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Funnel control for a class of nonlinear infinite-dimensional systems

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Abstract

An adaptive funnel control method is considered for the regulation of the output for a class of nonlinear infinite-dimensional systems on real Hilbert spaces. After a decomposition of the state space and some change of variables related to the Byrnes-Isidori form, it is shown that the funnel controller presented in (Berger et al., 2020) achieves the control objective under some assumptions on the nonlinear system dynamics, like well-posedness and Bounded-Input-State Bounded-Output (BISBO) stability. The theory is applied to the regulation of the temperature in a chemical plug-flow tubular reactor whose reaction kinetics are modeled by the Arrhenius nonlinearity. Furthermore a damped sine-Gordon model is shown to fit the required assumptions as well. The theoretical results are illustrated by means of numerical simulations.

Keywords: Nonlinear distributed parameter systems - Funnel control - Plug-flow tubular reactor - sine-Gordon equation

1. Introduction

An adaptive funnel controller is a model-free output error feedback controller with a time-varying gain that adapts according to the output error in order to ensure a predetermined transient behavior for the tracking error with no necessary asymptotic tracking performance. Note that a funnel controller is a non dynamic controller, which makes it different from other types of adaptive control methods, see e.g. Åström (1983) and Krstić et al. (1995) for surveys on adaptive control. This field of control has attracted a lot of attention in the last few years. It has been thoroughly developed in Ilchmann et al. (2002) for systems with relative degree one. In particular, they have shown that the funnel control approach is feasible for linear finite-dimensional systems, infinite-dimensional linear regular systems, nonlinear finite-dimensional systems, nonlinear delay systems and systems with hysteresis. Details for nonlinear systems with relative degree one are also available in Ilchmann et al. (2005). Few years later, funnel control for MIMO nonlinear systems with known strict relative degree has been widely developed in Berger et al. (2018). The new funnel controller they introduced involves the $r - 1$ derivatives of the tracking error, where r stands for the relative degree of the system. As an illustration of the method, they consider a mass-spring system mounted

on a car. Note that a new survey for funnel control for different types of systems can be found in Berger et al. (2021c). Funnel control has also been considered in several applications. For instance Ilchmann and Trenn (2004) developed a funnel controller to regulate the reaction temperature in chemical reactors. Later, the position of a moving water tank has been controlled via a funnel controller in Berger et al. (2022). A linearized version of the Saint-Venant Exner infinite-dimensional dynamics has been used as internal dynamics. This shows that funnel control becomes more and more attractive for systems driven by infinite-dimensional internal dynamics. In that way, this topic has been considered in Berger et al. (2020). The authors proved that some class of infinite-dimensional linear systems fits the required assumptions for funnel control to be feasible. Moreover, they proved that linear-infinite dimensional systems that can be written in Byrnes-Isidori form, see Ilchmann et al. (2016), are encompassed in that new class. More recently, funnel control has also been applied to a nonlinear infinite-dimensional reaction-diffusion equation coupled with the nonlinear Fitzhugh-Nagumo model, which represent together defibrillation processes of the human heart, see Berger et al. (2021a). Note also that funnel control has been lately coupled to model-predictive-control (Funnel MPC) for nonlinear systems with relative degree one, see Berger et al. (2021b).

The novelty in this paper consists in considering a class of nonlinear infinite-dimensional systems to which fun-

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nel control can be applied¹. Based on the Byrnes-Isidori form for linear systems, we introduce a change of variables that aims at extracting the output dynamics of the system, which is assumed to be finite-dimensional. Based on this transformation, funnel control is shown to be feasible provided that the remaining part of the dynamics satisfies some BISBO stability assumption. This constitutes the main result of the paper. Moreover, a way of getting this BISBO stability condition is presented for bounded nonlinearities. We illustrate our results on two nonlinear infinite-dimensional partial differential equations (PDEs) modeling the dynamics of a plug-flow tubular reactor and a damped sine-Gordon nonlinear equation, respectively.

1.1. Structure of the concepts and results

The paper is organized as follows: the system class for which funnel control is known to be feasible together with the control objective are presented in Section 2. In Section 3 a class of nonlinear controlled and observed infinite-dimensional systems with some appropriate assumptions is considered and is shown to fulfill the assumptions needed to apply funnel control. A way to get BISBO stability of the internal dynamics is presented. Applications of the main results with some numerical simulations are given in Section 4 for a chemical plug-flow tubular reactor and a damped sine-Gordon equation. Section 5 is dedicated to conclusions and perspectives.

1.2. Notations

Throughout this paper, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}^+$ is the set of positive real numbers and $\mathbb{N}$ is the set of natural numbers. By $L^\infty(\mathbb{R}^+, \mathbb{R})$ we mean the Lebesgue space of essentially bounded functions $f : \mathbb{R}^+ \rightarrow \mathbb{R}$. The subscript *loc* in $L_{loc}^\infty(\mathbb{R}, \mathbb{R})$ refers to locally, i.e. $L_{loc}^\infty(\mathbb{R}^+, \mathbb{R})$ is the space of locally essentially bounded real-valued functions defined on $\mathbb{R}^+$. The space $\mathcal{C}(\mathbb{R}^+, \mathbb{R})$ denotes the space of continuous functions $f : \mathbb{R}^+ \rightarrow \mathbb{R}$. The spaces $L^p(0, 1)$, $1 \leq p < \infty$, are the Lebesgue spaces of p -integrable functions defined on $[0, 1]$. The Sobolev space of $L^\infty(\mathbb{R}^+, \mathbb{R})$ -functions whose first order weak derivative is still in $L^\infty(\mathbb{R}^+, \mathbb{R})$ is denoted by $W^{1,\infty}(\mathbb{R}^+, \mathbb{R})$.

2. Input/Output system class and control objective

This section is dedicated to the introduction of the considered system class together with the control objective that is pursued.

2.1. General framework

We shall consider a differential equation that makes the connection between the input and the output of a dynamical system whose internal dynamics are not necessarily known (model-free). This equation reads as

$$\begin{cases} \dot{y}(t) = N(d(t), T(y)(t)) + \Gamma(d(t), T(y)(t))u(t), \\ y(0) = y_0, \end{cases} \quad (1)$$

where the following conditions are assumed to hold.

Assumption 2.1. *The disturbance $d \in L^\infty(\mathbb{R}^+, \mathbb{R})$, the nonlinear function N is in $\mathcal{C}(\mathbb{R}^2, \mathbb{R})$ and the gain function $\Gamma \in \mathcal{C}(\mathbb{R}^2, \mathbb{R})$ is positive in the sense that $\Gamma(d, \varrho) > 0$ for all $(d, \varrho) \in \mathbb{R}^2$.*

Assumption 2.2. *The map $T : \mathcal{C}(\mathbb{R}^+, \mathbb{R}) \rightarrow L_{loc}^\infty(\mathbb{R}^+, \mathbb{R})$ is a (possibly nonlinear) operator which satisfies the following conditions:*

1. *Bounded trajectories are mapped into bounded trajectories (BIBO property), i.e. for all $k_1 > 0$, there exists $k_2 > 0$ such that for all $y \in \mathcal{C}(\mathbb{R}^+, \mathbb{R})$,*

$$\sup_{t \in \mathbb{R}^+} |y(t)| \leq k_1 \Rightarrow \sup_{t \in \mathbb{R}^+} |T(y)(t)| \leq k_2 \quad (2)$$

2. *The operator T is causal, i.e. for any $t \in \mathbb{R}^+$ and any $y, \hat{y} \in \mathcal{C}(\mathbb{R}^+, \mathbb{R})$*

$$y|_{[0,t]} = \hat{y}|_{[0,t]} \Rightarrow T(y)|_{[0,t]} = T(\hat{y})|_{[0,t]}, \quad (3)$$

where $f|_I$ denotes the restriction of a function f to the interval I .

3. *T is locally Lipschitz in the sense that for all $t \in \mathbb{R}^+$ and all $y \in \mathcal{C}([0, t], \mathbb{R})$ there exist positive constants τ, δ and ρ such that for any $y_1, y_2 \in \mathcal{C}(\mathbb{R}^+, \mathbb{R})$ with $y_i|_{[0,t]} = y, i = 1, 2$ and $|y_i(s) - y(t)| < \delta$ for all $s \in [t, t + \tau]$ and $i = 1, 2$ it holds that*

$$\|(T(y_1) - T(y_2))|_{[t, t+\tau]}\|_\infty \leq \rho \|(y_1 - y_2)|_{[t, t+\tau]}\|_\infty, \quad (4)$$

where $\|f|_{[t, t+\tau]}\|_\infty := \sup_{s \in [t, t+\tau]} |f(s)|$.

Note that the nonlinear operator T encodes the internal dynamics of the system that is hidden in the differential relation between the input and the output (1). As highlighted in Berger et al. (2020), the map T allows for linear infinite-dimensional internal dynamics. It will be shown here that some class of nonlinear infinite-dimensional systems are permitted too.

The class of systems governed by (1) with Assumptions 2.1–2.2 is presented in (Berger et al., 2020, Section 1) for systems with (possible) memory and relative degree $r \in \mathbb{N}$. Here we consider systems with no memory and relative degree one, see (1). This class is quite general and encompasses systems with infinite-dimensional internal dynamics as shown in Berger et al. (2020) and Ilchmann et al. (2016) for instance. However it is still not clear which classes of distributed-parameter systems (DPS) may be written as the input-output equation (1). Such a compatible class is presented in Section 3.

¹Note that this class is restricted to systems with relative degree one.

2.2. Control objective

The objective here consists in developing an output error feedback $u(t) = \mathcal{G}(t, e(t))$ with $e(t) = y(t) - y_{\text{ref}}(t)$ for a reference signal $y_{\text{ref}} \in W^{1,\infty}(\mathbb{R}^+, \mathbb{R})$, such that, when connected to (1), it results in a closed-loop system for which the error $e(t)$ evolves in a prescribed performance funnel

$$\mathcal{F}_\phi := \{(t, e) \in \mathbb{R}^+ \times \mathbb{R}, \phi(t)|e(t)| < 1\}, \quad (5)$$

where the function ϕ is assumed to belong to

$$\Phi := \left\{ \phi \in \mathcal{C}(\mathbb{R}^+, \mathbb{R}), \phi, \dot{\phi} \in L^\infty(\mathbb{R}^+, \mathbb{R}), \right. \\ \left. \phi(t) > 0, \forall t \in \mathbb{R}^+ \text{ and } \liminf_{t \rightarrow \infty} \phi(t) > 0 \right\}. \quad (6)$$

This control objective is also considered in (Berger et al., 2020, Section 1). As described in Berger et al. (2020); Ilchmann et al. (2016) or Berger et al. (2018), a controller that achieves the output tracking performance described above is expressed as

$$u(t) = \frac{-e(t)}{1 - \phi^2(t)e^2(t)}, \quad (7)$$

with $\phi \in \Phi$ and $\phi(0)|e(0)| < 1$. The controller (7) is called a funnel controller and can be viewed as the output error feedback $u(t) = -k(t)e(t)$ with a time-varying (adaptive) gain $k(t) = \frac{1}{1 - \phi^2(t)e^2(t)}$. Let us consider the following theorem, coming from Berger et al. (2018) with $r = 1$, which characterizes the effectiveness of the controller (7) in terms of existence and uniqueness of solutions of the closed-loop systems and also in terms of output tracking performance.

Theorem 2.1. *Consider a system (1) with Assumptions 2.1–2.2. Let $y_{\text{ref}} \in W^{1,\infty}(\mathbb{R}^+, \mathbb{R})$, $\phi \in \Phi$ and $y_0 \in \mathbb{R}$ such that the condition $\phi(0)|e(0)| < 1$ holds true. Then the funnel controller (7) applied to (1) results in a closed-loop system whose solution $y : [0, \omega) \rightarrow \mathbb{R}$, $\omega \in (0, \infty]$, has the following properties:*

1. *The solution is global, i.e. $\omega = \infty$;*
2. *The input $u : \mathbb{R}^+ \rightarrow \mathbb{R}$, the gain function $k : \mathbb{R}^+ \rightarrow \mathbb{R}$ and the output $y : \mathbb{R}^+ \rightarrow \mathbb{R}$ are bounded;*
3. *The tracking error $e : \mathbb{R}^+ \rightarrow \mathbb{R}$ evolves in the funnel $\mathcal{F}_\phi$ and is bounded away from the funnel boundaries in the sense that there exists $\epsilon > 0$ such that, for all $t \geq 0$, $|e(t)| \leq \frac{1}{\phi(t)} - \epsilon$.*

This theorem will be a paramount tool in the next section, in order to prove output tracking control of the scalar output of some class of nonlinear DPS.

The main contribution here relies on the fact that we consider a class of nonlinear infinite-dimensional systems that admits an input-output differential description of the form (1). This constitutes also the difference with respect to e.g. Berger et al. (2020) wherein a class of operators T is introduced, which comes from systems modeled by linear infinite-dimensional internal dynamics. Our contribution

enlarges the class of systems for which funnel control is feasible since, to the best of our knowledge, our class of nonlinear infinite-dimensional systems is shown to be appropriate for funnel control for the first time here. However, examples in which nonlinear infinite-dimensional systems are considered have already been studied for funnel control in Berger et al. (2021a). Our work also extends the Byrnes Isidori forms studied in Ilchmann et al. (2016) to nonlinear infinite-dimensional systems that satisfy some relatively standard assumptions.

3. A class of state space single-input single-output nonlinear distributed parameter systems

In this section a quite general class of nonlinear infinite-dimensional systems is introduced for which output error control is considered. It is shown how the abstract differential equation governing the dynamics can be transformed into (1) by using a change of variables as it is made for linear systems in Ilchmann et al. (2016). Moreover, by stating some quite easily checkable assumptions, it is proved that the transformed equation satisfies Assumptions 2.1–2.2 introduced in Section 2, which implies that funnel control is feasible for such a class of systems. This constitutes the main contributions of the paper.

Let H be a real (separable) Hilbert space equipped with the inner product $\langle \cdot, \cdot \rangle_H$. The nonlinear systems that we consider are governed by the following controlled and perturbed abstract ordinary differential equation

$$\Sigma : \begin{cases} \dot{x}(t) = Ax(t) + f(x(t)) + b(u(t) + d(t)), \\ y(t) = \langle x(t), c \rangle_H, \\ x(0) = x_0 \in H, \end{cases} \quad (8)$$

for which we make the following three assumptions.

Assumption 3.1. *The (unbounded) linear operator $A : D(A) \subset H \rightarrow H$ is the infinitesimal generator of a strongly continuous (C_0) semigroup $(T(t))_{t \geq 0}$ of bounded linear operators on H . The growth bound of $(T(t))_{t \geq 0}$ is denoted by ω_0 .*

Assumption 3.2. *The nonlinear operator $f : H \rightarrow H$ is uniformly Lipschitz continuous on H .*

Assumption 3.3. *The vectors b and c are in $D(A)$ and $D(A^*)$, respectively, where $D(A^*)$ is the domain of the adjoint of the operator A . Moreover, it is assumed that $\langle b, c \rangle_H > 0$. The disturbance² $d \in L^\infty(\mathbb{R}^+, \mathbb{R})$.*

Note that the scalar functions u and y stand for the input and the output, respectively. According to (Curtain and Zwart, 2020, Theorem 11.1.5), Assumptions 3.1 and 3.2 implies that the homogenous³ part of (8) has a unique mild solution on $[0, \infty)$ which can even be a classical one provided that $x_0 \in D(A)$.

²By entering in the system (8) in this way, the disturbance d may be interpreted as modeling uncertainties on the input, that could be due to unmodelled actuator dynamics, for instance.

³By "homogenous" we mean uncontrolled and unperturbed.

Remark 3.1. We emphasize that, despite the fact that Assumption 3.3 may be seen as restrictive, if b and c are shape functions that do not lie in $D(A)$ and $D(A^*)$, it is always possible to approximate them, as accurately as desired, by functions in $D(A)$ and $D(A^*)$, since they are both dense subspaces of H .

The consequences of Remark 3.1 will be discussed later in this section.

Remark 3.2. At first glance, it is natural to think that the class of distributed parameter systems (8) seems to exclude boundary control and boundary observation. Formally, it does, but this does not mean that nothing is possible when such a situation occurs. Take for instance the example of a control operator b being the Dirac delta distribution at 0, i.e. $b = \delta_0$. This b clearly does not satisfy the required assumption. However, it is possible to replace this b by a term of a Dirac sequence lying in $D(A)$. A same comment could be formulated in the case of unbounded observation, like an output expressed as $y(t) = x(1, t)$ for instance. For this, the output could be approximated by a linear functional like $\langle x(t), \mathbf{h}_n \rangle_H$ with $\mathbf{h}_n$ being also a term of a Dirac sequence in $D(A^*)$ simulating the Dirac effect at 1. We do insist that such compromises entail approximation errors which should require thorough attention.

The following proposition reflects some decomposition of the state space H that is possible thanks to Assumption 3.3, see Ilchmann et al. (2016) and Byrnes et al. (1998).

Proposition 3.1. Let Assumption 3.3 hold. Then the state space H may be decomposed as

$$H = \text{span}\{c\} \oplus \{b\}^\perp =: \mathcal{O} \oplus \mathcal{I}, \quad (9)$$

where $\{b\}^\perp := \{f \in H, \langle f, b \rangle_H = 0\}$.

Proof. We start by showing that the intersection between $\mathcal{O}$ and $\mathcal{I}$ is reduced to the origin of H . Assume by contradiction that there exists $f \neq 0$ such that $f \in \mathcal{O}$ and $f \in \mathcal{I}$. Since $f \in \mathcal{O}$, there exists $\lambda \in \mathbb{R}_0$ such that $f = \lambda c$. As f is in $\mathcal{I}$ too, f is orthogonal to b , i.e. $\langle f, b \rangle_H = 0$. According to the relation $f = \lambda c$, there holds $\langle c, b \rangle_H = 0$, which contradicts Assumption 3.3. Now we show that the direct sum $\mathcal{O} \oplus \mathcal{I}$ is contained in H . This follows directly by noting that $c \in H$ and by the definition of $\{b\}^\perp$. Let us end the proof by proving the inclusion $H \subseteq \mathcal{O} \oplus \mathcal{I}$. Take any $f \in H$. Obviously f may be written as

$$f = \left(\frac{\langle f, b \rangle_H}{\langle c, b \rangle_H} c \right) + \left(f - \frac{\langle f, b \rangle_H}{\langle c, b \rangle_H} c \right), \quad (10)$$

where it is easy to see that $\frac{\langle f, b \rangle_H}{\langle c, b \rangle_H} c \in \mathcal{O}$ and $f - \frac{\langle f, b \rangle_H}{\langle c, b \rangle_H} c \in \mathcal{I}$. ■

Let us now consider the operator $P : H \rightarrow \mathbb{R}$ defined as

$$Px = \frac{\langle x, b \rangle_H}{\langle c, b \rangle_H}.$$

In what follows we shall denote the operator $I - cP$ by $P_{\mathcal{I}}$. A characterization of the subspaces $\mathcal{O}$ and $\mathcal{I}$ is highlighted in the next proposition.

Proposition 3.2. The subspaces $\mathcal{O}$ and $\mathcal{I}$ satisfy $\mathcal{O} = cPH$ and $\mathcal{I} = P_{\mathcal{I}}H$, respectively.

Proof. We start by showing that $\mathcal{O} = cPH$. First let us take $f \in \mathcal{O}$. Then there exists $\lambda \in \mathbb{R}$ such that $f = \lambda c$. Now observe that there exists $g \in H$ such that $f = cPg$. This g is given by λc . In a second step, let us consider $g \in cPH$. This entails that there exists $h \in H$ such that $g = cPh$. Then it is easy to see that $g \in \mathcal{O}$ since the constant λ such that $g = \lambda c$ is given by $\lambda = Ph$. Now we focus on the equality $\mathcal{I} = P_{\mathcal{I}}H$. Consider $f \in \mathcal{I}$. The question of existence of a function $g \in H$ such that $f = P_{\mathcal{I}}g$ is solved by taking $g = f$ because the relation $\langle f, b \rangle_H = 0$ holds. The inclusion $P_{\mathcal{I}}H \subseteq \mathcal{I}$ is proved as follows. Take any $f \in P_{\mathcal{I}}H$. Then there exists $h \in H$ such that $f = P_{\mathcal{I}}h$, that is $f = h - \frac{\langle h, b \rangle_H}{\langle c, b \rangle_H} c$. It is straightforward to show that this f is orthogonal to b , i.e. $f \in \mathcal{I}$. ■

Thanks to Propositions 3.1 and 3.2, any element $x \in H$ may be written uniquely as

$$x = cPx + P_{\mathcal{I}}x.$$

Moreover, observe that for any function $x \in H$ there holds $\langle P_{\mathcal{I}}x, b \rangle_H = 0$. The operator $P_{\mathcal{I}}$ is not an orthogonal projector on $\mathcal{I}$. Such a projector is denoted by $P^\perp : H \rightarrow \mathcal{I}$ and is defined as $P^\perp x = x - \frac{\langle x, b \rangle_H}{\langle b, b \rangle_H} b$. Now we shall introduce some operators that will be of importance in order to transform system (8). Let us consider the operator $U : H \rightarrow \mathbb{R} \times \mathcal{I}$ defined as

$$Ux = \begin{pmatrix} Px \\ P_{\mathcal{I}}x \end{pmatrix}. \quad (11)$$

This operator is boundedly invertible, see Ilchmann et al. (2016), with inverse $U^{-1} : \mathbb{R} \times \mathcal{I} \rightarrow H$ expressed as

$$U^{-1} \begin{pmatrix} \alpha \\ \eta \end{pmatrix} = \alpha c + \eta. \quad (12)$$

The adjoint of U^{-1} , denoted by $U^{-*} : H \rightarrow \mathbb{R} \times \mathcal{I}$, reads as follows

$$U^{-*}x = \begin{pmatrix} \langle x, c \rangle_H \\ P_{\mathcal{I}}^\perp x \end{pmatrix}. \quad (13)$$

Consequently, the operator $U^* : \mathbb{R} \times \mathcal{I} \rightarrow H$ is given by

$$U^* \begin{pmatrix} \alpha \\ \eta \end{pmatrix} = \alpha \frac{b}{\langle c, b \rangle_H} + \eta - \langle c, \eta \rangle_H \frac{b}{\langle c, b \rangle_H}. \quad (14)$$

Like the operators U and U^{-1} , the operators U^* and U^{-*} are linear and bounded. This allows us to perform the change of variables for system (8) defined by $x = U^*\xi$, $\xi \in \mathbb{R} \times \mathcal{I}$. By the invertibility of U^* , one gets

$$\xi = U^{-*}x = \begin{pmatrix} \langle x, c \rangle_H \\ P_{\mathcal{I}}^\perp x \end{pmatrix}. \quad (15)$$

By setting $\eta := P^\perp x$, one has that $\xi = \begin{pmatrix} y \\ \eta \end{pmatrix}$. Now we take a look at the dynamics of the variable ξ . In the new variables (15), the system (8) reads as

$$\tilde{\Sigma} : \begin{cases} \dot{\xi}(t) = U^{-*}AU^*\xi(t) + U^{-*}f(U^*\xi(t)) + U^{-*}bu(t) \\ \quad \quad \quad + U^{-*}bd(t), \\ y(t) = (1 \ 0)\xi(t), \\ \xi(0) = U^{-*}x_0 =: \xi_0. \end{cases} \quad (16)$$

Note that the systems Σ and $\tilde{\Sigma}$ are equivalent according to the state transformation induced by U^{-*} . By using the definitions of U^{-*} and U^* , see (13) and (14), the linear operator $U^{-*}AU^*$ can be rewritten as

$$\begin{aligned} U^{-*}AU^*\xi &= \begin{pmatrix} \langle AU^*\xi, c \rangle_H \\ P^\perp AU^*\xi \end{pmatrix} = \begin{pmatrix} \langle \xi, UA^*c \rangle_{\mathbb{R} \times \mathcal{I}} \\ P^\perp AU^*\xi \end{pmatrix} \\ &= \begin{pmatrix} y \frac{\langle A^*c, b \rangle_H}{\langle c, b \rangle_H} + \langle \eta, P_\mathcal{I}A^*c \rangle_H \\ y \frac{P^\perp Ab}{\langle c, b \rangle_H} + P^\perp A\eta - \frac{\langle c, \eta \rangle_H}{\langle c, b \rangle_H} P^\perp Ab \end{pmatrix} \\ &=: \begin{pmatrix} P_0 & S \\ R & Q \end{pmatrix} \xi, \end{aligned}$$

where the operators P_0, S, R and Q are defined as follows

$$\begin{aligned} P_0 : \mathbb{R} &\rightarrow \mathbb{R}, P_0 y = y \frac{\langle A^*c, b \rangle_H}{\langle c, b \rangle_H}, \\ S : \mathcal{I} &\rightarrow \mathbb{R}, S\eta = \langle \eta, P_\mathcal{I}A^*c \rangle_H, \\ R : \mathbb{R} &\rightarrow \mathcal{I}, Ry = y \frac{P^\perp Ab}{\langle c, b \rangle_H}, \\ Q : D(Q) &\subset \mathcal{I} \rightarrow \mathcal{I}, Q\eta = P^\perp A\eta - \frac{\langle c, \eta \rangle_H}{\langle c, b \rangle_H} P^\perp Ab, \end{aligned} \quad (17)$$

where $D(Q) = D(A) \cap \mathcal{I}$.

Remark 3.3. *The need of the boundedness of the control and the observation operators is put forward by looking for instance at the definition of the similarity transformation operator U , its adjoint U^* and the inverse of these two operators. For those to be bounded, it is necessary that b and c be bounded too.*

According to Ilchmann et al. (2016), the operator Q is the infinitesimal generator of a C_0 -semigroup⁴ $(T_Q(t))_{t \geq 0}$ on $\mathcal{I}$. Moreover, since the operators P_0, S and R are bounded, the operator $\begin{pmatrix} P_0 & S \\ R & Q \end{pmatrix}$ is still the generator of a C_0 -semigroup, see e.g. (Curtain and Zwart, 2020, Theorem 5.3.1).

By using (13) and (14), we can write the nonlinear part of (16) as

$$\begin{aligned} U^{-*}f(U^*\xi(t)) &= \begin{pmatrix} \langle f(U^*\xi(t)), c \rangle_H \\ f(U^*\xi(t)) - \frac{\langle f(U^*\xi(t)), b \rangle_H}{\langle c, b \rangle_H} b \end{pmatrix} \\ &=: \begin{pmatrix} \langle \tilde{f}(y(t), \eta(t)), c \rangle_H \\ \tilde{f}(y(t), \eta(t)) - \frac{\langle \tilde{f}(y(t), \eta(t)), b \rangle_H}{\langle c, b \rangle_H} b \end{pmatrix}, \end{aligned}$$

where the nonlinear operator $\tilde{f} : \mathbb{R} \times \mathcal{I} \rightarrow H$ is defined as $\tilde{f}(y, \eta) = f(y \frac{b}{\langle c, b \rangle_H} + \eta - \langle c, \eta \rangle_H \frac{b}{\langle c, b \rangle_H})$. By using Assumption 3.2 and the fact that U^{-*} and U^* are linear bounded operators, the nonlinear operator $U^{-*}f(U^*\cdot)$ is still uniformly Lipschitz continuous from $\mathbb{R} \times \mathcal{I}$ into $\mathbb{R} \times \mathcal{I}$. This entails that the homogeneous part of (16) possesses a unique mild solution on $[0, \infty)$. Taking any initial condition in $\mathbb{R} \times D(Q)$ implies that this solution is classical. Now observe that the term $U^{-*}b$ is expressed as $(\langle b, c \rangle_H \ 0)^T$. From these observations it follows that the dynamics of y and η may be written as

$$\dot{y}(t) = P_0 y(t) + S\eta(t) + \langle \tilde{f}(y(t), \eta(t)), c \rangle_H + \gamma u(t) + \gamma d(t) \quad (18)$$

and

$$\dot{\eta}(t) = Ry(t) + Q\eta(t) + \tilde{f}(y(t), \eta(t)) - \frac{\langle \tilde{f}(y(t), \eta(t)), b \rangle_H}{\langle b, b \rangle_H} b, \quad (19)$$

with initial condition $y(0) = y_0$ and $\eta(0) = \eta_0$, respectively, where $\gamma := \langle b, c \rangle_H$. This entails that (18) admits the representation (1), where

1. The gain function $\Gamma : \mathbb{R}^2 \rightarrow \mathbb{R}$ is defined as $\Gamma(d, \varrho) = \gamma > 0$;
2. The well-defined nonlinear operator $T : \mathcal{C}(\mathbb{R}^+, \mathbb{R}) \rightarrow L_{\text{loc}}^\infty(\mathbb{R}^+, \mathbb{R})$ has the form

$$T(y)(t) = P_0 y(t) + S\eta(t) + \langle \tilde{f}(y(t), \eta(t)), c \rangle_H, \quad (20)$$

where $\eta(t)$ is the mild solution of (19) for a fixed $\eta_0 \in \mathcal{I}$;

3. The function $N : \mathbb{R}^2 \rightarrow \mathbb{R}$ reads as $N(d, \varrho) = \gamma d + \varrho$.

This shows that Assumption 2.1 on system (1) is satisfied. It remains to show that the nonlinear operator T given by (20) possesses the three properties of Assumption 2.2. This constitutes the main result of this section. Before going into the details of this result, we shall denote by $\Sigma_{y \rightarrow \eta}$ the system which can be viewed as a system with input y and output η and whose dynamics are described by (19). Note also that the expression (20) of the operator T means that the latter is parametrized by the initial condition η_0 . All the reasonings hereafter will be performed by fixing first the initial condition η_0 resulting in a unique T . Moreover, the dependence of T to η_0 will not be made explicit. We make the following assumption on that system.

Assumption 3.4. *The system $\Sigma_{y \rightarrow \eta}$ whose dynamics are governed by (19) is BISBO stable in the following sense: for all $k > 0$ and all $\hat{k} > 0$, there exists $\tilde{k} > 0$ such that for all $y \in \mathcal{C}(\mathbb{R}^+, \mathbb{R})$ and all $\eta_0 \in \mathcal{I}$,*

$$\sup_{t \in \mathbb{R}^+} |y(t)| \leq k \text{ and } \|\eta_0\| \leq \hat{k} \Rightarrow \sup_{t \in \mathbb{R}^+} \|\eta(t)\| \leq \tilde{k}. \quad (21)$$

Theorem 3.1. *The operator T defined in (20), which arises from the nonlinear system (8) via the change of variables (15), satisfies Assumption 2.2.*

⁴Without loss of generality there exist constants $M \geq 1$ and $\omega \in \mathbb{R}$ such that $\|T_Q(t)\| \leq Me^{\omega t}$.

Proof. The proof is divided into three steps, according to the three items of Assumption 2.2.

Step 1: In order to show that T maps bounded trajectories into bounded ones, let us fix $k_1 > 0$, $k_* > 0$, $y \in \mathcal{C}(\mathbb{R}^+, \mathbb{R})$ and $\eta_0 \in \mathcal{I}$ such that $\sup_{t \in \mathbb{R}^+} |y(t)| \leq k_1$ and $\|\eta_0\| \leq k_*$. There exists a positive $\tilde{k}$ such that, for this y and this η_0 , the mild solution of (19) with initial condition η_0 satisfies $\sup_{t \in \mathbb{R}^+} \|\eta(t)\| \leq \tilde{k}$, according to Assumption 3.4. Thanks to the expression (20) of T , the boundedness of the operator S and the Cauchy-Schwarz inequality, one may write that

$$|T(y)(t)| \leq |P_0|y(t)| + \|S\|_{\mathcal{L}(\mathcal{I}, \mathbb{R})}\|\eta(t)\| + \|\tilde{f}(y(t), \eta(t))\|_H \|c\|_H.$$

Assumption 3.2 allows us to write

$$\begin{aligned} |T(y)(t)| &\leq |P_0|y(t)| + \|S\|_{\mathcal{L}(\mathcal{I}, \mathbb{R})}\|\eta(t)\| \\ &\quad + \|\tilde{f}(y(t), \eta(t)) - \tilde{f}(0, 0)\|_H \|c\|_H + \|\tilde{f}(0, 0)\|_H \|c\|_H \\ &\leq |P_0|y(t)| + \|S\|_{\mathcal{L}(\mathcal{I}, \mathbb{R})}\|\eta(t)\| \\ &\quad + (l_1|y(t)| + l_2\|\eta(t)\|)\|c\|_H + \sigma\|c\|_H, \end{aligned} \quad (22)$$

where $l_1 > 0$ and $l_2 > 0$ denote the Lipschitz constants of the operator $\tilde{f}$ associated with y and η , respectively, and where the positive constant σ is such that⁵ $\|f(0)\|_H \leq \sigma$. Consequently,

$$\sup_{t \in \mathbb{R}^+} |T(y)(t)| \leq |P_0|k_1 + \|S\|_{\mathcal{L}(\mathcal{I}, \mathbb{R})}\tilde{k} + \tilde{\sigma}\|c\|_H =: k_2,$$

where $\tilde{\sigma} = \sigma + l_1 k_1 + l_2 \tilde{k}$, which proves that T satisfies the BIBO condition required in Assumption 2.2.

Step 2: The causality can be easily established by noting that, for a fixed $y \in \mathcal{C}(\mathbb{R}^+, \mathbb{R})$ the corresponding mild solution of (19) is unique. This entails that for $y, \hat{y} \in \mathcal{C}(\mathbb{R}^+, \mathbb{R})$ such that $y|_{[0, t]} = \hat{y}|_{[0, t]}$, the corresponding mild solutions of (19), denoted by η and $\hat{\eta}$, respectively, satisfy $\eta|_{[0, t]} = \hat{\eta}|_{[0, t]}$. In view of the expression (20) of T , it follows that (3) holds.

Step 3: For the local Lipschitz continuity, let us consider $t \geq 0$ and $y \in \mathcal{C}([0, t], \mathbb{R})$. Now let us take $y_1, y_2 \in \mathcal{C}(\mathbb{R}^+, \mathbb{R})$ such that y_i coincides with y up to time t for $i = 1, 2$. The mild solutions of (19) with input y_i and starting at time t are given by

$$\begin{aligned} \eta_i(\tilde{t}) &= T_Q(\tilde{t} - t)\eta_{i,t} + \int_t^{\tilde{t}} T_Q(\tilde{t} - s)Ry_i(s)ds \\ &\quad + \int_t^{\tilde{t}} T_Q(\tilde{t} - s)P^\perp \tilde{f}(y_i(s), \eta_i(s))ds \end{aligned}$$

for any $\tilde{t} \in [t, t + \tau]$ with τ being an arbitrary positive constant. Note that the functions $\eta_{1,t}$ and $\eta_{2,t}$ correspond

to $\eta_1(t)$ and $\eta_2(t)$, respectively. Since, by assumption, $y_1(t) = y(t) = y_2(t)$ and $\eta_1(0) = \eta_0 = \eta_2(0)$ and by the uniqueness of the mild solution of (19), the relation $\eta_{1,t} = \eta_{2,t}$ holds true. Consequently,

$$\begin{aligned} \|\eta_1(\tilde{t}) - \eta_2(\tilde{t})\| &\leq \int_t^{\tilde{t}} \|T_Q(\tilde{t} - s)R(y_1(s) - y_2(s))\|ds \\ &\quad + \int_t^{\tilde{t}} \|T_Q(\tilde{t} - s)P^\perp(\tilde{f}(y_1(s), \eta_1(s)) - \tilde{f}(y_2(s), \eta_2(s)))\|ds. \end{aligned}$$

Assumption 3.2 together with the boundedness of the operator R implies that

$$\begin{aligned} \|\eta_1(\tilde{t}) - \eta_2(\tilde{t})\| &\leq (\|R\|_{\mathcal{L}(\mathbb{R}, \mathcal{I})} + 2l_1) \int_t^{\tilde{t}} M e^{|\omega|(\tilde{t}-s)} |y_1(s) - y_2(s)| ds \\ &\quad + 2l_2 \int_t^{\tilde{t}} M e^{|\omega|(\tilde{t}-s)} \|\eta_1(s) - \eta_2(s)\| ds, \end{aligned}$$

where l_1 and l_2 are the positive constants introduced in (22). We shall use the notation $\|R\|_{\mathcal{L}(\mathbb{R}, \mathcal{I})} + 2l_1 =: \mathbf{g}$ in what follows. Applying Gronwall's lemma to the function $e^{-|\omega|\tilde{t}}\|\eta_1(\tilde{t}) - \eta_2(\tilde{t})\|$ yields the inequality

$$\|\eta_1(\tilde{t}) - \eta_2(\tilde{t})\| \leq \mathbf{g} M e^{|\omega|\tilde{t}} e^{2Ml_2(\tilde{t}-t)} \int_t^{\tilde{t}} e^{-|\omega|s} |y_1(s) - y_2(s)| ds.$$

Taking the supremum over all $\tilde{t}$ in $[t, t + \tau]$ on both sides yields the estimate

$$\begin{aligned} \sup_{\tilde{t} \in [t, t + \tau]} \|\eta_1(\tilde{t}) - \eta_2(\tilde{t})\| &\leq \mathbf{g} M e^{(|\omega| + 2Ml_2)\tau} \tau \sup_{\tilde{t} \in [t, t + \tau]} |y_1(\tilde{t}) - y_2(\tilde{t})|. \end{aligned} \quad (23)$$

The notation $\mathbf{f}_\tau := \mathbf{g} M e^{(|\omega| + 2Ml_2)\tau} \tau$ will be used for the sake of simplicity. According to the definition (20) of the nonlinear operator T , it holds that

$$\begin{aligned} |T(y_1)(\tilde{t}) - T(y_2)(\tilde{t})| &\leq |P_0|y_1(\tilde{t}) - y_2(\tilde{t})| \\ &\quad + \|S\|_{\mathcal{L}(\mathcal{I}, \mathbb{R})}\|\eta_1(\tilde{t}) - \eta_2(\tilde{t})\| + l_1\|c\|_H |y_1(\tilde{t}) - y_2(\tilde{t})| \\ &\quad + l_2\|c\|_H \|\eta_1(\tilde{t}) - \eta_2(\tilde{t})\|, \end{aligned}$$

which, combined with (23), leads to

$$\sup_{\tilde{t} \in [t, t + \tau]} |T(y_1)(\tilde{t}) - T(y_2)(\tilde{t})| \leq \rho \sup_{\tilde{t} \in [t, t + \tau]} |y_1(\tilde{t}) - y_2(\tilde{t})|,$$

where $\rho := |P_0| + \|S\|_{\mathcal{L}(\mathcal{I}, \mathbb{R})}\mathbf{f}_\tau + l_1\|c\|_H + l_2\|c\|_H \mathbf{f}_\tau$. $\blacksquare$

This means that funnel control is feasible for a nonlinear infinite-dimensional system of the form (8) which satisfies Assumptions 3.1, 3.2, 3.3 and 3.4. Moreover the closed-loop system which consists of the interconnection of (8), described by (18)–(19) (system $\tilde{\Sigma}$), with the funnel controller (7) has the properties described in Theorem 2.1.

⁵This is valid since the nonlinear operator f maps the whole space H into itself, meaning that any point in H has a well-defined image by f in the H -norm.

This system is depicted in Figure 1. Note that in the case where the constant γ is negative, funnel control is still feasible but its sign has to be changed as

$$u(t) = \frac{e(t)}{1 - \phi^2(t)e^2(t)},$$

see Berger et al. (2018).

Remark 3.4. Here, we insist on the places where the global Lipschitz continuity of the nonlinear operator f is needed in the proof of Theorem 3.1. First, it is used to deduce the BIBO stability of the nonlinear map T . Secondly, it can be seen that it is needed in order to prove the local Lipschitz continuity of T . As it is also highlighted at different places in this paper, the solutions of the nonlinear abstract Cauchy problems are considered up to ∞ , which of course is allowed thanks to the global Lipschitz continuity. Perspectives regarding the smoothing of this assumption are raised in Section 5.

We state hereafter a useful criterion for checking Assumption 3.4, assuming that the nonlinearity is bounded.

Proposition 3.3. Assuming that the semigroup $(T_Q(t))_{t \geq 0}$ is exponentially stable on $\mathcal{I}$ and that the nonlinear operator f satisfies $\|f(x)\|_H \leq \hat{\sigma}$, for some constant $\hat{\sigma} > 0$ independent of x , is sufficient to ensure that Assumption 3.4 is satisfied.

Proof. Let us fix $k_1 > 0, k_2 > 0, y \in \mathcal{C}(\mathbb{R}^+, \mathbb{R})$ and $\eta_0 \in \mathcal{I}$ such that $\sup_{t \in \mathbb{R}^+} |y(t)| \leq k_1$ and $\|\eta_0\| \leq k_2$. As the semigroup $(T_Q(t))_{t \geq 0}$ is exponentially stable, the inequality $\|T_Q(t)\| \leq \tilde{M}e^{-\tilde{\omega}t}$ holds for some $\tilde{M} \geq 1$ and $\tilde{\omega} > 0$. Moreover the mild solution of (19) with initial condition η_0 and with the function y fixed above is given by

$$\begin{aligned} \eta(t) &= T_Q(t)\eta_0 + \int_0^t T_Q(t-s)Ry(s)ds \\ &\quad + \int_0^t T_Q(t-s)P^\perp \tilde{f}(y(s), \eta(s))ds. \end{aligned}$$

By taking the H -norm (restricted to $\mathcal{I}$) on both sides and by using the exponential stability of $(T_Q(t))_{t \geq 0}$, one gets that

$$\begin{aligned} \|\eta(t)\| &\leq \tilde{M}e^{-\tilde{\omega}t}\|\eta_0\| + \int_0^t \tilde{M}e^{-\tilde{\omega}(t-s)}\|Ry(s)\|ds \\ &\quad + \int_0^t \tilde{M}e^{-\tilde{\omega}(t-s)}\|P^\perp \tilde{f}(y(s), \eta(s))\|ds. \end{aligned}$$

The boundedness of the operator R combined with the definition of $P^\perp$ and the assumption on the operator f entails that

$$\begin{aligned} \|\eta(t)\| &\leq \tilde{M}e^{-\tilde{\omega}t}\|\eta_0\| + \tilde{M}e^{-\tilde{\omega}t}\|R\|_{\mathcal{L}(\mathbb{R}, \mathcal{I})} \int_0^t e^{\tilde{\omega}s}|y(s)|ds \\ &\quad + 2\hat{\sigma}\tilde{M}e^{-\tilde{\omega}t} \int_0^t e^{\tilde{\omega}s}ds. \end{aligned}$$

It follows, by using the estimate $\sup_{t \in \mathbb{R}^+} |y(t)| \leq k_1$, that

$$\|\eta(t)\| \leq \tilde{M}e^{-\tilde{\omega}t} \left(\|\eta_0\| + (\|R\|_{\mathcal{L}(\mathbb{R}, \mathcal{I})}k_1 + 2\hat{\sigma}) \int_0^t e^{\tilde{\omega}s}ds \right).$$

Hence

$$\begin{aligned} \|\eta(t)\| &\leq \tilde{M} \left(e^{-\tilde{\omega}t}\|\eta_0\| + \frac{\|R\|_{\mathcal{L}(\mathbb{R}, \mathcal{I})}k_1 + 2\hat{\sigma}}{\tilde{\omega}}(1 - e^{-\tilde{\omega}t}) \right) \\ &\leq \kappa, \end{aligned} \tag{24}$$

where $\kappa := \tilde{M} \left(k_2 + \frac{\|R\|_{\mathcal{L}(\mathbb{R}, \mathcal{I})}k_1 + 2\hat{\sigma}}{\tilde{\omega}} \right)$ does not depend on t . ■

We conclude this section with an analysis in line with Remark 3.1, in order to assess the impact of the approximation of the vectors b and c by elements of $D(A)$ and $D(A^*)$, respectively. We shall study here a bound on the error resulting from the approximation. Let us first remember that funnel control aims to be applied on a nonlinear infinite-dimensional system of the form (8). According to Remark 3.1, if b and c are not in $D(A)$ and $D(A^*)$, respectively, one may select two sequences $\{b_n\}_{n \in \mathbb{N}} \subset D(A)$ and $\{c_n\}_{n \in \mathbb{N}} \subset D(A^*)$ which converge towards b and c in H -norm, respectively. Let us select one element of each sequence, denoted by b_ℓ and $c_\ell, \ell \in \mathbb{N}$. For these elements, an approximation of (8) without any disturbance takes the form

$$\begin{cases} \dot{x}_\ell(t) = Ax_\ell(t) + f(x_\ell(t)) + b_\ell u_\ell(t), \\ y_\ell(t) = \langle x_\ell(t), c_\ell \rangle_H, \\ x_\ell(0) = x_0 \in H, \end{cases} \tag{25}$$

where the initial condition $x_\ell(0)$ is purposely chosen the same as the one of (8). According to the arguments that are developed in this section, funnel control is feasible for (25). A funnel controller for (25) is written as

$$u_\ell(t) = \frac{-e_\ell(t)}{1 - \phi^2(t)e_\ell^2(t)}, e_\ell(t) = y_\ell(t) - y_{ref}(t). \tag{26}$$

Moreover, according to Theorem 2.1, the control produced by controller (26) is bounded on $\mathbb{R}^+$, i.e. there exists $k_\ell > 0$ such that $|u_\ell(t)| \leq k_\ell$ for all $t \in \mathbb{R}^+$. Applying this control action to (8) without taking disturbances into account results in the closed-loop system

$$\begin{cases} \dot{x}(t) = Ax(t) + f(x(t)) + bu_\ell(t), \\ y(t) = \langle x(t), c \rangle_H, \\ x(0) = x_0 \in H. \end{cases}$$

An estimation on the state and the output approximation errors $\rho(t) := x(t) - x_\ell(t)$ and $\lambda(t) := y(t) - y_\ell(t)$, respectively, is given in the following proposition.

Proposition 3.4. The state and the output approximation errors $\rho(t)$ and $\lambda(t)$ satisfy the estimates

$$\|\rho(t)\|_H \leq \frac{M^*k_\ell(e^{\omega^*t} - 1)}{\omega^*} e^{M^*t_f} \|b - b_\ell\|_H, \tag{27}$$

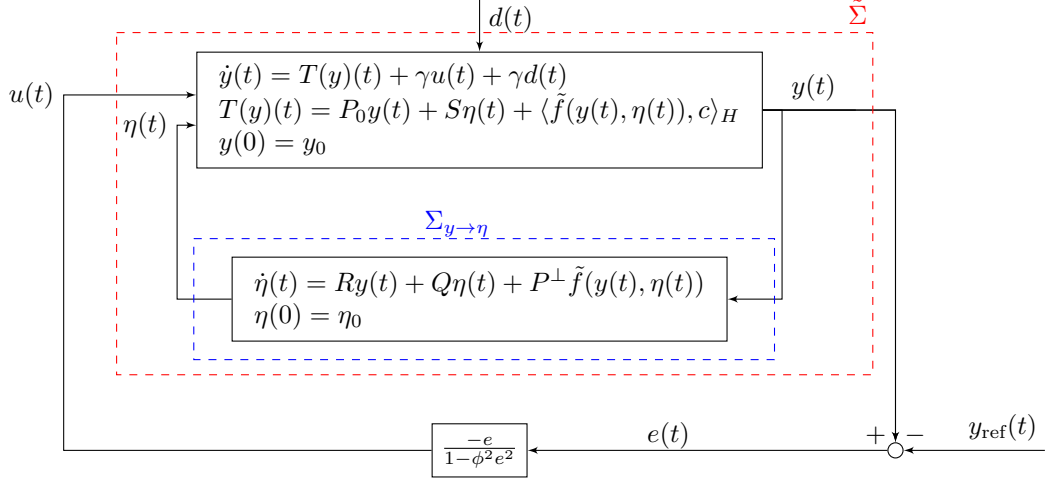


Figure 1: Interconnection of $\tilde{\Sigma}$ and the funnel controller (7).

and

$$|\lambda(t)| \leq \|\rho(t)\|_H \|c\|_H + \|x_\ell(t)\|_H \|c - c_\ell\|_H, \quad (28)$$

respectively, where $\omega^* > \max\{0, \omega_0\}$.

Proof. The time derivative of the variable $\rho(t)$ is subject to the following abstract ODE

$$\dot{\rho}(t) = A\rho(t) + f(x(t)) - f(x_\ell(t)) + (b - b_\ell)u_\ell(t), \rho(0) = 0,$$

which has a mild solution given by

$$\begin{aligned} \rho(t) &= \int_0^t T(t-s)[f(x(s)) - f(x_\ell(s))]ds \\ &+ \int_0^t T(t-s)[b - b_\ell]u_\ell(s)ds, \end{aligned} \quad (29)$$

for any positive t . Taking the H -norm of both sides of (29) yields the inequality

$$\begin{aligned} \|\rho(t)\|_H &\leq \int_0^t M^* e^{\omega^*(t-s)} l_f \|\rho(s)\|_H ds \\ &+ \int_0^t M^* e^{\omega^*(t-s)} \|b - b_\ell\|_H |u_\ell(s)| ds, \end{aligned}$$

where the inequality $\|T(t)\| \leq M^* e^{\omega^* t}$, $M^* \geq 1$, $\omega^* > \omega_0$ has been used and where the positive constant l_f is a Lipschitz constant associated to f . By using the boundedness of the input $u_\ell(t)$, it follows that

$$\begin{aligned} \|e^{-\omega^* t} \rho(t)\|_H &\leq M^* l_f \int_0^t \|e^{-\omega^* s} \rho(s)\|_H ds \\ &+ \|b - b_\ell\|_H M^* k_\ell \frac{1 - e^{-\omega^* t}}{\omega^*}. \end{aligned}$$

By using Gronwall's inequality and by choosing ω^* sufficiently large such that $\omega^* > \max\{0, \omega_0\}$, one gets that

$$\|\rho(t)\|_H \leq \frac{M^* k_\ell (e^{\omega^* t} - 1)}{\omega^*} e^{M^* l_f t} \|b - b_\ell\|_H, t \geq 0.$$

Let us have a look at the output approximation error $\lambda(t)$. Observe that

$$\begin{aligned} |\lambda(t)| &= |\langle x(t), c \rangle - \langle x_\ell(t), c_\ell \rangle| \\ &= |\langle x(t), c \rangle - \langle x_\ell(t), c \rangle + \langle x_\ell(t), c \rangle - \langle x_\ell(t), c_\ell \rangle| \\ &= |\langle \rho(t), c \rangle + \langle x_\ell(t), c - c_\ell \rangle| \\ &\leq \|\rho(t)\|_H \|c\|_H + \|x_\ell(t)\|_H \|c - c_\ell\|_H, \end{aligned}$$

where the Cauchy-Schwarz inequality has been used. $\blacksquare$

Remark 3.5. In view of the inequality (27), one could still go a step further in the case where the sequence $\{k_\ell\}_{\ell \in \mathbb{N}}$ was uniformly bounded. In such a situation, for any fixed time instant t , letting ℓ tend to ∞ would imply that $x_\ell(t)$ converges towards $x(t)$ in H -norm, since $\lim_{\ell \rightarrow \infty} k_\ell \|b - b_\ell\|_H = 0$. In other words, one should get the pointwise convergence of the sequence of functions $\{x_\ell\}_{\ell \in \mathbb{N}}$ towards the state trajectory x . Actually it follows from inequality (27) that this convergence should be uniform on any compact interval in $\mathbb{R}^+$. The same conclusion should also hold for the output approximation error $\lambda(t)$ by looking at (28). However, the uniform boundedness of $\{k_\ell\}_{\ell \in \mathbb{N}}$ is not guaranteed and an estimation of these constants may be difficult to compute.

4. Applications

Two applications are considered in this section. First we first shall use the framework developed in the previous section to regulate the average temperature in a nonisothermal plug-flow tubular reactor. As a second application, a nonlinear damped sine-Gordon equation is considered.

4.1. Nonisothermal plug-flow tubular reactor

The dimensionless nonlinear dynamics of a plug-flow tubular reactor without axial dispersion is governed by

the following PDE

$$\begin{cases} \partial_t \theta_1 = -\partial_z \theta_1 + \delta f(\theta_1, \theta_2) + \beta(1_{[0,1]}(z)\theta_w(t) - \theta_1) \\ \partial_t \theta_2 = -\partial_z \theta_2 + f(\theta_1, \theta_2) \\ \theta(t, 0) = 0, \theta_2(t, 0) = 0, \end{cases} \quad (30)$$

with $t \geq 0, z \in [0, 1]$ and where θ_1 and θ_2 represent the dimensionless temperature and concentration inside the reactor, respectively. The nonlinear part of the equation, encompassed in the function f , is due to the reaction kinetics. The Arrhenius law is considered here, which implies that the function f is expressed as

$$f(\theta_1, \theta_2) = \begin{cases} 0 & \text{if } \theta_1 < -1 \\ \alpha e^{\frac{\mu\theta_1}{1+\theta_1}} & \text{if } \theta_1 \geq -1 \text{ and } \theta_2 < 0 \\ \alpha(1 - \theta_2)e^{\frac{\mu\theta_1}{1+\theta_1}} & \text{if } \theta_1 \geq -1 \text{ and } 0 \leq \theta_2 \leq 1 \\ 0 & \text{otherwise.} \end{cases} \quad (31)$$

Note that this definition of f implies that the latter is uniformly Lipschitz as a pointwise function defined from $\mathbb{R}^2$ into $\mathbb{R}$. Moreover, it satisfies $|f(x, y)| \leq \alpha e^\mu$ for any $(x \ y)^T \in \mathbb{R}^2$. The positive constants α, β, μ and δ depend on the model parameters. An overview of these constants and their physical meaning can be found in Dochain (2018) among others. The scalar control variable, denoted by $\theta_w(t)$, is due to a heat exchanger that acts as a distributed control input along the reactor through the characteristic function $1_{[0,1]}(z)$ which takes the value 1 for $z \in [0, 1]$ and 0 elsewhere. The control objective that is pursued here consists in the tracking of the following output function which corresponds to the mean value of the temperature along the reactor:

$$y(t) = \int_0^1 \theta_1(t, z) dz. \quad (32)$$

In order to reach this goal, we shall develop a funnel controller producing an input $\theta_w(t)$ which will take the form (7) for some reference signal $y_{\text{ref}} \in W^{1,\infty}(\mathbb{R}^+, \mathbb{R})$. First observe that (30) admits the abstract representation

$$\begin{cases} \dot{x}(t) = Ax(t) + F(x(t)) + Bu(t) \\ x(0) = x_0 \in X \\ y(t) = \langle c, x(t) \rangle_X, \end{cases} \quad (33)$$

where the state space X is chosen as $X := L^2(0, 1) \times L^2(0, 1)$ equipped with the inner product

$$\langle \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \rangle_X := \langle x_1, w_1 \rangle_{L^2(0,1)} + \langle x_2, w_2 \rangle_{L^2(0,1)}, \quad (34)$$

for $(x_1 \ x_2)^T, (w_1 \ w_2)^T \in X$. Note that the inner product on $L^2(0, 1)$ (as a real Hilbert space) is defined by

$$\langle x_1, w_1 \rangle_{L^2(0,1)} = \int_0^1 x_1(z)w_1(z)dz, \quad (35)$$

where the functions are supposed to take values in $\mathbb{R}$. The unbounded linear operator A is given by $A =$

$\begin{pmatrix} -d_z - \beta I & 0 \\ 0 & -d_z \end{pmatrix}$ on the dense linear subspace

$$D(A) = \{(x_1 \ x_2)^T \in H^1(0, 1) \times H^1(0, 1), \\ x_1(0) = 0 = x_2(0)\}.$$

From that definition of $(A, D(A))$, the operator A is the infinitesimal generator of a contraction (and even exponentially stable) C_0 -semigroup on X . Hence Assumption 3.1 is satisfied. The nonlinear operator $F : X \rightarrow X$ is given by $F(x_1, x_2) = (\delta f(x_1, x_2) \ f(x_1, x_2))^T$. Due to the definition of f given in (31) and the properties of that function viewed as a pointwise function acting on $\mathbb{R}^2$, the nonlinear operator F is uniformly Lipschitz continuous on X . Moreover, it satisfies $\|F(x_1, x_2)\|_X \leq \alpha e^\mu \sqrt{\delta^2 + 1}$ for any $(x_1 \ x_2)^T \in X$. This implies that Assumption 3.2 is fulfilled. The control operator $B : \mathbb{R} \rightarrow X$ is given by $Bu = (\beta 1_{[0,1]}(z) \ 0)^T u$ while the observation operator $C : X \rightarrow \mathbb{R}$ takes the form

$$C(x_1 \ x_2)^T = \left\langle \begin{pmatrix} 1_{[0,1]} \\ 0 \end{pmatrix}, \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right\rangle_X.$$

Hence the functions b and c satisfy $b(z) = \beta c(z) = (\beta 1_{[0,1]}(z) \ 0)^T$. Note that the domain $D(A^*)$ of the adjoint operator of A is given by

$$\{(x_1 \ x_2)^T \in H^1(0, 1) \times H^1(0, 1), x_1(1) = 0 = x_2(1)\}.$$

Clearly b and c do not lie in $D(A)$ and $D(A^*)$, respectively. In order to overcome this difficulty, we approximate the function $1_{[0,1]}(z)$ by a linear combination of elements of the orthonormal basis $\{\sqrt{2} \sin(n\pi z)\}_{n \in \mathbb{N}_0}$ of $L^2(0, 1)$. For a fixed $N \in \mathbb{N}_0$, the N -th order approximation of $1_{[0,1]}(z)$, denoted by $1_N(z)$, is given by

$$\begin{aligned} 1_N(z) &= \sum_{n=1}^N \langle 1_{[0,1]}(\cdot), \sqrt{2} \sin(n\pi \cdot) \rangle_{L^2(0,1)} \sqrt{2} \sin(n\pi z) \\ &= \sum_{\substack{n=1 \\ n \text{ odd}}}^N \frac{4}{n\pi} \sin(n\pi z). \end{aligned} \quad (36)$$

As this approximation lies in $H^1(0, 1)$ and vanishes both for $z = 0$ and $z = 1$, the approximations of $b(z)$ and $c(z)$, denoted by $b_N(z)$ and $c_N(z)$ and whose expressions are given by $(\beta 1_N(z) \ 0)^T$ and $(1_N(z) \ 0)^T$ are in $D(A)$ and $D(A^*)$, respectively. Now observe that the inner product between b_N and c_N is given by

$$\langle b_N, c_N \rangle_X = \beta \langle 1_N, 1_N \rangle_{L^2(0,1)} = \beta \sum_{\substack{n=1 \\ n \text{ odd}}}^N \frac{8}{n^2 \pi^2} > 0,$$

which implies that Assumption 3.3 is satisfied. Now it remains to show that the system $\Sigma_{y \rightarrow \eta}$ is BIBO stable in the sense of Assumption 3.4. We shall use the characterization of Proposition 3.3 to reach this aim. To this end, let us

first introduce the decomposition of the state space X as in (9), which is given here by:

$$\begin{aligned} X &= \text{span}\{c_N\} \oplus \{b_N\}^\perp = \text{span}\{c_N\} \oplus \{c_N\}^\perp \\ &= \text{span}\left\{\begin{pmatrix} 1_N \\ 0 \end{pmatrix}\right\} \oplus \left\{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in X, \langle \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} 1_N \\ 0 \end{pmatrix} \rangle_X = 0\right\} \\ &= \mathcal{O} \oplus \mathcal{I}, \end{aligned}$$

where $\mathcal{I}$ may also be written as

$$\mathcal{I} = \{x_1 \in L^2(0, 1), \langle x_1, 1_N \rangle_{L^2(0,1)} = 0\} \times L^2(0, 1). \quad (37)$$

The operator $Q : D(Q) = D(A) \cap \mathcal{I} \rightarrow \mathcal{I}$ whose definition is given in (17) is expressed as

$$Q\eta = P^\perp A\eta - \frac{\langle c_N, \eta \rangle_X}{\langle c_N, b_N \rangle_X} P^\perp A b_N,$$

for $\eta = (\eta_1 \quad \eta_2)^T \in D(Q)$. Expanding the application of Q to η gives rise to

$$\begin{aligned} Q\eta &= A\eta - \frac{\langle A\eta, b_N \rangle_X}{\langle b_N, b_N \rangle_X} b_N \\ &= \begin{pmatrix} -d_z \eta_1 - \beta \eta_1 \\ -d_z \eta_2 \end{pmatrix} + \frac{\langle d_z \eta_1, 1_N \rangle_{L^2(0,1)}}{\langle 1_N, 1_N \rangle_{L^2(0,1)}} \begin{pmatrix} 1_N \\ 0 \end{pmatrix} \\ &=: \begin{pmatrix} Q_1 & 0 \\ 0 & Q_2 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}, \end{aligned}$$

where the relation $\langle \eta_1, 1_N \rangle_{L^2(0,1)} = 0$ has been used. The operator $Q_1 : D(Q_1) \subset \{1_N\}^\perp \rightarrow \{1_N\}^\perp$ is defined as

$$Q_1 \eta_1 = -d_z \eta_1 - \beta \eta_1 + \frac{\langle d_z \eta_1, 1_N \rangle_{L^2(0,1)}}{\langle 1_N, 1_N \rangle_{L^2(0,1)}} 1_N,$$

for $\eta_1 \in D(Q_1)$ given by

$$D(Q_1) = \{x_1 \in H^1(0, 1), x_1(0) = 0\} \cap \{1_N\}^\perp,$$

whereas $Q_2 \eta_2 = -d_z \eta_2$ on

$$D(Q_2) = \{x_2 \in H^1(0, 1), x_2(0) = 0\}.$$

According to Lyapunov's Theorem, see e.g. (Curtain and Zwart, 2020, Theorem 4.1.3), the semigroup generated by the operator Q is exponentially stable if and only if there exists a positive self-adjoint operator $P \in \mathcal{L}(\mathcal{I})$ such that

$$\langle Q\eta, P\eta \rangle_{\mathcal{I}} + \langle P\eta, Q\eta \rangle_{\mathcal{I}} \leq -\langle \eta, \eta \rangle_{\mathcal{I}}, \quad (38)$$

for all $\eta \in D(Q)$. Note that, for any two functions in $\mathcal{I}$, their inner product on $\mathcal{I}$ is the same as the one defined in (34). Let us propose the following form for the operator P

$$P = \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix}.$$

Let us define P_1 as $P_1 \eta_1 = \frac{1}{2\beta} \eta_1$ and P_2 as

$$(P_2 \eta_2)(z) = (1 - z) \eta_2(z), z \in [0, 1].$$

Now observe that, for $\eta_1 \in D(Q_1)$, there holds

$$\begin{aligned} \langle Q_1 \eta_1, P_1 \eta_1 \rangle + \langle P_1 \eta_1, Q_1 \eta_1 \rangle &= 2 \langle Q_1 \eta_1, P_1 \eta_1 \rangle \\ &= -\frac{1}{\beta} \langle d_z \eta_1, \eta_1 \rangle - \langle \eta_1, \eta_1 \rangle + \frac{1}{\beta} \frac{\langle d_z \eta_1, 1_N \rangle_{L^2(0,1)}}{\langle 1_N, 1_N \rangle_{L^2(0,1)}} \langle 1_N, \eta_1 \rangle \\ &= -\frac{1}{2\beta} \eta_1^2(1) - \langle \eta_1, \eta_1 \rangle \leq -\langle \eta_1, \eta_1 \rangle, \end{aligned}$$

where the fact that η_1 and 1_N are orthogonal has been used. Moreover, for any $\eta_2 \in D(Q_2)$, one has that

$$\begin{aligned} \langle Q_2 \eta_2, P_2 \eta_2 \rangle + \langle P_2 \eta_2, Q_2 \eta_2 \rangle &= -2 \int_0^1 (1 - z) \eta_2(d_z \eta_2) dz = [-(1 - z) \eta_2^2(z)]_0^1 - \langle \eta_2, \eta_2 \rangle \\ &= -\langle \eta_2, \eta_2 \rangle. \end{aligned}$$

Thus the semigroup generated by the operator Q is exponentially stable. This fact combined with the properties of the nonlinear operator F and Proposition 3.3 ensures that Assumption 3.4 is satisfied. Hence funnel control is feasible for the system (30), which means that considering a heat exchanger temperature expressed as

$$\theta_w(t) = \frac{-e(t)}{1 - \phi^2(t)e^2(t)}, \quad (39)$$

where $e(t) = y(t) - y_{\text{ref}}(t)$, $y_{\text{ref}} \in W^{1,\infty}(\mathbb{R}^+, \mathbb{R})$, $\phi \in \Phi$, and y is given by (32), yields a closed-loop system which possesses the properties stated in Theorem 2.1.

As an illustration of the results, we shall report numerical simulations hereafter. The parameters related to the model (30) have been chosen as follows: $\delta = 0.25$, $\alpha = 2.3248$, $\mu = 16.6607$, $\beta = 8$, see Aksikas (2005). The number of basis functions N in the approximation of the function b has been set at $N = 100$. As initial conditions for θ_1 and θ_2 we consider the following functions

$$\begin{aligned} \theta_1(0, z) &= 0.02(-z^3 + z^2 + z), \\ \theta_2(0, z) &= 0.7(-z^3 + z^2 + z). \end{aligned}$$

The reference signal that has to be tracked by the output (32) is $y_{\text{ref}}(t) = \frac{1}{20} + \frac{1}{20} \arctan(t)$. The funnel boundary is chosen as $\frac{1}{\phi(t)} = e^{-2t} + 2.5 * 10^{-3}$.

The output trajectory (32) and the reference signal $y_{\text{ref}}(t)$ are depicted in Figure 2 and the funnel control, $\theta_w(t)$, given by (39), is shown in Figure 3. The tracking error is depicted in Figure 4. The state trajectories corresponding to $\theta_1(t, z)$ and $\theta_2(t, z)$ are represented in Figures 5 and 6, respectively.

4.2. Damped sine-Gordon equation

Here we consider a damped and actuated sine-Gordon equation of the form

$$\begin{cases} \partial_{tt} x = \partial_{zz} x - \alpha \partial_t x + \nu \sin(x) + b(z)u(t) \\ x(t, 0) = 0, \partial_z x(t, 1) = 0, \end{cases} \quad (40)$$

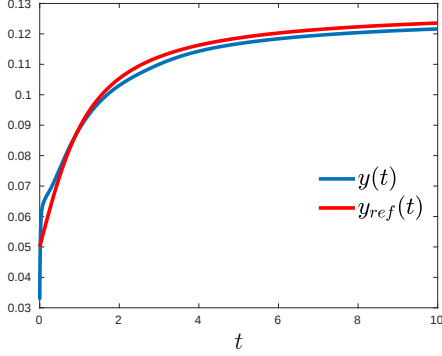


Figure 2: Output trajectory (32) with the reference signal $y_{\text{ref}}(t)$.

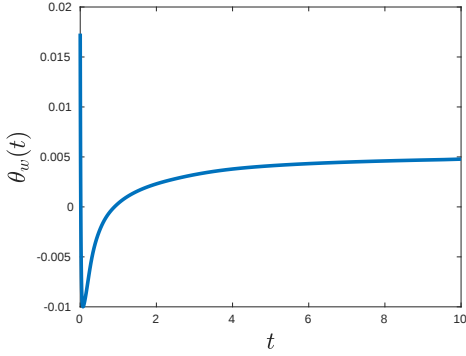


Figure 3: Input trajectory.

where the space variable $z \in [0, 1]$ and $t \in \mathbb{R}^+$ denotes the time variable. The parameters ν and α are such that $\nu \in \mathbb{R}_0$ and $\alpha > \pi$. This nonlinear PDE encompasses many phenomena in physics as the dynamics of a Josephson junction driven by a current source, see e.g. Temam (1997); Cuevas-Maraver et al. (2014), as well as the dynamics of mechanical transmission lines, see Cirillo et al. (1981) among others. The stability of the homogeneous dynamics ($u \equiv 0$) of (40) has been investigated for Dirichlet and Neumann boundary conditions in Dickey (1976) and Callegari and Reiss (1973). For control problems related to (40), we refer to Dolgopoli et al. (2016) and Efimov et al. (2019) for instance, where boundary energy control and robust input-to-state stability are developed.

Let us consider the operator $A_0 = -\frac{d^2}{dz^2}$ on the domain

$$D(A_0) = \left\{ x \in H^2(0, 1), x(0) = 0 = \frac{dx}{dz}(1) \right\}.$$

As the operator A_0 is self-adjoint and coercive⁶, it admits a unique nonnegative square-root, see e.g. (Curtain and

⁶By using Poincaré's inequality it can be seen that, for any $x \in D(A_0)$, the relation $\langle A_0 x, x \rangle_{L^2(0,1)} \geq \frac{\pi^4}{4} \|x\|_{L^2(0,1)}^2$ holds.

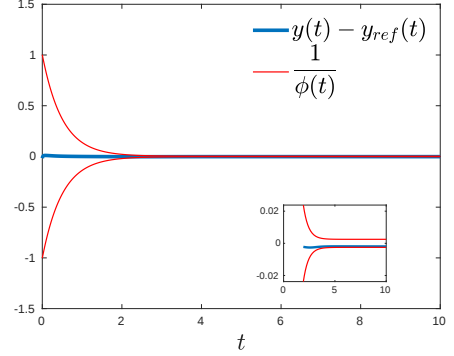


Figure 4: Output error trajectory $y(t) - y_{\text{ref}}(t)$ with the funnel boundaries $\frac{1}{\phi(t)}$ and $-\frac{1}{\phi(t)}$.

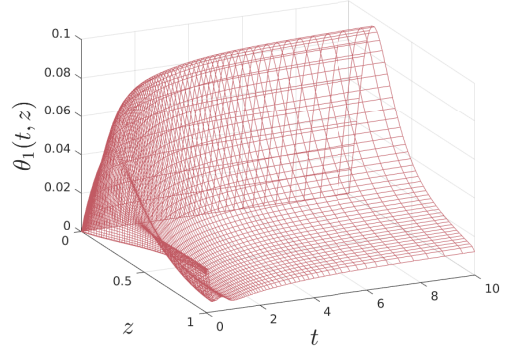


Figure 5: Closed-loop state trajectory $\theta_1(t, z)$.

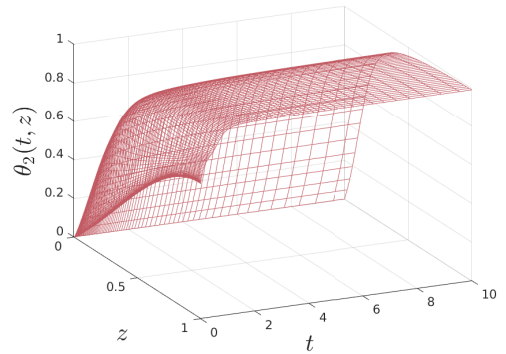


Figure 6: Closed-loop state trajectory $\theta_2(t, z)$.

Zwart, 2020, Lemma A.3.82), which satisfies

$$D(A_0^{\frac{1}{2}}) = \{x \in H^1(0, 1), x(0) = 0\}.$$

This allows us to consider the Hilbert state space $Z = D(A_0^{\frac{1}{2}}) \times X$ equipped with the inner product⁷

$$\left\langle \begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix}, \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \right\rangle_Z := \langle d_z \zeta_1, d_z w_1 \rangle_X + \langle \zeta_2, w_2 \rangle_X, \quad (41)$$

where $\zeta_1, w_1 \in D(A_0^{\frac{1}{2}})$ and $\zeta_2, w_2 \in X$ with $X := L^2(0, 1)$. The inner product on X is the same as the one defined in (35). By considering the state variable $\zeta = \begin{pmatrix} x \\ \partial_t x \end{pmatrix} =: \begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix}$, the PDE (40) may be written as (33), where the operator A is given by

$$A = \begin{pmatrix} 0 & I \\ -A_0 & -\alpha I \end{pmatrix} \quad (42)$$

on $D(A) = D(A_0) \times D(A_0^{\frac{1}{2}})$. According to (Curtain and Zwart, 2020, Example 2.3.5), the operator A is the generator of a contraction C_0 -semigroup on Z . Note that the adjoint operator of A , denoted by A^* , is expressed as $A^* = \begin{pmatrix} 0 & -I \\ A_0 & -\alpha I \end{pmatrix}$ on $D(A^*) = D(A)$. The nonlinear operator $F : Z \rightarrow Z$ is expressed as $F(\zeta_1, \zeta_2) = \begin{pmatrix} 0 \\ \nu \sin(\zeta_1) \end{pmatrix}$. The latter is uniformly Lipschitz continuous and satisfies $\|F(\zeta_1, \zeta_2)\|_Z \leq |\nu|$ for any $(\zeta_1 \ \zeta_2)^T \in Z$. Consequently, Assumptions 3.1 and 3.2 are satisfied. Here we consider that the operator $B : \mathbb{R} \rightarrow Z$ is defined for $u \in \mathbb{R}$ by $Bu = b(z)u = \begin{pmatrix} 0 \\ b_N(z) \end{pmatrix} u$ where $b_N(z) = 1_N(z) = \sum_{\substack{n=1 \\ n \text{ odd}}}^N \frac{4}{n\pi} \sin(n\pi z)$ is the function defined in (36). Obviously $b \in D(A)$. For the function c , we choose the expression

$$c(z) = \begin{pmatrix} \frac{2}{\pi^2}(\alpha + \sqrt{\alpha^2 - \pi^2}) \sin(\frac{\pi}{2}z) \\ \sin(\frac{\pi}{2}z) \end{pmatrix},$$

so that the output trajectory corresponding to (40) is given by

$$\begin{aligned} y(t) &= \langle c, \zeta(t) \rangle_Z \\ &= \frac{\alpha + \sqrt{\alpha^2 - \pi^2}}{\pi} \int_0^1 \cos\left(\frac{\pi}{2}z\right) \partial_z x(t, z) dz \\ &\quad + \int_0^1 \sin\left(\frac{\pi}{2}z\right) \partial_t x(t, z) dz. \end{aligned} \quad (43)$$

It can be easily seen that the function $c \in D(A^*)$. A straightforward computation shows that

$$\langle b, c \rangle_Z = \sum_{\substack{n=1 \\ n \text{ odd}}}^N \frac{16}{\pi^2(4n^2 - 1)} > 0,$$

which entails that Assumption 3.3 is satisfied. Before showing that funnel control is feasible in our context, let us introduce the decomposition of the state space

$$Z = \text{span}\{c\} \oplus \{b\}^\perp = \mathcal{O} \oplus \mathcal{I},$$

with $\mathcal{I} = D(A_0^{\frac{1}{2}}) \times \{x \in X, \langle x, b_N \rangle_{L^2(0,1)} = 0\}$. According to (15) the system (40) admits the representation (18)–(19), in which we shall focus on the linear part, i.e. the operator $\begin{pmatrix} P_0 & S \\ R & Q \end{pmatrix}$. In order to show that funnel control is feasible for (40), one could use the criterion of BIBO stability stated in Proposition 3.3. Therefore it remains to show that the C_0 -semigroup generated by the operator Q , see (17), is exponentially stable. First observe that the operator P_0 takes the form $P_0 y = p_0 y$, where

$$\begin{aligned} p_0 &= \frac{\langle A^* c, b \rangle_Z}{\langle b, c \rangle_Z} \\ &= \frac{\langle \frac{1}{2}(\alpha + \sqrt{\alpha^2 - \pi^2}) \sin(\frac{\pi}{2}z) - \alpha \sin(\frac{\pi}{2}z), b_N(z) \rangle_X}{\langle \sin(\frac{\pi}{2}z), b_N(z) \rangle_X} \\ &= -\frac{\alpha}{2} + \frac{1}{2} \sqrt{\alpha^2 - \pi^2} < 0, \end{aligned}$$

which means that the semigroup generated by P_0 , which is given by $\left(e^{(-\frac{\alpha}{2} + \frac{1}{2}\sqrt{\alpha^2 - \pi^2})t}\right)_{t \geq 0}$ is exponentially stable. Secondly, let us compute the operator $S : \mathcal{I} \rightarrow \mathbb{R}, S\eta = \langle \eta, P_{\mathcal{I}} A^* c \rangle_Z$. Observe that

$$\begin{aligned} P_{\mathcal{I}} A^* c &= A^* c - P_0 c \\ &= \begin{pmatrix} -\frac{\alpha}{2} + \frac{1}{2}\sqrt{\alpha^2 - \pi^2} \end{pmatrix} \sin\left(\frac{\pi}{2}z\right) \\ &\quad + \begin{pmatrix} \frac{\alpha}{2} - \frac{1}{2}\sqrt{\alpha^2 - \pi^2} \end{pmatrix} \begin{pmatrix} \frac{2}{\pi^2}(\alpha + \sqrt{\alpha^2 - \pi^2}) \\ 1 \end{pmatrix} \sin\left(\frac{\pi}{2}z\right) \\ &= \begin{pmatrix} 0 \end{pmatrix}. \end{aligned}$$

Consequently, the operator $\begin{pmatrix} P_0 & S \\ R & Q \end{pmatrix}$ is a triangular operator of the form $\begin{pmatrix} P_0 & 0 \\ R & Q \end{pmatrix}$. As it is similar to the operator A , the corresponding semigroups are also similar, i.e. denoting by $(S(t))_{t \geq 0}$ and by $(\tilde{S}(t))_{t \geq 0}$ the C_0 -semigroups generated by A and $\begin{pmatrix} P_0 & S \\ R & Q \end{pmatrix}$, respectively, the relation $\tilde{S}(t) = U^{-1} S(t) U^*$ holds for all $t \geq 0$. Consequently, $(S(t))_{t \geq 0}$ and $(\tilde{S}(t))_{t \geq 0}$ have the same growth bounds. In that way, let us have a look at the sign of the growth bound of the semigroup $(S(t))_{t \geq 0}$ via a Lyapunov equation approach. Therefore let us define the operator $\Pi \in \mathcal{L}(Z)$ by

$$\Pi = \begin{pmatrix} \frac{1}{\alpha} I + \frac{\alpha}{2} A_0^{-1} & \frac{1}{2} A_0^{-1} \\ \frac{1}{2} I & \frac{1}{\alpha} I \end{pmatrix},$$

where A_0^{-1} , the inverse of A_0 , is a bounded and linear operator, since A_0 is self-adjoint and coercive, see (Curtain and Zwart, 2020, Lemma A.3.85). The operator Π is self-adjoint for the inner product defined in (41). Moreover it is coercive since

$$\langle \Pi \zeta, \zeta \rangle_Z \geq \frac{1}{2\alpha} \|\zeta\|_Z^2, \text{ for all } \zeta \in Z.$$

Furthermore, by taking $\zeta = (\zeta_1 \ \zeta_2)^T \in D(A)$, one gets

⁷The equality $\langle d_z \zeta_1, d_z w_1 \rangle_X = \langle A_0^{\frac{1}{2}} \zeta_1, A_0^{\frac{1}{2}} w_1 \rangle_X$ holds.

that

$$\begin{aligned}
\langle A\zeta, \Pi\zeta \rangle_Z + \langle \Pi\zeta, A\zeta \rangle_Z &= 2\langle A\zeta, \Pi\zeta \rangle_Z \\
&= 2\left\langle \begin{pmatrix} \zeta_2 \\ -A_0\zeta_1 - \alpha\zeta_2 \end{pmatrix}, \begin{pmatrix} \frac{1}{\alpha}\zeta_1 + \frac{\alpha}{2}A_0^{-1}\zeta_1 + \frac{1}{2}A_0^{-1}\zeta_2 \\ \frac{1}{2}\zeta_1 + \frac{1}{\alpha}\zeta_2 \end{pmatrix} \right\rangle_Z \\
&= \frac{2}{\alpha}\langle A_0^{\frac{1}{2}}\zeta_2, A_0^{\frac{1}{2}}\zeta_1 \rangle_X + \alpha\langle A_0^{\frac{1}{2}}\zeta_2, A_0^{\frac{1}{2}}A_0^{-1}\zeta_1 \rangle_X \\
&\quad + \langle A_0^{\frac{1}{2}}\zeta_2, A_0^{\frac{1}{2}}A_0^{-1}\zeta_2 \rangle_X - \langle A_0\zeta_1, \zeta_1 \rangle_X - \frac{2}{\alpha}\langle A_0\zeta_1, \zeta_2 \rangle_X \\
&\quad - \alpha\langle \zeta_2, \zeta_1 \rangle_X - 2\langle \zeta_2, \zeta_2 \rangle_X. \tag{44}
\end{aligned}$$

As $A_0^{-1}\zeta_1$ and $A_0^{-1}\zeta_2$ are in $D(A_0)$ by construction and as the relation $\langle A_0\zeta_1, \zeta_1 \rangle_X = \langle A_0^{\frac{1}{2}}\zeta_1, A_0^{\frac{1}{2}}\zeta_1 \rangle_X$ holds for any $\zeta_1 \in D(A_0)$, the relation (44) can also be written as

$$\begin{aligned}
\langle A\zeta, \Pi\zeta \rangle_Z + \langle \Pi\zeta, A\zeta \rangle_Z &= \frac{2}{\alpha}\langle \zeta_2, A_0\zeta_1 \rangle_X + \alpha\langle \zeta_2, \zeta_1 \rangle_X \\
&\quad + \langle \zeta_2, \zeta_2 \rangle_X - \langle A_0^{\frac{1}{2}}\zeta_1, A_0^{\frac{1}{2}}\zeta_1 \rangle_X - \frac{2}{\alpha}\langle A_0\zeta_1, \zeta_2 \rangle_X \\
&\quad - \alpha\langle \zeta_2, \zeta_1 \rangle_X - 2\langle \zeta_2, \zeta_2 \rangle_X \\
&= -\langle A_0^{\frac{1}{2}}\zeta_1, A_0^{\frac{1}{2}}\zeta_1 \rangle_X - \langle \zeta_2, \zeta_2 \rangle_X = -\left\| \begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix} \right\|_Z^2.
\end{aligned}$$

According to (Curtain and Zwart, 2020, Theorem 4.3.1), it follows that the semigroup $(S(t))_{t \geq 0}$ whose infinitesimal generator is the operator A , is exponentially stable. This means that the growth bound of $(S(t))_{t \geq 0}$ is negative. The same holds for the growth bound of the semigroup generated by the operator $\begin{pmatrix} P_0 & 0 \\ R & Q \end{pmatrix}$. As the growth bound of the semigroup generated by P_0 is negative too, the growth bound of the semigroup generated by Q is also negative, showing that this semigroup is exponentially stable. Thanks to Proposition 3.3 the system $\Sigma_{y \rightarrow \eta}$ is BIBO stable in the sense of Assumption 3.4. As a consequence, funnel control is feasible for the sine-Gordon equation (40) with the output given in (43).

We shall now illustrate the feasibility of funnel control on (40) with some numerical simulations. Let us consider the following set of parameters: $\alpha = \pi + \frac{1}{6}$, $\nu = -1$. The initial conditions for the variables x and $\partial_t x$ have been chosen as

$$x(0, z) = \frac{1}{6} \sin\left(\frac{\pi}{2}z\right), \partial_t x(0, z) = \frac{1}{5}(2z^2 - z^4),$$

while the reference signal $y_{\text{ref}}(t) = \frac{1}{5} \cos(e^{-\frac{t}{4}})$. The function $\phi(t)$, whose inverse determines the funnel boundaries, is fixed to $\frac{1}{e^{-2t} + 2.5 \cdot 10^{-3}}$.

The output trajectory (43) with the reference signal $y_{\text{ref}}(t)$ are represented in Figure 7 whereas the corresponding funnel control is given in Figure 8. It can be seen that the output tracks the reference quite well. The tracking error is depicted in Figure 9 wherein one observes that it remains within the funnel boundaries, as was to be expected. The state trajectories corresponding to $x(t, z)$ and $\partial_t x(t, z)$ are shown in Figures 10 and 11, respectively.

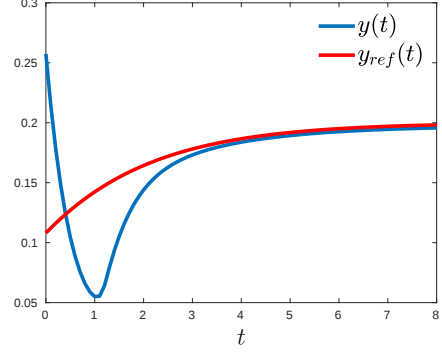


Figure 7: Output trajectory (43) with the reference signal $y_{\text{ref}}(t)$.

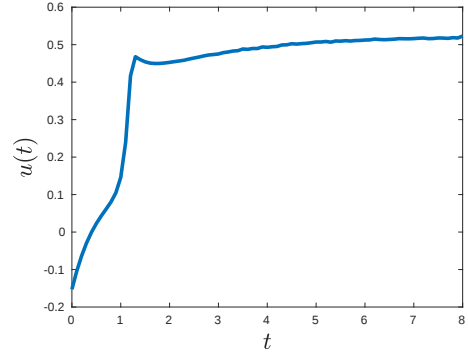


Figure 8: Input trajectory.

5. Conclusion and Perspectives

Funnel control for a class of nonlinear infinite-dimensional systems was considered. On the basis of the theory presented in Berger et al. (2020) and developed in Ilchmann et al. (2016) for linear systems with an arbitrary relative degree $r \in \mathbb{N}_0$, we have shown how to extend the feasibility of funnel control to the given system class which has to satisfy some well-posedness and BIBO stability assumptions. This constitutes the main result of the paper and is mainly based on a change of variables that enables to extract the output dynamics of the system. This is known as the Byrnes-Isidori form for linear systems, see e.g. Ilchmann et al. (2016). A way of getting the BISBO stability assumption was also established. Two examples whose dynamics are written as a nonlinear PDE and for which the results may be applied were considered. Some numerical simulations reinforce the theoretical results.

As perspectives for further research, the extension of the results to higher differential orders for the output equation, i.e. in some sense, to systems with higher relative degree could be of great interest. The two main references for tackling this question should probably be the works of Ilchmann et al. (2016) and Berger et al. (2020) in which

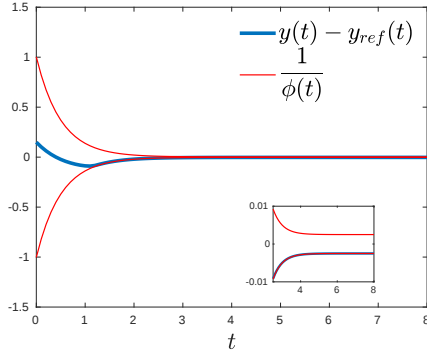


Figure 9: Output error trajectory $y(t) - y_{\text{ref}}(t)$ with the funnel boundaries $\frac{1}{\phi(t)}$ and $-\frac{1}{\phi(t)}$.

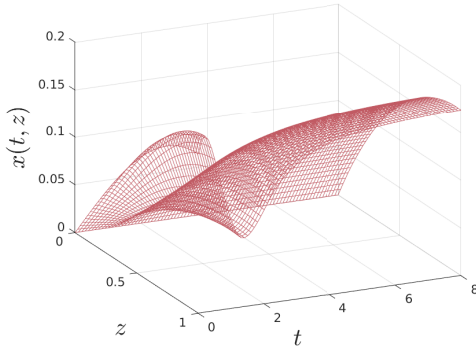


Figure 10: Closed-loop state trajectory $x(t, z)$.

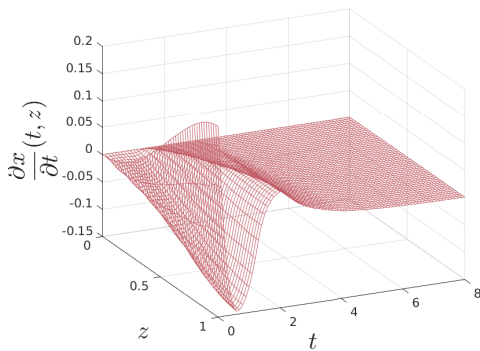


Figure 11: Closed-loop state trajectory $\partial_t x(t, z)$.

a general relative degree is considered. Another perspective aims at weakening the assumption of uniform Lipschitz continuity of the nonlinear operator to only local Lipschitz continuity. As further research the consideration of boundary control and observation could be also an interesting and valuable challenge. This study should include the formalisation of the concept of nonlinear boundary controlled and observed systems. What is meant by well-posedness in this context should also be properly defined and analyzed. For this, we refer to the preliminary works by Tucsnak and Weiss (2014), Hastir et al. (2019) and Schwenninger (2020). The enlargement of the class of nonlinear infinite-dimensional systems Σ for which funnel control is feasible could be interesting to investigate too.

Acknowledgments

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