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A frequency domain approach to Kalman filtering on Hilbert spaces: Application to Sturm-Liouville systems with pointwise measurement

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Abstract

The problem of optimal state estimation via deterministic Kalman filtering in the time and in the frequency domains is considered. The frequency domain method based on spectral factorization, which was developed previously for linear quadratic optimal control, is extended here to Kalman filtering. For a class of Riesz-spectral systems, it is shown that the spectral factorization problem can be solved by symmetric extraction of poles and zeros, which leads to a tractable computational method in order to calculate the optimal output injection in the Kalman filter problem. Then the class of Sturm-Liouville operators is considered on the space of square integrable functions on a finite interval. According to the Riesz-spectral property, the self-adjointness and the positivity of such unbounded linear operators on that space, a class of Hilbert spaces constructed as the domains of the positive (in particular, fractional) powers of any Sturm-Liouville operator is considered in a second time. Properties of Sturm-Liouville operators on these spaces are exhibited, together with properties of the C_0-semigroups that are generated by these operators. In addition, a characterization of approximate observability by means of point measurement operators is established for such systems. For the aforementioned class of Sturm-Liouville systems with pointwise measurement, the assumptions needed for applying the symmetric extraction method are shown to be satisfied, which entails that this class of systems is suitable for Kalman filtering with a pointwise measurement observation operator which is bounded on a well-chosen Hilbert state space. The great advantage of considering a new state space is pushed forward by this optimal state estimation problem, which would not make sense in the space of square integrable functions, notably in terms of Riccati equation. The main results are applied to the Kalman filtering of a diffusion system with mixed boundary conditions and pointwise measurement.

Keywords: Kalman filtering – Riccati equation – Spectral factorization – Sturm-Liouville operators – Pointwise measurement – Riesz-spectral operators – Fractional powers of operators

1. Introduction

Knowing exactly at any time the state of a dynamical system is of utopia, which is one of the main reasons why observer design is so important in systems and control theory, and in particular, for linear infinite-dimensional systems. The classical way of constructing an observer consists in copying the state equations and adding a correction term that is related to the error between the measured and the approximated outputs. One particular choice of observer is the well-known Luenberger observer, see e.g. (Curtain and Zwart, 1995, Chapter 5). In this approach, the correction term enters the dynamics linearly through an output injection operator. The ways of designing this linear operator are numerous. We refer to Yupanqui Tello et al. (2021) for a comprehensive overview on state estimation techniques for infinite-dimensional systems. For instance, in the presence of disturbances, both in the dynamics and in the known output, one particular approach relies on solving an optimization problem which is related to the solution of an operator Riccati equation. This problem is known as the Kalman-Bucy filter approach and was first presented in Kalman and Bucy (1961). This method is often considered in a stochastic setting but it has also interpretations in a deterministic context. In particular, it is extensively studied for stochastic linear infinite-dimensional systems in (Curtain and Pritchard, 1978, Chapter 6) while it is interpreted deterministicly in (Curtain and Zwart, 1995, Chapter 6).

As mentioned earlier, the solution of the Kalman filter problem relies on an operator Riccati equation often called the *filter Riccati equation.* This problem is therefore strongly related to the linear-quadratic optimal control problem and may be viewed as its dual. From a control point of view, this control method design has several solid theoretical foundations, both for finite- and infinite-dimensional systems. Two different approaches are available to tackle this control problem: a time-domain apraoch that relies on the solution of a Riccati equation and a frequency domain approach that is based on the spectral factorization of a coercive spectral density. We refer to the

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references Callier and Winkin (1990, 1992) and Sayed and Kailath (2001) for detailed developments of the spectral factorization approach to optimal control with bounded input and observation operators. A way of solving this problem is also presented in Winkin et al. (2004) where the method known as the symmetric extraction is developed extensively. In Weiss and Weiss (1997), the authors proposed a preliminary approach to linear quadratic optimal control based on spectral factorization for systems in which unbounded control and observation operators are considered. To the best of our knowledge, no further developments have been built on the frequency domain approach to optimal control or state observation with unbounded control and observation operators ever since.

Coming back to the Kalman filter problem, not so much attention has been paid to a frequency domain description of the solution to that problem. We refer to the reference Kučera (1978) in which this question is tackled for finite-dimensional systems, where the solution relies on a spectral factorization problem.

Moving to a particular class of infinite-dimensional systems, Sturm-Liouville systems have attracted a lot of attention in the last few years since they are able to describe many different phenomena that are modeled by parabolic partial differential equations (PDEs). Sturm-Liouville operators are defined in (Curtain and Zwart, 2020, Chapter 2) and in (Tucsnak and Weiss, 2009, Chapter 3) where properties like self-adjointeness and closedness are studied. In Delattre et al. (2003), the class of Sturm-Liouville systems is shown to be a class of Riesz-spectral systems for which properties of the spectrum are given. These properties are analyzed in Orlov (2017) too where further details are given about the eigenfunctions, notably their estimation in terms of lower and upper solutions to the corresponding two-point boundary value problem.

Quite a lot of research has been done concerning observer and controller design for Sturm-Liouville systems. We refer to the works Vries et al. (2010); Lhachemi and Prieur (2022); Schaum et al. (2017); Feketa et al. (2021), in which this problem is studied in the case of a point measurement observation operator. Feedback regulation is also a topic of strong interest for such systems, see e.g. Deutscher (2015) and references therein. For control aspects, we refer to Emirsjlow and Townley (2000). The state space associated to these problems is often chosen as the space of square integrable functions, which makes the design of the control law challenging since the pointwise measurement is described by an unbounded operator, and is even not well-defined in that Hilbert space.

The first main contribution in this paper is the adaptation of the existing results on spectral factorization and linear quadratic optimal control to the Kalman filter problem for deterministic infinite-dimensional systems. In particular, the central results of Callier and Winkin (1990, 1992) are extended to the Kalman filtering problem by using duality arguments. As a particularity, under some quite mild assumptions, it is shown that the spectral factorization method applied to the Kalman filter problem can be solved by using a computational method known as the symmetric extraction. In a second time, the other main contribution in this paper consists in using the properties of Sturm-Liouville operators on the space of square integrable functions in order to define them on a class of Hilbert spaces, constructed as the domains of their positive powers. The Riesz-spectral property of Sturm-Liouville operators is shown to hold on any of these Hilbert spaces. This constitues the second main result of this paper. Moreover, properties like generation of strongly continuous semigroup as well as compactness are shown too. Keeping in mind the problem of unboundedness of the pointwise measurement operator raised above, it is shown that such an operator becomes bounded on the domain of fractional powers of Sturm-Liouville operators, in different cases of boundary conditions, provided that the power is larger than $\frac{1}{2}$ and less than 1. Thanks to this important fact, approximate observability of the corresponding system is characterized via a simple checkable condition, independently of the chosen power between $\frac{1}{2}$ and 1. This is our third main result. As a consequence, it is shown that the aforementioned class of Sturm-Liouville systems with pointwise measurement is well-adapted to the symmetric extraction method for spectral factorization and hence it is suitable for Kalman filtering with an observation operator that is originally unbounded when looked on the Hilbert space of square integrable functions. We would like to insist a lot on this fact, which makes our approach powerful for the state estimation of infinite-dimensional systems with boundary observation. The approach that we follow is much more simpler since the considered observation operator is bounded, without having taken any approximation.

The paper is organized as follows: Notations and preliminary concepts are presented in Section 2. In Section 3 the deterministic Kalman filter problem is recalled and its solution is given both in time and frequency domains, where the spectral factorization method is generalized to this context. This method is detailed for a class of Riesz-spectral systems in Section 4 in which the symmetric extraction is shown feasible. Then, Section 5 aims at defining the considered class of Sturm-Liouville operators and at highlighting numerous of his properties. The general setting is given in Subsection 5.1. Then, the domain of the fractional powers of Sturm-Liouville operators are introduced in Subsection 5.2 in which Riesz-spectral property, generation of a strongly continuous semigroup and compactness of the latter are shown. The approximate observability is studied in subsection 5.3. The symmetric extraction method to solve the Kalman filter problem is shown to be well-adapted for the considered class of Sturm-Liouville systems with pointwise measurement in Section 6 in which an example is given. Conclusions together with some perspectives are described in Section 7.

2. Notations, nomenclature and preliminary definitions

Throughout this paper, $\mathbb{N}, \mathbb{R}$ and $\mathbb{C}$ denote the set of natural, real and complex numbers, respectively. The nonnegative (resp. nonpositive) real line is written $\mathbb{R}^+$ ($\mathbb{R}^-$, resp.). To denote the open left (resp. right) half plane of $\mathbb{C}$, we use $\mathbb{C}_0^-$ (resp. $\mathbb{C}_0^+$). The space of linear and bounded operators from the Hilbert space X into the Hilbert space Y is denoted by $\mathcal{L}(X,Y)$. When $X = Y$, the notation $\mathcal{L}(X)$ is used. The space of real-valued square integrable functions defined on the finite interval $[a,b], a < b$ is written $L^2(a,b)$. It is equipped with the inner product $\langle f, g\rangle_{L^2(a,b)} = \int_a^b f(z)g(z)dz$. The notation $H^k(a,b)$ is used for the Sobolev space of square integrable functions defined on $[a,b]$ whose weak derivatives are square integrable up to order $k \in \mathbb{N}$. Those are Hilbert spaces with the inner product $\langle f, g\rangle_{H^k(a,b)} = \sum_{n=0}^k \langle \frac{d^n f}{dz^n}, \frac{d^n g}{dz^n}\rangle_{L^2(a,b)}$. For a linear, closed, densely defined, self-adjoint and coercive operator A on a Hilbert space X, the space $D((-A)^\alpha) =: X_\alpha, \alpha \geq 0$, is a Hilbert space when equipped with the inner product $\langle f, g\rangle_\alpha = \langle (-A)^\alpha f, (-A)^\alpha g\rangle_X, f, g \in X_\alpha$. The following classes of impulse responses and transfer functions may be found in Callier and Winkin (1992) and Curtain and Zwart (1995) for instance.

Definition 2.1. *Let $\sigma \in \mathbb{R}$. The generalized function f with support on $\mathbb{R}^+$ is said to be in $\mathcal{A}(\sigma)$ if $f(t) = f_a(t) + f_{sa}(t)$ where the regular functional part $f_a \in L^1_\sigma$, i.e. $\int_0^\infty |f_a(t)|e^{-\sigma t}dt < \infty$ and the singular atomic part $f_{sa} := \sum_{m=0}^\infty f_m \delta(\cdot - t_m)$, where $t_0 = 0, t_m > 0$ for $m = 1, 2, \ldots$, and $f_m \in \mathbb{C}$ for $m = 0, 1, \ldots$, with $\sum_{m=0}^\infty |f_m|e^{-\sigma t_m} < \infty$. The impulse response f is said to be in $\mathcal{A}_-(\sigma)$ if and only if it is in $\mathcal{A}(\sigma_1)$ for some $\sigma_1 < \sigma$. The norm of the impulse response f in $\mathcal{A}(\sigma)$ is given by*

$$\|f\|_{\mathcal{A}(\sigma)} := \int_0^\infty |f_a(t)|e^{-\sigma t}dt + \sum_{m=0}^\infty |f_m|e^{-\sigma t_m}. \qquad (1)$$

$\mathcal{A}(\sigma)$ is called the Wiener class and generalizes the notion of integrability to distributions. Note that the class of transfer functions that are Laplace transforms of elements of[1] $\mathcal{A}_-(\sigma)$, denoted by $\hat{\mathcal{A}}_-(\sigma)$ (transfer functions that are stable), contains the multiplicative subset $\hat{\mathcal{A}}_-^\infty(\sigma)$ of transfer functions in $\hat{\mathcal{A}}_-(\sigma)$ that are bounded away from 0 at infinity on $\mathfrak{Re}(s) \geq 0$ (i.e. that are biproper stable), that is, that satisfy $\inf_{s\in\overline{\mathbb{C}_\sigma^+}, |s|\geq\rho} |\hat{f}(s)| > 0$, for some $\rho > 0$, see (Curtain and Zwart, 2020, Chapter 7). Let us now define the Callier-Desoer algebra of transfer functions, see Callier and Desoer (1978).

[1]For an impulse response f in $\mathcal{A}(\sigma)$, the $\hat{\mathcal{A}}(\sigma)-$norm of $\hat{f}$ is defined by $\|\hat{f}\|_{\hat{\mathcal{A}}(\sigma)} = \|f\|_{\mathcal{A}(\sigma)}$, where $\hat{f}$ is the Laplace transform of f.

Definition 2.2. *Let $\sigma \in \mathbb{R}$. A transfer function $\hat{f}$ is in the Callier-Desoer class, $\hat{\mathcal{B}}(\sigma)$, of transfer functions, if and only if $\hat{f} = \hat{n}\hat{d}^{-1}$ where $\hat{n} \in \hat{\mathcal{A}}_-(\sigma)$ and $\hat{d} \in \hat{\mathcal{A}}_-^\infty(\sigma)$.*

As a convention, a transfer function matrix is said to be in $\mathrm{Mat}(\hat{\mathcal{B}}(\sigma))$ if all its entries are in $\hat{\mathcal{B}}(\sigma)$. Multivariable transfer functions $\hat{P}$ in $\mathrm{Mat}(\hat{\mathcal{B}})$ are described by a left fraction $\hat{P} = \hat{D}^{-1}\hat{P}$ where $\hat{N}$ and $\hat{D}$ are in $\mathrm{Mat}(\hat{\mathcal{A}}_-)$ and det $(\hat{D})$ is in $\hat{\mathcal{A}}_-^\infty$. Moreover, if there exist proper stable transfer matrices $\hat{U}$ and $\hat{V}$ in $\mathrm{Mat}(\hat{\mathcal{A}}_-)$ such that the Bezout identity $\hat{D}\hat{V} + \hat{N}\hat{U} = I$, then $\hat{P}$ is said to have a l.c.f. (left coprime fraction) $(\hat{N}, \hat{D})$ in $\mathrm{Mat}(\hat{\mathcal{A}}_-)$, see e.g. Callier and Winkin (1990) and Callier and Winkin (1992).

3. Kalman filtering (KF) for optimal state estimation in infinite horizon

This section is devoted to a brief overview of the deterministic version of the Kalman-Bucy filtering problem for infinite-dimensional state-space systems, see e.g. (Curtain and Zwart, 1995, Chapter 6), and to its frequency domain solution. To the best of our knowledge, the way of solving this problem in the frequency domain is presented here for the first time. Note that a stochastic interpretation of the Kalman-Bucy filtering problem for infinite-dimensional systems may be found e.g. in Curtain and Pritchard (1978).

3.1. General framework

Let us consider the following infinite-dimensional system

$$\begin{cases} \dot{x}(t) = Ax(t) + Qw(t), x(0) = x_0 \\ y(t) = Cx(t) + S^{\frac{1}{2}}\eta(t), \end{cases} \qquad (2)$$

where the operator A is the infinitesimal generator of a C_0-semigroup on the Hilbert space X. The operators Q and C satisfy $Q \in \mathcal{L}(U, X)$ and $C \in \mathcal{L}(X, Y)$, respectively, and the operator $S \in \mathcal{L}(Y)$ is coercive, where $U = \mathbb{R}^m$ and $Y = \mathbb{R}^p$. The estimation problem that is considered consists in estimating the state of (2) on the basis of the known output $y(t)$ in the presence of the uncertain input signals $w \in L^2([0,\infty); U)$ and $\eta \in L^2([0,\infty); Y)$. By denoting by $\hat{x}(t)$ the estimation of the state at time $t \geq 0$, we seek for an operator $M(\cdot) : (-\infty, t] \to \mathcal{L}(Y, X)$ which, from the known output signal y (which is assumed to be available on $(-\infty, t]$), produces the estimated state $\tilde{x}(t)$. In other words, M is such that

$$\tilde{x}(t) = \int_{-\infty}^t M(t-\tau)y(\tau)d\tau, \qquad (3)$$

see e.g. (Curtain and Zwart, 1995, Chapter 6). Moreover, we assume that the pairs (A, Q) and (C, A) are exponentially stabilizable and exponentially detectable, respectively. As an additional requirement, we are looking for a map M whose expectation $E(M)$ is minimal, where

$$E(M)x := \int_{-\infty}^{t} (M(t-\tau)SM^*(t-\tau) + X^*(t-\tau)QQ^*X(t-\tau)x)d\tau, \quad (4)$$

with X being the mild solution of

$$\dot{X}(\tau) = -A^*X(\tau) + C^*M^*(\tau - t)x, X(t) = x.$$

3.2. Time-domain approach

According to (Curtain and Zwart, 1995, Chapter 6), the optimal estimated state at time $t \geq 0$, $\tilde{x}(t)$, solution of the optimization problem described in Subsection 3.1 is given by

$$\tilde{x}(t) = \int_{-\infty}^{t} T_{-PC^*S^{-1}C}(t-\tau)PC^*S^{-1}y(\tau)d\tau, \quad (5)$$

where $(T_{-PC^*S^{-1}C}(t))$ is the C_0-semigroup generated by the operator $A - PC^*S^{-1}C$ with P being the unique self-adjoint, nonnegative solution in $\mathcal{L}(X)$ of the following operator Riccati equation (ORE)

$$AP + PA^* + QQ^* - PC^*S^{-1}CP = 0, \quad (6)$$

on $D(A^*)$ with $P(D(A^*)) \subset D(A)$. Differentiating the estimated state with respect to time yields that $\tilde{x}(t)$ is the solution of the following Luenberger-type state estimator

$$\begin{cases} \dot{\tilde{x}}(t) = A\tilde{x}(t) + L(\tilde{y}(t) - y(t)), \\ \tilde{y}(t) = C\tilde{x}(t), \end{cases} \quad (7)$$

in which the optimal output injection operator $L \in \mathcal{L}(Y, X)$ is given by $L = -PC^*S^{-1}$.

3.3. Frequency domain approach

To describe the frequency domain version of the Kalman filter on an infinite-horizon, we shall rely on the references Callier and Winkin (1990), Callier and Winkin (1992) and Kučera (1978). In order to work in the framework of Callier and Winkin (1992), we shall consider the spaces U and Y as being $\mathbb{R}^m$ and $\mathbb{R}^p$, respectively, $m, p \in \mathbb{N}$. We recall that the pairs (A, Q) and (C, A) are still assumed to be exponentially stabilizable and exponentially detectable in this section. We shall also set the operator S as $S = I$, without loss of generality. We shall start with a general setting and then see how the results can be treated in the case where the operator A is a Riesz-spectral operator, when both the disturbance w and the output y are one-dimensional. The following theorem is the dual version of (Callier and Winkin, 1990, Theorem 2) and provides a frequency domain approach to compute the optimal output injection L described in Subsection 3.2.

Theorem 3.1. *Let us consider the infinite-dimensional system (2) together with the Kalman filter (7), under the exp. stabilizability and exp. detectability assumptions of the pairs (A, Q) and (C, A), respectively. The optimal output injection solution of the Kalman filter problem is given by $L = -PC^*$ where P is the unique self-adjoint nonnegative solution of the ORE (6). Let $NO(C, A)$ denote the non-observable subspace of the pair (C, A). Consider a (pre-)stabilizing output injection $L_0 \in \mathcal{L}(\mathbb{R}^p, X)$ such that the C_0-semigroup generated by the operator $A + L_0C$ is exp. stable. The transfer function of (2) mapping $\hat{w}$ to $\hat{y}$ is given by $\hat{P}(s) = C(sI - A)^{-1}Q$ and admits the following l.c.f in $(\hat{N}, \hat{D}) \in Mat(\hat{\mathcal{A}}_-)$*

$$(\hat{N}, \hat{D}) := (C(sI-A-L_0C)^{-1}Q, I+C(sI-A-L_0C)^{-1}L_0). \quad (8)$$

Let us also define

$$\hat{\mathcal{N}}(s) := C(sI - A - L_0C)^{-1} \in \mathcal{L}(X, \mathbb{R}^p). \quad (9)$$

Now let us consider any solution of the spectral factorization problem

$$\hat{N}\hat{N}_* + \hat{D}\hat{D}_* = \hat{R}\hat{R}_*, \quad (10)$$

where $\hat{R}$ is a spectral factor such that $\hat{R}, \hat{R}^{-1} \in Mat(\hat{\mathcal{A}}_-)$ and $\hat{R}_*(s) = \hat{R}^*(-\overline{s})$ denotes the parahermitian of $\hat{R}$. Then the following assertions hold:

a) Any spectral factor $\hat{R}$ of (10) is expressed as[2]

$$\hat{R}(s) = \hat{D}(s)\hat{\Gamma}(s)U, \quad (11)$$

where U is a unitary constant matrix and

$$\hat{\Gamma}(s) := I - C(sI - A)^{-1}L. \quad (12)$$

b) Consider any constant solution $(\mathcal{U}, \mathcal{V}) \in \mathcal{L}(\mathbb{R}^p) \times \mathcal{L}(\mathbb{R}^p, X)$ (with $\mathcal{U}$ being unitary) of the operator Diophantine equation

$$\hat{D}(s)\mathcal{U} + \hat{\mathcal{N}}(s)\mathcal{V} = \hat{R}(s), \quad (13)$$

which is induced by the operator left coprime fraction over $\hat{\mathcal{A}}_-$ of the state-output transfer function described by $C(sI - A)^{-1} = \hat{D}(s)^{-1}\hat{\mathcal{N}}(s)$, with operator Bezout identity over $\hat{\mathcal{A}}_-$ given by $\hat{D}(s) - \hat{\mathcal{N}}(s)L_0 = I$.

c) The observable projection of the optimal output injection L is given by

$$\Pi L = -\Pi\mathcal{V}\mathcal{U}^*, \quad (14)$$

where $\Pi \in \mathcal{L}(X)$ is the orthogonal projection onto the orthogonal complement of $NO(C, A)$, i.e. the reachable subspace[3] *of (A^*, C^*).*

[2]To make the spectral factor unique, one may look for the standard spectral factor, i.e. the spectral factor $\hat{R}(s)$ which has the property that $\hat{R}(\infty) = I$. This has the consequence that $U = I$ in that case.

[3]When the pair (C, A) is approximately observable in infinite-time instead of exp. detectable only, the operator $\Pi = I$ and the optimal output injection is completely known.

d) *The operator ΠL is stabilizing, i.e. the C_0-semigroup $(T_{\Pi LC}(t))_{t\geq 0}$ generated by the error dynamics operator $A+\Pi LC$ is exp. stable.*

e) *The transfer function $\hat{G}_f$ of the Kalman filter, mapping the measured output $\hat{y}$ to the estimated output $\hat{\tilde{y}}$, is given by*

$$\begin{aligned} \hat{G}_f(s) &:= C\hat{M}(s) = -C(sI-A-LC)^{-1}L \qquad (15)\\ &= I-\Gamma^{-1}(s) = I-U\hat{R}^{-1}(s)\hat{D}(s), \end{aligned}$$

where $\hat{\Gamma}$ is given by (12) and $\hat{R}$ is the spectral factor of (10) such that $\hat{R}(\infty)=U$. Hence the injection operator ΠL is also optimal from the input-output viewpoint, i.e. the filter transfer function $\hat{G}_f$ is unchanged if L is replaced by ΠL in (15).

Proof. *Mutatis mutandis*, the proof goes along the lines of the proofs of (Callier and Winkin, 1992, Theorems 3 and 4). See also the proof of (Callier and Winkin, 1990, Theorem 2). ∎

4. KF for a class of SISO Riesz-spectral systems

4.1. Setting

This section aims at showing how Theorem 3.1 applies to a class of operators A that admit a Riesz-spectral decomposition, in the presence of scalar disturbance $w(t)$ and scalar output $y(t)$. Note that, since the output is one-dimensional, the operator $C: X\to\mathbb{R}$ is a bounded linear functional that is expressed as $Cx=\langle c,x\rangle_X$ for all $x\in X$ and some $c\in X$. One shall consider the assumptions of Theorem 3.1 together with some additional assumptions listed below:

Assumption 4.1. *The operator A is self-adjoint and of Riesz-spectral type, i.e.*

$$Ax=\sum_{n=1}^{\infty}\lambda_n\langle x,\phi_n\rangle_X\phi_n,$$

for all $x\in D(A)$ such that $\sup_{n\geq 1}\lambda_n<\infty$ and $\sigma(A)\cap[\beta,\infty)$ is finite for some $\beta<0$, where $\{\lambda_n\}_{n\geq 1}$ and $\{\phi_n\}_{n\geq 0}$ are the (real) eigenvalues and the normalized eigenfunctions of A in X, respectively.

Assumption 4.2. *The disturbance and the output are collocated, which, in terms of operators, means that*

$$Q=C^*,\ \text{i.e. } Qw=cw \text{ for all } w\in\mathbb{R}.$$

Moreover, the pair (C,A) is approximately observable in infinite-time. In view of the Riesz-spectral nature of A and (Curtain and Zwart, 2020, Theorem 6.3.6), this is equivalent to the condition[4]

$$\langle c,\phi_n\rangle_X\neq 0 \text{ for all } n\geq 1.$$

[4]As $Q=C^*$ and $A^*=A$, a direct consequence of the approximate observability of (C,A) is the approximate controllability of (Q,A) in infinite-time.

Assumption 4.3. *The eigenvalues of A are such that*[5]

$$\sup_{m\in\mathbb{N}}\sum_{\substack{n=1\\n\neq m}}^{\infty}\frac{1}{|\lambda_n-\lambda_m|^2}=\kappa, \qquad (16)$$

for some $0<\kappa<\infty$.

Observe that it follows from Assumptions 4.1 and 4.2 that the pair (C,A) is exponentially detectable and the pair (A,Q) is exp. stabilizable. This setting entails that the optimal output injection L is a multiplication operator by a function in X. We will adopt the notation $L=\ell$ in what follows. Moreover, Assumption 4.3 together with Sun (1981) entails that the operator $A+\ell\langle c,\cdot\rangle_X$ is a Riesz-spectral operator.

Proposition 4.1. *Let us consider that Assumptions 4.1, 4.2 and 4.3 hold. Then the components of the optimal output injection $L=-\mathcal{V}\mathcal{U}^*=\ell$, see (14), in the orthonormal basis $\{\phi_n\}_{n\geq 1}$ of X, denoted by $\{\ell_n\}_{n\geq 1}:=\{\langle \ell,\phi_n\rangle_X\}_{n\geq 1}$, are given by*

$$\ell_n=\frac{-\mathfrak{Res}(\hat{D}^{-1}(s)\hat{R}(s);\lambda_n)}{\langle c,\phi_n\rangle_X}, \qquad (17)$$

where $\mathfrak{Res}(f;\lambda)$ is the residue of the function f at λ, $\hat{D}(s)=1+\langle c,(sI-A-\ell_0\langle c,\cdot\rangle)^{-1}\ell_0\rangle_X$, ℓ_0 is a pre-stabilizing output injection and $\hat{R}(s)$ is the spectral factor of (10) such that $\hat{R}(\infty)=1$.

Proof. First observe that, by Assumption 4.2, the projection operator $\Pi=I$, since $\mathcal{NO}(C,A)=\{0\}$. Hence, $L=\Pi L=-\mathcal{V}\mathcal{U}^*$. According to Theorem 3.1, the (scalar) spectral factor $\hat{R}(s)$ satisfies (11) with $U=1$. Equivalently, there holds

$$\hat{D}^{-1}(s)\hat{R}(s)=1-C(sI-A)^{-1}\ell. \qquad (18)$$

According to Assumption 4.1, (18) may be written as

$$\hat{D}^{-1}(s)\hat{R}(s)=1-\sum_{n=1}^{\infty}\frac{\langle c,\phi_n\rangle_X\langle \ell,\phi_n\rangle_X}{s-\lambda_n},$$

which implies that

$$\ell_n\langle c,\phi_n\rangle_X=-\mathfrak{Res}(\hat{D}^{-1}(s)\hat{R}(s);\lambda_n).$$

Since the pair (C,A) is approximately observable in infinite-time, none of the coefficients $\langle c,\phi_n\rangle_X$ is 0, which implies (17). ∎

4.2. Computational method

As it can be seen in Proposition 4.1, the key point in computing the components of the optimal output injection is the computation of the spectral factor $\hat{R}(s)$. We report

[5]This assumption implies that the condition $\inf_{n,m\in\mathbb{N},n\neq m}|\lambda_n-\lambda_m|=\mu>0$ holds true, see e.g. Abouzaïd et al. (2022).

hereafter the symmetric extraction method, which is based on the infinite-product representation of spectral densities that are meromorphic functions of finite-order, see Winkin et al. (2004). This method is described by considering that Assumptions 4.1, 4.2 and 4.3 are satisfied. The following proposition gives a way of computing the spectral factor $\hat{R}(s)$ as an infinite-product, based on the infinite-product representation of the spectral density given in (10).

Proposition 4.2. *Let us consider that Assumptions 4.1, 4.2 and 4.3 hold. Then the standard spectral factor $\hat{R}(s)$, such that $\hat{R}(\infty) = 1$, can be written as the following infinite-product*

$$\hat{R}(s) = \prod_{n=1}^{\infty} \frac{s-\mu_n}{s-\beta_n}, \tag{19}$$

where $\{\mu_n\}_{n\geq 1}$ and $\{\beta_n\}_{n\geq 1}$ are the real stable zeros and the real stable poles, respectively, of the spectral density

$$\hat{F}(s) = \hat{N}\hat{N}_* + \hat{D}\hat{D}_*, \tag{20}$$

where $(\hat{N}, \hat{D})$ is a l.c.f of the open-loop transfer function

$$\hat{P}(s) = \sum_{n=1}^{\infty} \langle c, \phi_n\rangle_X^2 (s-\lambda_n)^{-1}.$$

Proof. The proof is divided into two parts.

- First, note that the pre-stabilizing output injection L_0 in (8) can be chosen such that

$$\sigma(A + L_0C) = (\sigma(A) \cap \mathbb{C}_0^-) \,\dot{\cup}\, \Sigma, \tag{21}$$

  where $\Sigma \subset \mathbb{R}_0^-$ is a finite set having the same number of elements (counting multiplicities) as $\sigma(A) \cap \mathbb{C}^+$. Secondly, according to (Winkin et al., 2004, Lemma 4.7), the real coercive spectral density $\hat{F}(s)$ given by (20) is a meromorphic function of finite order that can be described as a fraction

$$\hat{F}(s) = \frac{\hat{\mathfrak{N}}(s)}{\hat{\mathfrak{D}}(s)}, \tag{22}$$

  where the functions $\hat{\mathfrak{N}} = \hat{\mathfrak{N}}_*$ and $\hat{\mathfrak{D}} = \hat{\mathfrak{D}}_*$ are entire functions with countable zero sets $Z[\hat{\mathfrak{N}}]$ and $Z[\hat{\mathfrak{D}}]$, respectively. By denoting by $P[\hat{F}]$ the poles of the spectral density $\hat{F}$, there holds

$$Z[\hat{\mathfrak{D}}] = P[\hat{F}] \subset \{p, -p \mid p \in (\sigma(A) \cap \mathbb{C}_0^-) \,\dot{\cup}\, \Sigma\} \tag{23}$$

  and

$$Z[\hat{\mathfrak{N}}] = Z[\hat{F}] \subset \{z, -z \mid z \in \sigma(A + LC)\}, \tag{24}$$

  where L is the optimal output injection given in (14). The inclusion (24) comes from the equality[6]

$$\begin{aligned}\hat{R}^{-1}(s) &= (1 + C(sI - A - LC)^{-1}L)\hat{D}^{-1}(s)\\ &=: (1 - \hat{G}_f(s))\hat{D}^{-1}(s)\end{aligned} \tag{25}$$

  together with the fact that the poles of $\hat{R}^{-1}$ coincide with the stable zeros of $\hat{F}$. The fact that these stable zeros are real comes from Assumptions 4.1 and 4.2. Indeed, in this case, the corresponding Hamiltonian operator is self-adjoint, hence its stable spectrum, which coincides with $\sigma(A+LC)$, is real (see e.g. Callier et al. (1995)). The inclusion (24) leads to the conclusion.

- The second part of the proof consists in showing that $\hat{R}(s)$ admits the infinite-product representation (19). This follows from (Winkin et al., 2004, Theorem 4.4) in which, thanks to Assumptions 4.1, 4.2 and 4.3, the symmetric extraction method is shown to be applicable to the meromorphic spectral density $\hat{F}(s)$. In other words, $\hat{F}(s)$ admits the infinite-product representation

$$\hat{F}(s) = \prod_{n=1}^{\infty} \frac{s^2-\mu_n^2}{s^2-\beta_n^2}$$

  and its inverse admits the representation

$$\hat{F}^{-1}(s) = \prod_{n=1}^{\infty} \frac{s^2-\beta_n^2}{s^2-\mu_n^2}$$

  where the elementary (second order) spectral densities $\hat{F}_n(s) := \frac{s^2-\mu_n^2}{s^2-\beta_n^2} = \hat{R}_n(s)\hat{R}_n(-s) := \frac{s-\mu_n}{s-\beta_n}\frac{-s-\mu_n}{-s-\beta_n}$ may be factorized with first order spectral factors since the zeros and the poles of $\hat{F}(s)$ are real. The spectral factor $\hat{R}(s)$ and its inverse are then expressed as $\hat{R}(s) = \prod_{n=1}^{\infty} \hat{R}_n(s)$ and $\hat{R}^{-1}(s) = \prod_{n=1}^{\infty} \hat{R}_n^{-1}(s)$, respectively. ■

Remark 4.1. *α) As a matter of convergence, we emphasize the fact that, by selecting $\sigma < 0$ such that $2|\sigma| \leq \min(|\mu_n|, |\beta_n|)$ for all $n \geq 1$, the sequence $(\prod_{n=1}^{N} \hat{R}_n)_{N\geq 1}$ of invertible approximated rational spectral factors converges to the exact invertible standard spectral factor $\hat{R} \in \hat{\mathcal{A}}_-$ of $\hat{F}$ in the $\hat{\mathcal{A}}(\sigma)-$norm given in (1). Moreover, the sequence $(\prod_{n=1}^{N} \hat{R}_n^{-1})_{N\geq 1}$ converges to the corresponding inverse spectral factor $\hat{R}^{-1} \in \hat{\mathcal{A}}_-$, see (Winkin et al., 2004, Theorem 4.4).*

β) In the case where the spectrum of the operator A is located in $\mathbb{C}_0^-$, the pre-stabilizing output injection L_0 may be chosen as being 0, which entails that $(\hat{N}, \hat{D}) = (\hat{G}, 1)$ is a l.c.f. of the transfer function $\hat{G}(s) = \langle c, (sI - A)^{-1}Q\rangle_X$. The associated spectral density $\hat{F}(s)$ is expressed as $\hat{F} = 1 + \hat{G}\hat{G}_*$. In that case, the negative numbers $\{\beta_n\}_{n\geq 1}$ are the eigenvalues of the operator A, namely $\{\lambda_n\}_{n\geq 1}$. Moreover, according to (17), the components of the optimal output injection in the basis $\{\phi_n\}_{n\geq 1}$ are expressed as

$$\ell_n = \frac{\mu_n - \lambda_n}{\langle c, \phi_n\rangle_X} \prod_{\substack{k=1\\k\neq n}}^{\infty} \frac{\lambda_n - \mu_k}{\lambda_n - \lambda_k}. \tag{26}$$

[6]The function $\hat{G}_f(s)$ is the transfer function of the Kalman filter (7).

γ) *Thanks to Assumption 4.3, the closed-loop operator $A + \ell\langle c, \cdot\rangle_X$ is a Riesz-spectral operator with a pure point spectrum, see Sun (1981). Since the coefficients $\{\mu_n\}_{n\geq 1}$ are its eigenvalues, the coefficients of its eigenfunctions in the orthonormal basis $\{\phi_n\}_{n\geq 1}$ can be computed explicitly. Indeed, denoting by $\{\phi_n^{cl}\}_{n\geq 1}$ the eigenfunctions of the closed-loop operator $A + \ell\langle c, \cdot\rangle_X$ and taking $k \in \mathbb{N}$, there holds*

$$(A + \ell\langle c, \cdot\rangle_X)\phi_k^{cl} = \mu_k \phi_k^{cl}.$$

Projecting the previous equation in the direction of the $m-th$ eigenfunction of the operator A yields that

$$\langle A\phi_k^{cl}, \phi_m\rangle_X + \langle \sum_{n=1}^{\infty} \ell_n \phi_n, \phi_m\rangle_X \langle c, \phi_k^{cl}\rangle_X = \mu_k \langle \phi_k^{cl}, \phi_m\rangle_X.$$

By using the self-adjointness of the operator A, the previous expression becomes

$$\lambda_m \langle \phi_k^{cl}, \phi_m\rangle_X + \ell_m \langle c, \phi_k^{cl}\rangle_X = \mu_k \langle \phi_k^{cl}, \phi_m\rangle_X.$$

As the eigenfunctions are defined up to multiplicative constants, one may assume without loss of generality that the coefficients $\langle c, \phi_k^{cl}\rangle_X, k \geq 1$ are equal to 1. This has the consequence that the $m-th$ component of the $k-th$ eigenfunction of the operator $A+\ell\langle c, \cdot\rangle_X$ in the orthonormal basis $\{\phi_n\}_{n\geq 1}$ of X is given by

$$\langle \phi_k^{cl}, \phi_m\rangle_X = \frac{\ell_m}{\mu_k - \lambda_m}. \tag{27}$$

As it will be highlighted later in this paper, the symmetric extraction method is well-adapted to the optimal state estimation of systems described by the well-known class of Sturm-Liouville operators with pointwise measurement on some appropriate Hilbert spaces. Therefore, a proper introduction and a dynamical analysis of such systems are needed. In the following section, the considered Hilbert spaces are introduced together with the studied class of Sturm-Liouville operators. Moreover, properties of these operators are analyzed and an approximate observability analysis of Sturm-Liouville systems with pointwise measurement is performed.

5. Dynamical analysis of a class of Sturm-Liouville systems

5.1. Systems class

Let us consider the class of Sturm-Liouville operators expressed as

$$(\mathcal{A}f)(z) = \frac{1}{\rho(z)}\left(\frac{d}{dz}\left(-p(z)\frac{df}{dz}\right) + q(z)f(z)\right), \tag{28}$$

where the functions $p, \frac{dp}{dz}, q$ and ρ are real-valued and continuous on $[a, b]$ with $\rho > 0$ and $p > 0$. The domain of the operator $\mathcal{A}$, $D(\mathcal{A})$, is expressed in the following way

$$\{f \in H^2(a,b), \begin{bmatrix} \alpha_a & \beta_a \end{bmatrix} \begin{bmatrix} \frac{df}{dz}(a) \\ f(a) \end{bmatrix} = 0 = \begin{bmatrix} \alpha_b & \beta_b \end{bmatrix} \begin{bmatrix} \frac{df}{dz}(b) \\ f(b) \end{bmatrix}\}, \tag{29}$$

where $(\alpha_a, \beta_a) \neq (0,0) \neq (\alpha_b, \beta_b)$. Let us now equip the space $L^2(a,b)$ with the inner product $\langle f, g\rangle_\rho := \langle f, \rho g\rangle_{L^2(a,b)}$ and corresponding norm $\|\cdot\|_\rho$. In the sequel, this space is denoted by $L^2_\rho(a,b) =: X$. The class of operators $\mathcal{A}$ given in (28) – (29) has already been considered in Delattre et al. (2003), in the context of Sturm-Liouville systems, involving observation and control. Therein, it is highlighted that the operator $A := -\mathcal{A}$ defined on $D(A) = D(\mathcal{A})$ is a Riesz-spectral operator whose spectrum is composed only of eigenvalues which are simple, see (Delattre et al., 2003, Lemma 1). An additional property of the operator A is its self-adjointness with respect to the inner product $\langle\cdot,\cdot\rangle_\rho$, see e.g. (Sagan, 2012, Chapter 2.4). Moreover, the spectrum of A is bounded from above, in the sense that there exists γ such that for any $n \in \mathbb{N}$, there holds $\lambda_n < \gamma < +\infty$, where $\{\lambda_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ denotes the set of eigenvalues of the operator A. The latter property implies that the operator A is the infinitesimal generator of a C_0-semigroup of bounded linear operators on X, see (Curtain and Zwart, 2020, Theorem 3.2.8). The set of (normalized) eigenfunctions of A is denoted by $\{\phi_n\}_{n\in\mathbb{N}}$. It forms a Riesz basis of X, which is actually an orthonormal basis of that space. These facts have the consequence that A admits the representation

$$Af = \sum_{n=1}^{\infty} \lambda_n \langle f, \phi_n\rangle_\rho \phi_n, \tag{30}$$

for $f \in D(A)$ given by

$$D(A) = \{f \in X, \sum_{n=1}^{\infty} \lambda_n^2 \langle f, \phi_n\rangle_\rho^2 < \infty\}. \tag{31}$$

5.2. Riesz-spectral property on a class of Hilbert spaces

First, we are interested in properties of the operator A when regarded as an operator on some spaces that are more regular than X. Observe that, except for Corollaries 5.1 and 5.2, the concepts and results of this section are valid for any Riesz-spectral operator whose spectrum is real and upper bounded.

Let us define the real constant $\mu := \gamma + \epsilon$, where $\epsilon > 0$ is fixed, and where γ is the upper bound on the eigenvalues $\{\lambda_n\}_{n\in\mathbb{N}}$ introduced in Section 5.1. It is clear from Section 5.1 that the operator $A - \mu I$ is a self-adjoint Riesz-spectral operator whose spectrum is given by $\{\lambda_n - \mu\}_{n\in\mathbb{N}} \subset \mathbb{R}_0^-$.

Therefore, for $f \in D(A)$,

$$\langle (\mu I - A)f, f\rangle_\rho = \sum_{n=1}^{\infty}(\mu - \lambda_n)\langle f, \phi_n\rangle_\rho^2 > \epsilon \sum_{n=1}^{\infty}\langle f, \phi_n\rangle_\rho^2 = \epsilon\|f\|_\rho^2,$$

which implies the coercivity of the operator $\mu I - A$. This property of $\mu I - A$ together with its self-adjointness implies that one may define the operators $(\mu I - A)^\alpha$ for any $\alpha > 0$, see e.g. (Pazy, 1983, Chapter 2) and (Curtain and Zwart, 2020, Chapter 3). In particular, according to (Curtain and Zwart, 2020, Lemma 3.2.11), the operators $(\mu I - A)^\alpha$ are of Riesz-spectral type and can be expressed as

$$(\mu I - A)^\alpha f = \sum_{n=1}^{\infty}(\mu - \lambda_n)^\alpha \langle f, \phi_n\rangle_\rho \phi_n, \tag{32}$$

for any $f \in D((\mu I - A)^\alpha)$ given by

$$D((\mu I - A)^\alpha) = \{f \in X, \sum_{n=1}^{\infty}(\mu - \lambda_n)^{2\alpha}\langle f, \phi_n\rangle_\rho^2 < \infty\}. \tag{33}$$

We shall now focus on the properties of the operator A on the spaces $D((\mu I - A)^\alpha)$. First observe that for $\alpha > 0$, the space $D((\mu I - A)^\alpha) =: X_\alpha$ is a real Hilbert space equipped with the inner product

$$\langle f, g\rangle_\alpha := \langle (\mu I - A)^\alpha f, (\mu I - A)^\alpha g\rangle_\rho, f, g \in X_\alpha, \tag{34}$$

see (Curtain and Zwart, 2020, Chapter 3). According to (Pazy, 1983, Chapter 2, Theorem 6.8), for $\alpha, \beta \in (0, 1), \alpha < \beta$, the following inclusions hold:

$$D(A) \subset X_\beta \subset X_\alpha \subset X. \tag{35}$$

Moreover, the space X_α may be seen as the completion of $D(A)$ with respect to the norm induced by $\langle \cdot, \cdot\rangle_\alpha$, which makes it an extension of the space $D(A)$ equipped with the graph norm associated to A, namely X_1, see e.g. (Tucsnak and Weiss, 2009, Chapter 2 and 3), when $\alpha \in (0, 1)$. Let us have a look at the properties of the operator A on X_α for $\alpha > 0$ fixed. To this end, for $n \geq 1$, let us define the function $\phi_{n,\alpha}$ as

$$\phi_{n,\alpha} = (\mu - \lambda_n)^{-\alpha}\phi_n. \tag{36}$$

It is easy to see that $\phi_{n,\alpha} \in X_\alpha$ for any $n \geq 1$. In the next result, it is shown that the rescaling (36) preserves the orthonormality of the basis $\{\phi_{n,\alpha}\}_{n\in\mathbb{N}}$ in the space X_α.

Proposition 5.1. *The set $\{\phi_{n,\alpha}\}_{n\in\mathbb{N}}$ is an orthonormal basis of X_α.*

Proof. First observe that the set $\{\phi_{n,\alpha}\}_{n\in\mathbb{N}}$ is orthonormal. Indeed, for $n, m \in \mathbb{N}$, there holds

$$\begin{aligned}&\langle \phi_{n,\alpha}, \phi_{m,\alpha}\rangle_\alpha \\ &= (\mu - \lambda_n)^{-\alpha}(\mu - \lambda_m)^{-\alpha}\langle (\mu I - A)^\alpha \phi_n, (\mu I - A)^\alpha \phi_m\rangle_\rho \\ &= (\mu - \lambda_n)^{-\alpha}(\mu - \lambda_m)^{-\alpha}\langle (\mu - \lambda_n)^\alpha \phi_n, (\mu - \lambda_m)^\alpha \phi_m\rangle_\rho \\ &= \langle \phi_n, \phi_m\rangle_\rho = \delta_{nm}.\end{aligned}$$

It remains to be shown that this set is complete in order to prove that it forms an orthonormal basis. Let us pick any $f \in X_\alpha$ orthogonal to $\phi_{n,\alpha}$ for all $n \geq 1$. As f lies in X, it admits the following representation

$$f = \sum_{k=1}^{\infty}\langle f, \phi_k\rangle_\rho \phi_k. \tag{37}$$

The orthogonality of f and $\phi_{n,\alpha}$ in X_α translates as $\langle f, \phi_{n,\alpha}\rangle_\alpha = 0$. By developing the left-hand side of the previous equality in which (37) is plugged, we have

$$\begin{aligned}\langle f, \phi_{n,\alpha}\rangle_\alpha &= \sum_{k=1}^{\infty}\langle f, \phi_k\rangle_\rho \langle \phi_k, \phi_{n,\alpha}\rangle_\alpha \\ &= \sum_{k=1}^{\infty}\langle f, \phi_k\rangle_\rho (\mu - \lambda_k)^\alpha \langle \phi_k, \phi_n\rangle_\rho = (\mu - \lambda_n)^\alpha \langle f, \phi_n\rangle_\rho.\end{aligned} \tag{38}$$

Hence, the identity $\langle f, \phi_{n,\alpha}\rangle_\alpha = 0$ is equivalent to $\langle f, \phi_n\rangle_\rho = 0$. Since the last identity holds for all $n \geq 1$ and $\{\phi_n\}_{n\in\mathbb{N}}$ is an orthonormal basis of X, it follows that f is identically 0. As a consequence, the set $\{\phi_{n,\alpha}\}_{n\in\mathbb{N}}$ is complete in X_α. ■

Now we are able to state our first central result.

Theorem 5.1. *The operator A is a self-adjoint Riesz-spectral operator on X_α with simple eigenvalues given by $\{\lambda_n\}_{n\in\mathbb{N}}$ associated to the orthonormal basis of eigenfunctions $\{\phi_{n,\alpha}\}_{n\in\mathbb{N}}$.*

Proof. First observe that one can apply the operator A to any function $\phi_{n,\alpha}$ for $n \geq 1$. There holds

$$A\phi_{n,\alpha} = (\mu - \lambda_n)^{-\alpha}A\phi_n = (\mu - \lambda_n)^{-\alpha}\lambda_n\phi_n = \lambda_n\phi_{n,\alpha},$$

which means that λ_n is an eigenvalue of A with $\phi_{n,\alpha}$ as eigenfunction. As the set $\{\phi_{n,\alpha}\}_{n\in\mathbb{N}}$ forms an orthonormal basis of X_α, it is also a Riesz basis. Together with the fact that the set of eigenvalues $\{\lambda_n\}_{n\in\mathbb{N}}$ has no accumulation point, this shows that the operator A is a Riesz-spectral operator on X_α. Let us now define the subspace $D(A)_\alpha$ of X_α by

$$D(A)_\alpha := \{f \in X_\alpha, \sum_{n=1}^{\infty}\lambda_n^2\langle f, \phi_{n,\alpha}\rangle_\alpha^2 < \infty\}. \tag{39}$$

For $f \in X_\alpha$, there holds $\langle f, \phi_{n,\alpha}\rangle_\alpha = (\mu - \lambda_n)^\alpha\langle f, \phi_n\rangle_\rho$, see (38). Consequently, for $f \in D(A)_\alpha$, there holds

$$\sum_{n=1}^{\infty}\lambda_n^2\langle f, \phi_{n,\alpha}\rangle_\alpha^2 = \sum_{n=1}^{\infty}\lambda_n^2(\mu - \lambda_n)^{2\alpha}\langle f, \phi_n\rangle_\rho^2 > \epsilon^{2\alpha}\sum_{n=1}^{\infty}\lambda_n^2\langle f, \phi_n\rangle_\rho^2,$$

which implies that $D(A)_\alpha \subset D(A)$, where $D(A)$ is given in (31). Consequently, for $f \in D(A)_\alpha$ one may write

$$\begin{aligned} Af &= \sum_{n=1}^{\infty} \lambda_n \langle f, \phi_n \rangle_\rho \phi_n \\ &= \sum_{n=1}^{\infty} \lambda_n \frac{(\mu-\lambda_n)^\alpha}{(\mu-\lambda_n)^\alpha} \langle f, \phi_n \rangle_\rho \phi_n = \sum_{n=1}^{\infty} \lambda_n \langle f, \phi_{n,\alpha} \rangle_\alpha \phi_{n,\alpha}. \end{aligned} \tag{40}$$

This last computation shows that A is self-adjoint on X_α and that its domain in that space is given by $D(A)_\alpha$. ■

Proposition 5.2. *The Riesz-spectral operator A defined on $D(A)_\alpha$ is the infinitesimal generator of a C_0−semigroup $(T(t))_{t\geq 0}$ of bounded linear operators on the space X_α.*

Proof. The property $\sup_{n\geq 1} \lambda_n < \infty$ recalled in Section 28 is sufficient to establish this result, see (Curtain and Zwart, 2020, Theorem 3.2.8). ■

Remark 5.1. *It follows from Proposition 5.2 that, for any initial condition $x_0 \in D(A)_\alpha$, the state trajectory $T(t)x_0$, i.e. the classical solution of the abstract Cauchy problem $\dot{x}(t) = Ax(t), x(0) = x_0$ exists on the whole nonnegative real axis, lies in $D(A)_\alpha$ and is continuously differentiable, that is, $x(t) \in C^1([0,\infty); X_\alpha)$, $x(t) \in D(A)_\alpha$, $t \geq 0$, see e.g. (Curtain and Zwart, 2020, Chapter 5).*

The following corollary, which makes the link between the operator $\mathcal{A}$ and the spaces $X_\alpha, \alpha > 0$, is straightforward.

Corollary 5.1. *The opposite $A = -\mathcal{A}$ of a Sturm-Liouville operator $\mathcal{A}$ of the form (28) on the domain (29) is a Riesz-spectral operator on any of the spaces $X_\alpha := D((\mu I - A)^\alpha)$ given by (33) for any $\alpha > 0$, where $X = L^2_\rho(a,b)$ and $\mu := \gamma + \epsilon, \epsilon > 0$, where, γ is any upper bound of the set of eigenvalues of the operator A.*

For the C_0−semigroup that is under consideration in Proposition 5.2, we may go a step further by characterizing its stability and its compactness.

Proposition 5.3. *The C_0−semigroup $(T(t))_{t\geq 0}$ generated by the operator A on the space $X_\alpha, \alpha > 0$, is exponentially stable if and only if* $\sup_{n\geq 1} \lambda_n < 0$.

Proof. This is a direct consequence of the fact that the spectrum of A is given by $\sigma(A) = \{\lambda_n\}_{n\in\mathbb{N}}$ on any $X_\alpha, \alpha > 0$, and of (Curtain and Zwart, 2020, Theorem 3.2.8). ■

Proposition 5.4. *The C_0−semigroup $(T(t))_{t\geq 0}$ generated by the operator A on the space $X_\alpha, \alpha > 0$, is compact, i.e. for all $t > 0$, $T(t)$ is a compact operator, if and only if*

$$\lim_{n\to\infty} e^{\lambda_n} = 0. \tag{41}$$

Proof. Let $t > 0$ be arbitrarily fixed. Observing that $\lim_{n\to\infty} e^{\lambda_n t} = 0$ iff $\lim_{n\to\infty} e^{\lambda_n} = 0$ and using the spectral mapping $\sigma(T(t)) = \{e^{\lambda_n t} : n \in \mathbb{N}\} \cup \{0\} = \exp[\sigma(A)t] \cup \{0\}$, *mutatis mutandis*, the proof goes along the lines of the proof of (Curtain and Zwart, 2020, Lemma 3.2.12). ■

Remark 5.2. *Note that (41) holds provided that the sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ possesses the following property:*

$$\forall K \in \mathbb{R}, \exists n_0 \in \mathbb{N}, \forall n \geq n_0, \lambda_n < K.$$

This may also be interpreted as the fact that the sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ must contain at most a finite number of positive terms for $(T(t))_{t\geq 0}$ to be compact.

Corollary 5.2. *The C_0−semigroup $(T(t))_{t\geq 0}$ generated by the opposite $A = -\mathcal{A}$ of a Sturm-Liouville operator, given by (28) – (29), on any space $X_\alpha, \alpha > 0$, is compact.*

Proof. Condition (41) trivially holds since $\lambda_n \to -\infty$ as n tends to infinity. ■

5.3. Observability analysis

In this section, approximate observability in infinite-time of the pair (C, A), where the operator A is given by (40), (39), is studied for a particular choice of output operator C, for different boundary conditions associated to the operator (28). Starting with the state space $X_{\frac{1}{2}}$, it will be highlighted notably that approximate observability with that choice is equivalent to approximate observability on any of the spaces $X_\alpha, \alpha \in [\frac{1}{2}, 1]$. As operator C, let us consider the point evaluation at $\eta \in [a,b]$, i.e.,

$$C : X_{\frac{1}{2}} \to \mathbb{R}, Cx = x(\eta), \tag{42}$$

for some $\eta \in [a,b]$. We insist on the fact that, on the original space $X = L^2_\rho(a,b)$, this observation operator is not well-defined, which makes the approach followed in this paper, necessary and meaningful. Let us now recall that $X_{\frac{1}{2}}$ is the completion of $D(A)$ with the norm $(\langle \cdot, (\mu I - A)\cdot \rangle_X)^{\frac{1}{2}}$. By considering the definition of the operator A, a straightforward computation reveals that, for any $x \in D(A)$,

$$\begin{aligned} \langle x, (\mu I - A)x \rangle_X &= \int_a^b [\mu\rho(z) + q(z)]\, x^2(z) dz \\ &- \left[x(z)p(z)\frac{dx}{dz}(z) \right]_a^b + \int_a^b p(z) \left(\frac{dx}{dz} \right)^2 dz. \end{aligned} \tag{43}$$

This fact shows that, even if the space $X_{\frac{1}{2}}$ is not explicitly known, it has to be a subspace of $H^1(0,1)$, in order that the right-hand side of (43) be well-defined. This highlights the fact that the considered operator C may be defined in this context. A technical but important detail here is the choice of the parameter μ. As the functions ρ and q are continuous on the interval $[a,b]$, those are bounded functions which admit a minimum and a maximum. Let us define

$$\rho_* := \min_{z\in[a,b]} \rho(z),\ \rho^* := \max_{z\in[a,b]} \rho(z),$$
$$q_* := \min_{z\in[a,b]} q(z),\ q^* := \max_{z\in[a,b]} q(z).$$

With these notations, one shall choose the parameter μ such that $\mu > \frac{-q_*}{\rho_*}$. This has the consequence that

$$0 < \mu\rho_* + q_* \leq \mu\rho(z) + q(z) \leq \mu\rho^* + q^*, z \in [a,b]. \quad (44)$$

Let us now consider three standard cases for the boundary conditions associated to the operator A, see (29).

1. Dirichlet boundary conditions: $\alpha_a = 0 = \alpha_b$
In that case, observe that the inner product (43) becomes

$$\langle x, (\mu I - A)x\rangle_X = \int_a^b [\mu\rho(z) + q(z)]\, x^2(z)dz + \int_a^b p(z)\left(\frac{dx}{dz}\right)^2 dz = \|x\|^2_{X_{\frac{1}{2}}}. \quad (45)$$

This inner product, which defines the norm on $X_{\frac{1}{2}}$, is equivalent to the standard $H^1(0,1)-$norm since condition (44) is satisfied and since the function p is positive. As a consequence, the operator $C : X_{\frac{1}{2}} \to \mathbb{R}$ defined by (42) is a bounded linear operator. Thanks to the Riesz Representation Theorem, see e.g. (Curtain and Zwart, 2020, Theorem A.3.55), there exists a unique $\mathfrak{h}_\eta \in X_{\frac{1}{2}}$ such that $Cx = \langle \mathfrak{h}_\eta, x\rangle_{X_{\frac{1}{2}}}$ for every $x \in X_{\frac{1}{2}}$. Since $X_{\frac{1}{2}}$ is a reproducing kernel Hilbert space, the operator C, and even any pointwise evaluation functional, is bounded from $X_{\frac{1}{2}}$ into $\mathbb{R}$. Now let us examine the approximate observability in infinite time[7] of the pair (C, A). In view of the Riesz-spectral property of the operator A and (Curtain and Zwart, 2020, Theorem 6.3.4), this can be deduced by looking at the conditions $\langle \mathfrak{h}_\eta, \phi_{n,\frac{1}{2}}\rangle_{X_{\frac{1}{2}}} \neq 0, n \geq 1$. Equivalently, the relation $\phi_{n,\frac{1}{2}}(\eta) \neq 0, n \geq 1$ has to hold. As the quantity $(\mu - \lambda_n)$ is different from 0 for any $n \geq 1$, the last condition is equivalent to $\phi_n(\eta) \neq 0, n \geq 1$.

2. Neumann boundary conditions: $\beta_a = 0 = \beta_b$
For that case of boundary conditions, remark that the norm on the space $X_{\frac{1}{2}}$ is again given by (45), which makes the approximate observability analysis the same as the one performed for the case of Dirichlet boundary conditions.

3. Mixed boundary conditions: $\alpha_a \neq 0 \neq \beta_a, \alpha_b \neq 0 \neq \beta_b, \frac{\beta_a}{\alpha_a} < 0, \frac{\beta_b}{\alpha_b} > 0$
First let us comment on the choice of the signs of the quantities $\frac{\beta_a}{\alpha_a}$ and $\frac{\beta_b}{\alpha_b}$. This is the most relevant situation from a physical point of view, see e.g. Orlov (2017). In that case, the inner product (43) becomes $\langle x, (\mu I - A)x\rangle_{X_{\frac{1}{2}}} =$

$$\int_a^b [\mu\rho(z) + q(z)]x^2(z)dz + \int_a^b p(z)\left(\frac{dx}{dz}(z)\right)^2 dz + \frac{\beta_b}{\alpha_b}p(b)x^2(b) + \left|\frac{\beta_a}{\alpha_a}\right| p(a)x^2(a) =: \|x\|^2_{X_{\frac{1}{2}}}. \quad (46)$$

According to the boundedness of the trace operator from $H^1(0,1)$ into $\mathbb{R}$, the inequality $\|x\|^2_{X_{\frac{1}{2}}} \leq \nu\|x\|^2_{H^1(0,1)}$ holds for some $\nu > 0$ independent of $x \in X_{\frac{1}{2}}$. Moreover, it follows from (46) that there holds

$$\|x\|^2_{X_{\frac{1}{2}}} \geq \min\{\mu\rho_* + q_*, p_*\}\|x\|^2_{H^1(0,1)},$$

which implies that the norm $\|\cdot\|_{X_{\frac{1}{2}}}$ defined in (46) is equivalent to the standard $H^1(0,1)-$norm. Hence, the approximate observability (necessary and sufficient) condition in infinite-time remains $\phi_n(\eta) \neq 0, n \geq 1$. We are now in position to state the main result of this section.

Theorem 5.2. *Let $\alpha \in [\frac{1}{2}, 1]$ and let us consider the operator $A = -\mathcal{A}$ given by (40), (28) with domain $D(A)_\alpha$ defined by (39), with boundary conditions of Dirichlet, Neumann or mixed type*[8]. *For any point observation operator $C : X_\alpha \to \mathbb{R}$ given by $Cx = x(\eta), \eta \in [a,b]$, the pair (C, A) is approximately observable in infinite-time on X_α if and only if $\phi_n(\eta) \neq 0$, for all $n \geq 1$.*

Proof. The result has already been shown for $\alpha = \frac{1}{2}$, see above. For $\alpha \in \left(\frac{1}{2}, 1\right]$, the inclusion $X_\alpha \subseteq X_{\frac{1}{2}}$ holds, see (35). Moreover, for $\alpha \left(\frac{1}{2}, 1\right]$, the inequality $\|x\|_{\frac{1}{2}} \leq \tilde{\nu}\|x\|_\alpha$ holds for all $x \in X_\alpha$ and some $\tilde{\nu} > 0$ independent of x. This means that the evaluation functional $C : X_\alpha \to \mathbb{R}$ defined by $Cx = x(\eta)$ is bounded from X_α into $\mathbb{R}$ for any $\alpha \in [\frac{1}{2}, 1]$. This implies that approximate observability in infinite-time of the pair (C, A) on X_α is equivalent to $\phi_{n,\alpha}(\eta) \neq 0, n \geq 1$. This is also equivalent to $\phi_n(\eta) \neq 0, n \geq 1$ since $(\mu - \lambda_n) \neq 0, n \geq 1$. ∎

Remark 5.3. *For checking the condition $\phi_n(\eta) \neq 0$ in practical applications, we refer to the book De Coster and*

[7]By approximate observability in infinite time, we mean the approximate observability on $[0, \tau]$ for all $\tau > 0$, see (Curtain and Zwart, 2020, Corollary 6.2.26).

[8]By mixed boundary conditions, we mean $\alpha_a \neq 0 \neq \beta_a, \alpha_b \neq 0 \neq \beta_b, \frac{\beta_a}{\alpha_a} < 0, \frac{\beta_b}{\alpha_b} > 0$.

Habets (2006) in which lower and upper solutions of two-point boundary value problems are described and characterized. With a focus on Sturm-Liouville systems, we also refer to Orlov (2017) in which estimations on the eigenfunctions of the operator A are performed.

6. KF for Sturm-Liouville systems with pointwise measurement

The Sturm-Liouville systems introduced in Section 5 are considered here on the Hilbert space $X_{\frac{1}{2}}$ with a pointwise measurement observation operator $C : X_{\frac{1}{2}} \to \mathbb{R}, Cx = x(\eta), \eta \in [a,b]$. For such systems, the assumptions needed to apply the spectral factorization method for the Kalman filtering problem described in Section 3 will be checked in such a way that the method can be applied directly to compute the optimal output injection. We emphasize that working on $X_{\frac{1}{2}}$ has the great advantage of having the operator C bounded, which would not be the case in $L^2(a,b)$ for instance. In such a space, a corresponding Riccati equation with unbounded operator C would not have any meaning. In such a case, one could possibly use an approach that would be "dual" to the one analyzed in Weiss and Weiss (1997), but that would be much more involved, theoretically and computationally.

6.1. Applicability of the spectral factorization method by symmetric extraction

First, observe that, according to Corollary 5.1, a Sturm-Liouville operator $A = -\mathcal{A}$ of the form (28) is a self-adjoint Riesz-spectral operator on $X_{\frac{1}{2}} := D((\mu I - A)^{\frac{1}{2}})$ for some $\mu \geq 0$ such that $\sigma(\mu I - A) \subset \mathbb{R}_0^+$. The corresponding (simple) eigenvalues and eigenfunctions on $X_{\frac{1}{2}}$ are denoted by $\{\lambda_n\}_{n\geq 1}$ and $\{\phi_n\}_{n\geq 1}$, respectively. This has the consequence that Assumption 4.1 is satisfied. The point $\eta \in [a,b]$ at which the observation is made is chosen such that $\phi_n(\eta) \neq 0$ for all $n \geq 1$. As it is highlighted in Section 5.3, the space $X_{\frac{1}{2}}$ is a linear subspace of the Sobolev space $H^1(0,1)$ which entails that the pointwise observation at $\eta \in [a,b]$ is represented by a bounded linear operator $Cx = \langle c_\eta, x\rangle_{\frac{1}{2}}$ for some $c_\eta \in X_{\frac{1}{2}}$ that is independent of x. Moreover, the disturbance operator Q is chosen as $Q = C^* = c_\eta$. Hence, the assumption that $\phi_n(\eta) \neq 0$ for all $n \geq 1$ implies that the pairs (C, A) and (A, Q) are approximately observable and approximately controllable in infinite-time, which makes Assumption 4.2 satisfied. The last assumption that is needed, namely Assumption 4.3, follows from the asymptotic behavior of the eigenvalues of a Sturm-Liouville operator $\mathcal{O}(\lambda_n^{-1}) = \mathcal{O}(\frac{1}{n^2})$, see e.g. (Orlov, 2017, Section II). The following proposition highlights the applicability of the symmetric extraction method to the spectral factorization for Sturm-Liouville systems.

Proposition 6.1. *Let us consider the infinite-dimensional system (2) together with the Kalman filter (7). Moreover let us consider that the opposite of the operator A has the form (28) – (29). Its eigenvalues and its orthonormal eigenfunctions are given by $\{\lambda_n\}_{n\geq 1}$ and $\{\phi_n\}_{n\geq 1}$, respectively. The associated state space is $X_{\frac{1}{2}} =: D((\mu I - A)^{\frac{1}{2}})$ where $\mu \geq 0$ is such that $\sigma(\mu I - A) \subset \mathbb{R}_0^+$. The operator $C : X_{\frac{1}{2}} \to \mathbb{R}$ is the point evaluation at $\eta \in [a,b]$, i.e. $Cx = x(\eta) =: \langle c_\eta, x\rangle_{\frac{1}{2}}$ for all $x \in X_{\frac{1}{2}}$ and some $c_\eta \in X_{\frac{1}{2}}$, where η is chosen such that $\phi_n(\eta) \neq 0$ for all $n \geq 1$. The operator Q is taken as $Q = C^*$. Moreover, let us consider an operator $L_0 : \mathbb{R} \to X_{\frac{1}{2}}$ given by $L_0 c = \ell_0 c$ for some function $\ell_0 \in X_{\frac{1}{2}}$ such that the spectrum of the operator $A + \ell_0\langle c_\eta, \cdot\rangle_{\frac{1}{2}}$ is real and is located in $\mathbb{C}_0^-$. Then the symmetric extraction method for solving the spectral factorization problem in order to find the optimal output injection is convergent in the sense of Remark 4.1.α. Moreover, the components of the optimal output injection $L =: \ell$ in the orthonormal basis $\{\phi_n\}_{n\geq 1}$ of $X_{\frac{1}{2}}$, which is the solution of the Kalman filter problem of Subsection 3.1, are given by*

$$\ell_n = \frac{-\mathfrak{Res}(\hat{D}^{-1}(s)\hat{R}(s);\lambda_n)}{\phi_n(\eta)}, \tag{47}$$

where $\hat{D}(s) = 1 + [(sI - A - \ell_0\langle c_\eta, \cdot\rangle_{\frac{1}{2}})^{-1}\ell_0](\eta)$ and the standard spectral factor $\hat{R}(s)$ is given by the infinite-product representation

$$\hat{R}(s) = \prod_{k=1}^{\infty} \frac{s - \mu_k}{s - \beta_k} \tag{48}$$

with $\{\mu_n\}_{n\geq 1}$ and $\{\beta_n\}_{n\geq 1}$ being the stable zeros of the coercive spectral density $\hat{F} = \hat{N}\hat{N}_* + \hat{D}\hat{D}_*$, where $\hat{N}(s) = \langle c_\eta, (sI - A - \ell_0\langle c_\eta, \cdot\rangle_{\frac{1}{2}})^{-1}c_\eta\rangle_{\frac{1}{2}} = [(sI - A - \ell_0\langle c_\eta, \cdot\rangle_{\frac{1}{2}})^{-1}c_\eta](\eta)$, and the eigenvalues of the Riesz-spectral operator $A + \ell_0\langle c_\eta, \cdot\rangle_{\frac{1}{2}}$, respectively.

Proof. As it has been explained above in this section, the Sturm-Liouville nature of the operator A together with the fact that $\phi_n(\eta) \neq 0, n \geq 1$, implies that Assumptions 4.1, 4.2 and 4.3 are satisfied. Hence, Proposition 4.2 implies that the spectral factor $\hat{R}(s)$ admits the infinite-product representation (48). Note that, thanks to the expression of the operator $L_0 = \ell_0$ and Assumption 4.3, the operator $A + \ell_0\langle c_\eta, \cdot\rangle_{\frac{1}{2}}$ is a Riesz-spectral operator whose spectrum is composed only of the eigenvalues $\{\beta_n\}_{n\geq 1}$. Then, Proposition 4.1 has the consequence that the components $\{\ell_n\}_{n\geq 1}$ are given by (47). ■

6.2. Case study

In order to show the efficiency of Proposition 6.1, we consider the following illustrative example. The operator A is chosen as the one-dimensional diffusion operator on the interval $[0,1]$ with mixed boundary conditions: $Ax = \frac{d^2x}{dz^2}$ with $D(A) = \{x \in H^2(0,1), \frac{dx}{dz}(0) = \chi_0 x(0), \frac{dx}{dz}(1) = -\chi_1 x(1)\}, \chi_0, \chi_1 > 0$. It is easy to see that the opposite of A may be written as (28) – (29) with

$\rho(z) = 1, p(z) = 1, q(z) = 0$ on $[a, b] = [0, 1]$. As an additional feature, one may observe that the operator A is self-adjoint on $L^2(0, 1) =: X$ and its opposite is coercive. A spectral analysis on $L^2(0, 1)$ reveals that the spectrum of A, composed only of simple eigenvalues is given by

$$\sigma(A) = \{-s_n^2\}_{n\geq 1} =: \{\lambda_n\}_{n\geq 1}, \tag{49}$$

where the coefficients $\{s_n\}_{n\geq 1}$ are the positive solutions of the resolvent equation

$$\tan(s_n) = \frac{(\chi_0 + \chi_1)s_n}{s_n^2 - \chi_0\chi_1}. \tag{50}$$

The eigenfunctions of the operator A are given by

$$\{\tilde{\phi}_n(z)\}_{n\geq 1} = \left\{ B_n \left[\frac{s_n}{\chi_0} \cos(s_n z) + \sin(s_n z) \right] \right\}_{n\geq 1}, \tag{51}$$

where $\{B_n\}_{n\geq 1}$ are such that $\|\tilde{\phi}_n\|^2_{L^2(0,1)} = 1$ for all $n \geq 1$. According to Section 5, this setting allows to consider the Hilbert spaces given by $X_\alpha = D((-A)^\alpha), \alpha > 0$. As a particular Hilbert space, one shall consider the space $X_{\frac{1}{2}}$. That space satisfies $X_{\frac{1}{2}} = H^1(0, 1)$ and is equipped with the inner product (46) where $\mu = 0$, i.e.

$$\langle f, g\rangle_{\frac{1}{2}} = \int_0^1 \frac{df}{dz}\frac{dg}{dz}dz + \chi_0 f(0)g(0) + \chi_1 f(1)g(1), \tag{52}$$

for all $f, g \in X_{\frac{1}{2}}$. According to Proposition 5.1, the set

$$\{\phi_n\}_{n\geq 1} = \left\{ \frac{\tilde{\phi}_n}{s_n} \right\}_{n\geq 1} \tag{53}$$

is an orthonormal basis of $X_{\frac{1}{2}}$. Moreover, the operator A is a self-adjoint Riesz-spectral operator on $X_{\frac{1}{2}}$ with eigenvalues $\{\lambda_n\}_{n\geq 1}$ and eigenfunctions $\{\phi_n\}_{n\geq 1}$, see Theorem 5.1. As operator C on $X_{\frac{1}{2}}$, we consider the pointwise evaluation at $z = 0$, i.e. $C : X_{\frac{1}{2}} \to \mathbb{R}, Cx = x(0)$. As it is highlighted in Section 5.3, $X_{\frac{1}{2}}$ is a reproducing kernel Hilbert space, which has the consequence that the pointwise evaluation is a bounded linear functional represented by a unique element of $X_{\frac{1}{2}}$, i.e. $Cx = \langle c_0, x\rangle_{\frac{1}{2}}$ for all $x \in X_{\frac{1}{2}}$ and some function $c_0 \in X_{\frac{1}{2}}$. In order to get an explicit expression for the function c_0, observe that by looking for $c_0 \in D(A)$, there holds

$$\langle c_0, x\rangle_{\frac{1}{2}} = \langle (-A)^{\frac{1}{2}} c_0, (-A)^{\frac{1}{2}} x\rangle_X = \langle (-A)c_0, x\rangle_X,$$

for all $x \in X_{\frac{1}{2}}$. Since the function c_0 produces the Dirac delta effect on $X_{\frac{1}{2}}$, one gets that $\langle (-A)c_0, x\rangle_X = \langle \delta_0, x\rangle_X$, where δ_0 is the Dirac delta distribution at 0. Hence, the function c_0 is formally computed as $c_0(z) = ((-A)^{-1}\delta_0)(z)$. This implies that c_0 is the restriction to $[0, 1]$ of the function

$$\tilde{c}_0(z) = \begin{cases} \frac{(1+\chi_1)(1+\chi_0 z)}{\chi_0+\chi_1+\chi_0\chi_1}, \text{ if } z \in \left[-\frac{1}{\chi_0}, 0\right], \\ \frac{1+\chi_1}{\chi_0+\chi_1+\chi_0\chi_1} - \frac{\chi_1}{\chi_0+\chi_1+\chi_0\chi_1} z, \text{ if } z \in \left[0, 1 + \frac{1}{\chi_1}\right]. \end{cases} \tag{54}$$

Note that, in this context, the derivative of c_0 at $z = 0$ should be interpreted as $\frac{dc_0}{dz}(0) = \frac{d\tilde{c}_0}{dz}(0^-)$. According to Subsection 5.3 and to (53), the pair (C, A) is approximately observable in infinite-time since $\phi_n(0) = \frac{B_n}{\chi_0} \neq 0$ for all $n \geq 1$. Now the transfer function $\hat{g}(s) = \langle c_0, (sI - A)^{-1}c_0\rangle_{\frac{1}{2}}$ is given by

$$\hat{g}(s) = \frac{\hat{n}(s)}{\hat{d}(s)}, \tag{55}$$

where $\hat{n}$ and $\hat{d}$ are given by

$$\hat{n}(s) = (\nu - \mu\chi_1)\left(\frac{1 - \cosh(\sqrt{s})}{s}\right) + \mu\frac{\sinh(\sqrt{s})}{\sqrt{s}} + \nu\chi_1\left(\frac{\sqrt{s} - \sinh(\sqrt{s})}{s\sqrt{s}}\right) \tag{56}$$

and

$$\hat{d}(s) = \frac{\sqrt{s}(\chi_0 + \chi_1)\cosh(\sqrt{s}) + (s + \chi_0\chi_1)\sinh(\sqrt{s})}{\sqrt{s}}, \tag{57}$$

respectively, where μ and ν are expressed as $\mu = \frac{1+\chi_1}{\chi_0+\chi_1+\chi_0\chi_1}, \nu = \frac{\chi_1}{\chi_0+\chi_1+\chi_0\chi_1}$, respectively. It is easy to see that the denominator $\hat{d}(s)$ has no poles and that its stable zeros are expressed as $\lambda_n = -s_n^2$ where $\{s_n\}_{n\geq 1}$ are solutions of the resolvent equation (50). Moreover, observe that the numerator $\hat{n}(s)$ has no poles. Thus, looking for the stable zeros of the coercive spectral density $\hat{F}(s) = 1 + \hat{g}(s)\hat{g}(-s)$ is equivalent to looking for the stable zeros of the function $\hat{n}(s)\hat{n}(-s) + \hat{d}(s)\hat{d}(-s)$. Writing these zeros as $\mu_n = -\beta_n^2, n \geq 1$ entails that $\{\beta_n\}_{n\geq 1}$ are the solutions of the following equation

$$\hat{n}(-\beta_n^2)\hat{n}(\beta_n^2) + \hat{d}(-\beta_n^2)\hat{d}(\beta_n^2) = 0. \tag{58}$$

As an illustration, we consider $\chi_0 = \frac{1}{2} = \chi_1$. The optimal output injection solution of the Kalman filter problem computed via the symmetric extraction method by means of (47) – (48) is depicted in Figure 1. Therein, a comparison is made with the optimal output injection computed numerically via the Hamiltonian operator. The solutions of the resolvent equation (50) are depicted in Figure 2 whereas the coefficients $\{\beta_n\}_{n\geq 1}$ solving (58) are represented in 3. In those two figures, the coefficients $s_n, n \geq 1$ and $\beta_n, n \geq 1$ have to be seen as intersections of the curves $\sin(s_n)$ and $\cos(s_n)(\chi_0 + \chi_1)s_n/(s_n^2 - \chi_0\chi_1)$ (for the s_n's) and $\hat{n}(-\beta_n^2)\hat{n}(\beta_n^2)/\hat{d}(\beta_n^2)$ and $-\hat{d}(-\beta_n^2)$ (for the β_n's), respectively. Figure 4 shows the absolute difference between the eigenvalues of the operator A and those of the operator $A + \ell\langle c_0, \cdot\rangle_{\frac{1}{2}}$, namely $\lambda_n = -s_n^2$ and $\mu_n = -\beta_n^2, n \geq 1$, respectively. The convergence towards 0 may be observed as is explained in (Winkin et al., 2004, Lemma 4.6).

7. Conclusion and perspectives

In this paper, the spectral factorization method originally introduced to solve linear-quadratic optimal control

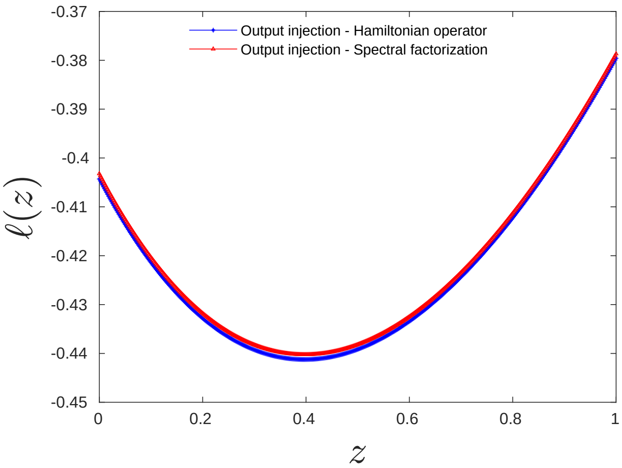

Figure 1: Comparison between the optimal output injection obtained via the spectral factorization by symmetric extraction and via the Hamiltonian operator.

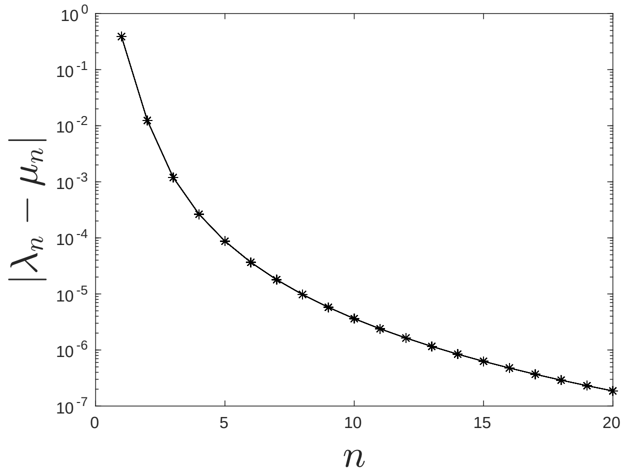

Figure 4: Distance between the eigenvalues of the operator A and the eigenvalues of the operator $A + \ell\langle c_0, \cdot\rangle_{\frac{1}{2}}$.

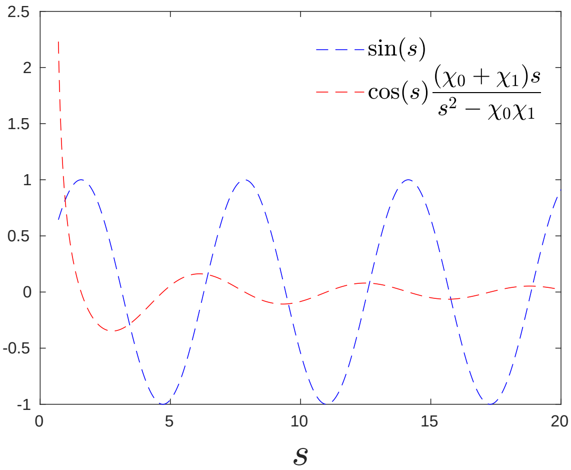

Figure 2: Solutions of the resolvent equation (50).

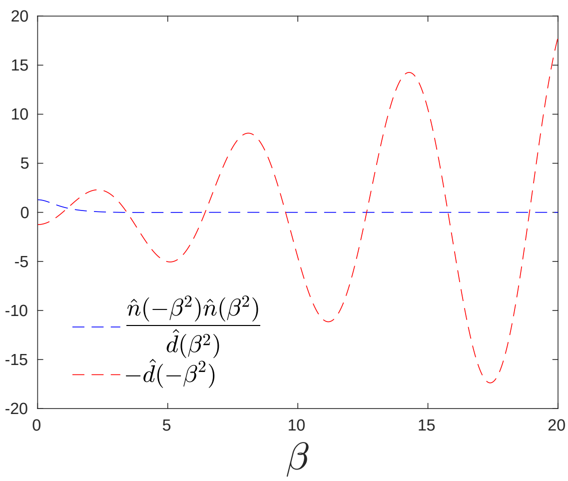

Figure 3: Solutions of equation (58).

problems has been extended to the Kalman filtering of linear infinite-dimensional systems. For a specific class of Riesz-spectral systems, it has been shown how to solve this problem by symmetric extraction based on the infinite product representation of a coercive spectral density.

It was also proved that a class of Sturm-Liouville operators satisfies the Riesz-spectral property on a class of Hilbert spaces expressed as the domains of positive powers of some self-adjoint positive operators. Moreover, the advantage of such an approach was highlighted by establishing in Theorem 5.2 a simple way of checking the approximate observability via some observation operator that was originally undefined by considering $L^2(0,1)$ as state space.

As an additional contribution, it was shown that the considered class of Sturm-Liouville operators with boundary observation was conducive to the symmetric extraction method for the resolution of the spectral factorization method. Then, the new results have been illustrated on a particular case study to show the efficiency of the approach.

As future research, we think of the investigation of the properties of the operator A on the domain of the negative powers of $\mu I - A$. At first glance, this could allow to treat the question of approximate controllability in infinite time for originally unbounded control operators, such as the Dirac delta distribution. Simple modal tests for this question could be expected. This could open the path to solve efficiently linear-quadratic optimal control problems for systems with originally unbounded control operators.

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systems theory and in particular for his famous algebra of transfer functions that is used in this paper.

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