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Exploring weak measurements within the Einstein-Dirac cosmological framework

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Our study applies the two-state formalism alongside weak measurements within a spatially homogeneous and isotropic cosmological framework, wherein Dirac spinors are intricately coupled to classical gravity. We compute the weak values of the energy-momentum tensors, the Z component of spin, and the pure states. Weak measurements are a generalization and extension of the computation already made by Finster and Hainzl [A spatially homogeneous and isotropic Einstein–Dirac cosmology, J. Math. Phys. (N.Y.) **52**, 042501 (2011)] in a spatially homogeneous and isotropic Einstein-Dirac cosmology. Our analysis reveals that the acceleration of the Universe expansion can be understood as an outcome of postselection, underscoring the effectiveness of weak measurement as a discerning approach for gauging cosmic acceleration.

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I. INTRODUCTION

Measuring involves assigning numerical values to the attributes or characteristics of a phenomenon. The impact of the observer on measurement results is a complex consideration in scientific research. In quantum mechanics, the alteration of a quantum state after its measurement is well known, but its interpretation is not obvious. For example, the famous double-slit experiment demonstrates that entities like electrons can behave as particles and waves and measurement prescribes their behavior [1,2].

Weak measurements offer a procedure for measuring the system state through a weak coupling between the measurement apparatus and the system state, leaving the measured system almost undisturbed [3]. The paid cost is that the pointer variable is not sharp but instead has a broad spread. Weak measurements have enabled, for instance, the characterization of a particle's average trajectory in the double-slit experiment without disrupting the interference pattern [2,4–6]. Unlike classical or ideal measurements, which result in the stochastic collapse of the system state into one of the eigenstates of the measured observable [7], weak measurements do not induce a collapse of the state vector. Instead, they introduce a small bias to the state vector, and the measurement device exhibits a superposition of multiple values rather than a clear eigenvalue [4]. Consequently, weak measurements unveil unconventional weak values, including complex numbers.

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In the literature, a few authors have explored weak measurements and the *two-state vector formalism* within the theoretical framework of cosmology and ontology. George Ellis and Rotman [8] proposed a paradigm that challenges conventional notions of time and reality, termed the *crystallizing block Universe* (CBU), an extension of the *emergent block Universe*. According to their perspective, past, present, and future coalesce, amalgamating spacetime into a singular entity. The future is conceptualized as a superposition of a myriad of possibilities, while the past remains immutable. The transition between these temporal states predominantly unfolds in the present, albeit in a nonuniform manner. Within the CBU framework, certain discrete patches of quantum indeterminacy endure and are resolved later on. Furthermore, CBU models integrate the *two-time interpretation* [3] of quantum mechanics. Nevertheless, this theoretical and ontological paradigm lies beyond the purview of our current study.

Davies [9] was the first to propose applying the weak measurements theory, combined with pre- and postselection, to quantum cosmology and to explore the potential large-scale cosmological effects arising from this new sector of quantum mechanics. He illustrated the theory with a two-dimensional spacetime toy model of a scalar field with mass m propagating in an expanding Universe with the scale factor $a(t)$ and metric $ds^2 = dt^2 - a(t)dx^2$. He resolved a debate regarding the coupling of pre- and postselected quantum fields to gravity and proposed an experimental test for weak values in cosmology. He observed that, in the literature, the equation $G_{\mu\nu}$ + higher order terms in curvature $= \frac{\langle 0_{\rm fin}|T_{\mu\nu}|0_{\rm in}\rangle}{\langle 0_{\rm fin}|0_{\rm in}\rangle}$, originally derived by DeWitt, who adapted the Schwinger effective action theory of quantum electrodynamics to the gravitational case, the source term is nothing else than the weak value of the stress-energy-momentum tensor $T_{\mu\nu}$, [10]. This equation was widespread in 1970 [11–13].

After Davies's paper [9], no research papers have been pursued in the direction of the theory of weak measurements in cosmology until now. The subject seems to arouse interest. Charis Anastopoulos has just published on arXiv a paper on *Final States in Quantum Cosmology: Cosmic Acceleration as a Quantum Post-selection Effect* [14], wherein he argues that there is no compelling physical reason to preclude a probability assignment with a final quantum state at the cosmological level and analyzes its implications in quantum cosmology. One significant result is that cosmic acceleration emerges as a quantum post-selection effect.

We aim to examine weak measurements of Dirac particles within the framework of time symmetry as applied to the Einstein-Dirac system in a homogeneous and isotropic space such as the Friedman-Lemaître-Robertson-Walker (FLRW) space. To our knowledge, no prior research has explored this direction. Drawing upon insights from the paper of Finster and Hainzl, titled *A Spatially Homogeneous and Isotropic Einstein-Dirac Cosmology* [15], we endeavor to extend specific findings to the domain of weak measurements. Among the outcomes of our investigation are the computations of weak values, including those about energy-momentum, pure states, and the Z component of spin. Additionally, we demonstrate that cosmic acceleration in the Universe may be regarded as a consequence of postselection. This corroborates earlier studies that have explored the potential of spinor fields to elucidate phenomena such as the inflationary period in the early Universe and, subsequently, the concept of dark energy. Notably, Anastopoulos arrived at a similar conclusion without invoking a spinor field or dark energy, thus emphasizing alternative avenues for understanding the observed cosmological acceleration. Ribas reached the same conclusion without resorting to weak measurements or postselection techniques. References to relevant literature supporting these assertions include [16–21].

In the time-symmetric formulation of quantum mechanics [22], a quantum system is characterized by two state vectors at a specific time instead of a single one. The preselected state is determined by measurements conducted on the system prior to t_0, while the postselected state is by measurements at times $t > t_0$. The postselected state is conceptualized as a quantum state evolving backward in time. Weak measurements occur in the interval between these conventional measurements through a weak coupling between the measurement apparatus and the system [4].

Formally, if $\hat{A}$ is a Hermitian operator on the system S, $|\psi_{\text{in}}\rangle$ and $|\psi_{\text{out}}\rangle$ (or $|\psi_{\text{fin}}\rangle$) are the preselected and postselected state vectors in the Hilbert space of S, and $|\phi(q)\rangle$ is the state vector of the needle of the measurement device, $g(t)$ a weak coupling impulse such that $\int_0^T g(t)dt = g_0 T$, T the coupling time, and $\hat{P}_d$ the momentum operator conjugated to the position operator $\hat{Q}_d$, then weakly measuring an ensemble of preselected vectors states $|\psi_{\text{in}}\rangle$ with the interaction Hamiltonian $\hat{H} = g(t)\hat{A} \otimes \hat{P}_d$ will yield

$$\begin{aligned} |\psi_w\rangle &= e^{(-i\hat{H}T)}|\psi_{\text{in}}\rangle \otimes |\phi(x)\rangle \\ &= \sum_i \alpha_i |a_i\rangle \otimes |\phi(q - g_0 T a_i)\rangle, \end{aligned}$$

wherein $|\psi_{\text{in}}\rangle = \sum_i \alpha_i |a_i\rangle$ is expressed in vector basis of the observable $\hat{A}$. Making strong measurement on the device pointer state will yield $\langle q|\psi_w\rangle = \sum_i \alpha_i \phi(q - g_0 T a_i)|a_i\rangle$.

Weak measurement with postselection is obtained by making a projection with the postselected state $\langle\psi_{\text{fin}}|$. In detail, let $\hat{P} = |\psi_{\text{fin}}\rangle\langle\psi_{\text{fin}}| \otimes I$ be an operator which projects onto the postselected state $|\psi_{\text{fin}}\rangle$, and $|\psi_w\rangle = e^{(-i\hat{H}T)}|\psi_{\text{in}}\rangle \otimes |\phi(x)\rangle$, the entangled state of the system with the state of the measurement device [4,23], then we get

$$\hat{P}|\psi_w\rangle = |\psi_{\text{fin}}\rangle\langle\psi_{\text{fin}}|e^{(-i\hat{H}T)}|\psi_{\text{in}}\rangle \otimes |\phi(x)\rangle.$$

Hence, weak coupling between the system and the device will yield

$$\begin{aligned} \hat{P}|\psi_w\rangle &\approx |\psi_{\text{fin}}\rangle\langle\psi_{\text{fin}}|(1 - i\hat{A} \otimes \hat{P}_d \cdot g_0 T)|\psi_{\text{in}}\rangle \otimes |\phi(q)\rangle \\ &= |\psi_{\text{fin}}\rangle\langle\psi_{\text{fin}}|\psi_{\text{in}}\rangle e^{-i(\langle\hat{A}\rangle_w \hat{P}_d T)} \otimes |\phi(q)\rangle, \end{aligned}$$

where $\langle\hat{A}\rangle_w = \frac{\langle\psi_{\text{fin}}|\hat{A}|\psi_{\text{in}}\rangle}{\langle\psi_{\text{fin}}|\psi_{\text{in}}\rangle}$ is the so-called weak value of the observable $\hat{A}$. The probabilities to get this result are $|\langle\psi_{\text{fin}}|\psi_{\text{in}}\rangle|^2$.

Weak values are weak measurement results and must be understood as statistical averages. They are also members of a decomposition of eigenvalues weighted by the relative amplitudes of finding the out-states among the ensemble of state vectors identically prepared in $|\psi_{\text{in}}\rangle$. A weak value should then be understood as a conditional probability where one of the events undergoes a perturbation. Dressel [24] clarifies this probabilistic understanding of weak values. He shows that weak values characterize the relative correction to a detection probability $|\langle\psi_{\text{fin}}|\psi_{\text{in}}\rangle|^2$, due to a small intermediate perturbation $\hat{U}(\epsilon)$, that results in a modified detection probability $|\langle\psi_{\text{fin}}|\hat{U}(\epsilon)|\psi_{\text{in}}\rangle|^2$. Alternatively, in a simple way, he defines weak values as complex numbers that one can assign to the powers of a quantum observable operator $\hat{A}$ using two states: an initial state $|\psi_{\text{in}}\rangle$ and a final state $|\psi_{\text{fin}}\rangle$. Weak values might lie outside the spectrum of eigenvalues and can be complex numbers. One interesting phenomenon is the amplification obtained when the preselected state and the postselected one are almost orthogonal [4,9].

In cosmology [9], all observations and measurements are considered weak in the quantum sense, due to the nature of the processes involved. For instance, when observing the redshift of a galaxy, the measurement is conducted by observing the light emitted from a large number of photons originating from numerous sources within the Galaxy. While the emission of a single photon from an atom may not be considered weak from the atom's perspective, the use of a large ensemble of photons to measure a property of the entire Galaxy constitutes a weak measurement in the quantum sense. This is because the quantum backreaction of the photons on the relevant physical variable of the entire Galaxy, such as its momentum, is negligible. Therefore, the large-scale nature of cosmological observations and measurements, involving statistical averages over a large ensemble of photons, results in weak values in the quantum sense.

This idea of weakness of all observations and measurements in cosmology is rooted in classical field. Dressel, in his paper [25], has shown that, given two spacetime hypersurfaces σ_I and σ_F on which field states $|I\rangle$ and $\langle F|$ are defined as boundary conditions, the classical background field ϕ has the form of a weak value of the quantum field operator $\phi = \frac{\langle F|\hat{\phi}|I\rangle}{\langle F|I\rangle}$. Any weak interaction, that is any weak measurement that does not appreciably perturb the classical background field or its boundary conditions, will result in a quantum weak measurement whether or not the measurement coupling is weak for every field excitation.

In the laboratory, the two-state vectors require human interventions. In quantum cosmology, there are many proposals, as noticed by Davies: Hartle and Hawking came up with the no-boundary wave function, which is the state of the Universe before the Planck epoch [26]. In a semiclassical approach, in the framework of the theory of quantum fields, the initial state is taken to be a vacuum state. As for the final state, this can be anything. In weak measurement, one will ensure that the final state is not orthogonal to the in-state.

For the Einstein-Dirac in FLRW context, the preselected state will be the spinor solution of the Einstein-Dirac equation in the limit of the massless Universe, that is, the radiation Universe, whereas the final state will be the spinor solution of the Einstein-Dirac equation for the dust Universe. The implementation of a weak measurement in this specific context is intended to acquire geometrical information about the Universe. The subsequent computational analysis is made feasible by the association between weak measurement and the Berry phase [7]. This has never been explored before, as far as we know.

Regarding the Einstein-Dirac system, it investigates the interaction between particles with a spin of $1/2$ and the gravitational field. The Einstein field equations essentially relate the geometry of spacetime to the distribution of matter and energy within that spacetime. In simpler terms, they describe how matter and energy, represented here by the stress tensor applied on spinors, influence the curvature of spacetime and how the curvature of spacetime influences the motion of matter and energy. Finster [15] investigated the nonlinear coupling of gravity to matter in a time-dependent spherically symmetric Einstein-Dirac system. He showed that quantum oscillations of the Dirac wave functions can prevent the formation of a big bang or big crunch singularity.

Weak measurements present numerous advantages, foremost among them being the capability to detect exceedingly subtle effects while minimizing disturbance to the system state. Onur Hosten and Paul Kwiat [27] exemplified the utility of weak measurement techniques in amplifying the spin Hall effect of light, Meng-Jun Hu [28] discussed in his paper the proposal for weak measurements amplification based Laser Interferometer Gravitational Wave Observatory to detect gravitational waves using weak measurements to amplify ultrasmall phase signals, while Dixon *et al.* [29] applied these methods to enhance the detection of minute transverse deflections in an optical beam. This facet of weak measurements proves highly advantageous in cosmology, enabling the amplification and measurement of wave functions originating in the remote past, close to singularities.

This paper is organized as follows: after the Introduction, we give a summary of the Einstein-Dirac system and then find the spinor solutions for the radiation Universe and the dust one. In Sec. III, we give the results of our main computations on weak measurements on the Einstein-Dirac solutions. In Sec. IV, we give a conclusion. The appendixes provide more details of our computations.

II. THE EINSTEIN-DIRAC SOLUTION IN FRIEDMANN-LEMAÎTRE-ROBERTSON-WALKER SPACE

In this section, we provide an overview of the Einstein-Dirac solution within the Friedmann-Lemaître-Robertson-Walker (FLRW) space framework. For a more comprehensive treatment, readers are encouraged to refer to [15] and [30].

The Einstein-Dirac equations are given by

$$R^i_j - \frac{1}{2} R\delta^i_j = 8\pi\kappa T^i_j, \qquad (\mathcal{D} - m)\Psi = 0, \tag{1}$$

where κ represents the gravitational constant, R^i_j the Ricci tensor, R the scalar curvature, T^i_j the energy-momentum tensor of the Dirac particles, $\{i, j\}$ the indices representing the coordinates $\{t, r, \theta, \phi\}$, $\mathcal{D}$ the Dirac operator, m the mass of the Dirac particles, and Ψ the Dirac wave function.

Our Einstein-Dirac equations are formulated within the FLRW space, a spacetime manifold with a (1, 3) signature. This simple cosmological model is characterized by a metric described by the homogeneous and isotropic line element

$$ds^2 = dt^2 - R^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right), \qquad (2)$$

where $t \in \mathbb{R}^+$ represents time, $(\theta, \phi) \in (0, \pi) \times [0, 2\pi)$ are angular coordinates, r is the radial coordinate, $R(t)$ the scale function, $d\Omega^2$ the line element on S^2, and k can take values of -1, 0, or 1, corresponding to open, flat, and closed universes, respectively. We adopt "Planck units" where $c = \hbar = G = 1$.

The derivation of the Dirac equation in this homogeneous and isotropic spacetime is conducted in Appendix A. The further analysis of the system is made possible by separating the spatial from time dependence via the ansatz

$$\Psi = \frac{1}{R(t)^{\frac{3}{2}}}\begin{pmatrix} \alpha(t)\psi_\lambda(r,\theta,\phi) \\ \beta(t)\psi_\lambda(r,\theta,\phi) \end{pmatrix}. \qquad (3)$$

This ansatz enables the construction of a coupled system of ordinary differential equations (ODEs) for the complex-valued functions $\alpha(t)$ and $\beta(t)$:

$$i\frac{d}{dt}\begin{pmatrix} \alpha(t) \\ \beta(t) \end{pmatrix} = \begin{pmatrix} m & -\frac{\lambda}{R(t)} \\ -\frac{\lambda}{R(t)} & -m \end{pmatrix}\begin{pmatrix} \alpha(t) \\ \beta(t) \end{pmatrix}. \qquad (4)$$

The spinors are normalized according to

$$|\alpha|^2 + |\beta|^2 = \lambda^2 - \frac{1}{4}. \qquad (5)$$

These spinors enter the Einstein equation via the energy-momentum tensor of the wave function, ensuring a coupling with the Dirac equation. The nonvanishing components of the energy-momentum tensor are computed in Appendix B of [15] and are given by

$$T^0_0 = R^{-3}\left[m(|\alpha|^2 - |\beta|^2) - \frac{2\lambda}{R}\mathrm{Re}(\alpha\bar{\beta})\right], \qquad (6)$$

$$T^r_r = T^\theta_\theta = T^\phi_\phi = R^{-3}\frac{2\lambda}{3R}\mathrm{Re}(\alpha\bar{\beta}). \qquad (7)$$

A short calculation for the Einstein tensor G^j_k, given in [15], yields

$$G^0_0 = 3\left(\frac{\dot{R}^2+1}{R^2}\right), \qquad (8)$$

$$G^r_r = G^\theta_\theta = G^\phi_\phi = 2\frac{\ddot{R}}{R} + \frac{\dot{R}^2+1}{R^2}. \qquad (9)$$

Knowing that

$$G^i_j = 8\pi\kappa T^i_j, \quad \text{where } G^i_j = R^i_j - \frac{1}{2}R\delta^i_j,$$

we set $\kappa = \frac{3}{8\pi}$ for a convenient computation as in [15]. This yields the following expression, the Einstein equation,

$$\dot{R}^2 + 1 = R^2 T^0_0, \qquad (10)$$

which in terms of the two-level system is given as

$$\dot{R}^2 + 1 = \frac{m}{R}(|\alpha|^2 - |\beta|^2) - \frac{2\lambda}{R^2}\mathrm{Re}(\alpha\bar{\beta}). \qquad (11)$$

We derive also the acceleration of the Universe represented here by the second derivative with respect to time of the scale function given firstly in terms of the energy-momentum tensors as

$$\ddot{R} = \frac{R}{2}(3T^j_j - T^0_0). \qquad (12)$$

Then in terms of spinors we shall have

$$\ddot{R} = -\frac{m}{2R^2}(|\alpha|^2 - |\beta|^2) + \frac{2\lambda}{R^3}\mathrm{Re}(\alpha\bar{\beta}). \qquad (13)$$

One sees immediately that the acceleration of the Universe is caused by the spinor field, and this is consistent along the lines of [16–21]. We will later on show some behaviors of the acceleration of the scale function.

At this stage, we are left with two differential equations, Eqs. (4) and (11), in which the spinor $\binom{\alpha}{\beta}$ is considered as a two-level quantum state.

As suggested by Finster and Hainzl in [15], the Einstein-Dirac equation can be rewritten in terms of a *Bloch vector* $\vec{v}$, where

$$\vec{v} = \begin{pmatrix} \langle\xi|\sigma^1|\xi\rangle \\ \langle\xi|\sigma^2|\xi\rangle \\ \langle\xi|\sigma^3|\xi\rangle \end{pmatrix}, \qquad (14)$$

$$\vec{b} = 2\frac{\lambda}{R}\vec{e}_1 - 2m\vec{e}_3. \qquad (15)$$

Here, ξ represents the spinor $\binom{\alpha}{\beta}$, σ^i the Pauli matrices, and $\vec{e}_i$ the standard basis vectors in $\mathbb{R}^3$ $(i \in \{1,2,3\})$.

The Einstein-Dirac equations can then be expressed as

$$\dot{\vec{v}} = \vec{b} \wedge \vec{v}, \qquad \dot{R}^2 + 1 = -\frac{1}{2R}\vec{b}\cdot\vec{v}. \qquad (16)$$

Here, "$\wedge$" and "$\cdot$" denote the cross and scalar product in Euclidean space $\mathbb{R}^3$, respectively. We rewrite also the scale acceleration in terms of the Bloch vector components as

$$\ddot{R} = -\frac{m}{2R^2}v^3 + \frac{\lambda}{R^3}v^1. \quad (17)$$

To simplify the analysis, a rotation"U" around the $\vec{e}_2$ axis is applied to $\vec{b}$, making $\vec{b}$ parallel to $\vec{e}_1$. This results in a transformed vector $\vec{w}$,

$$\vec{w} = U\vec{v} = \begin{pmatrix} w^1 \\ w^2 \\ w^3 \end{pmatrix}, \quad (18)$$

where the components of $\vec{w}$ are given by

$$w^1 = \frac{1}{\sqrt{\lambda^2 + m^2R^2}}(2\lambda \mathrm{Re}(\alpha\bar{\beta}) - mR(|\alpha|^2 - |\beta|^2)), \quad (19)$$

$$w^2 = 2\mathrm{Im}(\alpha\bar{\beta}), \quad (20)$$

$$w^3 = \frac{1}{\sqrt{\lambda^2 + m^2R^2}}(2m\mathrm{Re}(\alpha\bar{\beta}) + \lambda(|\alpha|^2 - |\beta|^2)). \quad (21)$$

At this stage, the Einstein-Dirac equations reduce to a system of ODEs involving the scale function $R(t)$ and the complex functions $\alpha(t)$ and $\beta(t)$,

$$\dot{\vec{w}} = \vec{d} \wedge \vec{w}, \qquad \dot{R}^2 + 1 = -\frac{1}{R^2}\sqrt{\lambda^2 + m^2R^2}w^1, \quad (22)$$

where $\vec{d}$ is defined as

$$\vec{d} := \frac{2}{R}\sqrt{\lambda^2 + m^2R^2}\vec{e}_1 - \frac{\lambda m R}{\lambda^2 + m^2R^2}\frac{\dot{R}}{R}\vec{e}_2. \quad (23)$$

The length of the Bloch vector is constant, and the normalization convention is

$$|\vec{w}| = \lambda^2 - \frac{1}{4} = N. \quad (24)$$

Within weak measurement and the *two vectors state formalism*, this research primarily explores solutions to the Einstein-Dirac equation under specific conditions. These conditions involve either the mass parameter ($m = 0$), in which case the solution of (22) is the preselected state, or the eigenvalue ($\lambda = 0$), in which case the solution of the same equation is the postselected state.

In the case where $m = 0$, the Einstein-Dirac equation simplifies to the well-known FLRW equation describing the Universe's behavior in the radiation-dominated phase. Conversely, in the second scenario, $\lambda = 0$ corresponds to a dust-dominated Universe. Notably, for sufficiently large-scale factor R, the Universe exhibits classical behavior similar to the dust-dominated case. Figure 1 gives a pictorial description of weak measurement or weak value computation in the Einstein-Dirac cosmological context, which we aim to explore in this article.

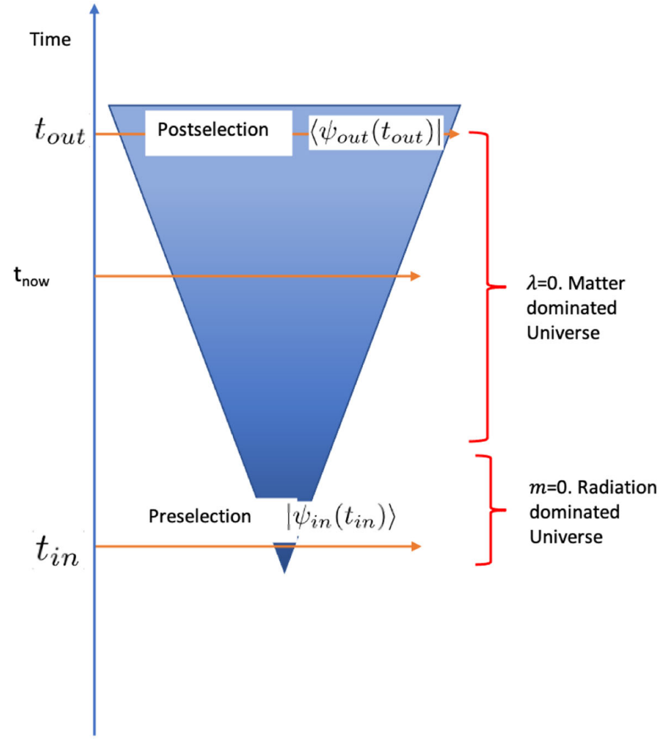

FIG. 1. Preselection and postselection à t_{in} and t_{out}.

The classical Friedmann general equation with $k = 1, 0, -1$ corresponding to closed, flat, and opened Universe, respectively, is given by

$$\dot{R}^2 + k = -\left(\frac{\rho_m}{R} + \frac{\rho_r}{R^2} - \Lambda R^2\right), \quad (25)$$

where ρ_m is the density of matter, ρ_r is the density of radiation, and Λ is the vacuum energy. For $R \ll 1$, in the right side of expression (25), the radiative term dominates in the evolution of the Universe. The expression becomes

$$\dot{R}^2 + k = -\left(\frac{\rho_r}{R^2}\right). \quad (26)$$

While ignoring the vacuum energy, if $R \gg 1$, then the matter term dominates and expression (25) becomes

$$\dot{R}^2 + k = -\left(\frac{\rho_m}{R}\right). \quad (27)$$

This equation describes a dust-dominated Universe.

It is essential to note that the radiation-dominated Universe pertains to a brief period near the initial singularity. Our work considers the radiation-dominated period beginning at the end of the reheating phase, which comes after the cosmic inflation. During reheating, the Universe is repopulated with particles, and the energy density of the Universe becomes dominated by radiation (photons and relativistic particles) (see [31] on page 388 and [32] on page 335). From that moment, the phenomena of the creation of particles become negligible. During this period, the rate of cosmological expansion slows down compared to the inflationary period, and the dynamics of the Universe become much more gradual (see [33,34]). In contrast, the dust-dominated Universe extends over a much longer time frame, including the present day.

A. $|\Psi_{\mathrm{in}}\rangle$: Solution of the Einstein-Dirac equation for $m=0$

When $m = 0$, the ODEs in (22) become

$$\begin{cases} \dot{w}^1 = 0 \\ \dot{w}^2 = -\frac{2\lambda}{R} w^3 \\ \dot{w}^3 = \frac{2\lambda}{R} w^2, \end{cases} \tag{28}$$

$$\left(\frac{dR}{dt}\right)^2 + 1 = -\frac{1}{R^2}\lambda w^1. \tag{29}$$

We set the following conditions. The expression (29) must verify that at time $t = t_B, \frac{dR}{dt}|_{t_B} = 0$ and $R(t_B) = R_0$. This implies that $w^1 = -\frac{R_0^2}{\lambda}$ is a constant. The solution to the differential scale function equation is

$$R(t) = \sqrt{-(t - R_0)^2 + R_0^2}. \tag{30}$$

The form of this solution is confirmed by scale function for the radiation dominated Universe proposed by Plebanski and Krasinski in [35] on page 289. It reads as

$$R(t) = \sqrt{-k\left(t - t_B - \frac{1}{k}\sqrt{\frac{\epsilon_0}{3}}\right)^2 + \frac{\epsilon_0}{3k}}, \quad \text{when } k \neq 0,$$
$$R(t) = \sqrt{\frac{\epsilon_0}{3}}(t - t_B), \quad \text{when } k = 0,$$

where ϵ_0 is the radiation density, k takes values 1 or -1.

Having the initial condition to the Bloch vector differential equation,

$$\vec{w}(t_B) = \left(-\frac{R_0^2}{\lambda}, \sqrt{N^2 - \frac{R_0^4}{\lambda^2}}\sin(\phi_0), \sqrt{N^2 - \frac{R_0^4}{\lambda^2}}\cos(\phi_0)\right), \tag{31}$$

and substituting $R(t)$ into (28), and based on the relationship between spherical and hyperbolic trigonometry, we can find the subsequent solutions:

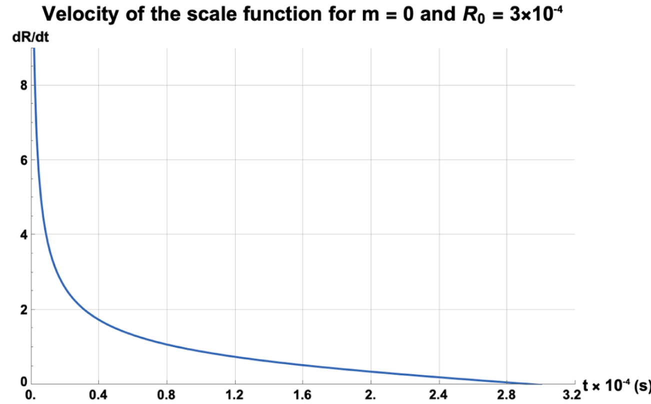

FIG. 2. $\frac{dR(t)}{dt}$ from time $t = 0$ to $t = R_0$. If one considers the today scale function is set at $R = 1$, the scale function being without unity, we may calibrate time so that, when the radiative forces cease to be the driving one, then the corresponding time would be $t = R_0$. In the literature we know that the radiation dominated Universe epoch elapsed ≈72000 years [36], page 331.

$$\begin{cases} w^1 = -\frac{R_0^2}{\lambda} \\ w^2 = \sqrt{N^2 - \frac{R_0^4}{\lambda^2}}\sin(\Phi_{\mathrm{in}}(t)) \\ w^3 = \sqrt{N^2 - \frac{R_0^4}{\lambda^2}}\cos(\Phi_{\mathrm{in}}(t)), \end{cases} \tag{32}$$

where

$$\Phi_{\mathrm{in}}(t) = 4\lambda G(t) - \lambda\pi - \phi_0, \tag{33}$$

$G(t) = \tan^{-1}(\frac{t}{R(t)})$, and ϕ_0, a phase. We can notice that the Bloch vector is a function of the scale function $R(t)$.

Substituting $m = 0$ into the components of $\vec{w}$ in (19)–(21), we obtain

$$\begin{cases} w^1 = 2\mathrm{Re}(\alpha\bar{\beta}) \\ w^2 = 2\mathrm{Im}(\alpha\bar{\beta}) \\ w^3 = |\alpha|^2 - |\beta|^2. \end{cases} \tag{34}$$

Replacing α by $\rho_{in_1}e^{i\eta_{in_1}}$ and β by $\rho_{in_2}e^{i\eta_{in_2}}$ into (34), where ρ is the modulus of α and η its argument, we differentiate modulus and argument from preselected and postselected states with indices. We obtain

$$\begin{cases} w^1 = 2\rho_{in_1}\rho_{in_2}\cos(\eta_{in_1} - \eta_{in_2}) \\ w^2 = 2\rho_{in_1}\rho_{in_2}\sin(\eta_{in_1} - \eta_{in_2}) \\ w^3 = \rho_{in_1}^2 - \rho_{in_2}^2. \end{cases} \tag{35}$$

These results constitute additional constraints for the solutions (32). Consequently, w^1, w^2, w^3 must be real numbers. $G(t)$ is real for $t \neq 0$.

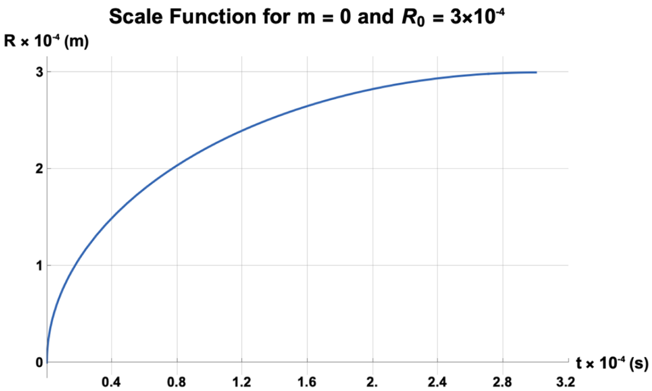

FIG. 3. $R(t)$ increasing from time $t = 0$ to $t = R_0$.

From expression (35) we obtain

$$\eta_{in_1} - \eta_{in_2} = -\arctan\left(\frac{\sqrt{\lambda^2 N^2 - R_0^4}\sin\left(\Phi_{\text{in}}(t)\right)}{R_0^2}\right). \tag{36}$$

Also from (35) and using the normalization condition from (24), we obtain

$$\rho_{in_1} = \frac{\sqrt{2}}{2}\sqrt{N + \sqrt{N^2 - \frac{R_0^4}{\lambda^2}}\cos\left(\Phi_{\text{in}}(t)\right)}, \tag{37}$$

$$\rho_{in_2} = \frac{\sqrt{2}}{2}\sqrt{N - \sqrt{N^2 - \frac{R_0^4}{\lambda^2}}\cos\left(\Phi_{\text{in}}(t)\right)}. \tag{38}$$

To sum up, $|\Psi_{\text{in}}\rangle$ is represented by the knowledge of the components of the three-vector $\vec{w}$ or $\alpha_{\text{in}} = \rho_{in_1} e^{i\eta_{in_1}}$ and $\beta_{in} = \rho_{in_2} e^{i\eta_{in_2}}$ at a certain t_{in} chosen such that the scale function is real.

The time in the scale function is defined within the interval $[0, R_0]$, but we will consider t_{in} in the interval $]0, R_0]$ for the Bloch vector to be well defined. It is worth noting that, the radiation-dominated Universe starts after the reheating phase; this is around 10^{-12} s. Within this period the scale function increases and stops at $t = R_0$. But in the general picture, the driving force due to matter takes over, and the Universe continues to expand. Let us point out also that the period within the radiation-dominated Universe is very short compared to the lifespan of the Universe. This period is approximated as 72000 years, compared to the current age of 13,9 Gyrs today [36]. We also remind that within this period, the scale function is very small. The radiative period ceases where the scale function R approximately attains the value $R \approx 3.10^{-4}$. Finster in [15] shows a bouncing behavior of the scale function around singularity. Figures 2 and 3 represent the velocity of the scale factor and the scale factor itself over that short period, respectively.

B. $|\Psi_{\text{out}}\rangle$: Solution of the Einstein-Dirac equation for $\lambda = 0$

When $\lambda = 0$, the ODEs in (22) become

$$\begin{cases} \dot{w}^1 = 0, \\ \dot{w}^2 = -2mw^3, \\ \dot{w}^3 = 2mw^2, \end{cases} \tag{39}$$

$$\left(\frac{dR}{dt}\right)^2 + 1 = -\frac{1}{R}mw^1. \tag{40}$$

We differ from Finster in [15], and we set that at $t_{\max}, \frac{dR}{dt}|_{t_{\max}} = 0$, and $R(t_{\max}) = R_{\max}$. This implies that $w^1 = -\frac{R_{\max}}{m}$. One easily sees that the two differential equations decouple. The Bloch vector solution will not depend on the scale function. These equations yield the following solutions:

$$R(\tau) = \frac{R_{\max}}{2}(1 - \cos(\tau)), \tag{41}$$

$$t(\tau) = \frac{R_{\max}}{2}(\tau - \sin(\tau)), \tag{42}$$

where τ is a parameter varying between 0 and 2π. The above parametric equations, represented by Fig. 4, describe a cycloid. Figure 5 represents its rate of change. The parametric equations represent a cycloid.

The solution to the Bloch vector differential equation for the following conditions,

$$\vec{w}(t_{\max}) = (w^1_{\max}, \rho\cos(\phi_{\max}), \rho\sin(\phi_{\max})), \tag{43}$$

gives the following solution:

$$\begin{cases} w^1 = -\frac{R_{\max}}{m} \\ w^2 = \sqrt{N^2 - \frac{R^2_{\max}}{m^2}}\cos\left(\Phi_{\text{out}}(t)\right) \\ w^3 = \sqrt{N^2 - \frac{R^2_{\max}}{m^2}}\sin\left(\Phi_{\text{out}}(t)\right), \end{cases} \tag{44}$$

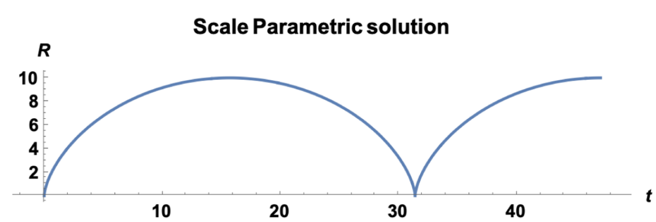

FIG. 4. This parametric solution is set for $R_{\max} = 10$. This value is taken from Finster in [15]. The time is derived from the formula $t(\tau) = \frac{R_{\max}}{2}(\tau - \sin((\tau))$.

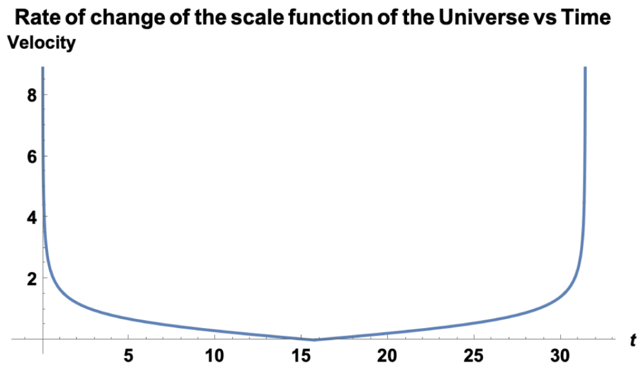

FIG. 5. Rate of change: the scale function in a universe dominated by matter.

where

$$\Phi_{\text{out}}(t) = 2mt + \phi_{\max} - 2mt_{\max}, \tag{45}$$

m is the mass of Dirac particle, and $\phi_{\max}$, a phase.

Substituting $\lambda = 0$ into the components of $\vec{w}$ in (19)–(21), we obtain

$$\begin{cases} w^1 = & -(|\alpha|^2 - |\beta|^2) \\ w^2 = & 2\text{Im}(\alpha\bar{\beta}) \\ w^3 = & 2\text{Re}(\alpha\bar{\beta}). \end{cases} \tag{46}$$

Replacing α by $\rho_{out_1} e^{i\eta_{out_1}}$ and β by $\rho_{\text{out}_2} e^{i\eta_{\text{out}_2}}$ into (47), we obtain

$$\begin{cases} w^1 = & -(\rho^2_{\text{out}_1} - \rho^2_{\text{out}_2}), \\ w^2 = & 2\rho_{\text{out}_1}\rho_{\text{out}_2}\sin(\eta_{\text{out}_1} - \eta_{\text{out}_2}), \\ w^3 = & 2\rho_{\text{out}_1}\rho_{\text{out}_2}\cos(\eta_{\text{out}_1} - \eta_{\text{out}_2}). \end{cases} \tag{47}$$

These results constitute additional constraints for the solutions (32). Consequently, w^1, w^2, w^3 must be real numbers.

From expression (47) we obtain

$$\eta_{\text{out}_1} - \eta_{\text{out}_2} = \frac{\pi}{2} - \Phi_{\text{out}}(t). \tag{48}$$

We also obtain from (47) and the norm (24),

$$\rho_{\text{out}_1} = \frac{\sqrt{2}}{2}\sqrt{N + \frac{R_{\max}}{m}}, \tag{49}$$

$$\rho_{\text{out}_2} = \frac{\sqrt{2}}{2}\sqrt{N - \frac{R_{\max}}{m}}. \tag{50}$$

To sum up, $|\Psi_{\text{out}}\rangle$ is represented by the knowledge of the components of the three-vector $\vec{w}$ or $\alpha_{\text{out}} = \rho_{\text{out}_1} e^{i\eta_{\text{out}_1}}$ and $\beta_{\text{out}} = \rho_{\text{out}_2} e^{\eta_{\text{out}_2}}$ at a certain t_{out}, chosen such that the scale function is real.

III. WEAK MEASUREMENT ON THE EINSTEIN-DIRAC SOLUTIONS

A weak value of any observable is given by

$$\langle\hat{A}\rangle_w = \frac{\langle\psi_{\text{out}}(t)|\hat{A}|\psi_{\text{in}}(t)\rangle}{\langle\psi_{\text{out}}(t)|\psi_{\text{in}}(t)\rangle}, \tag{51}$$

wherein $\langle\psi_{\text{out}}(t)| = \langle\psi_{\text{out}}(t_{\text{out}})|U^\dagger(t, t_{\text{out}})$, and $|\psi_{\text{in}}(t)\rangle = U(t, t_{\text{in}})|\psi_{\text{in}}(t_{\text{in}})\rangle$.

Weak values are results of weak measurements. They are also derivable independently from weak measurements based on classical expectation values [37]. For instance, if $\hat{A}$ is an observable, then its expectation value is

$$\langle\hat{A}\rangle = \int \langle\hat{A}\rangle_w P_{\phi\psi} d\phi,$$

wherein $\langle\hat{A}\rangle = \langle\psi|\hat{A}|\psi\rangle$, $\langle\hat{A}\rangle_w = \frac{\langle\psi|\hat{A}|\phi\rangle}{\langle\psi|\phi\rangle}$ is the so-called weak value of the observable $\hat{A}$, and $P_{\phi\psi} = |\langle\phi|\psi\rangle|^2$ is the detection probability of event ϕ out of the prepared initial state ψ. This probability is that of measuring $\langle\hat{A}\rangle_w$.

Alternatively, if one sees $|\psi\rangle$ as a sum or difference of two other state vectors, let us say $|\psi\rangle = |\psi_1\rangle \pm |\psi_2\rangle$, then we will have

$$\langle\hat{A}\rangle = \langle\hat{A}_1\rangle + \langle\hat{A}_2\rangle \pm 2\text{Re}\langle\hat{A}_{1,2}\rangle_w\langle\psi_2|\psi_1\rangle,$$

wherein $\langle\hat{A}_1\rangle = \langle\psi_1|\hat{A}|\psi_1\rangle$, $\langle\hat{A}_2\rangle = \langle\psi_2|\hat{A}|\psi_2\rangle$, and $\langle\hat{A}_{1,2}\rangle_w = \frac{\langle\psi_1|\hat{A}|\psi_2\rangle}{\langle\psi_1|\psi_2\rangle}$.

In (51), state vectors are multiplied by evolutionary operators. A pictorial description of weak measurement in our context is represented in Fig. 6.

The solutions of Einstein-Dirac equations $|\psi_{\text{in}}(t_{\text{in}})\rangle$ and $|\psi_{\text{out}}(t_{\text{out}})\rangle$ are defined at the respective times t_{in} and t_{out}. We must construct the evolutionary operator to allow the weak measurement to occur at time t between t_{in} and t_{out}.

Because of the complexity involving the construction of the exact evolutionary operator due to the time dependency of the Dirac Hamiltonian, we are going to approximate it by using the *Wentzel-Kramers-Brillouin* (WKB) method. Also, following Finster [30], the smallness of the Compton wavelength compared to the lifetime of the Universe justifies the use of WKB-type approximation. The approximated unitary evolutionary operator will be expressed as

$$U(t, t_0) = U^{-1}(t)\begin{pmatrix} e^{-iF(t,t_0)} & 0 \\ 0 & e^{iF(t,t_0)} \end{pmatrix}U(t_0), \tag{52}$$

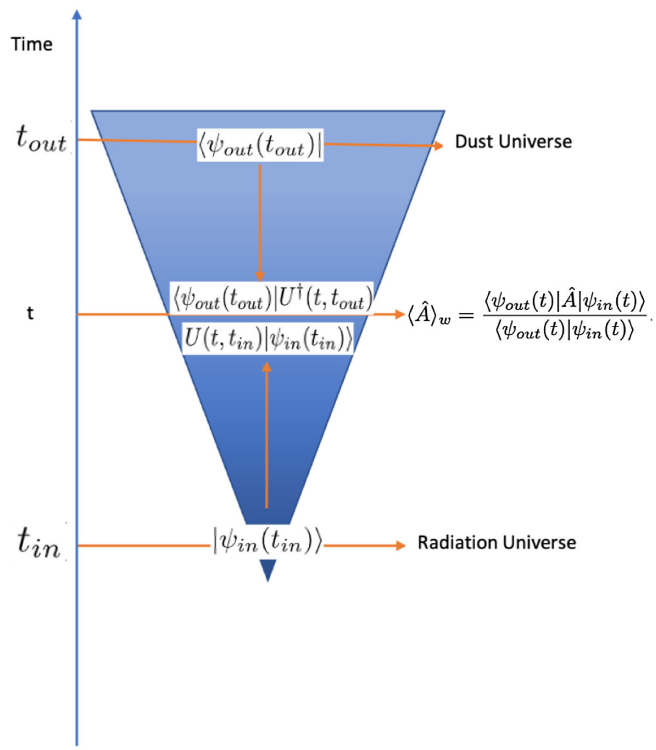

FIG. 6. Weak measurement at time t.

wherein $F(t,t_0) = -F(t_0,t) = \int_{t_0}^{t} \frac{\sqrt{R^2(t')m^2+\lambda^2}}{R(t')} dt'$. $U(t)$ is defined such that

$$U(t)\begin{pmatrix} m & -\frac{\lambda}{R(t)} \\ -\frac{\lambda}{R(t)} & -m \end{pmatrix} U(t)^{-1} = \frac{\sqrt{R^2(t)m^2+\lambda^2}}{R(t)}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{53}$$

Equation (53) yields $g(t) = \frac{1}{2}\arctan(\frac{\lambda}{mR(t)})$ for the unitary matrix given by

$$U(t) = \begin{pmatrix} \cos(g(t)) & -\sin(g(t)) \\ \sin(g(t)) & \cos(g(t)) \end{pmatrix}. \tag{54}$$

The computation of $\begin{pmatrix} e^{-iF(t,t_0)} & 0 \\ 0 & e^{iF(t,t_0)} \end{pmatrix}$ follows Finster in [30].

The unitary operator applied to the preselected state $U(t,t_{\rm in})$ is given as

$$U^{-1}(t)\begin{pmatrix} e^{-iF(t,t_{\rm in})} & 0 \\ 0 & e^{iF(t,t_{\rm in})} \end{pmatrix} U(t_{\rm in}). \tag{55}$$

Since the preselected state is taken from the radiation-dominated Universe, where $m=0$, this will yield $g(t_{\rm in}) = \frac{\pi}{4} + k\pi$. For $k=0$,

$$g(t_{\rm in}) = \frac{\pi}{4}, \tag{56}$$

and the matrix will be given by

$$U(t_{\rm in}) = \frac{\sqrt{2}}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \tag{57}$$

whereas the postselected state is taken from the dust-dominated Universe wherein $\lambda = 0$. This will yield $g(t_{\rm out}) = 0 + k\pi$ for $\lambda = 0$, for $k = 0$,

$$g(t_{\rm in}) = 0, \tag{58}$$

and the unitary matrix $U(t_{\rm out})$ is the identity matrix

$$U(t_{\rm out}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{59}$$

Knowing that $U^{\dagger}(t,t_{\rm out}) = U(t_{\rm out},t)$, this later is given by

$$U^{-1}(t_{\rm out})\begin{pmatrix} e^{-iF(t_{\rm out},t)} & 0 \\ 0 & e^{iF(t_{\rm out},t)} \end{pmatrix} U(t). \tag{60}$$

The unitary operator approximated, we can now compute the weak value of some observables starting with the energy-momentum tensor.

A. Weak measurement of the energy-momentum tensor

The computation of the weak value of the energy-momentum tensor (see Appendix B) given as

$$\langle T_{\mu\nu}\rangle_w = \frac{\langle\psi_{\rm out}(t)|T_{\mu\nu}|\psi_{\rm in}(t)\rangle}{\langle\psi_{\rm out}(t)|\psi_{\rm in}(t)\rangle}, \tag{61}$$

yields that its nonvanishing components, because of the homogeneity and the isotropy of the FLRW space, are

$$\langle T_0^0\rangle_w = \frac{\left[mR(t)(\alpha_{\rm in}(t)\overline{\alpha_{\rm out}}(t) - \beta_{\rm in}(t)\overline{\beta_{\rm out}}(t)) - \lambda(\alpha_{{\rm in}(t)}\overline{\beta_{\rm out}}(t) + \beta_{\rm in}(t)\overline{\alpha_{\rm out}}(t))\right]}{R(t)(\overline{\alpha_{\rm out}}(t)\alpha_{\rm in}(t) + \overline{\beta_{\rm out}}(t)\beta_{\rm in}(t))}, \tag{62}$$

and

$$\langle T^j_j\rangle_w = \frac{\lambda(\alpha_{\rm in}(t)\overline{\beta_{\rm out}}(t) + \beta_{\rm in}(t)\overline{\alpha_{\rm out}}(t))}{3R(t)(\overline{\alpha_{\rm out}}(t)\alpha_{\rm in}(t) + \overline{\beta_{\rm out}}(t)\beta_{\rm in}(t))}, \quad (63)$$

where j represents r, θ, ϕ. The space diagonal energy momentum tensors are all equal, that is $T^r_r = T^\theta_\theta = T^\phi_\phi$.

One sees here that if $|\psi_{\rm in}(t)\rangle = |\psi_{\rm out}(t)\rangle$, then the weak value of the energy-momentum tensor will coincide with the one computed in [15]. In this regard, weak values or measurements generalize classical averages or measurements.

Another quick observation is the amplification phenomenon. The occurrence of the almost orthogonality condition occurs in a practical way if

$$\overline{\alpha_{\rm out}}(t)\alpha_{\rm in}(t) + \overline{\beta_{\rm out}}(t)\beta_{\rm in}(t) \approx 0.$$

This is realized when $\overline{\alpha_{\rm out}}(t) \approx -k\beta_{\rm in}(t)$ and $\overline{\beta_{\rm out}}(t) \approx k\alpha_{\rm in}(t)$, where k is a factor or a phase.

It might be interesting to have an explicit expression of the orthogonality condition for measurement and better analysis. To this end, let us construct a *complex Bloch vector* in relation to the density operator, as introduced by Aharonov and Gruss in their paper *Two-Time Interpretations* [3]. One should note that if the preselected state is equal to the postselected one, then we shall get the same construction as in the paper of Finster and Hainzl on page 16 in [15]. Our complex Bloch vector will be given as

$$\vec{v}_{f,i} = \begin{pmatrix} v^1_{f,i} \\ v^2_{f,i} \\ v^3_{f,i} \end{pmatrix} = \begin{pmatrix} \langle\xi_{\rm out}(t)|\sigma^1|\xi_{\rm in}(t)\rangle \\ \langle\xi_{\rm out}(t)|\sigma^2|\xi_{\rm in}(t)\rangle \\ \langle\xi_{\rm out}(t)|\sigma^3|\xi_{\rm in}(t)\rangle \end{pmatrix}, \quad (64)$$

wherein $\xi_{\rm out}(t) = \begin{pmatrix} \alpha_{\rm out}(t) \\ \beta_{\rm out}(t) \end{pmatrix}$ and $\xi_{\rm in}(t) = \begin{pmatrix} \alpha_{\rm in}(t) \\ \beta_{\rm in}(t) \end{pmatrix}$.

We also define $v^0_{f,i}$ as $\langle\xi_{\rm out}(t)|\xi_{\rm in}(t)\rangle$. This will yield $\overline{\alpha_{\rm out}}(t)\alpha_{\rm in}(t) + \overline{\beta_{\rm out}}(t)\beta_{\rm in}(t)$. We can now redefine the weak energy-momentum tensor as

$$\langle T^0_0\rangle_w = \frac{\left[mR(t)\left(v^3_{f,i}\right) - \lambda\left(v^1_{f,i}\right)\right]}{R(t)\left(v^0_{f,i}\right)},$$

$$\langle T^j_j\rangle_w = \frac{\lambda\left(v^1_{f,i}\right)}{3R(t)\left(v^0_{f,i}\right)}. \quad (65)$$

Let us express the energy-momentum tensor weak values in *terms of* $\alpha_{\rm in}(t_{\rm in})$ *and* $\alpha_{\rm out}(t_{\rm out})$, wherein $A = e^{i(F(t_{\rm out},t_{\rm in}))}$, $\overline{A} = e^{-i(F(t_{\rm out},t_{\rm in}))}$, $B = e^{i(F(t,t_{\rm in})+F(t,t_{\rm out}))}$, and $\bar{B} = e^{-i(F(t,t_{\rm in})+F(t,t_{\rm out}))}$. Because of the lengthiness of the expression, we break it as follows:

$$v^0_{f,i} = \frac{\sqrt{2}}{2}(\overline{\alpha_{\rm out}}(t_{\rm out})\alpha_{\rm in}(t_{\rm in}) - \overline{\alpha_{\rm out}}(t_{\rm out})\beta_{\rm in}(t_{\rm in}))\overline{A} + \frac{\sqrt{2}}{2}(\overline{\beta_{\rm out}}(t_{\rm out})\beta_{\rm in}(t_{\rm in}) + \overline{\beta_{\rm out}}(t_{\rm out})\alpha_{\rm in}(t_{\rm in}))A, \quad (66)$$

$$\begin{aligned} v^3_{f,i} = {} & \frac{\sqrt{2}}{2}\alpha_{\rm in}(t_{\rm in})\overline{\alpha_{\rm out}}(t_{\rm out})[\sin(2g(t))B + \cos(2g(t))\bar{A}] + \frac{\sqrt{2}}{2}\beta_{\rm in}(t_{\rm in})\overline{\beta_{\rm out}}(t_{\rm out})[-\sin(2g(t))\bar{B} - \cos(2g(t))A] \\ & + \frac{\sqrt{2}}{2}\alpha_{\rm in}(t_{\rm in})\overline{\beta_{\rm out}}(t_{\rm out})[\sin(2g(t))\bar{B} - \cos(2g(t))A] + \frac{\sqrt{2}}{2}\beta_{\rm in}(t_{\rm in})\overline{\alpha_{\rm out}}(t_{\rm out})[\sin(2g(t))B - \cos(2g(t))\bar{A}], \end{aligned} \quad (67)$$

$$\begin{aligned} v^1_{f,i} = {} & \frac{\sqrt{2}}{2}\alpha_{\rm in}(t_{\rm in})\overline{\alpha_{\rm out}}(t_{\rm out})[-\sin(2g(t))\bar{A} + \cos(2g(t))B] + \frac{\sqrt{2}}{2}\beta_{\rm in}(t_{\rm in})\overline{\beta_{\rm out}}(t_{\rm out})[\sin(2g(t))A - \cos(2g(t))\bar{B}] \\ & + \frac{\sqrt{2}}{2}\alpha_{\rm in}(t_{\rm in})\overline{\beta_{\rm out}}(t_{\rm out})[\sin(2g(t))A + \cos(2g(t))\bar{B}] + \frac{\sqrt{2}}{2}\beta_{\rm in}(t_{\rm in})\overline{\alpha_{\rm out}}(t_{\rm out})[\sin(2g(t))\bar{A} + \cos(2g(t))B]. \end{aligned} \quad (68)$$

We can deduce the orthogonality condition from $v^0_{f,i} \approx 0$. This is obtained if $\overline{\alpha_{\rm out}}(t_{\rm out}) \approx$

$$\begin{aligned} & -\frac{\sqrt{2}}{2}\alpha_{\rm in}(t_{\rm in})[-\sin(g(t_{\rm out}))e^{-iF(t_{\rm out},t_{\rm in})} + \cos(g(t_{\rm out}))e^{iF(t_{\rm out},t_{\rm in})}] \\ & -\frac{\sqrt{2}}{2}\beta_{\rm in}(t_{\rm in})[\sin(g(t_{\rm out}))e^{-iF(t_{\rm out},t_{\rm in})} + \cos(g(t_{\rm out}))e^{iF(t_{\rm out},t_{\rm in})}], \end{aligned} \quad (69)$$

and $\overline{\beta_{\rm out}}(t_{\rm out}) \approx$

$$\begin{aligned} & \frac{\sqrt{2}}{2}\alpha_{\rm in}(t_{\rm in})[\cos(g(t_{\rm out}))e^{-iF(t_{\rm out},t_{\rm in})} + \sin(g(t_{\rm out}))e^{iF(t_{\rm out},t_{\rm in})}] \\ & + \frac{\sqrt{2}}{2}\beta_{\rm in}(t_{\rm in})[-\cos(g(t_{\rm out}))e^{-iF(t_{\rm out},t_{\rm in})} + \sin(g(t_{\rm out}))e^{iF(t_{\rm out},t_{\rm in})}]. \end{aligned} \quad (70)$$

As it is difficult to have a simple expression of the weak value of the energy-momentum tensor to make an interpretation, let us calculate a concrete example using values that simplify the computation. For the preselected states, let us have

$$\begin{cases} \lambda = 3/2, \text{the eigenvalue of } \mathcal{D}_{\mathcal{H}} \\ g(t_{\rm in}) = \frac{\pi}{4}, \text{see (56)} \\ U(t_{\rm in}) = \frac{\sqrt{2}}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \text{see (57)} \\ R_0 \text{ is of order } 10^{-4}, \text{we set } R_0^4 \approx 0. \end{cases} \tag{71}$$

These values will yield

$$\begin{cases} \Phi_{\rm in}(t) = 3G(t) - \frac{\pi}{4} - \phi_0, \text{see (33)} \\ \eta_{in_1} - \eta_{in_2} = -\arctan\left(\frac{3\sin(\Phi_{\rm in}(t))}{R_0^2}\right), \text{see (36)} \\ \text{we set } \eta_{in_2} = 0 \\ \rho_{in_1} = \frac{\sqrt{2}}{2}\sqrt{2 + 2\cos(\Phi_{\rm in}(t))}, \text{see (37)} \\ \rho_{in_2} = \frac{\sqrt{2}}{2}\sqrt{2 - 2\cos(\Phi_{\rm in}(t))}, \text{see (38).} \end{cases} \tag{72}$$

Therefore the preselected state at $t_{\rm in}$ will be given by

$$\begin{cases} \alpha_{\rm in}(t_{\rm in}) = \frac{\sqrt{2}}{2}\sqrt{2 + 2\cos(\Phi_{\rm in}(t_{\rm in}))}e^{-i\arctan\left(\frac{3\sin(\Phi_{\rm in}(t_{\rm in}))}{R_0^2}\right)} \\ \beta_{\rm in}(t_{\rm in}) = \frac{\sqrt{2}}{2}\sqrt{2 - 2\cos(\Phi_{\rm in}(t_{\rm in}))}. \end{cases} \tag{73}$$

As for the postselected state vector, the scale function and the time expressed as $R(\tau) = \frac{R_{\max}}{2}(1 - \cos(\tau))$, $t(\tau) = \frac{R_{\max}}{2}(\tau - \sin(\tau))$, respectively, we have the following observations:

(i) A maximal $R(\tau)$ will be $R_{\max}$, and the maximal t, that is $t_{\rm out}$, will be $R_{\max}k\frac{\pi}{2}$.

(ii) We shall have $\rho_{{\rm out}_1} = \sqrt{2 + \frac{R_{\max}}{m}}$, $\rho_{{\rm out}_2} = \sqrt{2 - \frac{R_{\max}}{m}}$ and $\eta_{{\rm out}_1} - \eta_{{\rm out}_2} = \frac{\pi}{2} - \phi_{\max}$.

We remind that the postselected state is taken from the dust-dominated Universe wherein $\lambda = 0$. This will yield $g(t_{\rm out}) = \frac{1}{2}\arctan(\frac{\lambda}{m*R(t_{\rm out})}) = 0 + k\pi$. This will lead, for $k = 0$, to $U(t_{\rm out}) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, an identity matrix.

The postselected state $t_{\rm out}$ will be

$$\alpha_{\rm out}(t_{\rm out}) = \sqrt{2 + \frac{R_{\max}}{m}}e^{i(\frac{\pi}{2} - \phi_{\max})},$$

$$\beta_{\rm out}(t_{\rm out}) = \sqrt{2 - \frac{R_{\max}}{m}}. \tag{74}$$

For simplicity, let us consider a weak measurement taking place at time $t_{\rm out}$ and $R_{\max} = 10$ and $m = 21.5$. This implies that

$$\begin{pmatrix} \alpha_{\rm in}(t_{\rm out}) \\ \beta_{\rm in}(t_{\rm out}) \end{pmatrix} = U(t_{\rm out}, t_{\rm in})\begin{pmatrix} \alpha_{\rm in}(t_{\rm in}) \\ \beta_{\rm in}(t_{\rm in}) \end{pmatrix},$$

where

$$U(t_{\rm out}, t_{\rm in}) = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{-i\pi mR_{\max}} & -e^{-i\pi mR_{\max}} \\ e^{i\pi mR_{\max}} & e^{i\pi mR_{\max}} \end{pmatrix},$$

and $U(t_{\rm out}, t_{\rm out}) = I$. Note that $F(t_{\rm out}, t_{\rm in}) = mt_{\rm out}$. These elements yield the following state vectors:

$$\alpha_{\rm in}(t_{\rm out}) = \frac{\sqrt{2}}{2}(\alpha_{\rm in}(t_{\rm in}) - \beta_{\rm in}(t_{\rm in}))e^{-i\pi mR_{\max}},$$

$$\beta_{\rm in}(t_{\rm out}) = \frac{\sqrt{2}}{2}(\alpha_{\rm in}(t_{\rm in}) + \beta_{\rm in}(t_{\rm in}))e^{i\pi mR_{\max}}. \tag{75}$$

Since the weak measurement is taking place at time $t_{\rm out}$, the postselected state is fixed.

The values of the weak measurement of the energy-momentum tensor in the expression (65) will be

$$v^0_{f,i} = \mathrm{C}\left((A - B)\cos\left(\frac{\Phi_{\rm in}}{2}\right)\mathrm{D} + (A + B)\sin\left(\frac{\Phi_{\rm in}}{2}\right)\right)$$

$$v^1_{f,i} = \mathrm{C}(AB + 1)\left(\cos\left(\frac{\Phi_{\rm in}}{2}\right)D - \sin\left(\frac{\Phi_{\rm in}}{2}\right)\right)$$

$$v^3_{f,i} = C\left((A + B)\cos\left(\frac{\Phi_{\rm in}}{2}\right)D + (A - B)\sin\left(\frac{\Phi_{\rm in}}{2}\right)\right), \tag{76}$$

wherein we have set $A = e^{2i\pi mR_{\max}}$, $B = e^{i\phi_{\max}}$, $C = e^{-i\pi mR_{\max}}\sqrt{\frac{R_{\max}}{m} + 2}$ and $D = e^{i\tan^{-1}(\frac{3\sin(\phi_{\rm in})}{R_0^2})}$.

If we set $\Phi_{\rm in}$ such that $\Phi_{\rm in} = \pi$, then we will have

$$v^0_{f,i} = \mathrm{C}(A + B)$$

$$v^1_{f,i} = -\mathrm{C}(AB + 1)$$

$$v^3_{f,i} = C(A - B). \tag{77}$$

Such configuration happens if $t_{\rm in} = t_B$ and $\phi_0 = \pi$.

What we can conclude is that weak measurements reveal unusual values, such as complex numbers. In our case, for fixed m and $R_{\max}$, the numerical result will depend on $\phi_{\max}$.

In (12), the acceleration of the Universe is given in terms of the energy momentum tensor. In (17), it is given in terms of the Bloch vector, but in the *two-state formalism*, the Hopf transformation yields a complex Bloch vector. The *two-state formalism* generalizes the *one-state formalism* since it suffices to consider the same state vector in the *two-state formalism* to have the classical Bloch vector. We propose to express the acceleration of the Universe in terms of weak values of the energy-momentum tensors and derive it in terms of a complex Bloch vector. The acceleration of the Universe will then be given as

$$\ddot{R} = \frac{R}{2}(3\langle T^j_j\rangle_w - \langle T^0_0\rangle_w),$$
$$\ddot{R} = \frac{\left[-mR\left(v^3_{f,i}\right) + 2\lambda\left(v^1_{f,i}\right)\right]}{2\left(v^0_{f,i}\right)}. \tag{78}$$

In this formula, one clearly sees that the acceleration of the Universe may be comprehended from the *two-state formalism* and weak measurement theories. Assuming a different postselected state vector may change the shape of the acceleration of the Universe. We remind here that the real and the imaginary parts of a weak value can be measured. The real part of a weak value may be interpreted as the best estimation of the conditioned average associated with an observable in two vector formalism [24,38], and they represent the shift of the average detected position due to postselection. The imaginary part of the weak value represents the shift of the average impulsion due to postselection.

The computation of the acceleration of the Universe with the above condition yields

$$\ddot{R} = \mathrm{Re}\left(\frac{\left[-mR\left(v^3_{f,i}\right) + 2\lambda\left(v^1_{f,i}\right)\right]}{2\left(v^0_{f,i}\right)}\right). \tag{79}$$

This example shows that the acceleration is not zero and depends here on the postselection.

Figure 7 is about the weak measurement of the acceleration of the Universe as a function of time in a matter dominated Universe.

The difference between Figs. 7 and 8 shows how the acceleration of the Universe may change with a different postselection. We have postselected a state vector with $\phi_{\max} = 0$ in Fig. 7 and $\phi_{\max} = \frac{\pi}{4}$ in Fig. 8. It is clear that this difference causes a greater amplitude of acceleration in Fig. 8 than in Fig. 7.

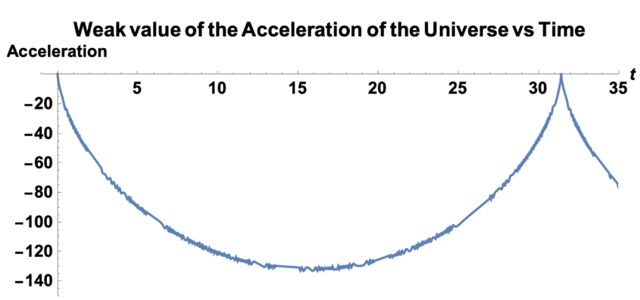

FIG. 7. The real part of the weak values of the acceleration of the Universe in function of time for $m = 21.5, R_{\max} = 10$, $\phi_{\max} = 0$, and $\lambda = \frac{3}{2}$.

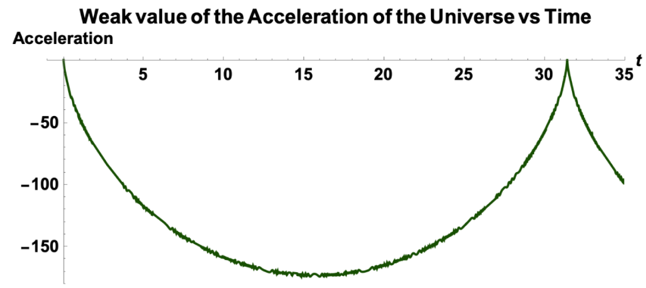

FIG. 8. The real part of the weak values of the acceleration of the Universe in function of time for $m = 21, R_{\max} = 10$, $\phi_{\max} = \frac{\pi}{4}$, and $\lambda = \frac{3}{2}$.

In conclusion, we have shown how to compute the weak value of the energy-momentum tensor. We derived how to amplify weak values of the energy-momentum tensor using an appropriate state vector almost orthogonal to the preselected one. We have finally derived the weak values of the acceleration of the Universe and have showed how sensitive it is compared to classical acceleration.

B. Weak value of σ^z operator

In this section, we delve into the weak measurement of the σ^z operator, which is given by the following expression:

$$\sigma^z_w = \frac{\langle\psi_{\text{out}}(t)|\sigma^z|\psi_{\text{in}}(t)\rangle}{\langle\psi_{\text{out}}(t)|\psi_{\text{in}}(t)\rangle}. \tag{80}$$

Following Ferraz, Kofman, and Cormann in [7,39,40], respectively, the weak value of the generator of $SU(2)$ is given by

$$\sigma_{r,w} = \frac{\vec{f}.\vec{r} + \vec{r}.\vec{i} + i\vec{f}.(\vec{r}\wedge\vec{i})}{\frac{1}{2}(1+\vec{f}.\vec{i})}. \tag{81}$$

In our case, $\vec{r} = (0\,0\,1)$.

The vectors $\vec{f}$ and $\vec{i}$ are real unit vectors on Bloch sphere applied to the postselected state for $\vec{f}$ and to the preselected state for $\vec{i}$. The vector $\vec{f}$ is equal to $\frac{\vec{w_f}}{|\vec{w_f}|}$, and $\vec{i} = \frac{\vec{w_i}}{|\vec{w_i}|}$. The length of both Bloch vectors are

$$|\vec{w_f}| = |\vec{w_i}| = \lambda^2 - \frac{1}{4}.$$

The vector $\vec{w_f}$ is built as in (14) as follows:

$$\vec{w_f} = \begin{pmatrix} \langle\xi_{\text{out}}(t)|\sigma^1|\xi_{\text{out}}(t)\rangle \\ \langle\xi_{\text{out}}(t)|\sigma^2|\xi_{\text{out}}(t)\rangle \\ \langle\xi_{\text{out}}(t)|\sigma^3|\xi_{\text{out}}(t)\rangle \end{pmatrix} \tag{82}$$

and

$$\vec{w_i} = \begin{pmatrix} \langle \xi_{\rm in}(t)|\sigma^1|\xi_{\rm in}(t)\rangle \\ \langle \xi_{\rm in}(t)|\sigma^2|\xi_{\rm in}(t)\rangle \\ \langle \xi_{\rm in}(t)|\sigma^3|\xi_{\rm in}(t)\rangle \end{pmatrix}. \tag{83}$$

In terms of the components, the σ^z weak measurement is given by

$$\sigma_{r,w} = \frac{{w_f}^3 + {w_i}^3 + i(-{w_f}^1{w_i}^2 + {w_f}^2{w_i}^1)}{\frac{1}{2}(1 + {w_f}^1{w_i}^1 + {w_f}^2{w_i}^2 + {w_f}^3{w_i}^3)}. \tag{84}$$

For more details in the computations, see Appendix C.

Choosing $\Phi_{\rm in}(t_{\rm in}) = 0$ made possible for $t_{\rm in} = t_B$ [see (29)] and $\phi_0 = 0$; we find, respectively, the real part, the imaginary part of the numerator, and the denominator of the weak value of σ^z operator in terms of the preselected and postselected vector components,

$$\begin{aligned} {w_f}^3 + {w_i}^3 = {}& \sin(2g(t))\left(\sqrt{N^2 - \frac{R_0^4}{\lambda^2}}\cos(2F(t,t_{\rm in})) + \sqrt{N^2 - \frac{R_{\rm max}^2}{m^2}}\cos(\eta_{{\rm out}_1} - \eta_{{\rm out}_2} - 2F(t_{\rm out},t))\right) \\ & + \cos(2g(t))\left(\frac{R_{\rm max}}{m} - \frac{R_0^2}{\lambda}\right), \end{aligned} \tag{85}$$

$$\begin{aligned} -{w_f}^1{w_i}^2 + {w_f}^2{w_i}^1 = {}& \sin(2g(t))\left(-\frac{R_{\rm max}}{m}\sqrt{N^2 - \frac{R_0^4}{\lambda^2}}\sin(2F(t,t_{\rm in})) + \sqrt{N^2 - \frac{R_{\rm max}^2}{m^2}}\frac{R_0^2}{\lambda}\sin(\eta_{{\rm out}_1} - \eta_{{\rm out}_2} - 2F(t_{\rm out},t)\right) \\ & + \sqrt{N^2 - \frac{R_{\rm max}^2}{m^2}}\sqrt{N^2 - \frac{R_0^4}{\lambda^2}}\cos(2g(t))\sin(\eta_{{\rm out}_1} - \eta_{{\rm out}_2} + 2F(t,t_{\rm in}) + 2F(t,t_{\rm out})), \end{aligned} \tag{86}$$

and the denominator $(1 + {w_f}^1{w_i}^1 + {w_f}^2{w_i}^2 + {w_f}^3{w_i}^3)$

$$= \sqrt{N^2 - \frac{R_0^4}{\lambda^2}}\sqrt{N^2 - \frac{R_{\rm max}^2}{m^2}}\cos(2F(t,t_{\rm in}) + 2F(t,t_{\rm out}) + \eta_{{\rm out}_1} - \eta_{{\rm out}_2}) - \frac{R_0^2 R_{\rm max}}{\lambda m} + 1. \tag{87}$$

One can see that the amplification phenomenon is obtained in many ways. One way is

$$\cos(a) \approx -\frac{1 - \frac{R_0^2 R_{\rm max}}{\lambda m}}{\sqrt{N^2 - \frac{R_0^4}{\lambda^2}}\sqrt{N^2 - \frac{R_{\rm max}^2}{m^2}}}, \tag{88}$$

wherein $a = 2F(t,t_{\rm in}) + 2F(t,t_{\rm out}) + \eta_{{\rm out}_1} - \eta_{{\rm out}_2}$. By adjusting adequately the parameters appearing in a, one can amplify the weak values.

It is essential to note that any arbitrary observable in a two-level system can be expressed as $\hat{A} = a_I\hat{I}_2 + a_L\vec{\alpha}.\hat{\vec{\sigma}}$, where a_I and a_L are constants, $\vec{\alpha}$ is a unit vector in three dimensions, and σ denotes the Pauli matrices.

In this subsection we have derived the computation of the weak value of the Z component of the spin of fermions in the context of cosmology. We have shown how it is possible to amplify the signal by choosing adequate parameters.

C. Berry phase from the weak value of a pure state

Let us now compute the weak value of a pure state which is given as follows:

$$\hat{\Pi}_{r,w} = \frac{\langle \psi_{\rm out}(t)|\hat{\Pi}_r|\psi_{\rm in}(t)\rangle}{\langle \psi_{\rm out}(t)|\psi_{\rm in}(t)\rangle}, \tag{89}$$

where $\hat{\Pi}_r$ is a pure state given as $\frac{1}{2}(\hat{I} + \vec{r}\,\hat{\vec{\sigma}})$. The complete formula to compute the weak values of a pure state is found in [39] or the Appendix of [40] and [7]. It reads as

$$\hat{\Pi}_{r,w} = \frac{1 + \vec{f}.\vec{r} + \vec{r}.\vec{i} + \vec{f}.\vec{i} + i\vec{f}.(\vec{r}\wedge\vec{i})}{\frac{1}{2}(1 + \vec{f}.\vec{i})}. \tag{90}$$

The vector $\vec{r}$ is defined as

$$\begin{pmatrix} \sin(\theta)\cos(\phi) \\ \sin(\theta)\sin(\phi) \\ \cos(\theta) \end{pmatrix}. \tag{91}$$

We obtain the real part of the numerator computed for $\Phi_{\text{in}}(t_{\text{in}}) = 0$ and $\theta = \varphi = \frac{\pi}{2}$,

$$1 + A\sin(\eta_{\text{out}_1} - \eta_{\text{out}_2} - 2F(t_{\text{out}}, t) - \frac{R_0^2 R_{\max}}{\lambda m} + (A\cos(a) - \sin(2F(t_{\text{out}}, t)))\sqrt{N^2 - \frac{R_0^4}{\lambda^2}}, \quad (92)$$

wherein $A = \sqrt{N^2 - \frac{R_{\max}^2}{m^2}}$ and $a = 2F(t, t_{\text{in}}) + 2F(t, t_{\text{out}}) + \eta_{\text{out}_1} - \eta_{\text{out}_2}$.

The imaginary part of numerator of the $\hat{\Pi}_{r,w}$ is given as

$$-\frac{R_{\max}}{m}\sqrt{N^2 - \frac{R_0^4}{\lambda^2}}\cos(2F(t, t_{\text{in}})) - \frac{R_0^2}{\lambda}\sqrt{N^2 - \frac{R_{\max}^2}{m^2}}\cos(2F(t_{\text{out}}, t) - (\eta_{\text{out}_1} - \eta_{\text{out}_2})). \quad (93)$$

The denominator, $1 + \vec{f}.\vec{i}$, remains the same as computed previously,

$$\sqrt{N^2 - \frac{R_0^4}{\lambda^2}}\sqrt{N^2 - \frac{R_{\max}^2}{m^2}}\cos(a) - \frac{R_0^2 R_{\max}}{\lambda m} + 1. \quad (94)$$

The amplification phenomenon is obtained as explained in the case of the weak value of σ^z.

Let us compute now the argument of the weak value of the pure state. It is given from [7] as

$$\arg \hat{\Pi}_{r,w} = \arctan \frac{\vec{f}.(\vec{r} \wedge \vec{i})}{1 + \vec{f}.\vec{r} + \vec{r}.\vec{i} + \vec{f}.\vec{i}}. \quad (95)$$

The berry phase will be given by $\arg \hat{\Pi}_{r,w} =$

$$\arctan\left(\frac{-\frac{R_{\max}}{m}\sqrt{N^2 - \frac{R_0^4}{\lambda^2}}\cos(2F(t, t_{\text{in}})) - \frac{R_0^2}{\lambda}A\cos(2F(t_{\text{out}}, t) - (\eta_{\text{out}_1} - \eta_{\text{out}_2}))}{1 - A\sin(2F(t_{\text{out}}, t) - (\eta_{\text{out}_1} - \eta_{\text{out}_2})) - \frac{R_0^2 R_{\max}}{\lambda m} + (A\cos(a) - \sin(2F(t_{\text{out}}, t)))\sqrt{N^2 - \frac{R_0^4}{\lambda^2}}}\right). \quad (96)$$

A detailed computation is found in Appendix D. As a conclusion, this berry phase confirms the fact that the Universe considered here is not flat since its argument is not 0 or a multiple of 2π in all cases.

IV. CONCLUSION

Our objectives in this paper were to apply the theory of weak measurement in the Einstein-Dirac system in the *Friedman-Lemaître-Robertson-Walker* space. We began by noticing, with the help of Davies [9], that all observations and measurements are weak in the quantum sense due to the nature of the processes involved. Then, after computing the two asymptotic state vectors, solutions of the Einstein-Dirac equations around the big bang and the Universe at its maximal value of the scale factor and finding the evolutionary operator, we computed the weak values of the energy momentum tensor and two other operators.

In this work, we have elucidated numerous conclusions, particularly regarding the enlarged scope of weak measurements in contrast to classical measurement paradigms. Notably, our analysis revealed that the weak value of the energy-momentum tensor extends beyond what is described by classical energy-momentum concepts, exhibiting unusual values as complex numbers in certain instances.

We have also shown that it is possible to amplify measurements with less expensive equipments through the weak measurements process by strategically assuming appropriate initial and final state vector. Furthermore, weak measurements serve as a powerful tool for shedding some light on the geometric characteristics of the underlying space. By virtue of its association with the Berry phase, weak measurements enabled us to ascertain that the underlying space exhibits nonflat geometry, as evidenced by the weak measurement of fermion wave functions.

Our analytical framework also demonstrates the effectiveness of weak measurements in detecting the acceleration of the Universe, leveraging the two-state vector formalism and weak measurement techniques. Notably, we established that the scale function may be understood as an outcome of the postselection process.

We end this paper by noticing that it is possible to use this tool experimentally in cosmology. This should be the next step for some future researches.

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APPENDIX A: THE DIRAC EQUATION AND ITS SEPARATION

In this article, we specifically consider the case of a closed universe with $k = 1$, although similar computations can be extended to other cases. In this scenario, the line element simplifies to

$$ds^2 = dt^2 - R^2(t)\left(\frac{dr^2}{1-r^2} + r^2 d\Omega^2\right). \tag{A1}$$

In this case, the coordinate r will vary in the interval (0, 1).

We follow Finster in [15] and [30] to construct the Dirac operator. Our Dirac matrices in FLRW spacetime are expressed as

$$\begin{aligned}
G^0 &= \gamma^0,\\
G^r &= \frac{f(r)}{R(t)}(\cos\theta\gamma^3 + \sin\theta\cos\phi\gamma^1 + \sin\theta\sin\phi\gamma^2),\\
G^\theta &= \frac{1}{rR(t)}(-\sin\theta\gamma^3 + \cos\theta\cos\phi\gamma^1 + \cos\theta\sin\phi\gamma^2),\\
G^\phi &= \frac{1}{rR(t)\sin\theta}(-\sin\phi\gamma^1 + \cos\phi\gamma^2),
\end{aligned} \tag{A2}$$

wherein

$$f(r) = \sqrt{1-r^2}, \qquad r \in (0, 1), \tag{A3}$$

and the Dirac matrices are given as

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \qquad \gamma^\alpha = \begin{pmatrix} 0 & \sigma^\alpha \\ -\sigma^\alpha & 0 \end{pmatrix}.$$

I is the 2×2 identity matrix, and σ^α are the Pauli matrices,

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{A4}$$

The differential operator D in $(\mathcal{D} - m)\Psi = 0$ is given by $D_\mu = \partial_\mu - iE_\mu$, and G^μ obeys to the anticommutation rule

$$\{G^\mu, G^\nu\} \equiv G^\mu G^\nu + G^\nu G^\mu = 2g^{\mu\nu}\mathbf{1}_4.$$

The so-called spin coefficients are given by E_μ.

For an orthogonal metric, the combination $G^\mu E_\mu$ takes the simple form (for more details see [15])

$$G^\mu E_\mu = \frac{i}{2\sqrt{|g|}}\partial_\mu\left(\sqrt{|g|}G^\mu\right) \quad \text{with} \quad g = \det g_{ij},$$

making it unnecessary to compute the spin connection coefficients or even the Christoffel symbols. The constructed Dirac equation yields

$$\left[i\gamma^0\left(\partial_t + \frac{3\dot{R}}{2R}\right) - R(t)m + \begin{pmatrix} 0 & \mathcal{D}_\mathcal{H} \\ -\mathcal{D}_\mathcal{H} & 0 \end{pmatrix}\right]\Psi = 0, \tag{A5}$$

where $\mathcal{D}_\mathcal{H}$ represents the purely spatial operator on S^3,

$$\mathcal{D}_\mathcal{H} = i\sigma^r\left(\partial_r + \frac{f-1}{rf}\right) + i\sigma^\theta\partial_\theta + \sigma^\phi\partial_\phi. \tag{A6}$$

The Pauli matrices σ^r, σ^θ, and σ^ϕ are linear combinations of the Pauli matrices defined in (A4),

$$\begin{cases}
\sigma^r := f(r)(\cos\theta\sigma^3 + \sin\theta\cos\phi\sigma^1 + \sin\theta\sin\phi\sigma^2)\\
\sigma^\theta := \frac{1}{r}(-\sin\theta\sigma^3 + \cos\theta\cos\phi\sigma^1 + \cos\theta\sin\phi\sigma^2)\\
\sigma^\phi := \frac{1}{r\sin\theta}(-\sin\phi\sigma^1 + \cos\phi\sigma^2),
\end{cases} \tag{A7}$$

and the function $f(r)$ is given in (A3).

Note that $G_\mu = g_{\mu\nu}G^\nu$. This spatial operator $\mathcal{D}_\mathcal{H}$ has a purely discrete spectrum

$$\sigma(\mathcal{D}_\mathcal{H}) = \left\{\pm\frac{3}{2}, \pm\frac{5}{2}, \pm\frac{7}{2}, \ldots\right\}, \tag{A8}$$

and the dimension of the corresponding eigenspace is given as follows:

$$\dim\ \ker(\mathcal{D}_\mathcal{H} - \lambda) = \lambda^2 - \frac{1}{4}. \tag{A9}$$

An orthonormal eigenvector basis, in terms of spherical harmonics and Jacobi polynomials, is provided in the Appendix of [30] and is denoted by $\psi^\pm_{njk}$, where $n \in \mathbb{N}_0$, $j \in \mathbb{N}_0 + \frac{1}{2}$, and $k \in \{-j, -j+1, \ldots, j\}$.

The spatial differential operator on the eigenvector can be represented as follows:

$$\mathcal{D}_\mathcal{H}\psi^\pm_{njk} = \lambda\psi^\pm_{njk}, \quad \text{where } \lambda = \pm(n+j+1). \tag{A10}$$

We represent the normalized eigenfunction of $\mathcal{D}_\mathcal{H}$ related to the eigenvalue λ by $\psi_\lambda \in L^2(S^3)^2$.

For the separation of the Dirac equation

$$(D - m)\Psi = 0, \tag{A11}$$

one has to consider an eigenspinor ψ_λ of the spatial Dirac operator

$$\mathcal{D}_\mathcal{H}\psi_\lambda = \lambda\psi_\lambda,$$

and use

$$\Psi = \frac{1}{R(t)^{\frac{3}{2}}}\begin{pmatrix}\alpha(t)\psi_\lambda(r,\theta,\phi)\\ \beta(t)\psi_\lambda(r,\theta,\phi)\end{pmatrix}. \tag{A12}$$

APPENDIX B: WEAK MEASUREMENTS OF THE ENERGY MOMENTUM

In this section, we compute the weak value of the energy-momentum tensor given by

$$\langle T_{\mu\nu}\rangle_w = \frac{\langle\psi_{\rm out}(t)|T_{\mu\nu}|\psi_{\rm in}(t)\rangle}{\langle\psi_{\rm out}(t)|\psi_{\rm in}(t)\rangle}. \tag{B1}$$

More explicitly, we have

$$\langle T_{\mu\nu}\rangle_w = \frac{1}{2}\frac{\langle\psi_{\rm out}(t)|(iG_\mu D_\nu + iG_\nu D_\mu)|\psi_{\rm in}(t)\rangle}{\langle\psi_{\rm out}(t)|\psi_{\rm in}(t)\rangle}, \tag{B2}$$

where G^μ are the linear combinations of the Dirac matrices of Minkowski space given by (A2).

To compute the numerator part of the weak value of the energy-momentum tensor, following Finster in [30], we adapt the object $P(t,\vec{x};t',\overrightarrow{x'})$

$$= (R(t)R(t'))^{-\frac{3}{2}}E_\lambda(\vec{x},\overrightarrow{x'})\otimes\begin{pmatrix}\alpha_{\rm in}(t)\overline{\alpha_{\rm out}(t')} - \alpha_{\rm in}(t)\overline{\beta_{\rm out}(t')}\\ \beta_{\rm in}(t)\overline{\alpha_{\rm out}(t')} - \beta_{\rm in}(t)\overline{\beta_{\rm out}(t')}\end{pmatrix} \tag{B3}$$

to our need, wherein E_λ denote the spectral projectors of $\mathcal{D}_{S^3}$, given by $E_{\pm|\lambda|} = \sum_{n=0}^{|\lambda|-\frac{3}{2}}\sum_{k=-j}^{j}\psi^{\pm}_{njk}(x)\overline{\psi^{\pm}_{njk}(\acute{x})}$. Then, the numerator part of weak value of the energy-momentum tensor, $\langle\psi_{\rm out}|T_{\mu\nu}|\psi_{\rm in}\rangle(t,\vec{x})$, can be expressed in terms of P by

$$= \frac{1}{2}\mathrm{Tr}_{\mathbb{C}^4}\left\{(iG_\mu D_\nu + iG_\nu D_\mu)P(t,\vec{x};t',\vec{x}')\right\}\Big|_{t'=t,\vec{x}'=\vec{x}}. \tag{B4}$$

Because of the homogeneity and the isotropy of the FLRW space, only the diagonal energy-momentum tensor components are different from 0. We get the following results after computations and normalization,

$$\langle T^0_0\rangle_w = \frac{\langle\psi_{\rm out}(t)|T^0_0|\psi_{\rm in}(t)\rangle}{\langle\psi_{\rm out}(t)|\psi_{\rm in}(t)\rangle} = \frac{\left[mR(t)(\alpha_{\rm in}(t)\overline{\alpha_{\rm out}}(t) - \beta_{\rm in}(t)\overline{\beta_{\rm out}}(t)) - \lambda(\alpha_{{\rm in}(t)}\overline{\beta_{\rm out}}(t) + \beta_{\rm in}(t)\overline{\alpha_{\rm out}}(t))\right]}{R(t)(\overline{\alpha_{\rm out}}(t)\alpha_{\rm in}(t) + \overline{\beta_{\rm out}}(t)\beta_{\rm in}(t))}, \tag{B5}$$

and

$$\langle T^j_j\rangle_w = \frac{\langle\psi_{\rm out}(t)|T^j_j|\psi_{\rm in}(t)\rangle}{\langle\psi_{\rm out}(t)|\psi_{\rm in}(t)\rangle} = \frac{\lambda(\alpha_{\rm in}(t)\overline{\beta_{\rm out}}(t) + \beta_{\rm in}(t)\overline{\alpha_{\rm out}}(t))}{3R(t)(\overline{\alpha_{\rm out}}(t)\alpha_{\rm in}(t) + \overline{\beta_{\rm out}}(t)\beta_{\rm in}(t))}, \tag{B6}$$

where j represents r,θ,ϕ. The space diagonal energy momentum tensors are all equal, that is $T^r_r = T^\theta_\theta = T^\phi_\phi$.

1. Computation of the preselected state at time t

The spinor at time t is obtained by applying the evolutionary operator on a spinor defined at a specific time. The case of preselected state at time t is the preselected state at time $t_{\rm in}$ on which the unitary operator is applied. It is given by

$$\begin{pmatrix}\alpha_{\rm in}(t)\\ \beta_{\rm in}(t)\end{pmatrix} = U(t,t_{\rm in})\begin{pmatrix}\alpha_{\rm in}(t_{\rm in})\\ \beta_{\rm in}(t_{\rm in})\end{pmatrix},$$

wherein $U(t,t_{\rm in})$ is given in (55). This will lead to

$$\alpha_{\rm in}(t) = \frac{\sqrt{2}}{2}\alpha_{\rm in}(t_{\rm in})\left[\cos(g(t))e^{-iF(t,t_{\rm in})} + \sin(g(t))e^{iF(t,t_{\rm in})}\right] + \frac{\sqrt{2}}{2}\beta_{\rm in}(t_{\rm in})\left[-\cos(g(t))e^{-iF(t,t_{\rm in})} + \sin(g(t))e^{iF(t,t_{\rm in})}\right], \tag{B7}$$

and

$$\beta_{\rm in}(t) = \frac{\sqrt{2}}{2}\alpha_{\rm in}(t_{\rm in})\left[-\sin(g(t))e^{-iF(t,t_{\rm in})} + \cos(g(t))e^{iF(t,t_{\rm in})}\right] + \frac{\sqrt{2}}{2}\beta_{\rm in}(t_{\rm in})\left[\sin(g(t))e^{-iF(t,t_{\rm in})} + \cos(g(t))e^{iF(t,t_{\rm in})}\right]. \tag{B8}$$

We remind here that $F(t,t_{\rm in}) = \int_{t_{\rm in}}^{t}\frac{\sqrt{R^2(t)m^2+\lambda^2}}{R(t)}dt$.

2. Computation of the postselected state at time *t*

The postselected state is taken from the dust dominated Universe, wherein $\lambda = 0$. Knowing that $U^\dagger(t, t_{\text{out}}) = U(t_{\text{out}}, t)$, wherein $U(t_{\text{out}}, t)$ is specified in (60), we have

$$\left(\overline{\alpha_{\text{out}}}(t)\,\overline{\beta_{\text{out}}}(t)\right) = \left(\overline{\alpha_{\text{out}}}(t_{\text{out}})\,\overline{\beta_{\text{out}}}(t_{\text{out}})\right) U(t_{\text{out}}, t).$$

This will lead to

$$\overline{\alpha_{\text{out}}}(t) = \overline{\alpha_{\text{out}}}(t_{\text{out}}) \cos(g(t)) e^{-iF(t_{\text{out}},t)} + \overline{\beta_{\text{out}}}(t_{\text{out}}) \sin(g(t)) e^{iF(t_{\text{out}},t)}, \tag{B9}$$

and

$$\overline{\beta_{\text{out}}}(t) = -\overline{\alpha_{\text{out}}}(t_{\text{out}}) \sin(g(t)) e^{-iF(t_{\text{out}},t)} + \overline{\beta_{\text{out}}}(t_{\text{out}}) \cos(g(t)) e^{iF(t_{\text{out}},t)}. \tag{B10}$$

3. Computation of numerators and denominators of the energy-momentum tensor

In terms of $\alpha_{\text{in}}(t_{\text{in}})$ and $\alpha_{\text{out}}(t_{\text{out}})$, where

$$A = e^{i(F(t_{\text{out}},t_{\text{in}}))}, \qquad \overline{A} = e^{-i(F(t_{\text{out}},t_{\text{in}}))},$$
$$B = e^{i(F(t,t_{\text{in}})+F(t,t_{\text{out}}))}, \qquad \bar{B} = e^{-i(F(t,t_{\text{in}})+F(t,t_{\text{out}}))},$$

(i) $(\overline{\alpha_{\text{out}}}(t)\alpha_{\text{in}}(t) + \overline{\beta_{\text{out}}}(t)\beta_{\text{in}}(t))$ is equal to

$$\frac{\sqrt{2}}{2}(\overline{\alpha_{\text{out}}}(t_{\text{out}})\alpha_{\text{in}}(t_{\text{in}}) - \overline{\alpha_{\text{out}}}(t_{\text{out}})\beta_{\text{in}}(t_{\text{in}}))\bar{A} + \frac{\sqrt{2}}{2}\left(\overline{\beta_{\text{out}}}(t_{\text{out}})\beta_{\text{in}}(t_{\text{in}}) + \overline{\beta_{\text{out}}}(t_{\text{out}})\alpha_{\text{in}}(t_{\text{in}})\right)A. \tag{B11}$$

(ii) $(\alpha_{\text{in}}(t)\overline{\alpha_{\text{out}}}(t) - \beta_{\text{in}}(t)\overline{\beta_{\text{out}}}(t))$ is equal to

$$\begin{aligned}&\frac{\sqrt{2}}{2}\alpha_{\text{in}}(t_{\text{in}})\overline{\alpha_{\text{out}}}(t_{\text{out}})[\sin(2g(t))B + \cos(2g(t))\bar{A}] \\ &+\frac{\sqrt{2}}{2}\beta_{\text{in}}(t_{\text{in}})\overline{\beta_{\text{out}}}(t_{\text{out}})[-\sin(2g(t))\bar{B} - \cos(2g(t))A] \\ &+\frac{\sqrt{2}}{2}\alpha_{\text{in}}(t_{\text{in}})\overline{\beta_{\text{out}}}(t_{\text{out}})[\sin(2g(t))\bar{B} - \cos(2g(t))A] \\ &+\frac{\sqrt{2}}{2}\beta_{\text{in}}(t_{\text{in}})\overline{\alpha_{\text{out}}}(t_{\text{out}})[\sin(2g(t))B - \cos(2g(t))\bar{A}].\end{aligned} \tag{B12}$$

(iii) $(\alpha_{\text{in}}(t)\overline{\beta_{\text{out}}}(t) + \beta_{\text{in}}(t)\overline{\alpha_{\text{out}}}(t))$ is equal to

$$\begin{aligned}&\frac{\sqrt{2}}{2}\alpha_{\text{in}}(t_{\text{in}})\overline{\alpha_{\text{out}}}(t_{\text{out}})[-\sin(2g(t))\bar{A} + \cos(2g(t))B] \\ &+\frac{\sqrt{2}}{2}\beta_{\text{in}}(t_{\text{in}})\overline{\beta_{\text{out}}}(t_{\text{out}})[\sin(2g(t))A - \cos(2g(t))\bar{B}] \\ &+\frac{\sqrt{2}}{2}\alpha_{\text{in}}(t_{\text{in}})\overline{\beta_{\text{out}}}(t_{\text{out}})[\sin(2g(t))A + \cos(2g(t))\bar{B}] \\ &+\frac{\sqrt{2}}{2}\beta_{\text{in}}(t_{\text{in}})\overline{\alpha_{\text{out}}}(t_{\text{out}})[\sin(2g(t))\bar{A} + \cos(2g(t))B].\end{aligned} \tag{B13}$$

APPENDIX C: WEAK MEASUREMENTS OF σ^z OPERATOR

The weak measurement of the generator of $SU(2)$ is given by

$$\sigma_{r,w} = \frac{\vec{f}.\vec{r} + \vec{r}.\vec{i} + i\vec{f}.(\vec{r}\wedge\vec{i})}{\frac{1}{2}(1 + \vec{f}.\vec{i})}. \tag{C1}$$

In our case, $\vec{r} = (0\,0\,1)^T$ to get σ^z. Following Kofman in [39], the vectors $\vec{f}$ and $\vec{i}$ are computed as follows:

$$\vec{w_f} = \begin{pmatrix}\langle\xi_{\text{out}}(t)|\sigma^1|\xi_{\text{out}}(t)\rangle \\ \langle\xi_{\text{out}}(t)|\sigma^2|\xi_{\text{out}}(t)\rangle \\ \langle\xi_{\text{out}}(t)|\sigma^3|\xi_{\text{out}}(t)\rangle\end{pmatrix} = \begin{pmatrix}2\mathrm{Re}(\alpha_{\text{out}}(t)\overline{\beta}_{\text{out}}(t)) \\ 2\mathrm{Im}(\alpha_{\text{out}}(t)\overline{\beta}_{\text{out}}(t)) \\ |\alpha_{\text{out}(t)}|^2 - |\beta_{\text{out}(t)}|^2\end{pmatrix},$$

wherein $\xi_{\text{out}}(t) = \begin{pmatrix}\alpha_{\text{out}}(t) \\ \beta_{\text{out}}(t)\end{pmatrix}$.

The postselected state components $\alpha_{\text{out}}(t)$ and $\beta_{\text{out}}(t)$ are given in (B9) and (B10).

The products $\alpha_{\text{out}}(t)\bar{\beta}_{\text{out}}(t)$ yield

$$\begin{aligned}\alpha_{\text{out}}(t)\bar{\beta}_{\text{out}}(t) &= -(|\alpha_{\text{out}}(t_{\text{out}})|^2 - |\beta_{\text{out}}(t_{\text{out}})|^2)\sin(g(t))\cos(g(t)) \\ &\quad+ \overline{\alpha_{\text{out}}}(t_{\text{out}})\beta_{\text{out}}(t_{\text{out}})\cos^2(g(t))e^{-2iF(t_{\text{out}},t)} \\ &\quad- \overline{\beta_{\text{out}}}(t_{\text{out}})\alpha_{\text{out}}(t_{\text{out}})\sin^2(g(t))e^{2iF(t_{\text{out}},t)}.\end{aligned} \tag{C2}$$

The real part will be given in terms of $\alpha_{\text{out}}(t_{\text{out}}) = \rho_{\text{out}_1} e^{i\eta_{\text{out}_1}}$ and $\beta_{\text{out}}(t_{\text{out}}) = \rho_{\text{out}_2} e^{i\eta_{\text{out}_2}}$ as

$$\begin{aligned}\mathrm{Re}(\alpha_{\text{out}}(t)\bar{\beta}_{\text{out}}(t)) &= \frac{1}{2}(\rho^2_{\text{out}_2} - \rho^2_{\text{out}_1})\sin(2g(t)) \\ &\quad+ \rho_{\text{out}_1}\rho_{\text{out}_2}\cos(2g(t))\cos(\eta_{\text{out}_2} - \eta_{\text{out}_1} - 2F(t_{\text{out}}, t)),\end{aligned} \tag{C3}$$

and the imaginary part will be

$$\mathrm{Im}(\alpha_{\mathrm{out}}(t)\bar{\beta}_{\mathrm{out}}(t)) = \frac{1}{2}(\rho^2_{\mathrm{out}_2} - \rho^2_{\mathrm{out}_1})\sin(2g(t)) + \rho_{\mathrm{out}_1}\rho_{\mathrm{out}_2}\sin(\eta_{\mathrm{out}_2} - \eta_{\mathrm{out}_1} - 2F(t_{\mathrm{out}}, t)). \tag{C4}$$

We obtain the following vector $\vec{w_f}$ components:

$$\begin{cases} {w_f}^1 = -\Delta\rho^2_{\mathrm{out}}\sin(2g(t)) + 2\rho_{\mathrm{out}_1}\rho_{\mathrm{out}_2}\cos(2g(t))\cos(\Delta_{\mathrm{out}}) \\ {w_f}^2 = 2\rho_{\mathrm{out}_1}\rho_{\mathrm{out}_2}\sin(\Delta_{\mathrm{out}}) \\ {w_f}^3 = \Delta\rho^2_{\mathrm{out}}\cos(2g(t)) + 2\rho_{\mathrm{out}_1}\rho_{\mathrm{out}_2}\sin(2g(t))\cos(\Delta_{\mathrm{out}}), \end{cases} \tag{C5}$$

where $(\eta_{\mathrm{out}_1} - \eta_{\mathrm{out}_2} + 2F(t_{\mathrm{out}}, t)) = \Delta_{\mathrm{out}}$, $\rho^2_{\mathrm{out}_1} - \rho^2_{\mathrm{out}_2} = \Delta\rho^2_{\mathrm{out}}$. Its norm is $|\vec{w_f}| = \lambda^2 - \frac{1}{4}$.

This implies that the unit vector $\vec{f}$ would be given by

$$\vec{f} = \frac{1}{\lambda^2 - \frac{1}{4}}\vec{w_f}.$$

In the same way, the preselected state vector is computed as follows:

$$\vec{w_i} = \begin{pmatrix} \langle\xi_{\mathrm{in}}(t)|\sigma^1|\xi_{\mathrm{in}}(t)\rangle \\ \langle\xi_{\mathrm{in}}(t)|\sigma^2|\xi_{\mathrm{in}}(t)\rangle \\ \langle\xi_{\mathrm{in}}(t)|\sigma^3|\xi_{\mathrm{in}}(t)\rangle \end{pmatrix} = \begin{pmatrix} 2\mathrm{Re}(\alpha_{\mathrm{in}}(t)\bar{\beta}_{\mathrm{in}}(t)) \\ 2\mathrm{Im}(\alpha_{\mathrm{in}}(t)\bar{\beta}_{\mathrm{in}}(t)) \\ |\alpha_{\mathrm{in}(t)}|^2 - |\beta_{\mathrm{in}(t)}|^2 \end{pmatrix},$$

wherein $\xi_{\mathrm{in}}(t) = \begin{pmatrix} \alpha_{\mathrm{in}}(t) \\ \beta_{\mathrm{in}}(t) \end{pmatrix}$,

$$\begin{aligned} \alpha_{\mathrm{in}}(t)\bar{\beta}_{\mathrm{in}}(t) = {} & (|\alpha_{\mathrm{in}}(t_{\mathrm{in}})|^2 - |\beta_{\mathrm{in}}(t_{\mathrm{in}})|^2)\left(\cos^2(g(t))e^{-2iF(t,t_{\mathrm{in}})} - \sin^2(g(t))e^{2iF(t,t_{\mathrm{in}})}\right) \\ & + \alpha_{\mathrm{in}}(t_{\mathrm{in}})\bar{\beta}_{\mathrm{in}}(t_{\mathrm{in}})\left(\sin(2g(t)) + \sin^2(g(t))e^{2iF(t,t_{\mathrm{in}})} + \cos^2(g(t))e^{-2iF(t,t_{\mathrm{in}})}\right) \\ & + \beta_{\mathrm{in}}(t_{\mathrm{in}})\bar{\alpha}_{\mathrm{in}}(t_{\mathrm{in}})\left(\sin(2g(t)) - \sin^2(g(t))e^{2iF(t,t_{\mathrm{in}})} - \cos^2(g(t))e^{-2iF(t,t_{\mathrm{in}})}\right). \end{aligned} \tag{C6}$$

The real part of the above product in terms of $\alpha_{\mathrm{in}}(t_{\mathrm{in}}) = \rho_{in_1}e^{i\eta_{in_1}}$ and $\beta_{i\mathrm{in}}(t_{\mathrm{in}}) = \rho_{in_2}e^{\eta_{in_2}}$ will be

$$\begin{aligned} \mathrm{Re}(\alpha_{\mathrm{in}}(t)\bar{\beta}_{\mathrm{in}}(t)) = {} & (\rho^2_{in_1} - \rho^2_{in_2})\cos(2g(t))\cos(2F(t, t_{\mathrm{in}})) + 2\rho_{in_1}\rho_{in_2}[\sin(2g(t))\cos(\eta_{in_1} - \eta_{in_2}) \\ & - \cos(2g(t))\sin(\eta_{in_1} - \eta_{in_2})\sin(2F(t, t_{\mathrm{in}}))], \end{aligned} \tag{C7}$$

and its imaginary part will be

$$\mathrm{Im}(\alpha_{\mathrm{in}}(t)\bar{\beta}_{\mathrm{in}}(t)) = (\rho^2_{in_2} - \rho^2_{in_1})\sin(2F(t, t_{\mathrm{in}})) + 2\rho_{in_1}\rho_{in_2}\sin(\eta_{in_1} - \eta_{in_2})\cos(2F(t, t_{\mathrm{in}})). \tag{C8}$$

In the same way, the preselected state components are given as

$$\begin{cases} {w_i}^1 = \Delta\rho^2_{\mathrm{in}}\cos(2g(t))\cos(2F(t, t_{\mathrm{in}})) + 2\rho_{in_1}\rho_{in_2}(\sin(2g(t))\cos(\Delta_{\mathrm{in}}) + \cos(2g)\sin(\Delta_{\mathrm{in}})\sin(2F(t, t_{\mathrm{in}}))) \\ {w_i}^2 = -\Delta\rho^2_{\mathrm{in}}\sin(2F(t, t_{\mathrm{in}})) + 2\rho_{in_1}\rho_{in_2}\sin(\Delta_{\mathrm{in}})\cos(2F(t, t_{\mathrm{in}})) \\ {w_i}^3 = \Delta\rho^2_{\mathrm{in}}\sin(2g(t))\cos(2F(t, t_{\mathrm{in}})) + 2\rho_{in_1}\rho_{in_2}(-\cos(2g(t))\cos(\Delta_{\mathrm{in}}) + \sin(2g(t))\sin(\Delta_{\mathrm{in}})\sin(2F(t, t_{\mathrm{in}}))), \end{cases} \tag{C9}$$

wherein $\eta_{in_1} - \eta_{in_2} = \Delta_{\rm in}$, $\rho^2_{in_1} - \rho^2_{in_2} = \Delta\rho^2_{\rm in}$. The vector $\vec{i}$ is equal to

$$\vec{i} = \frac{1}{\lambda^2 - \frac{1}{4}}\overrightarrow{w_i}.$$

The vectors $\vec{i}$ and $\vec{f}$ represent the unit vectors on the Bloch sphere of preselected and postselected state and $\vec{r} = (0\,0\,1)$. In terms of the components, the σ^z weak value is given by

$$\sigma_{r,w} = \frac{{w_f}^3 + {w_i}^3 + i(-{w_f}^1{w_i}^2 + {w_f}^2{w_i}^1)}{\frac{1}{2}(1 + {w_f}^1{w_i}^1 + {w_f}^2{w_i}^2 + {w_f}^3{w_i}^3)}. \tag{C10}$$

The real and the imaginary parts of the numerator of the weak measurement of σ^z operator $\sigma_{r,w}$ are given, respectively, by

$$\begin{aligned} {w_f}^3 + {w_i}^3 = {} & \Delta\rho^2_{\rm in}\cos(2F(t,t_{\rm in}))\sin(2g(t)) + \Delta\rho^2_{\rm out}\cos(2g(t)) \\ & + 2\rho_{in_1}\rho_{in_2}[-\cos(2g(t))\cos(\Delta_{\rm in}) + \sin(2F(t,t_{\rm in}))\sin(2g(t))\sin(\Delta_{\rm in})] + 2\rho_{\rm out_1}\rho_{\rm out_2}\sin(2g(t))\cos(\Delta_{\rm out}), \end{aligned} \tag{C11}$$

$$\begin{aligned} -{w_f}^1{w_i}^2 + {w_f}^2{w_i}^1 = {} & 2\rho_{in_1}\rho_{in_2}\{\Delta\rho^2_{\rm out}\cos(2F(t,t_{\rm in}))\sin(2g(t))\sin(\Delta_{\rm in}) \\ & + 2\rho_{\rm out_1}\rho_{\rm out_2}[-\cos(2g(t))\cos(2F(t,t_{\rm in}) + \Delta_{\rm out})\sin(\Delta_{\rm in}) \\ & + \cos(\Delta_{\rm in})\sin(2g(t))\sin(\Delta_{\rm out})]\} + \Delta\rho^2_{\rm in}[-\Delta\rho^2_{\rm out}\sin(2F(t,t_{\rm in}))\sin(2g(t)) \\ & + 2\rho_{\rm out_1}\rho_{\rm out_2}\cos(2g(t))\sin(2F(t,t_{\rm in}) + \Delta_{\rm out})], \end{aligned} \tag{C12}$$

and $1 + {w_f}^1{w_i}^1 + {w_f}^2{w_i}^2 + {w_f}^3{w_i}^3$ is given by

$$1 - 2\Delta\rho^2_{\rm out}\rho_{in_1}\rho_{in_2}\cos(\Delta_{\rm in}) + 2\rho_{\rm out_1}\rho_{\rm out_2}[\Delta\rho^2_{\rm in}\cos(2F(t,t_{\rm in}) + \Delta_{\rm out}) + 2\rho_{in_1}\rho_{in_2}\sin(\Delta_{\rm in})\sin(2F(t,t_{\rm in}) + \Delta_{\rm out})]. \tag{C13}$$

APPENDIX D: BERRY PHASE

The weak value of a pure state is given by

$$\hat{\Pi}_{r,w} = \frac{1 + \vec{f}.\vec{r} + \vec{r}.\vec{i} + \vec{f}.\vec{i} + i\vec{f}.(\vec{r}\wedge\vec{i})}{\frac{1}{2}(1 + \vec{f}.\vec{i})}. \tag{D1}$$

We obtain from $\vec{f} = \frac{\vec{w_f}}{|\vec{w_f}|}$ and $\vec{i} = \frac{\vec{w_i}}{|\vec{w_i}|}$, already computed in general terms, the real part $1 + \vec{f}.\vec{r} + \vec{r}.\vec{i} + \vec{f}.\vec{i} =$

$$\begin{aligned} & 1 + \cos(\varphi)\sin(\theta)\{\Delta\rho^2_{\rm in}\cos(2g(t))\cos(2F(t,t_{\rm in})) - \Delta\rho^2_{\rm out}\sin(2g(t)) \\ & + 2\rho_{in_1}\rho_{in_2}[\cos(2g(t))\sin(\Delta_{\rm in})\sin(2F(t,t_{\rm in})) + \sin(2g(t))\cos(\Delta_{\rm in})] + 2\rho_{\rm out_1}\rho_{\rm out_2}\cos(2g(t))\cos(\Delta_{\rm out})\} \\ & + \cos(\theta))\{\Delta\rho^2_{\rm in}\sin(2g(t))\cos(2F(t,t_{\rm in})) + \Delta\rho^2_{\rm out}\cos(2g(t)) + 2\rho_{in_1}\rho_{in_2}[-\cos(2g(t))\sin(2g(t)) \\ & \times\cos(\Delta_{\rm in})\sin(\Delta_{\rm in}) + \sin(2F(t,t_{\rm in}))] + 2\rho_{\rm out_1}\rho_{\rm out_2}\sin(2g(t))\cos(\Delta_{\rm out})\} \\ & + \sin(\theta)\sin(\varphi)\{2\rho_{in_1}\rho_{in_2}\sin(\Delta_{\rm in})\cos(2F(t,t_{\rm in})) - \Delta\rho^2_{\rm in}\sin(2F(t,t_{\rm in})) + 2\rho_{\rm out_1}\rho_{\rm out_2}\sin(\Delta_{\rm out})\}. \end{aligned} \tag{D2}$$

The imaginary part $\vec{f}.(\vec{r}\wedge\vec{i})$ is given by

$$\begin{aligned}
&\sin(\theta)\sin(\varphi)\{-\Delta\rho_{\mathrm{in}}^2\Delta\rho_{\mathrm{out}}^2\cos(2F(t,t_{\mathrm{in}}))-2\rho_{in_1}\rho_{in_2}[\Delta\rho_{\mathrm{out}}^2\sin(\Delta_{\mathrm{in}})\sin(2F(t,t_{\mathrm{in}}))+2\rho_{out_1}\rho_{\mathrm{out}_2}\cos(\Delta_{\mathrm{in}})\cos(\Delta_{\mathrm{out}})]\}\\
&+\cos(\theta)\{2\rho_{in_1}\rho_{in_2}\Delta\rho_{\mathrm{out}}^2[\sin(2g(t))\sin(\Delta_{\mathrm{in}})\cos(2F(t,t_{\mathrm{in}}))+2\rho_{\mathrm{out}_1}\rho_{\mathrm{out}_2}[-\cos(2g(t))\sin(\Delta_{\mathrm{in}})\cos(2F(t,t_{\mathrm{in}})+\Delta_{\mathrm{out}})\\
&+\sin(2g(t))\cos(\Delta_{\mathrm{in}})\sin(\Delta_{\mathrm{out}})]]+\Delta\rho_{\mathrm{in}}^2[2\rho_{\mathrm{out}_1}\rho_{\mathrm{out}_2}\cos(2g(t))\sin(2F(t,t_{\mathrm{in}})+\Delta_{\mathrm{out}})-\Delta\rho_{\mathrm{out}}^2\sin(2g(t))\sin(2F(t,t_{\mathrm{in}}))]\}\\
&+\cos(\varphi)\sin(\theta)\{2\rho_{in_1}\rho_{in_2}\Delta\rho_{\mathrm{out}}^2[\cos(2g(t))\sin(\Delta_{\mathrm{in}})\cos(2F(t,t_{\mathrm{in}}))+2\rho_{\mathrm{out}_1}\rho_{\mathrm{out}_2}[\sin(2g(t))\sin(\Delta_{\mathrm{in}})\cos(2F(t,t_{\mathrm{in}})+\Delta_{\mathrm{out}})\\
&+\cos(2g(t))\cos(\Delta_{\mathrm{in}})\sin(\Delta_{\mathrm{out}})]]-\Delta\rho_{\mathrm{in}}^2[2\rho_{\mathrm{out}_1}\rho_{\mathrm{out}_2}\sin(2g(t))\sin(2F(t,t_{\mathrm{in}})+\Delta_{\mathrm{out}})+\Delta\rho_{\mathrm{out}}^2\cos(2g(t))\sin(2F(t,t_{\mathrm{in}}))]\}.
\end{aligned}\tag{D3}$$

The denominator is the same as computed for the weak measurements of σ^z.

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