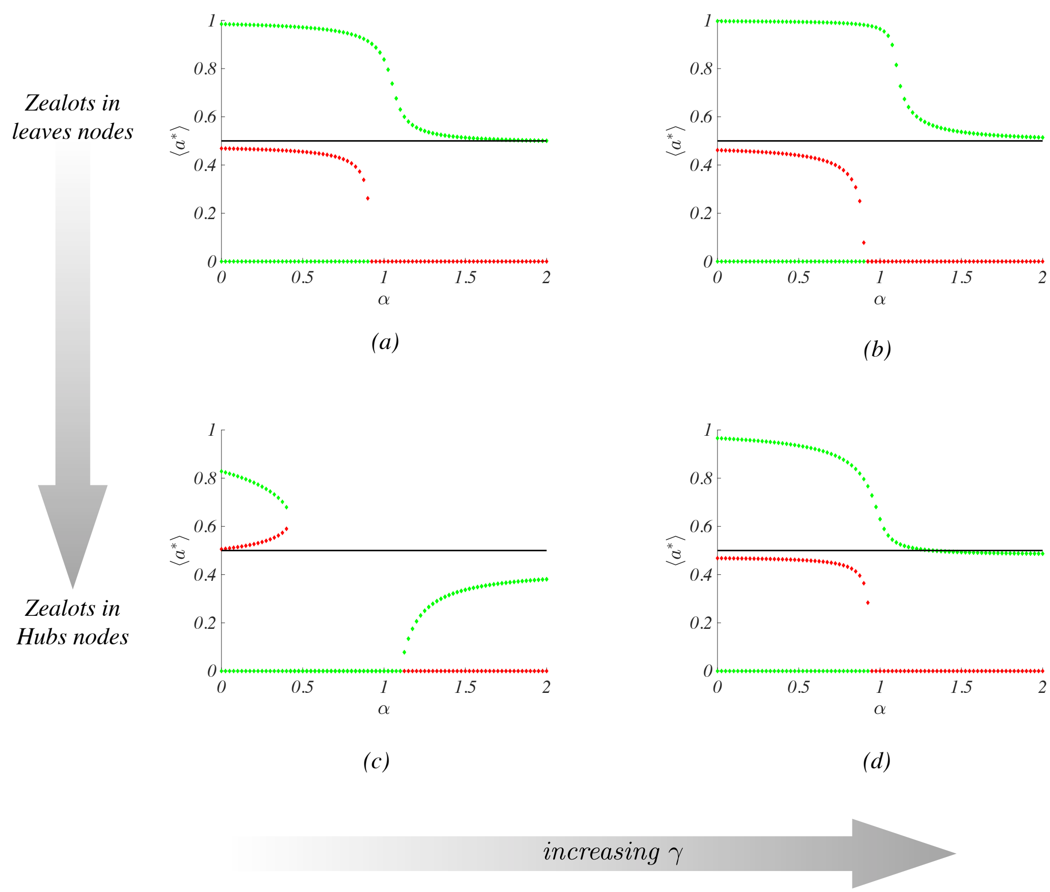

FIGURE 4: Bifurcation diagrams of the HMF. We report the equilibria $\langle a^* \rangle$ of Eq. (9) as a function of the parameter α for $Q = 0.9$. Stable equilibria, i.e., associated to $f'_\alpha(\langle a^* \rangle) < 0$, are coloured in green while unstable ones, i.e., associated to $f'_\alpha(\langle a^* \rangle) > 0$, are coloured in red. Top panels (a) and (b), correspond to zealots set into leaves nodes, i.e., $k_i = 8$, while bottom panels (c) and (d) correspond to zealots set into the large degree nodes, more precisely into nodes with $k_i \geq 60$. The underlying support is a scale–free network, with $\gamma = 2.5$ (panels (a) and (c)) and $\gamma = 3.5$ (panels (b) and (d)).

the higher-quality opinion, there is also a significant proportion of agents with opinion B. This is shown in Fig. 4c by the stable equilibrium point $0.5 < \langle a^* \rangle < 1$. Furthermore, when the network structure becomes more homogeneous (e.g., $\gamma = 3.5$), the presence of zealots does not prevent the system from reaching a consensus towards the higher-quality option, thereby reaching the stable equilibrium point $\langle a^* \rangle = 1$ (see Fig. 4d).

In Fig. 5 we compare the results obtained by numerically simulate the agent-based model and those following form the analytical model shown in Fig. 4a and 4c in a scale-free network with $\gamma = 3$. Two scenarios are considered. In the first case, a large number of zealots, $Z = 160$, are placed on low-degree nodes, $k_{min} = 8$, in Fig. 5a we report the average final fraction of agents holding opinions A as a function of α for the HMF (red and green diamonds) and for the ABM (blue dots). In the insets we show the distribution of the final fraction of agents holding opinions A for $\alpha = 0$ (see Fig. 5b) and $\alpha = 1.5$ (see Fig. 5c) arising from 100 independent simulations. In the second case, few zealots ($Z = 22$) are set into nodes with degrees $k_i \geq 60$ (see Fig. 5d), blue dots correspond to the ABM while red/green to the HMF ; the insets show again the final distribution for two values of the cognitive load, $\alpha = 0$ (Fig. 5e) and $\alpha = 1.5$ (Fig. 5f), resulting from for 100 independent simulations. In both cases red diamonds denote an unstable equilibrium while green diamonds a stable one ; one can appreciate a good agreement between the theory and the simulations, the few disagreement points in Fig. 5a are due to the fact that the system has been initialised with $n_A(0) = 0.6N$, namely "below" the unstable branch of the HMF and for this reason the ABM converged to $n_A/N = 0$. If we would have used a larger initial fraction of agents committed to option A, then the ABM would have converged to the (almost) all-A configuration. The main conclusion one can draw from observing these results is that once zealots are placed onto large degree nodes, then there is a large interval of α values for which the population find a consensus toward the option with the lower quality, B in this case.

To achieve a broader understanding of the complex relationship between the parameters, we studied the equilibrium $\langle a^* \rangle$ as a function of the cognitive load and the exponent γ, while maintaining constant the ratio of the opinion qualities $Q = 0.9$ (we report the results in Fig. 6). Furthermore, for each examined scenario, we assessed the influence of the strategy involving the placement of zealots on leaf nodes (see Fig. 6a and 6b) or on hubs nodes (see Fig. 6c and 6d). In Fig. 6a and 6c we represented by a colour code (yellow high values of $\langle a^* \rangle$ close to 1 and blue $\langle a^* \rangle \sim 0$) the equilibrium reached by the system starting from an initial population with half agents holding opinion A and half opinion B (note that half of agents holding opinion A are all susceptible agents while the other half, holding opinion B, also includes the zealots). One can observe a striking difference between Fig. 6a

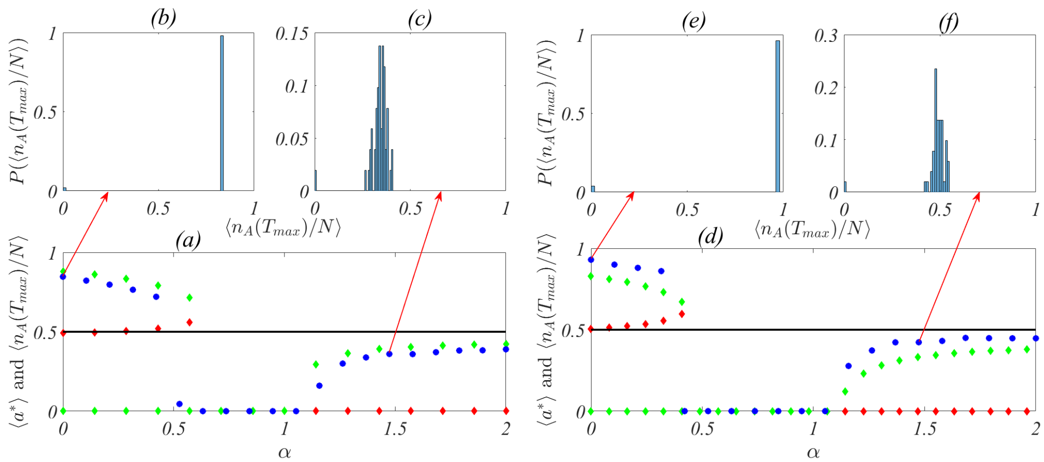

FIGURE 5: Comparison of the analytical results obtained by using the HMF and the numerical ones resulting from stochastic numerical simulation. In the main panels (a) and (d), we report the final fraction of agents with opinion A from the ABM simulations (blue dots) and the HMF equilibrium $\langle a^* \rangle$ (red and green diamonds) as a function of α, once the underlying support is a scale-free network with $\gamma = 3$ and $N = 1000$ nodes. In the left panels $Z = 160$ zealots have been set onto leaf nodes $k_i = 8$ while in the right panels a small number of zealots ($Z = 22$) are set onto nodes with large degree, $k_i \geq 60$. The insets display the distribution of the final fraction of agents committed to opinion A, for $\alpha = 0$ (panels (b) and (e)), and $\alpha = 1.5$ (panels (c) and (f)), for 100 independent simulations. The option qualities have been fixed to $(Q_A, Q_B) = (1, 0.9)$.

corresponding to zealots placed into leaves nodes, here $k_{min} = 8$, with respect to the Fig. 6c, where the zealots have been set into hubs nodes, here $k_M \geq 60$. In the former case, the equilibrium $\langle a^* \rangle$ is almost independent from γ and the system exhibits two main behaviours : for $\alpha \lesssim 1$ the whole group converges to a consensus to A, while for $\alpha \gtrsim 1$ the population faces a deadlock where agents with opinion A and B coexist. On the other hand, once zealots are placed into hubs nodes a third type of dynamics can manifest (see Fig. 6c) : the population can converge toward a consensus for the opinion with the lower quality. As shown in Fig. 6c, this happens for $\alpha \sim 1$, $Q = 0.9$, and for $\gamma \lesssim \gamma_* = 3.31$ (the value of γ^* has been computed numerically). To better emphasise this behaviour, we report in Fig. 6b and 6d the position and the stability of the equilibrium $\langle a^* \rangle$ as a function of the cognitive load for the value $\gamma = 2.2 < \gamma_* \sim 3.31$. Fig. 6b corresponds to the case where zealots are set into the leaves nodes and the population converges to a (almost) consensus to A for $\alpha \lesssim 1.0$, while for larger values of α the population faces a decision deadlock. On the other hand, once zealots are set into hubs (Fig. 6d), there exists an interval of values of α for which the group chooses the opinion with the lower quality. Those results support the claim that a population of agents adopting a voter model strategy for social exchange can be driven to adopt the opinion with the lower quality, by zealots placed into hubs of a sufficiently

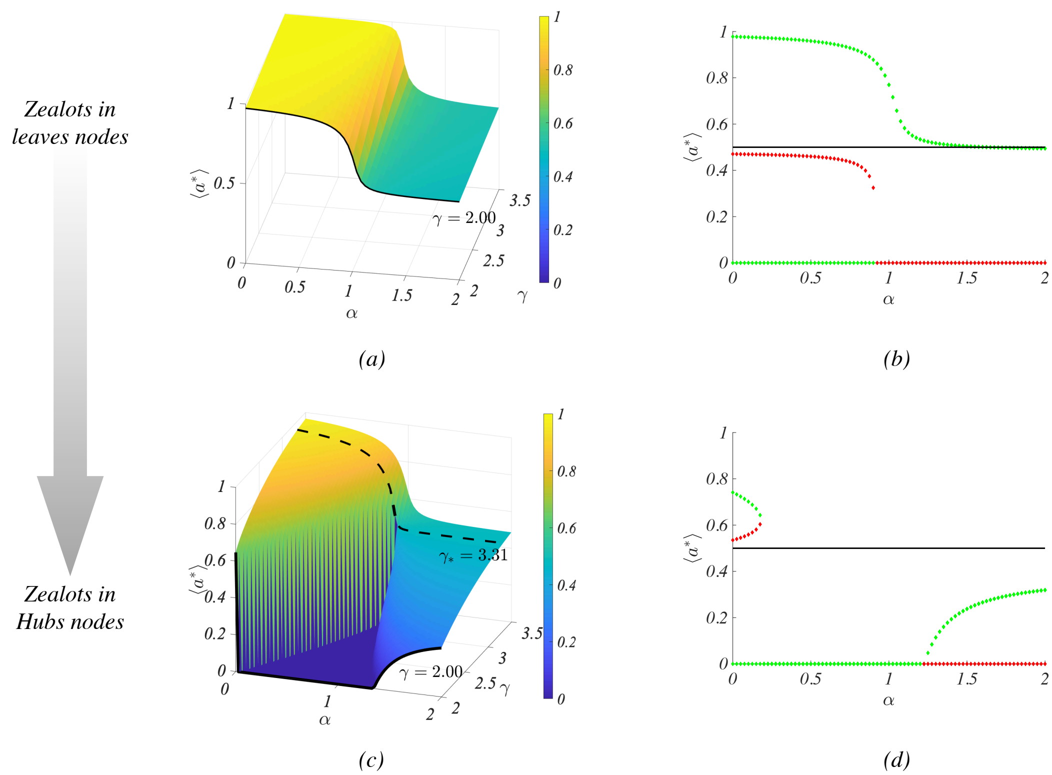

FIGURE 6: Bifurcation diagrams of the HMF. We report the equilibrium $\langle a^* \rangle$ given by Eq. (9) as a function of (α,γ) for a fixed value of $Q = 0.9$. Top panels (a) and (b) correspond to the strategy of placing zealots into leaves nodes, i.e., $k_i = 8$, while in the bottom panels (c) and (d), zealots are assigned to large degree nodes, more precisely to nodes such that $k_i \geq 60$. In panel (c) we can observe the existence of a critical value γ_* of the power-low exponent γ, below which the equilibrium $\langle a^* \rangle = 0$ is the unique stable one, for a given range of α. Panels (b) and (d) show the equilibria for a fixed value of $\gamma = 2.2 < \gamma_* \sim 3.31$.

Fig. 7 shows that similar dynamics can be observed by considering equilibrium $\langle a^* \rangle$ as a function of (α,Q) for a fixed value of γ. The results presented in Fig. 7b show that in the case of zealots located into hubs, for Q close enough to 1 (i.e., opinions with very similar qualities) and $\alpha \sim 1$, the population converges to a majority for the option with the lower quality ; by decreasing α, the system undergoes an abrupt bifurcation passing to a population with a large majority of agents holding an opinion in favour of the best quality. This behaviour cannot be observed if zealots are placed into the

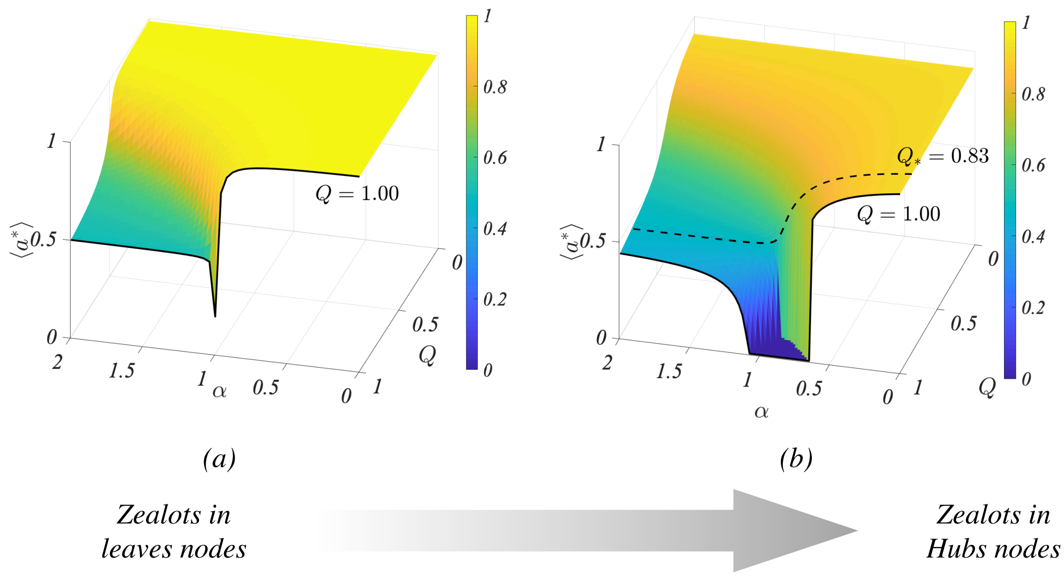

FIGURE 7: Bifurcation diagrams of the HMF. We report the equilibrium $\langle a^* \rangle$ given by Eq. (9) as a function of (α,Q) for a fixed value of $\gamma = 3.0$. Left panel (a) corresponds to zealots set into leaves nodes, i.e., $k_i = 8$, while the panel (b) to the strategy of placing the zealots into the hubs, i.e., nodes with $k_i \geq 60$. Let us observe the presence in panel (b) of a critical value of $Q = Q_* \sim 0.83$, below which the equilibrium $\langle a^* \rangle = 0$ is the unique stable one.

Our analytical framework confirmed the numerical results shown in Sec. 3 and generalises the analysis to any scale-free network, under the assumption of heterogeneous mean field.

6. How much quality is needed to compensate for the influence of a zealot ?

In the previous sections, we have considered the quality of the options as *a priori* measure of the value of one choice over the other, or the strength with which an option is transmitted. In the case of humans the latter role can be played by, e.g., mass media penetration. In the same framework, zealots, can thus be considered as individuals (or bots) repeating constantly one option with the goal of influencing the population.

It can thus be interesting to consider the following problem : can a change in option quality "compensate" the impact of the presence of zealots in the decision outcome ? Assume thus option A to be better than option B, i.e., $Q_A > Q_B$, then we can find values of $\alpha \geq 1$ such that in absence of zealots the system converges to an equilibrium $a^*_{\text{ini}} \lesssim 1$, namely the group almost achieve a total consensus for the option with the highest quality [**?**]. Then a fraction of zealots with opinion B, $0 < \zeta < 1$, is added to the population, and as shown above, this induces a shift in the system equilibrium : the asymptotic fraction of agents with opinion A, a^*_{tmp}, is now smaller than without zealots, $a^*_{\text{tmp}} < a^*_{\text{ini}} \lesssim 1$. The question we are interested in is thus : would it be possible to increase the quality of opinion A, $Q'_A > Q_A$, to compensate for the presence of zealots and restore a collective agreement to a^*_{ini} ?

To answer this question we decided to simplify the model by replacing the social network with an all-to-all coupling. This is not restrictive and allows us to focus on the main point ; a similar conclusion can be drawn also for a generic social network. In this setting we have thus only three variables, $a(t)$ the fraction of agents with opinion A, $b(t)$ the fraction of susceptible agents with opinion B and ζ, the constant fraction of zealots committed to opinion B, observe that at any time t, the relation $a(t) + b(t) + \zeta = 1$ holds true.

In the absence of zealots, we have previously shown [**?**] that the evolution of the fraction of agents holding opinion A can be described by the following ODE in the context of a complete network.

$$\frac{da}{dt} = -a + P_\alpha\left(\frac{a}{a(1-Q)+Q}\right) . \tag{11}$$

Let us denote by $a^*_{\text{ini}} < 1$ the stationary solution of Eq. (11), namely it satisfies the condition :

$$P_\alpha\left(\frac{a^*_{\text{ini}}}{a^*_{\text{ini}}(1-Q)+Q}\right) = a^*_{\text{ini}} .$$

By considering the presence of zealots, $0 < \zeta < 1$, we obtain a similar equation ruling the evolution of $a(t)$

$$\frac{da}{dt} = -a + (1-\zeta) P_\alpha \left(\frac{a}{a(1-Q)+Q} \right) . \tag{12}$$

The inclusion of zealots determines a new equilibrium, $a^*_{\text{tmp}} < a^*_{\text{ini}}$ solution of

$$(1-\zeta) P_\alpha \left(\frac{a^*_{\text{tmp}}}{a^*_{\text{tmp}}(1-Q)+Q} \right) = a^*_{\text{tmp}} .$$

Let now assume to increase the quality of option A, Q'_A, and to look for a stationary solution whose value equals again a^*_{ini} but allowing for the same fraction ζ of zealots. This means to be able to solve

$$(1-\zeta) P_\alpha \left(\frac{a^*_{\text{ini}}}{a^*_{\text{ini}}(1-Q')+Q'} \right) = a^*_{\text{ini}} . \tag{13}$$

The latter can be solved with respect to Q' to obtain the following explicit expression

$$\begin{cases} Q' = \left[\frac{2}{\left(\frac{2a^*_{\text{ini}}}{1-\zeta} - 1 \right)^{\frac{1}{\alpha}} + 1} - 1 \right] \dfrac{a^*_{\text{ini}}}{1-a^*_{\text{ini}}} & \text{if } \dfrac{a^*_{\text{ini}}}{a^*_{\text{ini}}(1-Q)+Q} > \frac{1}{2} \\ Q' = \left[\frac{2}{1 - \left(1 - \frac{2a^*_{\text{ini}}}{1-\zeta} \right)^{\frac{1}{\alpha}}} - 1 \right] \dfrac{a^*_{\text{ini}}}{1-a^*_{\text{ini}}} & \text{if } \dfrac{a^*_{\text{ini}}}{a^*_{\text{ini}}(1-Q)+Q} < \frac{1}{2} . \end{cases} \tag{14}$$

Eq. (14) provides insight into the relationship between the new quality $Q'_A = Q_B/Q'$ and the fraction of zealots ζ required to drive the system to the same equilibrium point a^*_{ini} that could be achieved in the absence of zealots and for the same cognitive load α. First of all, let us stress the non linear dependence of Q'_A with respect to ζ, starting from $Q'_A = Q_A$ if $\zeta = 0$, the function Q'_A rapidly increases and diverges for $\hat{\zeta} = 1 - a^*_{ini}$ (see Fig.8). Stated differently, the larger the fraction of zealots the larger should be the new quality of the option A to be able to compensate the presence of zealots.

In Fig. 9 we report the numerical results corresponding to the case $\alpha = 1.16$ for a generic set of parameters. The initial configuration is associated to $Q_A = 1$, $Q_B = 0.9$ and no zealots, $\zeta = 0$, resulting into the equilibrium $a^*_{\text{ini}} \sim 0.602$ and one can observe that the numerical simulation (blue line)

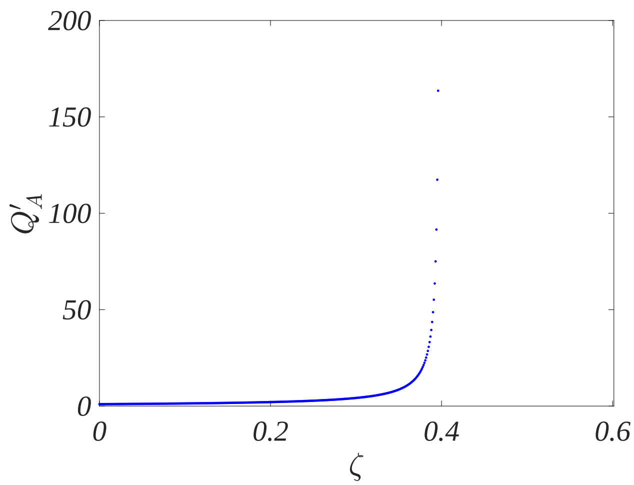

FIGURE 8: The nonlinear dependence of Q'_A as a function of the variable ζ.

oscillates about such equilibrium (black horizontal line) because of the finite size effect induced by the discrete population. We then add a fraction $\zeta = 0.2$ of zealots committed to B and the system now converges (red line) to the new equilibrium $a^*_{\text{tmp}} \sim 0.102$ (black dashed horizontal line). Finally, by using Eq. (14), we compute the value of $Q'_A \sim 2.076$ which represent the new quality of option A, or simply the frequency with which opinions for A are spread. By increasing option A's quality to Q'_A, the system returns to the original equilibrium a^*_{ini} (green line).

One can prove [**?**] that if $Q < 1$ and $\alpha < 1$, in absence of zealots the system, intialized with half of agents committed to A and half to B, will converge to a full consensus to A, namely $a^*_{\text{ini}} = 1$. In the previous section we have shown that in presence of zealots, the system cannot converge to a full A consensus, hence we can never find a new value for the quality Q_A able to return the equilibrium $a^*_{\text{ini}} = 1$ once zealots are present. To avoid this issue, we presented the above example by assuming $\alpha \geq 1$, values for which an equilibrium $a^*_{\text{ini}} < 1$ can be achieved.

7. Conclusions

We have presented the results of a study focused on a best-of-n collective decision-making problem, with $n = 2$ options of different quality. We analysed this problem through agent-based simulations and a mathematical model based on the heterogeneous mean-field approach. The interactions among

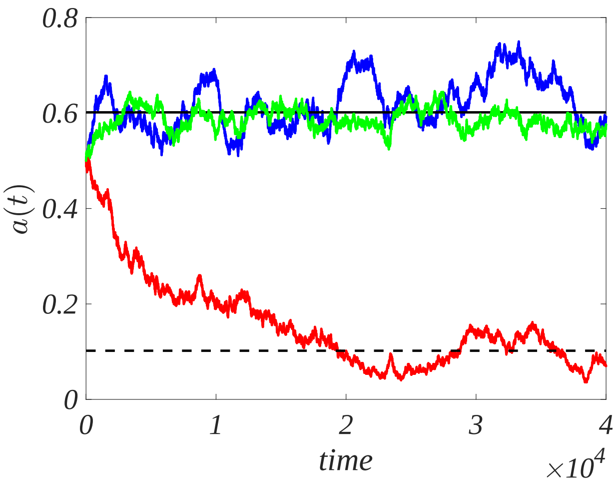

FIGURE 9: Numerical study of the relevance of quality versus zealots. We fixed $\alpha = 1.16$ and $Q_B = 0.9$, the blue line correspond to $Q_A = 1$ and $\zeta = 0$, the red line to $Q_A = 1$ and $\zeta = 0.2$ and the green line to $Q'_A \sim 2.0756$ and $\zeta = 0.2$.

the individuals are defined using a social network whose nodes are agents, or decision-markers, and edges the possible interactions among them. The original contribution of this paper is in highlighting the mutual effects existing between the following parameters of the model : i) the lower or higher heterogeneity of the scale-free network modelling the interactions between the agents ; ii) the number of agents within the population that never change opinion (i.e., the zealots) ; iii) the agents cognitive load, namely the cost required to gather the group information ; iv) the ratio in quality between the two options corresponding to the combination of the cost and benefit to each option, and v) the degree centrality (or the number of social connections) of the zealots. We have studied populations in which individuals are connected according to a scale-free network, and they select their opinion using decision mechanisms that differ in terms of their cognitive load, allowing to interpolate among existing models such as the voter model and the majority model. For each case, we have varied the number (or, in the mean-field model, the fraction) of zealots, all committed to the lowest quality option. The mathematical analysis of the combined effect of the considered parameters has been performed by determining the system equilibria and their stability.

The results have shown that the combined effect of these parameters generates an articulated landscape characterised by different outcomes of the collective decision-making process. We have shown that, when susceptible agents employ opinion selection mechanisms characterised by cognitive load $1/\alpha \sim 1$ (representing the voter model), both the number and the degree centrality of the zealots are elements that can induce the population to converge to the lowest quality option. In particular, the higher the number of zealots or the larger their degree centrality, the stronger their influence on the opinion dynamics. We have also shown that these effects are influenced by the nature of the opinion selection mechanisms employed by the susceptible agents. For example, the effect of the number of zealots on the opinion dynamics is not observed when susceptible agents have a large cognitive load, i.e., $1/\alpha \gg 1$ (representing the majority model). We have also shown that the connectivity structure modulates the influence of the zealots, indeed when the network becomes increasingly sparse and less heterogeneous, the effect of zealots is mitigated or largely reduced. In the future, we aim to extend the proposed analytical model to best-of-n decision-making by incorporating the concept of group interactions, also known as higher-order interactions [**?**], among decision-makers and explore how these higher-order interactions can influence the dynamics of the option selection. It could also be interesting to study heterogeneous values for the qualities across the population, to mimic noise in the personal appreciation of one option over the other.

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Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.