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A linear dissipativity approach to incremental input-to-state stability for a class of positive Lur'e systems

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ABSTRACT
Incremental stability properties are considered for certain systems of forced, nonlinear differential equations with a particular positivity structure. An incremental stability estimate is derived for pairs of input/state/output trajectories of the Lur'e systems under consideration, from which a number of consequences are obtained, including the incremental exponential input-to-state stability property and certain input–output stability concepts with linear gain. Our results show that an incremental version of the real Aizerman conjecture is true for positive Lur'e systems when an incremental gain condition is imposed on the nonlinear term, as we describe. Our argumentation is underpinned by linear dissipativity theory – a property of positive linear control systems.

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1. Introduction

Consider the following system of nonlinear controlled and observed differential equations

$$\left.\begin{aligned} \dot{x}(t) &= Ax(t) + B_1 f(t, C_1 x(t)) + B_2 w(t), \\ y(t) &= \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} x(t), \end{aligned}\right\} \tag{1}$$

a so-called forced four-block Lur'e (also Lurie or Lurye) system. Here A, B_i, C_i are appropriately-sized matrices, and f is a (time-varying) nonlinear function. The variables x, w and y in (1) denote the state, forcing term, and output, respectively. Depending on the context, the forcing term may represent a noise, disturbance or control variable. Here, we focus on incremental input-to-state stability (ISS) and incremental input–output stability properties of (1), under the assumption that it admits a certain positivity structure. Our approach appeals to linear dissipativity theory. We give background to each of these areas.

Positive dynamical systems are dynamical systems that leave a positive cone invariant. The academic literature has different conventions over the use of the words 'positive' and 'non-negative' in this context, and they are often used interchangeably. The most natural and obvious positive cone is arguably the non-negative orthant in real Euclidean space, equipped with the partial ordering of componentwise inequality. There are a myriad of physically motivated examples where the state variables denote necessarily non-negative quantities, such as abundances, densities, concentrations, flow rates or prices. Positive dynamical systems typically arise from mass balance relationships of these quantities – indeed, quoting Haddad et al. (2010, p. xv) '*[they] are widespread in biology, chemistry, ecology, economics, genetics, medicine, sociology, and engineering, and play a key role in the understanding of these disciplines.*' There are a number of monographs and textbooks devoted (either in whole or partially) to the study of positive systems, including Berman et al. (1989), Berman and Plemmons (1994) and Haddad et al. (2010). Positive dynamical systems are closely related to the concept of monotone dynamical systems (Hirsch & Smith, 2005), where the state variables inherit any ordering of the initial states.

Augmenting positive or monotone dynamical systems with input and output variables leads to so-called positive and monotone control systems, see the texts (Haddad et al., 2010) or (Angeli & Sontag, 2003), respectively. Again the nomenclature is confusing, as linear positive control systems, often simply called positive systems, are not strictly speaking linear since subtraction cannot always be performed within positive cones. Consequently, there are distinctions between results for positive systems and usual linear control systems, including on reachability and controllability (Caccetta & Rumchev, 2000), observability (Härdin & van Schuppen, 2007) and realisability (Benvenuti & Farina, 2004). However, the additional structure associated with positive systems is often helpful and simplifies matters. As an example, the analysis of positive systems benefits from readily constructed classes of scalable Lyapunov functions (Rantzer, 2015). A recent and thorough review of positive systems appears in Rantzer and Valcher (2021). Another feature of positive systems, which we utilise presently, is that of so-called *linear dissipativity*. Dissipativity (or passivity) theory as commonly used in systems and control theory dates back to

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the seminal work of Willems (1972a), where the concepts of supply rate and storage function were introduced. These capture, and generalise, the notion of a system storing and dissipating energy over time. Dissipativity theory is central to control, as consideration of energy is a natural framework for addressing stability. Multiple notions of power are equal to the product of two power-conjugate variables, leading to much attention being devoted to the study of *quadratic* supply rates (Willems, 1972b). Two famous results, sometimes called the Bounded Real Lemma and Positive Real Lemma, provide a complete state-space characterisation of the quadratic notions of dissipativity respectively called scattering passive and impedance passive; see, for example, Anderson and Vongpanitlerd (1973). It is known from the work of Haddad *et al.* (2010, Chapter 5), there in the context of absolute stability, and that of Briat (2013) and Rantzer (2015) in the context of robust control, that positive linear systems may be dissipative with respect to a *linear* supply rate and a *linear* storage function, called *linearly dissipative*. These ideas seemingly trace their roots back to Ebihara et al. (2011) and have been extended to linear control systems on general positive cones in Shen and Lam (2017). Linear dissipativity captures temporal mass (im)balance, such as via decay or dissipation, in positive systems.

Lur'e systems arise as the feedback connection of a linear control system and a nonlinear output feedback, and a block diagram of this arrangement is shown in Figure 1. Here the input variable to the linear plant comprises a nonlinear output feedback term and a forcing term, and the two components y_1 and y_2 of the measured variable y are used for feedback and are a performance output, respectively. As such, the terminology four-block relates to the four mappings from the two input- to the two output-variables in (1). Lur'e systems are a common and important class of nonlinear control systems, and are consequently well-studied objects. The study of stability properties of Lur'e systems, meaning a variety of possible notions, is called *absolute stability theory*, tracing its roots to the 1940s and the Aizerman conjecture, and ordinarily adopts the perspective that the nonlinearity f in (1) is uncertain, being specified rather in terms of given norm or sector bounds. An absolute stability criterion is a sufficient condition for stability, usually formulated in terms of the interplay of frequency properties of the linear system and norm or sector bounds of the nonlinearity. Absolute stability theory traditionally comes in one of two strands: the first adopts Lyapunov approaches to deduce global asymptotic stability of unforced (that is, $w = 0$) Lur'e systems, as is the approach in Hinrichsen and Pritchard (2005). The second strand is an input–output approach, pioneered by Sandberg and Zames in the 1960s, to infer L^2- and L^∞-stability; see, for example, Desoer and Vidyasagar (1975). The extensive literature reviewed in Liberzon (2006) evidences the interest that absolute stability theory has generated.

Input-to-State Stability (ISS) and its variants are a suite of stability concepts for controlled (or forced) systems of nonlinear differential equations and, roughly speaking, ensure natural boundedness properties of the state, in terms of (functions of) the initial conditions and inputs. ISS dates back to the 1989 work of Sontag (1989), and is nowadays supported by a vast literature, with surveys including Dashkovskiy et al. (2011), as well as the more recent text Mironchenko (2023). One strength of the input-to-state stability paradigm is that it both encompasses and unifies asymptotic and input–output approaches to stability, and to quote Mironchenko (2023, Preface): '…revolutionised our view on stabilisation of nonlinear systems…'. The practical importance of ISS is that 'small' external noise, disturbance or unmodelled dynamics result in a correspondingly 'small' effect on the resulting state. Considerable effort has been denoted to establishing ISS properties for Lur'e systems, originating in the work of Arcak and Teel (2002), and continued by Logemann and his students and collaborators, such as Jayawardhana et al. (2011). Incremental stability broadly refers to bounding the difference of two arbitrary trajectories of a given system in terms of the difference between initial states and, if included, input terms; see, for instance Angeli (2002) and Sepulchre et al. (2022). These are morally the same aims of contraction analysis or contraction theory (Cisneros-Velarde et al., 2022; Lohmiller & Slotine, 1998; Tsukamoto et al., 2021), and also of convergent systems; see (Rüffer et al., 2013) and the references therein. We refer to Jouffroy and Fossen (2010) and Rüffer et al. (2013) for comparisons between these concepts and to Bullo (2023) as a recent text on contraction analysis. Of course, for linear control systems, incremental stability concepts are equivalent to their corresponding stability concepts via the superposition principle, and this need not be the case for nonlinear systems. Incremental ISS properties have been studied across (Angeli, 2002, 2009) where, amongst a range of results, the somewhat surprising differences between usual ISS properties and their incremental versions are noted. It has recently been commented in Sepulchre et al. (2022) that incremental stability concepts are at least as important as stability notions alone. One motivation for studying incremental ISS properties is as a toolbox for nonlinear observer design and synchronisation-type problems (see Giaccagli et al., 2023 and the references therein), as well as the analysis of convergent or (almost) periodic inputs in forced differential equations (Gilmore et al., 2021), sometimes called entrainment. Other recent applications of incremental stability include to infer robustness properties of certain neural network architectures (Centorrino et al., 2023; Drummond et al., 2024; Revay et al., 2020). Here we establish a range of incremental (exponential) ISS and input–output stability properties for (1) with certain positivity structure. The main result is Theorem 4.1 which shows that a linear dissipativity hypothesis on the underlying linear system, and a certain incremental bound for the nonlinearity f, are together sufficient for an incremental stability estimate, which may roughly be interpreted as a weighted mass (im)balance (or dissipation) of differences of input/state/output trajectories of (1). A number of exponential stability and convergence properties are presented in Corollary 4.2 and Proposition 5.2, respectively. Three strengths of the linear dissipativity approach adopted are: (i) it

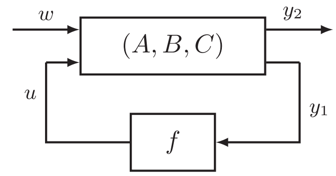

Figure 1. Block diagram of forced four-block Lur'e system (1).

affords concrete stability estimates (in terms of constants and rates), (ii) is well-suited for use with large scale systems, (iii) the treatment of multivariable (or multi-input multi-output systems) systems falls within the scope of our results, with little additional difficulty compared to the single-input single-output (SISO) case, which is much more prominent in the literature. Our work contributes to the recent and growing body of literature recognising the importance of incremental stability concepts, including Forni and Sepulchre (2018), Ofir et al. (2024) and Sepulchre et al. (2022). Our argumentation appeals to comparison arguments and a careful analysis of the interplay between the hypotheses on the linear and nonlinear components of (1).

We highlight the contribution and novelty of the present work by distinguishing it from other results available in the literature. There are a number of works which consider stability properties of positive Lur'e systems. The paper (Haddad & Chellaboina, 2005) and results in Haddad et al. (2010, Chapter 5) use a linear dissipativity framework to construct sufficient conditions for Lyapunov and global asymptotic stability for positive Lur'e systems, as well as more general nonlinear structures, but do not consider ISS properties. The papers Churilova (2010) and Drummond et al. (2023) demonstrate that the real Aizerman conjecture for certain positive Lur'e systems is true, and Drummond et al. (2023) also contains exponential ISS results for such systems. A number of absolute stability criteria for externally positive Lur'e systems (recall, characterised as those with componentwise non-negative impulse response of the linear component) appear in Gil' (2005, Chapter 6), such as Gil' (2005, Theorem 6.3.1). Various ISS results for forced, positive Lur'e differential equations and inclusions are obtained in Bill et al. (2016) and Guiver, Logemann, Rüffer (2019), respectively. None of these papers consider incremental stability properties, and so whilst some of the approaches are similar (indeed our current work is somewhat inspired by the linear dissipativity framework of Haddad and Chellaboina (2005); Haddad et al. (2010)), the overlap with the present work is limited.

In terms of incremental stability and contraction analysis, a suite of incremental stability results for general forced Lur'e systems appear in Gilmore et al. (2020) and Guiver, Logemann, Opmeer (2019), summarised as building on incremental refinements of the complex Aizerman conjecture and Circle Criterion. These papers do not consider positive systems, and so do not leverage the additional structure afforded in this setting. Roughly speaking, the papers (Gilmore et al., 2021; Guiver, Logemann, Opmeer, 2019) impose norm conditions which, in the multivariable case, may be more restrictive than the partial ordering conditions available for positive systems; see Drummond et al. (2023, Section III. E). The paper Proskurnikov et al. (2023) presents a suite of sufficient (and sometimes necessary) conditions for exponential and exponential incremental stability of general SISO Lur'e systems using a a contractivity approach based around a non-polynomial S-Lemma and non-Euclidean (weighted ℓ_p) norms. This latter paper contains a treatment of positive Lur'e systems as a special case, but considers the absolute stability problem only. So whilst both the present work and Proskurnikov et al. (2023) have aspects in common, neither paper contains the results of the other. Another feature which distinguishes this work from those listed above is consideration of the four-block problem, and our approach affords an explicit handle on the input–output gain from w to y_2 (and differences thereof) which, in this case, is linear.

The paper is organised as follows. After gathering preliminaries in Sections 2, 3 introduces the Lur'e systems under consideration, respectively. Our main incremental stability results appear in Sections 4 and 5 presents a consequence in terms of the response of the Lur'e system (1) to convergent forcing terms. Section 6 contains an example and summarising remarks appear in Section 7. Some technical material appears in the Appendix.

2. Preliminaries

2.1 Notation

The mathematical notation we use is standard, and we mention only a few items. As usual, let $\mathbb{N}$, $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{R}_+$ denote the set of positive integers, the fields of real and complex numbers and the set of non-negative real numbers, respectively. For $\mathrm{n}, \mathrm{m} \in \mathbb{N}$, $\mathbb{R}^{\mathrm{n}}$ and $\mathbb{R}^{\mathrm{n}\times\mathrm{m}}$ denote n-dimensional real Euclidean space and the set of $\mathrm{n} \times \mathrm{m}$ matrices with real entries, respectively. We equip $\mathbb{R}^{\mathrm{n}}$ with a (any) vector norm, denoted $\|\cdot\|$, and use the same symbol to represent the corresponding induced matrix norm.

2.2 Matrices, vectors and norms

Let $M = (m_{ij}) \in \mathbb{R}^{\mathrm{n}\times\mathrm{m}}$ and $N = (n_{ij}) \in \mathbb{R}^{\mathrm{n}\times\mathrm{m}}$. With the convention that $\mathbb{R}^{\mathrm{n}\times 1} = \mathbb{R}^{\mathrm{n}}$, we write $M \geq N$ if $m_{ij} \geq n_{ij}$ for all i, j, and $M \gg N$ if $m_{ij} > n_{ij}$ for all i, j, with corresponding respective conventions for $\leq$ and $\ll$. We say that a vector $v \in \mathbb{R}^{\mathrm{n}}$ is non-negative or strictly positive if $v \geq 0$ or $v \gg 0$, respectively, noting also that the symbol $\ggg$ is used in the literature for $\gg$. We use the same terminology for matrices. The symbols $\mathbb{R}^{\mathrm{n}}_+$ and $\mathbb{R}^{\mathrm{n}\times\mathrm{m}}_+$ denote the set of non-negative vectors and matrices, respectively. The symbols $>$ and $<$ are here only used for scalar inequalities.

The superscript $^\top$ denotes both matrix and vector transposition. As usual, the identity matrix is denoted by I, the size of which shall be consistent with the context.

Let $M \in \mathbb{R}^{\mathrm{n}\times\mathrm{n}}$ denote a square matrix. Recall that M is called Metzler if all its off-diagonal entries are non-negative. We let $\rho(M)$ denote the spectral radius of M, and M is called Hurwitz (sometimes also asymptotically stable) if every eigenvalue of M has negative real part. We shall make extensive use of weighted one-norms. For $\mathrm{n} \in \mathbb{N}$ and a vector $z \in \mathbb{R}^{\mathrm{n}}$ with ith component z_i, we let $|z|$ denote the vector with ith component equal to $|z_i|$, with the corresponding convention for matrices. For $v \in \mathbb{R}^{\mathrm{n}}_+$, define the semi-norm $|\cdot|_v$ on $\mathbb{R}^{\mathrm{n}}$ by

$$|z|_v := v^\top |z| \quad \forall\, z \in \mathbb{R}^{\mathrm{n}}.$$

Observe that $|z|_v := v^\top z$ for all $z \in \mathbb{R}^{\mathrm{n}}_+$. It is routine to show that $|\cdot|_v$ is a norm if, and only if, v is strictly positive. The terminology weighted one-norms stems from the equality $\|z\|_1 = |z|_{\mathbb{1}}$ for all $z \in \mathbb{R}^n$, where $\|\cdot\|_1$ denotes the usual Euclidean one-norm on $\mathbb{R}^{\mathrm{n}}$ and the symbol $\mathbb{1}$ denotes the vector with every component equal to one.

2.3 Function spaces

As usual, for $1 \leq s \leq \infty$, we let $L^s(\mathbb{R}_+, \mathbb{R}^{\mathrm{n}})$ denote the normed space of (equivalence classes of) measurable s-integrable (s finite), or essentially bounded (s infinite), functions $f : \mathbb{R}_+ \to \mathbb{R}^{\mathrm{n}}$. We let $L^s_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^{\mathrm{n}})$ denote the usual localised version. For $\alpha \in \mathbb{R}$ we define the weighted L^s space

$$L^s_\alpha(\mathbb{R}_+, \mathbb{R}^{\mathrm{n}}) := \Big\{ f \in L^s_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^{\mathrm{n}}) \; : \; \int_0^\infty \| \mathrm{e}^{\alpha t} f(t) \|^s \,\mathrm{d}t < \infty \; (s < \infty) \\ \text{or } \operatorname{ess\,sup}_{t \geq 0} \| \mathrm{e}^{\alpha t} f(t) \| < \infty \Big\},$$

which is a Banach space when equipped with the norm

$$\|f\|_{L^s_\alpha(\mathbb{R}_+)} := \begin{cases} \left(\int_0^\infty \| \mathrm{e}^{\alpha t} f(t) \|^s \,\mathrm{d}t \right)^{\frac{1}{s}} & s < \infty \\ \operatorname{ess\,sup}_{t \geq 0} \| \mathrm{e}^{\alpha t} f(t) \| & s = \infty. \end{cases}$$

For non-negative $r \in \mathbb{R}^{\mathrm{n}}_+$, and $0 \leq t_0 \leq t_1 < \infty$, we write for $f \in L^s_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^{\mathrm{n}})$

$$\|f\|_{L^s_\alpha(t_0, t_1\,;\, r)} := \begin{cases} \left(\int_{t_0}^{t_1} | \mathrm{e}^{\alpha t} f(t) |_r^s \,\mathrm{d}t \right)^{\frac{1}{s}} & s < \infty \\ \operatorname{ess\,sup}_{t \in [t_0, t_1]} \mathrm{e}^{\alpha t} |f(t)|_r & s = \infty, \end{cases}$$

simplifying $L^s_0(t_0, t_1\,;\, r)$ to $L^s(t_0, t_1\,;\, r)$. These are semi-norms in general, and are norms when r is strictly positive. By norm equivalence on $\mathbb{R}^{\mathrm{n}}$, they are equivalent to the usual L^s_α-norms on the interval (t_0, t_1), respectively.

2.4 A positive linear dissipation inequality

Whilst our overall focus of the present work relates to Lur'e systems, their analysis is underpinned by properties of the underlying linear system. Particularly relevant here are linear control systems which are positive with a certain linear dissipativity property, and in this section we gather required material.

For which purpose, consider the following controlled and observed system of linear differential equations

$$\dot{x} = Ax + Bu, \quad y = Cx + Du, \tag{2}$$

where $A \in \mathbb{R}^{\mathrm{n} \times \mathrm{n}}$, $B \in \mathbb{R}^{\mathrm{n} \times \mathrm{m}}$, $C \in \mathbb{R}^{\mathrm{p} \times \mathrm{n}}$ and $D \in \mathbb{R}^{\mathrm{p} \times \mathrm{m}}$ for $\mathrm{n}, \mathrm{m}, \mathrm{p} \in \mathbb{N}$.

The linear system (2) is called (internally) positive if A is Metzler (recall, meaning all off-diagonal entries are non-negative) and $B, C, D \geq 0$. Metzler matrices go by a number of other terms in the literature, such as essentially non-negative in Berman et al. (1989, p. 146) or quasi-positive in Smith (1995, p. 60). They are the continuous-time analogue of non-negative matrices which arise naturally in discrete-time non-negative dynamical systems.

We recall the following well-known facts related to the Metzler matrix $M \in \mathbb{R}^{n \times n}$.

(F1) e^{Mt} is a non-negative matrix for all $t \geq 0$.
(F2) M is Hurwitz if, and only if, $M^{-1} \leq 0$.

For proofs of these statements, we refer to, for example (Schneider & Vidyasagar, 1970, Theorem 3), and (Berman & Plemmons, 1994, characterisation N38 in Section 6.2) or (Plemmons, 1977, characterization F15), respectively. Fact one is a characterisation of Metzler matrices.

The transfer function associated with (2) is $\mathbf{G}(z) = C(zI - A)^{-1}B + D$ which is a proper rational function of the complex variable z. For positive systems with A Hurwitz, it follows that $\mathbf{G}(0) \geq 0$.

We next state a so-called linear dissipativity lemma.

Lemma 2.1: *Consider the positive linear control system (2), let $\xi \geq 0$ be given and assume that $A + \xi I$ is Hurwitz. Let $(r, q) \in \mathbb{R}^m_+ \times \mathbb{R}^p_+$ be given. The following statements are equivalent.*

(i) *there exist $p, l \in \mathbb{R}^n_+$ and $k \in \mathbb{R}^m_+$ satisfying*

$$\begin{aligned} p^\top A + \xi p^\top + q^\top C + l^\top &= 0, \\ p^\top B + q^\top D - r^\top + k^\top &= 0. \end{aligned} \tag{3}$$

(ii) $r^\top - q^\top \mathbf{G}(-\xi) \geq 0$

If either statement above holds and, additionally, $q^\top C(-A)^{-1}$ is assumed strictly positive, that is,

$$q^\top C(-A)^{-1} \gg 0, \tag{4}$$

then $p \gg 0$.

Some remarks are in order. Observe that the condition in statement (ii) is a simple componentwise inequality of vectors which, for given $\mathbf{G}$ can always be satisfied if r, q are sufficiently (componentwise) large or small, respectively. Furthermore, if the pair (r, q) satisfy $r^\top - q^\top \mathbf{G}(-\xi) \geq 0$, then so does $(\alpha r, \alpha q)$ for all $\alpha > 0$, that is, a choice of positive multiplicative scaling is available in the pair (r, q). Lemma 2.1 is part of a larger result, which one would naturally call a (r, q)-linearly dissipative lemma as it provides a state-space characterisation of the (r, q)-linear dissipative property for positive linear control systems. Indeed, it can be shown that the above statements are additionally equivalent to the inequality

$$\|y\|_{L^1(t_1, t_2\,;\, q)} \leq \|u\|_{L^1(t_1, t_2\,;\, r)} \quad 0 \leq t_1 \leq t_2,$$

holding for all trajectories (u, x, y) of (2) with non-negative $u \in L^1_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^{\mathrm{m}})$ and $x(0) = 0$.

Expanded versions of the linear dissipativity lemma which essentially (but not exactly) include Lemma 2.1 appear in Ebihara et al. (2011, Theorem 3), Briat (2013, Lemma 1), Rantzer (2011, Theorems 4 and 5), and Rantzer (2015, Proposition 5), broadly in the context of robust control of positive linear systems. In each of these cases $r = \mathbb{1}$ and $q = \gamma\,\mathbb{1}$ are taken as (scaled) vectors of ones, for scalar $\gamma > 0$. The authors of Haddad and Chellaboina (2005); Haddad et al. (2010) refer to the $(\mathbb{1}, \mathbb{1})$-linear dissipative property as *non-accumulative*, cf. Haddad et al. (2010, Definition 5.4). Another linear dissipativity lemma appears in Haddad and Chellaboina (2005, Theorem 6.2); see also (Haddad et al., 2010, Theorem 5.3, p.149), broadly with the focus of providing sufficient conditions for asymptotic stability of positive Lur'e systems, as in Haddad and Chellaboina (2005, Theorem 7.2). The condition (4) plays a role in the work (Franco et al., 2021) and a number of characterisations of (4) appear in Franco et al. (2021, Proposition 3.2). Intuitively, it is a necessary and sufficient condition for the pair $(q^\top C, A)$ to be *positive trajectory observable*, meaning if $y(t) = q^\top Cx(t)$, and $\dot{x}(t) = Ax(t)$ with $x(0) \in \mathbb{R}^n_+$, then the condition $y(t) = 0$ for all $t \geq 0$ implies that $x(0) = 0$. The positive trajectory observability property is called *zero-state observable* in Haddad and Chellaboina (2005, Definition 5.3), and appears for discrete-time control systems in Härdin and van Schuppen (2007, Definition 4.4). We comment that (4) is equivalent to

$$\ker \mathcal{O}(q^\top C, A) \cap \mathbb{R}^n_+ = \{0\},$$

where $\mathcal{O}(q^\top C, A)$ denotes the usual observability matrix of the pair $(q^\top C, A)$, but does not enforce (usual) observability of the pair $(q^\top C, A)$.

Proof of Lemma 2.1: From fact (F2) we have that $(-(A + \xi I))^{-1} \geq 0$. If statement (i) holds, then the first equality in (3) shows that

$$p^\top = \left(q^\top C + l^\top\right)(-(A + \xi I))^{-1},$$

which, when substituted into the second equality gives statement (ii). Conversely, it suffices to set $p^\top := q^\top C(-(A + \xi I))^{-1}$ which is readily shown to satisfy (3) with $l{:} = 0$ and $k := r - \mathbf{G}(-\xi)^\top q \geq 0$.

The final claim follows from multiplying the first equation in (3) by A^{-1} on the right, and rearranging to give $p^\top \geq q^\top C(-A)^{-1} \gg 0$. Here we are using that $\ell^\top, p^\top \geq 0$ and $(-A)^{-1} \geq 0$ by fact (F2). ■

3. Forced, four-block, positive Lur'e systems

We consider the forced four-block Lur'e system (1). Here $A \in \mathbb{R}^{n\times n}$, $B_i \in \mathbb{R}^{n\times m_i}$ and $C_i \in \mathbb{R}^{p_i\times n}$ for $i = 1, 2$, and $f : \mathbb{R}_+ \times \mathbb{R}^{p_1} \to \mathbb{R}^{m_1}$ is a time-varying nonlinearity. The variables x, w and y denote the state, forcing term and output of system (1), taking values in $\mathbb{R}^n$, $\mathbb{R}^{m_2}$ and $\mathbb{R}^{p_1+p_2}$, all respectively.

Lur'e system (1) arises as the feedback connection of the linear control system (2) with

$$B = \begin{pmatrix} B_1 & B_2 \end{pmatrix}, \quad C = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}, \quad D = 0,$$

and input variable (u, w), and the static, time-varying nonlinear feedback $u(t) = f(t, y_1)$. As such, the two inputs u and w are a control variable and an external forcing (or disturbance) term, respectively. The variables y_1 and y_2 denote the components of the measured variable y used for feedback and as a performance variable, respectively.

The transfer function $\mathbf{G}$ associated with the linear control system in (1) is given by

$$\mathbf{G}(s) := \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}(sI - A)^{-1}\begin{pmatrix} B_1 & B_2 \end{pmatrix},$$

and we let $\mathbf{G}_{ij}(s) := C_i(sI - A)^{-1}B_j$ for $i, j = 1, 2$. The following assumptions on the model data in (1) are imposed throughout.

(A1) The matrix A is Metzler, and B_1, B_2, C_1 and C_2 are nonnegative matrices.

1. The function f satisfies:
   - $t \mapsto f(t, \zeta)$ is in $L^1_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^{m_1})$ for every $\zeta \in \mathbb{R}^{p_1}$;
   - there exists $\zeta_0 \in \mathbb{R}^{p_1}$ such that $t \mapsto f(t, \zeta_0)$ is essentially bounded, and;
   - $$\operatorname{ess\,sup}_{t\geq 0} |f(t, \zeta_1) - f(t, \zeta_2)| \leq \Delta|\zeta_1 - \zeta_2| \quad \forall\, \zeta_1, \zeta_2 \in \mathbb{R}^{p_1}, \tag{5}$$

     for some fixed $\Delta \in \mathbb{R}^{m_1\times p_1}_+$.

The first two items in hypothesis (A2) are mild regularity requirements, and are always satisfied if f is independent of its first variable. Observe that the inequality in (5) is a componentwise inequality of vectors in $\mathbb{R}^{m_1}$. A consequence of hypothesis (A2) is that f is measurable in its first variable, for each fixed second variable, and globally Lipschitz in its second variable, uniformly with respect to its first variable. As usual, we call the $m_1 = p_1 = 1$ case the single-input single-output setting. Here, the inequality (5) reduces to the so-called *slope restriction condition*

$$-\delta \leq \sup_{\substack{t\geq 0\\ z_1,z_2\in\mathbb{R}\\ z_1\neq z_2}} \frac{f(t, z_1) - f(t, z_2)}{z_1 - z_2} \leq \delta, \tag{6}$$

for $\delta := \Delta > 0$. On the one hand, slope restriction conditions are somewhat strong. Indeed, when f is continuously differentiable in its second variable, then (6) is equivalent to boundedness of $\partial f/\partial z$ uniformly with respect to its first variable. On the other hand, slope restriction conditions are common in nonlinear control theory; see, for example (Haddad & Kapila, 1995; Turner & Drummond, 2019). Indeed, slope-restricted nonlinearities are an essential ingredient of the celebrated Kalman conjecture which, although false in general, has generated significant interest (see, for instance, Kuznetsov et al. (2018) and the literature therein).

We shall assume throughout that the forcing term $w : \mathbb{R}_+ \to \mathbb{R}^{m_2}$ in (1) is locally integrable, that is, $w \in L^1_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^{m_2})$. Given such a w, we call the pair (w, x) a trajectory of (1) if x is a locally absolutely continuous function $x : \mathbb{R}_+ \to \mathbb{R}^n$ which satisfies the differential equation in (1) almost everywhere. The collection of all trajectories of (1) is called the behaviour of (1) and is denoted $\mathcal{B}$. If $f(\cdot, 0) = 0$, then $(0, 0)$ is a constant trajectory of (1), but we do not impose this assumption throughout. In light of the combined assumptions in (A2), it follows from known results in the theory of ordinary differential equations (see, for example (Sontag, 2013, Appendix C)) that, for every $z \in \mathbb{R}^n$, there exists a unique trajectory (w, x) of (1) with $x(0) = z$ so that, in particular, the set of trajectories $\mathcal{B}$ is not empty.

It is straightforward to show that trajectories of (1) are characterised as those locally absolutely continuous functions x which satisfy the so-called variation of parameters formula: for all $t_1, t_0 \geq 0$,

$$x(t_1 + t_0) = \mathrm{e}^{At_1}x(t_0) + \int_{t_0}^{t_1+t_0} \mathrm{e}^{A(t_1+t_0-t)} \times \left(B_1 f(t, C_1x(t)) + B_2w(t)\right) \mathrm{d}t, \tag{7}$$

which, note, is well defined when $w \in L^1_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^{\mathrm{m}_2})$. It is for this reason that we only impose local integrability of w, rather than local essential boundedness, which is the more common assumption in ISS literature when considering general nonlinear control systems. Physically motivated forcing terms will typically belong to $L^\infty_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^{\mathrm{m}_2})$, though.

Given $(w, x) \in \mathcal{B}$, for convenience we call x and y in (1) a state trajectory and output trajectory, respectively, which we also write as $x(\cdot\,; x_0, w)$ and $y(\cdot\,; x_0, w)$ when $x(0) = x_0$ for $x_0 \in \mathbb{R}^{\mathrm{n}}$. The triplet (w, x, y) is called an input/state/output trajectory of Lur'e system (1) if $(w, x) \in \mathcal{B}$ and $y = y(\cdot\,; x(0), w)$.

Observe that we do not impose that $f(t, \cdot)$ is non-negative valued (this allows for negative feedback by putting the minus sign on the f, rather than the B_1 term). However, the typical situation for positive systems is that $f(t, \cdot)$ maps $\mathbb{R}^{\mathrm{p}_1}_+$ into $\mathbb{R}^{\mathrm{m}_1}_+$ and, in this case, a consequence of hypothesis (A1), fact (F1) and the variation of parameters formula (7) with $t_0 = 0$ is that $x(t) \geq 0$ for all $t \geq 0$ whenever $x(0) \geq 0$ and $w \geq 0$ almost everywhere. This positive invariance property may also be shown by invoking, for example, Haddad et al. (2010, Proposition 2.2). Moreover, if A is Hurwitz, then $\mathbf{G}_{ij}(0)$ is well defined and non-negative for all $i, j = 1, 2$.

We record the following hypotheses which will be used throughout, but not necessarily simultaneously in any of the following results.

(H1) There exist $\xi > 0$ and strictly positive $p \in \mathbb{R}^{\mathrm{n}}_+$ such that

$$p^\top (A + B_1 \Delta C_1) \leq -\xi p^\top.$$

(H2) There exist $\xi > 0$, strictly positive $p \in \mathbb{R}^{\mathrm{n}}_+$, non-negative $l \in \mathbb{R}^{\mathrm{n}}_+$, $q \in \mathbb{R}^{\mathrm{p}_2}_+$ and $r, k \in \mathbb{R}^{\mathrm{m}_2}_+$ such that

$$\left.\begin{aligned} p^\top (A + B_1 \Delta C_1) + \xi p^\top + q^\top C_2 + l^\top &= 0 \\ p^\top B_2 - r^\top + k^\top &= 0. \end{aligned}\right\} \tag{8}$$

Hypotheses (H1) and (H2) are essentially both stability assumptions, with characterisations provided in the following two lemmas, respectively. To minimise disruption, we relegate the proofs to the Appendix.

Lemma 3.1: *Let A, B_1, C_1 be as in* (A1), *and fix $\Delta \in \mathbb{R}^{m_1 \times p_1}_+$ as in* (A2). *The following statements are equivalent.*

(i) (H1) *holds*
(ii) *$A + B_1 \Sigma C_1$ is Hurwitz for all $\Sigma \in \mathbb{R}^{m_1 \times p_1}$ with $0 \leq \Sigma \leq \Delta$.*
(iii) *A is Hurwitz and $\rho(\mathbf{G}_{11}(0)\Delta) < 1$.*

Lemma 3.2: *Let A, B_i, C_i for $i = 1, 2$ be as in* (A1), *fix $\Delta \in \mathbb{R}^{m_1 \times p_1}_+$ as in (A2), and let $r \in \mathbb{R}^{m_2}_+$, $q \in \mathbb{R}^{p_2}_+$ be given. Assume that $A + B_1 \Delta C_1$ is Hurwitz and*

$$q^\top C_2 (-(A + B_1 \Delta C_1))^{-1} \gg 0.$$

Hypothesis (H2) *is equivalent to $A + B_1 \Delta C_1 + \xi I$ is Hurwitz and*

$$r^\top - q^\top C_2 (-\xi I - (A + B_1 \Delta C_1))^{-1} B_2 \geq 0. \tag{9a}$$

Moreover, inequality (9a) *is equivalent to*

$$\begin{aligned} r^\top - q^\top \big(\mathbf{G}_{22}(-\xi) + \mathbf{G}_{21}(-\xi)(I - \Delta \mathbf{G}_{11}(-\xi))^{-1} \\ \times \Delta \mathbf{G}_{12}(-\xi)\big) \geq 0. \end{aligned} \tag{9b}$$

By way of commentary, Lemmas 3.1 and 3.2 are valid for all $\Delta \in \mathbb{R}^{\mathrm{m}_1 \times \mathrm{p}_1}_+$, not just for those as in (A2). Further, statement (ii) of Lemma 3.1 is reminiscent of the key stability hypothesis in the real Aizerman conjecture for positive Lur'e systems, as considered in Drummond et al. (2023). Of course, in the single-input single-output setting, the condition $\rho(\mathbf{G}_{11}(0)\Delta) < 1$ reduces to the small-gain inequality $\mathbf{G}_{11}(0)\Delta < 1$. Although statement (iii) of Lemma 3.1 essentially replaces one eigenvalue condition, namely on the eigenvalues of $A + B_1 \Delta C_1$, with another eigenvalue condition $\rho(\Delta \mathbf{G}(0)) = \rho(\mathbf{G}_{11}(0)\Delta) < 1$, in usual applications $\mathrm{m}_1, \mathrm{p}_1 \ll \mathrm{n}$ so that at least one of the second eigenvalue problems is smaller than the first. We relate hypothesis (H1) to a logarithmic norm assumption appearing in Proskurnikov et al. (2023) in Remark 4.4.

It is clear that hypothesis (H2) implies (H1). Assumption (H2) is a linear dissipativity hypothesis, and equalities of the form (8) are the focus of Lemma 2.1. Whilst the current formulation allows for q, r, k, l to all equal zero, in practice we shall desire r and q in (8) to be non-zero as they shall induce weighted one-norms for estimates between the input and output variables of (1). Hypothesis (H2) may be stated with the slack variables k and l omitted by replacing the equations in (8) with componentwise inequalities.

4. Incremental stability properties

The following theorem is the main result of this work, and shows that hypotheses (H1) and (H2) are sufficient for the Lur'e system (1) to admit various incremental bounds which we present. A number of subsequent stability and convergence properties are underpinned by this result.

Theorem 4.1: *Consider the four-block Lur'e system* (1), *and let (w_a, x_a, y_a) and (w_b, x_b, y_b) denote two input/state/output trajectories of* (1). *If* (H1) *is satisfied, then the estimate*

$$\begin{aligned} |(x_a - x_b)(t_1)|_p &\leq \mathrm{e}^{-\xi(t_1 - t_0)} |(x_a - x_b)(t_0)|_p \\ &\quad + \mathrm{e}^{-\xi t_1} \|w_a - w_b\|_{L^1_\xi(t_0, t_1\,;\, B_2^\top p)} \\ &\qquad \forall\, t_1 \geq t_0 \geq 0, \end{aligned} \tag{10}$$

holds. If (H2) *is satisfied, then the following estimate holds:*

$$\begin{aligned} &\int_{t_0}^{t_1} \mathrm{e}^{\xi t} \big(|(y_{2,a} - y_{2,b})(t)|_q + |(w_a - w_b)(t)|_k \\ &\quad + |(x_a - x_b)(t)|_l \big)\, \mathrm{d}t + \mathrm{e}^{\xi t_1} |(x_a - x_b)(t_1)|_p \\ &\leq \mathrm{e}^{\xi t_0} |(x_a - x_b)(t_0)|_p + \int_{t_0}^{t_1} \mathrm{e}^{\xi t} |(w_a - w_b)(t)|_r\, \mathrm{d}t \\ &\qquad \forall\, t_1 \geq t_0 \geq 0. \end{aligned} \tag{11}$$

Here $y_{2,k} := C_2 x_k$ for $k = a, b$. We enumerate input/state/output trajectories of (1) using letters to reduce potential confusion between pairs of trajectories and the segments y_1 and y_2

of the output of (1). Observe that if (H2) holds, then the estimate (10) holds with $B_2^\top p$ replaced by r, as an immediate consequence of (11). We shall use this fact in certain later arguments.

Proof of Theorem 4.1: The derivations of (10) and (11) share the same start. Let (w_a, x_a, y_a) and (w_b, x_b, y_b) denote two input/state/output trajectories of (1). The functions x_a and x_b are locally absolutely continuous, and hence so is $t \mapsto |x_a(t) - x_b(t)|_p$ as the composition of a locally absolutely continuous function and a globally Lipschitz one. Therefore, $t \mapsto |x_a(t) - x_b(t)|_p$ is differentiable almost everywhere on $\mathbb{R}_+$ and the fundamental theorem of Lebesgue integral calculus holds; see, for example (Leoni, 2017, Theorem 3.20, p. 77). In particular, for all $0 \le t_0 < t_1$,

$$\begin{aligned}
&\left[e^{\xi t} |(x_a - x_b)(t)|_p \right]_{t_0}^{t_1} \\
&= \int_{t_0}^{t_1} \frac{d}{dt} \left(e^{\xi t} |(x_a - x_b)(t)|_p \right) dt \\
&= \int_{t_0}^{t_1} e^{\xi t} \left(\frac{d}{dt} |(x_a - x_b)(t)|_p + \xi |(x_a - x_b)(t)|_p \right) dt.
\end{aligned} \tag{12}$$

We proceed to derive an estimate for the derivative of $|x_a - x_b|_p$ which appears in (12). For which purpose, let $t \in (t_0, t_1)$ be a point of differentiability for $|x_a - x_b|_p$ (and so almost every $t \in (t_0, t_1)$ is suitable). The variation of parameters formula (7) yields, for all $h > 0$

$$\begin{aligned}
(x_a - x_b)(t + h) = {} & e^{Ah}(x_a - x_b)(t) \\
& + \int_t^{t+h} e^{A(t+h-\theta)} B_1 \left(f(C_1 x_a) - f(C_1 x_b) \right) d\theta \\
& + \int_t^{t+h} e^{A(t+h-\theta)} B_2 (w_a - w_b)\, d\theta,
\end{aligned} \tag{13}$$

where we have suppressed the first variable of f and the argument θ in the integrand for clarity. For brevity, set $z := x_a - x_b$. Routine estimates of (13) yield that, for all $h > 0$,

$$\begin{aligned}
|z(t+h)|_p \le {} & p^\top e^{Ah} |z(t)| + p^\top \int_t^{t+h} e^{A(t+h-\theta)} \\
& \times \left(B_1 \left| f(C_1 x_a) - f(C_1 x_b) \right| + B_2 |w_a - w_b| \right) d\theta.
\end{aligned}$$

From an application of the incremental bound (5) and the non-negativity of C_1, we estimate that

$$\begin{aligned}
|z(t+h)|_p \le {} & p^\top e^{Ah} |z(t)| + p^\top \int_t^{t+h} e^{A(t+h-\theta)} \\
& \times (B_1 \Delta C_1 |z| + B_2 |w_a - w_b|)\, d\theta
\end{aligned}$$

again, for all $h > 0$. Therefore,

$$\begin{aligned}
\frac{|z(t+h)|_p - |z(t)|_p}{h} \le {} & p^\top \left(\frac{e^{Ah} - I}{h} \right) |z(t)| \\
& + p^\top \frac{1}{h} \int_t^{t+h} e^{A(t+h-\theta)} \\
& \times (B_1 \Delta C_1 |z| + B_2 |w_a - w_b|)\, d\theta \\
& \forall\, h > 0.
\end{aligned} \tag{14}$$

We take the limit $h \searrow 0$ in (14). For which purpose, we record that

$$\frac{e^{Ah} - I}{h} \to A \quad \text{as } h \searrow 0, \tag{15}$$

and

$$\begin{aligned}
& e^{A(t+h)} \frac{1}{h} \int_t^{t+h} e^{-A\theta} \big(B_1 \Delta C_1 |(x_a - x_b)(\theta)| \\
& \qquad + B_2 |(w_a - w_b)(\theta)| \big) d\theta \\
& \to B_1 \Delta C_1 |z(t)| + B_2 |(w_a - w_b)(t)| \quad \text{as} \quad h \searrow 0,
\end{aligned} \tag{16}$$

by the Fundamental Theorem of Calculus. Substituting (15) and (16) into (14), and letting $h \searrow 0$, gives that

$$\begin{aligned}
\frac{d}{dt} |x_a(t) - x_b(t)|_p \le {} & p^\top (A + B_1 \Delta C_1) |(x_a - x_b)(t)| \\
& + p^\top B_2 |(w_a - w_b)(t)|.
\end{aligned} \tag{17}$$

Since $t \in (t_0, t_1)$ was arbitrary, we conclude that the estimate (17) holds for almost all $s \in (t_0, t_1)$. Consequently, the conjunction of (12) and (17) yields

$$\begin{aligned}
& e^{\xi t_1} |(x_a - x_b)(t_1)|_p - e^{\xi t_0} |(x_a - x_b)(t_0)|_p \\
& \le \int_{t_0}^{t_1} e^{\xi t} (p^\top (A + B_1 \Delta C_1) + \xi p^\top) |(x_a - x_b)(t)|\, dt \\
& \quad + \int_{t_0}^{t_1} e^{\xi t} p^\top B_2 |(w_a - w_b)(t)|\, dt.
\end{aligned} \tag{18}$$

If hypothesis (H1) holds, then the inequality (10) follows directly from (18). If instead hypothesis (H2) holds, then we invoke this in (18) to obtain

$$\begin{aligned}
& e^{\xi t_1} |(x_a - x_b)(t_1)|_p - e^{\xi t_0} |(x_a - x_b)(t_0)|_p \\
& \le \int_{t_0}^{t_1} e^{\xi t} (-l^\top - q^\top C_2) |(x_a - x_b)(t)|\, dt \\
& \quad + \int_{t_0}^{t_1} e^{\xi t} (r^\top - k^\top) |(w_a - w_b)(t)|\, dt.
\end{aligned} \tag{19}$$

Therefore, inequality (19), when combined with

$$\begin{aligned}
q^\top |(y_{2,a} - y_{2,b})(t)| &= q^\top |C_2 (x_a - x_b)(t))| \\
&\le q^\top C_2 |(x_a - x_b)(t)| \quad \forall\, t \ge 0,
\end{aligned}$$

establishes inequality (11), as required. ■

The next result is our first consequence of Theorem 4.1 and is an exponential incremental stability estimate for the four-block Lur'e system (1).

Corollary 4.2: *Consider the forced Lur'e system (1), and let (w_a, x_a, y_a) and (w_b, x_b, y_b) denote two input/state/output trajectories of (1). Let $1 \le s \le \infty$. If (H1) is satisfied, then* *there exists $\alpha_s > 0$ (independent of the trajectories) such that for all $t_1 \geq t_0 \geq 0$:*

$$|(x_a - x_b)(t_1)|_p \leq \mathrm{e}^{-\xi(t_1-t_0)}|(x_a - x_b)(t_0)|_p + \alpha_s\|w_a - w_b\|_{L^s(t_0,t_1\,;\,B_2^\top p)}. \quad (20)$$

If (H2) *is satisfied, then for all $t_1 \geq t_0 \geq 0$:*

$$\begin{aligned}&\|y_{2,a} - y_{2,b}\|_{L^1_\xi(t_0,t_1\,;\,q)} + \|w_a - w_b\|_{L^1_\xi(t_0,t_1\,;\,k)} \\ &\quad + \|x_a - x_b\|_{L^1_\xi(t_0,t_1\,;\,l)} + \mathrm{e}^{\xi t_1}|(x_a - x_b)(t_1)|_p \\ &\leq \mathrm{e}^{\xi t_0}|(x_a - x_b)(t_0)|_p + \|w_a - w_b\|_{L^1_\xi(t_0,t_1\,;\,r)}. \end{aligned} \quad (21)$$

The inequality (20) with $s = \infty$ is an incremental exponential input-to-state-stability estimate for (1), whilst estimate (21) entails incremental (weighted) L^1-input/output stability of system (1) with a linear (unity) gain, that is, when $x_a(t_0) = x_b(t_0)$,

$$\|y_{2,a} - y_{2,b}\|_{L^1_\xi(t_0,t_1\,;\,q)} \leq \|w_a - w_b\|_{L^1_\xi(t_0,t_1\,;\,r)}. \quad (22)$$

Observe that r and q in hypothesis (H2) appear as weights in (22), and that $q \gg 0$ is required for the left-hand side of (22) to contain a norm of $y_{2,a} - y_{2,b}$. (The right-hand side of (22) may always be bounded from above by some norm in the case that $r \not\gg 0$.) The conclusions of Corollary 4.2 illustrate the different consequences of hypotheses (H1) and (H2). Both of these hypotheses contain an 'internal stability' component – namely that $A + B_1\Delta C_1$ is Hurwitz. In the context of positive Lur'e systems, this assumption is necessary and sufficient for Lyapunov stability notions (see, for example (Drummond et al., 2023)). Hypothesis (H2) ensures that the incremental input–output estimates (21) and (22) hold, with bounds given in terms of the linear data arising in (8). Hypothesis (H1) by itself does seemingly not entail any such input–output estimate, other than by directly estimating $y_{2,a} - y_{2,b} = C_2(x_a - x_b)$ and invoking (20).

Proof of Corollary 4.2: Inequality (20) follows from (10), the cases $s = 1$ and $s = \infty$ being trivial, indeed, $\alpha_1 = 1$ and $\alpha_\infty = 1/\xi$. For notational simplicity set $\tilde{r} := B_2^\top p$. For finite $s \in (1, \infty)$, let $1 < s_0 < \infty$ be conjugate to s, that is, $1/s + 1/s_0 = 1$. Then, for all $t_1 \geq t_0 \geq 0$, an application of Hölder's inequality yields

$$\begin{aligned}&\mathrm{e}^{-\xi t_1}\int_{t_0}^{t_1} \mathrm{e}^{\xi t}|(w_a - w_b)(t)|_{\tilde{r}}\,\mathrm{d}t \\ &\leq \mathrm{e}^{-\xi t_1}\left(\int_{t_0}^{t_1} \mathrm{e}^{s_0\xi t}\,\mathrm{d}t\right)^{\frac{1}{s_0}}\left(\int_{t_0}^{t_1} |(w_a - w_b)(t)|_{\tilde{r}}^{s}\,\mathrm{d}t\right)^{\frac{1}{s}} \\ &\leq \mathrm{e}^{-\xi t_1}\left(\frac{\mathrm{e}^{s_0\xi t_1}}{s_0\xi}\right)^{\frac{1}{s_0}}\|w_a - w_b\|_{L^s(t_0,t_1\,;\,\tilde{r})} \\ &= (s_0\xi)^{-\frac{1}{s_0}}\|w_a - w_b\|_{L^s(t_0,t_1\,;\,\tilde{r})},\end{aligned}$$

that is, the desired second term on the right-hand side of (20).

The inequality (21) is simply a rewriting of (11) in terms of weighted L^1-norms, and follows immediately from Theorem 4.1. ■

We conclude this section with commentary on variations of our results.

Remark 4.3: (i) Let $K \in \mathbb{R}^{\mathrm{m}_1\times\mathrm{p}_1}$. The so-called loop shifted Lur'e system associated with (1) is given by

$$\begin{aligned}\dot{x}(t) &= (A + B_1KC_1)x(t) \\ &\quad + B_1\left(f(t, C_1x(t)) - KC_1x(t)\right) + B_2w(t), \\ y(t) &= \begin{pmatrix} y_1(t) \\ y_2(t)\end{pmatrix} = \begin{pmatrix} C_1 \\ C_2\end{pmatrix} x(t),\end{aligned}$$

which is itself an instance of (1) with A and f replaced by $A_K := A + B_1KC_1$ and $f_K(t, z) := f(t, z) - Kz$, respectively. The present results are applicable in this setting when A_K and f_K satisfy assumptions (A1) and (A2).

Loop shifting may be used to 'recentre' slope restricted nonlinearities with unequal slope limits in the form of (5). Indeed, restricting to the single-input single-output ($\mathrm{m}_1 = \mathrm{p}_1 = 1$), time-invariant case, consider f such that

$$\alpha_1 \leq \frac{f(z_1) - f(z_2)}{z_1 - z_2} \leq \alpha_2 \quad \forall\, z_1, z_2 \in \mathbb{R},\ z_1 \neq z_2, \quad (23)$$

for $\alpha_1 < \alpha_2$. By setting

$$\beta := \frac{(\alpha_1 + \alpha_2)}{2}, \quad \delta := \frac{\alpha_2 - \alpha_1}{2} > 0 \text{ and } f_\beta(z) := f(z) - \beta z,$$

it is routine to show that

$$-\delta \leq \frac{f_\beta(z_1) - f_\beta(z_2)}{z_1 - z_2} \leq \delta \quad \forall\, z_1, z_2 \in \mathbb{R},\ z_1 \neq z_2,$$

that is, the function f_β satisfies (5) with $\Delta = \delta$.

(ii) Consider next the following Lur'e system with output disturbance, that is, where the differential equation in (1) is replaced by

$$\dot{x}(t) = Ax(t) + B_1f(t, C_1x(t) + w_1(t)) + B_2w_2(t).$$

By introducing an error term, we rewrite the above as the differential equation in (1) with B_2w there replaced by

$$\tilde{w}(t) := B_2w_2(t) + B_1\left(f(t, C_1x(t) + w_1(t)) - f(t, C_1x(t))\right),$$

and apply the current results to the trajectory $(\tilde{w}, x)$ of (1) with B_2 replaced by I. In light of the incremental sector condition in (A2), observe that the componentwise absolute value $|\tilde{w}(t)|$ may be bounded from above in terms of $|w_1(t)|$ and $|w_2(t)|$, independently of x.

(iii) Now suppose that hypothesis (A2) is replaced by, for given non-empty subsets $Z_1, Z_2 \subseteq \mathbb{R}^{\mathrm{p}_1}$,

(A2)' The function f satisfies:

- there exist $a_1, a_2 > 0$ such that

$$\operatorname{ess\,sup}_{t\geq 0}\|f(t,\zeta)\| \leq a_1\|\zeta\| + a_2 \quad \forall\, \zeta \in \mathbb{R}^{\mathrm{p}_1};$$

- there exists $\Delta \in \mathbb{R}_+^{\mathrm{m}_1\times\mathrm{p}_1}$ such that

$$\operatorname{ess\,sup}_{t\geq 0}|f(t,\zeta_1) - f(t,\zeta_2)| \leq \Delta|\zeta_1 - \zeta_2| \quad \forall\, y_1 \in Z_1,\ \forall\, \zeta_2 \in Z_2. \quad (24)$$

The first item in hypothesis (A2)' is imposed to ensure existence of trajectories of (1) (in particular, existence of state variables defined on all of $\mathbb{R}_+$) and could be replaced by other suitable conditions. Note that Δ in hypothesis (A2)' may, in general, depend on the subsets $Z_1, Z_2 \subseteq \mathbb{R}^{p_1}$.

Then, the conclusions of Theorem 4.1 and Corollary 4.2 remain valid, only now for input/state/output trajectories (w_a, x_a, y_a) and (w_b, x_b, y_b) with $y_a \in Z_1$ and $y_b \in Z_2$ (inclusions understood pointwise almost everywhere). The only changes to the proofs are that references to (5) are replaced by references to (24). Observe that Theorem 4.1 and Corollary 4.2 are recovered when $Z_1 = Z_2 = \mathbb{R}^{p_1}$. One special case of interest is when Z_2 is a singleton, and (w_b, x_b, y_b) is equal to a constant trajectory with $y_b \in Z_2$. In summary, on the one hand, the inequality (24) is much less restrictive than (5), and permits 'localised' versions of our results. On the other hand, the results are only valid in the somewhat restrictive case that $y_a \in Z_1$ and $y_b \in Z_2$, which will in general require prior knowledge of the input/state/output trajectories to verify.

Remark 4.4: We connect our results to some in Proskurnikov et al. (2023) which, recall, amongst a wealth of material, provides sufficient conditions for absolute and incremental stability of SISO Lur'e systems in terms of contractivity conditions in non-Euclidean norms. Let $\Delta = \delta > 0$. We claim that (H1) holds with given $\xi > 0$ if, and only if, for some positive definite diagonal $R \in \mathbb{R}^{n\times n}$ and $\tau \geq 0$ it follows that

$$\mu_1\left(\begin{pmatrix} RAR^{-1} + \xi I & RB_1 \\ \tau C_1 R^{-1} & -\tau\delta^{-1} \end{pmatrix}\right) \leq 0, \tag{25}$$

where μ_1 is the logarithmic norm associated with the Euclidean one-norm; see, for instance (Proskurnikov et al., 2023, Table 1). Inequality (25) is recognised as Proskurnikov et al. (2023, Equation (4.18)). The proof of the above claim is given in the Appendix.

The above equivalence suggests that, in the SISO case with $w_a = w_b = 0$, the inequality (10) may be derived from Proskurnikov et al. (2023, Theorem 4.10), and in the special case considered presently (Proskurnikov et al., 2023, Theorem 4.10 (iii)) follows from the first part of Theorem 4.1. We write suggests as the authors of Proskurnikov et al. (2023) focus on asymmetric sector bounds – $\alpha_1 = 0$ and $\alpha_2 = \delta$ in (23) – whilst (A2) entails $\alpha_1 = -\delta$. This difference may be addressed by loop shifting as discussed above, but we do not give the details here.

5. A consequence of incremental stability – convergence properties

Here we investigate the state response of the forced Lur'e system (1) to forcing terms which are convergent, with main result Proposition 5.2 below. We assume that

$$f(t,z) = f(z) \quad \text{that is, } f \text{ does not depend explicitly on } t.$$

Since we focus on qualitative properties of the response of the state x to the external forcing term, from which analogous properties of the output $y = Cx$ readily follow, hypothesis (H1) is sufficient for the next result.

For reasons of presentation, we shall from hereon in make a slight abuse of notation and associate to each $z \in \mathbb{R}^n$ the constant function $\mathbb{R}_+ \to \mathbb{R}^n$ with value z.

The next lemma gathers some technical material relating to the existence of constant trajectories of (1). To minimise disruption, the proof of the next lemma appears in the Appendix.

Lemma 5.1: *Consider the forced Lur'e system* (1) *and assume that* (H1) *holds. If there exist strictly positive $v \in \mathbb{R}^{p_1}$ and $\rho \in (0,1)$ such that*

$$v^\top \mathbf{G}_{11}(0)\Delta \leq \rho v^\top, \tag{26}$$

then, for every $w_* \in \mathbb{R}^{m_2}$, there exists a unique $x_* \in \mathbb{R}^n$ such that

$$0 = Ax_* + B_1 f(C_1 x_*) + B_2 w_*. \tag{27}$$

In particular, $(w_*, x_*) \in \mathcal{B}$. If f maps $\mathbb{R}^{p_1}_+$ into $\mathbb{R}^{m_1}_+$ and $w_* \geq 0$, then $x_* \geq 0$.

When the nonlinearity f in (1) is scalar valued (that is, $m_1 = p_1 = 1$), then condition (26) simplifies to the scalar inequality $\mathbf{G}_{11}(0)\Delta < 1$ (the positive constant v cancels from both sides of (26)), which is a consequence of Lemma 3.1 as hypothesis (H1) holds. More generally, a simple sufficient condition for (26) is that $\mathbf{G}_{11}(0)\Delta$ is irreducible, in which case (26) holds with equality for some strictly positive v and $\rho = \rho(\mathbf{G}_{11}(0)\Delta) < 1$ by the Perron-Frobenius theorem and Lemma 3.1.

For the next result, recall that a subset $V \subseteq L^\infty(\mathbb{R}_+, \mathbb{R}^{m_2})$ is said to be *equi-convergent* to $v_* \in \mathbb{R}^{m_2}$ if, for all $\varepsilon > 0$, there exists $\tau \geq 0$ such that

$$\|\sigma_\tau v - v_*\|_{L^\infty} \leq \varepsilon \quad \forall\, v \in V,$$

where σ_τ for $\tau \in \mathbb{R}$ is the shift- or translation-operator $(\sigma_\tau)f(t) = f(t+\tau)$.

Proposition 5.2: *Consider the Lur'e system* (1), *and assume that* (H1) *is satisfied. Let $1 \leq s < \infty$, and let $(w_a, x_a), (w_b, x_b) \in \mathcal{B}$. The following statements hold.*

(i) *If $w_a(t) - w_b(t) \to 0$ as $t \to \infty$, or $w_a - w_b \in L^s(\mathbb{R}_+, \mathbb{R}^{m_2})$, then $x_a(t) - x_b(t) \to 0$ as $t \to \infty$.*
(ii) *If $w_a - w_b \in L^s_\gamma(\mathbb{R}_+, \mathbb{R}^{m_2})$ for some $\gamma > 0$, then $x_a(t) - x_b(t) \to 0$ exponentially as $t \to \infty$.*
(iii) *Assume that (26) holds. If $w_a(t) \to w_*$ as $t \to \infty$, then $x_a(t) \to x_*$ as $t \to \infty$. Moreover, if V is equi-convergent to v_*, then for every $L > 0$, the set of solutions*

$$\left\{x(\cdot\,; x^0, v) \;:\; (x^0, v) \in \mathbb{R}^n_+ \times V \text{ with } \|x^0\| + \|v\|_{L^\infty} \leq L\right\},$$

is equi-convergent to x_*.

Proof: Let $(w_a, x_a), (w_b, x_b) \in \mathcal{B}$ denote two trajectories of (1). Define $\tilde{r} := B_2^\top p \in \mathbb{R}^{m_2}_+$.

(i) Define $h : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$h(t) := \mathrm{e}^{-\xi t} \int_0^t \mathrm{e}^{\xi\theta} |(w_a - w_b)(\theta)|_{\tilde{r}} \, \mathrm{d}\theta.$$

If $w_a(t) - w_b(t) \to 0$ as $t \to \infty$, then a standard argument writing h as

$$h(t) = \mathrm{e}^{-\xi t} \left(\int_0^{t/2} \mathrm{e}^{\xi\theta} |(w_a - w_b)(\theta)|_{\tilde{r}} \, \mathrm{d}\theta + \int_{t/2}^t \mathrm{e}^{\xi\theta} |(w_a - w_b)(\theta)|_{\tilde{r}} \, \mathrm{d}\theta \right) \quad \forall\, t \geq 0,$$

and estimating both terms above separately shows that $h(t) \to 0$ as $t \to \infty$. That $(x_a - x_b)(t) \to 0$ as $t \to \infty$ now follows from inequality (10). If instead $w_a - w_b \in L^s(\mathbb{R}_+)$, then the estimate (20) shows that $x_a - x_b$ is bounded. Therefore, another application of the estimate (20), now with $t_0 = t/2$, gives

$$\begin{aligned} |(x_a - x_b)(t)|_p &\leq \mathrm{e}^{-\xi t/2} |(x_a - x_b)(t/2)|_p + \alpha_s \|w_a - w_b\|_{L^s(t/2,t;\tilde{r})} \\ &\leq \mathrm{e}^{-\xi t/2} \|x_a - x_b\|_{L^\infty(0,\infty;p)} + \alpha_s \|w_a - w_b\|_{L^s(t/2,\infty;\tilde{r})} \to 0 \quad \text{as } t \to \infty. \end{aligned} \tag{28}$$

(ii) Since $\|w_a - w_b\|_{L^s(t_0,t;\tilde{r})} \leq \mathrm{e}^{-\gamma t_0} \|w_a - w_b\|_{L^s_\gamma(t_0,t;\tilde{r})}$ for all $t_0 \geq 0$, it follows from (20) that $x_a - x_b$ is bounded. Moreover, the inequality (28) now yields that

$$|(x_a - x_b)(t)|_p \leq \mathrm{e}^{-\xi t/2} \|x_a - x_b\|_{L^\infty(0,\infty;p)} + \alpha_s \, \mathrm{e}^{-\gamma t/2} \|w_a - w_b\|_{L^s_\gamma(t/2,t;\tilde{r})} \quad \forall\, t \geq 0,$$

giving the claimed exponential convergence.

(iii) An application of Lemma 5.1 ensures the existence of a constant trajectory (w_*, x_*) of (1). The claimed convergence now follows from statement (i) with $(w_b, x_b) = (w_*, x_*)$. The claimed equi-convergence follows from (10) as, under the given hypotheses, for all $\varepsilon, L > 0$, there exists $\tau \geq 0$ such that $h(t) \leq \varepsilon = \varepsilon(L)$ for all $t \geq \tau$ and all $w_a \in V$ with $\|w_a\|_{L^\infty} \leq L$. ■

6. An example

We illustrate our results with an example.

Consider (1) with $\mathtt{n} = 3$, $\mathtt{m}_1 = \mathtt{p}_1 = 2$, $\mathtt{m}_2 = \mathtt{p}_2 = 1$ and model data

$$A := \begin{pmatrix} -1 & 0 & 0 \\ 0 & -10 & 0 \\ 0 & 0 & -100 \end{pmatrix} + 0.01 \mathbb{1}\mathbb{1}^\top,$$

$$B_1 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad B_2 := \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix},$$

$$C_1 := \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad C_2 := \begin{pmatrix} 1 & 0 & 0 \end{pmatrix},$$

which satisfies (A1), and nonlinearities $f : \mathbb{R}^2 \to \mathbb{R}^2$ of the form

$$f(t, y) = f(y) := \begin{pmatrix} f_1(y_1) + f_2(y_2) \\ f_2(y_2) \end{pmatrix} \quad \forall\, y \in \mathbb{R}^2.$$

Here f_1 and f_2 are assumed to be Lipschitz functions with (positive) Lipschitz constants δ_1 and δ_2, respectively. In light of

$$|f(y) - f(z)| = \begin{pmatrix} |f_1(y_1) - f_1(z_1) + f_2(y_2) - f_2(z_2)| \\ |f_2(y_2) - f_2(z_2)| \end{pmatrix} \leq \begin{pmatrix} \delta_1 & \delta_2 \\ 0 & \delta_2 \end{pmatrix} \begin{pmatrix} |y_1 - z_1| \\ |y_2 - z_2| \end{pmatrix} \quad \forall\, y, z \in \mathbb{R}^2,$$

it is clear that (A2) holds with Δ as above. Writing

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \mathbf{G}_{11}(0) = \frac{1}{100} \begin{pmatrix} 1.11 & 10.0 \\ 1.01 & 0.001 \end{pmatrix} \gg 0,$$

a calculation shows that the spectral radius condition $\rho(\mathbf{G}_{11}(0)\Delta) = 1$ yields

$$H(\delta_1, \delta_2) := (ad - bc)\delta_1\delta_2 - a\delta_1 - (c + d)\delta_2 + 1 = 0.$$

which is readily checked as describing a hyperbola in the variables (δ_1, δ_2) since $ad - bc < 0$. By monotonicity and Lemma 3.1, it follows that (H1) holds for all

$$(\delta_1, \delta_2) \in \left\{ (z_1, z_2) \in \mathbb{R}^2_+ \, : \, H(z_1, z_2) > 0 \right\}.$$

Recall that hypothesis (H1) is sufficient for the estimates (10) and (20), and is a key hypothesis in Proposition 5.2. Since $\mathbf{G}_{11}(0)\Delta \gg 0$, so that $\Delta\mathbf{G}_{11}(0)$ is *a forteriori* irreducible, it follows that hypothesis (26) holds as well. As a numerical example, we take

$$f_1(z) := \frac{L_1}{4.5}(0.5z + 4\sin(z)) \quad \text{and}$$
$$f_2(z) := \frac{L_2}{3.5}(-0.5z - 1.5\sin(2z)) \; \forall\, z \in \mathbb{R}.$$

These functions are globally Lipschitz with Lipschitz constants L_1 and L_2, respectively. Taking $L_1 = \delta_1 = 20$ and $L_2 = \delta_2 = 25$, it follows that $H(L_1, L_2) > 0$, and so hypothesis (H1) is satisfied. Computing the spectral abscissa of $A + B_1\Delta C_1$, denoted α, and a corresponding left eigenvector gives that (H1) holds with

$$\xi = -\alpha = 0.1887 \quad \text{and} \quad p^\top = \begin{pmatrix} 0.0166 & 0.9044 & 0.4264 \end{pmatrix}.$$

For the following simulations we use integer k as an index and amplitude parameter. We define initial conditions and forcing functions

$$x^k = k \begin{pmatrix} 1 & -1 & 1 \end{pmatrix}^\top \quad \text{and}$$
$$w^k(t) = \sin(1.5t) + 5kt\,\mathrm{e}^{-t} \; k \in \{1, 2, 3, 4\},$$

respectively, and write $(x(\cdot\,; k), w^k)$ for the trajectory of (1) subject to initial condition x^k. Our results are plotted in Figure 2.

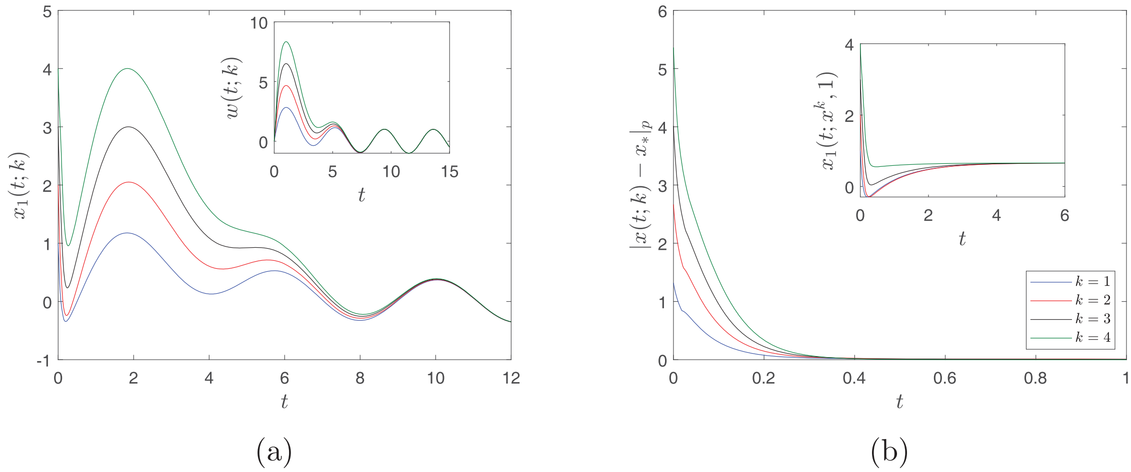

Figure 2. Simulation results from example of Section 6. Different curves correspond to different index k. See main text for a description. The legend in panel (b) applies to both panels.

Panel 2(a) plots the first component of $x(t;k)$ against t, with the inset containing graphs of $w(t;k)$ against t. Convergence of this state component to a common trajectory is observed. We do not show plots of $|x(t;k) - x(t;1)|_p$ against t, but these errors do indeed converge to zero.

Next, we consider the constant (and hence convergent) $w(t) = 1$, and plot the graphs of $|x(t;x^k,1) - x_*|_p$ against t in panel 2(b). Here $x_* \in \mathbb{R}^3$ is as in (27) and given by

$$x_* = \begin{pmatrix} 0.6577 & -0.0158 & 0.0066 \end{pmatrix}^\top .$$

In each case, convergence of the error to zero is observed, in accordance with Proposition 5.2. The inset contains plots of the first component $x(t;x^k,1)$.

Finally, we comment on (H2). Evidently, $A + B_1 \Delta C_1 + \xi I$ is Hurwitz for all positive $\xi \in (0, -\alpha)$. Since $\mathrm{m}_2 = \mathrm{p}_2 = 1$, the terms r and q in the linear dissipativity equations (8) are scalars. Therefore, by Lemma 3.2, particularly (9a), hypothesis (H2) is satisfied for $\xi \in (0, \alpha)$ with q and r related by

$$r = \left(\mathbf{G}_{22}(-\xi) + \mathbf{G}_{21}(-\xi)\left(I - \Delta \mathbf{G}_{11}(-\xi)\right)^{-1} \Delta \mathbf{G}_{12}(-\xi)\right) q,$$

recalling that there is a choice of positive scaling of pairs (r,q) which satisfy (8). For such (r,q), the inequality (21) holds for all $0 \le t_0 \le t_1$ and all pairs of input/state/output trajectories (w_a, x_a, y_a) and (w_b, x_b, y_b) of (1) with $x_a(t_0) = x_b(t_0)$ and model data as described in this section.

7. Summary

Incremental stability properties have been considered for a class of systems of forced, four-block, positive Lur'e differential equations. The approach to incremental stability is via linear dissipativity theory, explored across (Briat, 2013; Haddad & Chellaboina, 2005; Haddad et al., 2010; Rantzer, 2015), which is natural when the underlying linear system is positive, and distinguishes the work from other incremental stability works by the authors, including (Gilmore et al., 2020, 2021; Guiver, Logemann, Opmeer, 2019). In another vein, the hypotheses of the real Aizerman conjecture for positive Lur'e systems, when combined with a linear incremental gain condition on the nonlinear component (thus, recognised as the hypotheses of the Kalman conjecture), are sufficient for various incremental stability notions. Our main incremental stability result is Theorem 4.1, which may be viewed as an 'incremental (r,q)-mass imbalance or dissipation' for input/state/output trajectories of (1), and leads to incremental stability- and convergence-properties presented as Corollary 4.2 and Proposition 5.2, respectively. A strength of our approach is that the inequalities in (H1) and (H2) on the underlying linear data of the Lur'e system scale well with problem size. However, one limitation of our work is the linear incremental inequality in (A2) which necessitates that the nonlinear term f is globally Lipschitz in its second variable. Whilst we have not given formal results here, Theorem 4.1 can be shown to lead to desirable and predictable behaviour in terms of the state- and output-response of (1) to convergent and (almost) periodic forcing terms, by imitating the arguments used in Gilmore et al. (2020) and Gilmore et al. (2021).

We comment that one motivation for the current work is the paper (Revay et al., 2020). There, roughly speaking, the authors use IQCs to derive incremental input–output estimates for so-called equilibrium neural networks, which may be used to quantify their robustness. Briefly, and from a control theoretic perspective, equilibrium neural networks admit a description as the equilibrium of a forced Lur'e system with nonlinear term which satisfies a Lipschitz condition. Whilst our work makes essential use of the underlying positivity structure, should the networks considered in applications satisfy this structure, then our results become applicable.

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Statements and declarations

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Appendix

We provide proofs not given in the main text.

Proof of Lemma 3.1: Since $A + B_1\Delta C_1$ is Metzler, hypothesis (H1) is equivalent to $A + B_1\Delta C_1$ being Hurwitz by, for example, Haddad et al. (2010, Lemma 2.2, p. 31). This latter property is equivalent to statement (ii) by Drummond et al. (2023, Lemma 1).

We now prove the equivalence of statements (i) and (iii). Assume first that (H1) holds. In particular, since $p^\top B_1\Delta C_1 \geq 0$ by hypothesis and (A1), we have that

$$p^\top A \leq p^\top(A + B_1\Delta C_1) \leq -\xi p^\top,$$

so that A is Hurwitz and, hence, invertible. Seeking a contradiction, assume that $\rho(\mathbf{G}_{11}(0)\Delta) \geq 1$. In particular, there exists $\Sigma \in \mathbb{R}_+^{\mathrm{m}_1\times \mathrm{p}_1}$ with $\Sigma \leq \Delta$ and nonzero $w \in \mathbb{R}_+^{\mathrm{p}_1}$ such that $\mathbf{G}_{11}(0)\Sigma w = w$. Necessarily it must be the case that $(-A)^{-1}B_1\Sigma w > 0$, as otherwise the previous equality gives the contradiction $w = 0$.

On the one hand, we now compute that

$$\begin{aligned} p^\top(A + B_1\Sigma C_1)(-A)^{-1}B_1\Sigma w &= p^\top(-B_1\Sigma w + B_1\Sigma\mathbf{G}_{11}(0)\Sigma w) \\ &= p^\top(-B_1\Sigma w + B_1\Sigma w) = 0. \end{aligned}$$

On the other hand, by hypothesis

$$\begin{aligned} p^\top(A + B_1\Sigma C_1)(-A)^{-1}B_1\Sigma w &\leq p^\top(A + B_1\Delta C_1)(-A)^{-1}B_1\Sigma w \\ &\leq -\xi p^\top(-A)^{-1}B_1\Sigma w < 0, \end{aligned}$$

since $p \gg 0$ and $(-A)^{-1}B_1\Sigma w > 0$, which is a contradiction.

Conversely, assume that A is Hurwitz and that $\rho(\mathbf{G}_{11}(0)\Delta) < 1$. If $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) \geq 0$ and $v \in \mathbb{C}^{\mathrm{n}}$ are such that

$$(A + B_1\Delta C_1)v = \lambda v,$$

then λ is not an eigenvalue of A, since A is Hurwitz, so that $(\lambda I - A)^{-1}B_1\Delta C_1 v = v$ and, consequently,

$$\mathbf{G}_{11}(\lambda)\Delta C_1 v = C_1 v. \tag{A1}$$

The positivity hypotheses in (A1) ensure that

$$\begin{aligned} |\mathbf{G}_{11}(\lambda)| &= \left|\int_0^\infty \mathrm{e}^{-\lambda t}C_1\,\mathrm{e}^{At}B_1\,\mathrm{d}t\right| \leq \int_0^\infty |\,\mathrm{e}^{-\lambda t}| \cdot |C_1\,\mathrm{e}^{At}B_1|\,\mathrm{d}t \\ &\leq \int_0^\infty C_1\,\mathrm{e}^{At}B_1\,\mathrm{d}t = \mathbf{G}_{11}(0). \end{aligned}$$

Hence,

$$|\mathbf{G}_{11}(\lambda)\Delta| \leq |\mathbf{G}_{11}(\lambda)| \cdot |\Delta| \leq \mathbf{G}_{11}(0)\Delta.$$

Since $\rho(M) \leq \rho(|M|)$ for all square matrices M, we have that

$$\rho(\mathbf{G}_{11}(\lambda)\Delta) \leq \rho(|\mathbf{G}_{11}(\lambda)\Delta|) \leq \rho(\mathbf{G}_{11}(0)\Delta) < 1,$$

and we conclude from (A1) that $C_1 v = 0$. It now follows that $v = 0$, since A is Hurwitz. ■

Proof of Lemma 3.2: The claim follows from an application of Lemma 2.1. The given strictly positive hypothesis ensures that $p \gg 0$. For the second condition, we invoke the Sherman–Morrison–Woodbury formula to give

$$\begin{aligned} (-\xi I - (A + B_1\Delta C_1))^{-1} &= ((-\xi I - A)^{-1} + (-\xi I - A)^{-1}B_1 \\ &\quad \times \left(I - \Delta C_1(-\xi I - A)^{-1}B_1\right)^{-1} \\ &\quad \times \Delta C_1(-\xi I - A)^{-1}). \end{aligned}$$

Pre- and post-multiplying the right-hand side of the above by $q^\top C_2$ and B_2, respectively, shows that the left-hand side of (9a) equals the left-hand side of (9b). In particular, condition (9a) holds if, and only if, condition (9b) does. ■

Proof of Remark 4.4: Let M be a Metzler matrix. Recall, for example from Bullo (2023, Equation (2.43), p. 28), that the logarithmic norm μ_1 satisfies

$$\mu_1(M) = \max \mathbb{1}^\top M. \tag{A2}$$

The block matrix in (25) is Metzler by construction and hypothesis (A1). Suppose that (H1) holds. Define R as the diagonal matrix with ith diagonal entry equal to p_i, and set $\tau := \delta p^\top B_1$. Observe that $\mathbb{1}^\top R = p^\top$ and $\mathbb{1}^\top = p^\top R^{-1}$. We compute that

$$\begin{aligned} &\begin{pmatrix}\mathbb{1}^\top & 1\end{pmatrix}\begin{pmatrix} RAR^{-1} + \xi I & RB_1 \\ \tau C_1 R^{-1} & -\tau\delta^{-1}\end{pmatrix} \\ &= \begin{pmatrix}\mathbb{1}^\top RAR^{-1} + \xi\mathbb{1}^\top + \tau C_1R^{-1} & \mathbb{1}^\top RB_1 - \tau\delta^{-1}\end{pmatrix} \\ &= \begin{pmatrix}p^\top(A + B_1\delta C_1 + \xi I)R^{-1} & 0\end{pmatrix} \leq 0, \end{aligned}$$

by (H1) and as $R^{-1} \geq 0$. We conclude that (25) holds. Conversely, suppose that (25) holds for some positive definite diagonal R and $\tau \geq 0$. Setting

$p^\top := \mathbb{1}^\top R$, the conjunction of (25) and (A2) gives that

$$p^\top AR^{-1} + p^\top R^{-1}\xi + \tau C_1 R^{-1} \leq 0 \quad \text{so that } p^\top A + p^\top \xi + \tau C_1 \leq 0,$$

since $R \geq 0$, as well as

$$p^\top B_1 - \tau\delta^{-1} \leq 0.$$

Thus,

$$p^\top (A + B_1\delta C_1 + \xi I) \leq -\tau C_1 + p^\top B_1 \delta C_1 \leq -\tau C_1 + \tau C_1 = 0,$$

that is, (H1) holds. ■

Proof of Lemma 5.1: Let $w_* \in \mathbb{R}^{\mathrm{m}_2}$ be given. We proceed in two steps.

Step 1: construction of x_*. Consider the function

$$F : \mathbb{R}^{\mathrm{p}_1} \to \mathbb{R}^{\mathrm{p}_1}, \quad F(y) := \mathbf{G}_{11}(0)f(y) - C_1A^{-1}B_2w_* \quad \forall\, y \in \mathbb{R}^{\mathrm{p}_1}.$$

The function F is a contraction in the $|\cdot|_v$ norm, as seen from the estimates

$$\begin{aligned} |F(y_1) - F(y_2)|_v &= v^\top |\mathbf{G}_{11}(0)f(y_1) - \mathbf{G}_{11}(0)f(y_2)| \\ &\leq v^\top \mathbf{G}_{11}(0)\Delta|y_1 - y_2| \\ &\leq \rho|y_1 - y_2|_v \quad \forall\, y_1, y_2 \in \mathbb{R}^{\mathrm{p}_1}, \end{aligned}$$

where we have invoked that $\mathbf{G}_{11}(0) \geq 0$, the incremental bound (A2), and the inequality (26). Therefore, by the contraction mapping theorem, F has a unique fixed point, which we denote y_*. Now define

$$x_* := -A^{-1}B_1f(y_*) - A^{-1}B_2w_*, \tag{A3}$$

and observe that

$$\begin{aligned} &Ax_* + B_1f(C_1x_*) + B_2w_* \\ &\quad = -B_1f(y_*) + B_1f\left(-C_1A^{-1}B_1f(y_*) - C_1A^{-1}B_2w_*\right) \\ &\quad = -B_1f(y_*) + B_1f(F(y_*)) = 0, \end{aligned}$$

since y_* is the fixed point of F. We see that x_* satisfies (27)

Step 2: uniqueness of x_*. To establish uniqueness of x_* as solution of (27), assume that $x_\dagger \in \mathbb{R}^{\mathrm{n}}$ satisfies (27). Multiplying both sides of this expression by C_1 gives that

$$C_1x_\dagger = -C_1A^{-1}B_1f(C_1x_\dagger) - C_1A^{-1}B_2w_* = F(C_1x_\dagger),$$

and so, from the uniqueness of y_* as a fixed point of F, it follows that $C_1x_\dagger = y_*$. Therefore, from (A3),

$$x_* = -A^{-1}B_1f(y_*) - A^{-1}B_2w_* = -A^{-1}B_1f(C_1x_\dagger) - A^{-1}B_2w_* = x_\dagger,$$

as required.

To prove the non-negativity of x_*, we note that $y_* = \lim_{n\to\infty} F^n(y_0)$ for any $y_0 \in \mathbb{R}^{\mathrm{p}_1}$. Since F maps $\mathbb{R}^{\mathrm{p}_1}_+$ into $\mathbb{R}^{\mathrm{p}_1}_+$ by hypothesis on f, taking $y_0 \geq 0$ gives $y_* \geq 0$. Now $x_* \geq 0$ by (A3). ■