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IS MATHEMATICAL HISTORY WRITTEN BY THE VICTORS?

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Abstract. We examine prevailing philosophical and historical views about the origin of infinitesimal mathematics in light of modern infinitesimal theories, and show the works of Fermat, Leibniz, Euler, Cauchy and other giants of infinitesimal mathematics in a new light. We also detail several procedures of the historical infinitesimal calculus that were only clarified and formalized with the advent of modern infinitesimals. These procedures include Fermat's adequality; Leibniz's law of continuity and the transcendental law of homogeneity; Euler's principle of cancellation and infinite integers with the associated infinite products; Cauchy's "Dirac" delta function. Such procedures were interpreted and formalized in Robinson's framework in terms of concepts like the standard part principle, the transfer principle, and hyperfinite products. We evaluate the critiques of historical and modern infinitesimals by their foes from Berkeley and Cantor to Bishop and Connes. We analyze the issue of the consistency, as distinct from the issue of the rigor, of historical infinitesimals, and contrast the methodologies of Leibniz and Nieuwentijt in this connection.

Contents

Key words and phrases. Adequality; Archimedean axiom; infinitesimal; law of continuity; mathematical rigor; standard part principle; transcendental law of homogeneity; variable quantity.

1. The ABC's of the history of infinitesimal mathematics

The ABCs of the history of infinitesimal mathematics are in need of clarification. To what extent does the famous dictum "history is always written by the victors" apply to the history of mathematics, as well? A convenient starting point is a remark made by Felix Klein in his book *Elementary mathematics from an advanced standpoint* (Klein [75, p. 214]). Klein wrote that there are not one but two separate tracks for the development of analysis:

(A) the Weierstrassian approach (in the context of an *Archimedean* continuum); and

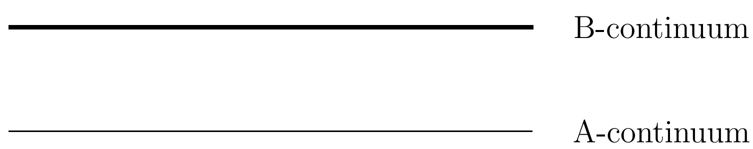

FIGURE 1. Parallel tracks: a thick continuum and a thin continuum

(B) the approach with indivisibles and/or infinitesimals (in the context of what we will refer to as a *Bernoullian* continuum).[1]

Klein's sentiment was echoed by the philosopher G. Granger, in the context of a discussion of Leibniz, in the following terms:

> Aux yeux des détracteurs de la nouvelle Analyse, l'insurmontable difficulté vient de ce que de telles pratiques font violence aux règles ordinaires de l'Algèbre, tout en conduisant à des résultats, exprimables en termes finis, dont on ne saurait contester l'exactitude. Nous savons aujourd'hui que deux voies devaient s'offrir pour la solution du problème:
>
> [A] Ou bien l'on élimine du langage mathématique le terme d'infiniment petit, et l'on établit, en termes finis, le sens à donner à la notion intuitive de 'valeur limite'. . . .
>
> [B] Ou bien l'on accepte de maintenir, tout au long du Calcul, la présence d'objets portant ouvertement la marque de l'infini, mais en leur conférant un statut propre qui les insère dans un système dont font aussi partie les grandeurs finies. . . .
>
> C'est dans cette seconde voie que les vues philosophiques de Leibniz l'ont orienté (Granger 1981 [43, pp. 27-28]).[2]

Thus we have two parallel tracks for conceptualizing infinitesimal calculus, as shown in Figure 1.

At variance with Granger's appraisal, some of the literature on the history of mathematics tends to assume that the A-approach is the ineluctably "true" one, while the infinitesimal B-approach was, at best, a kind of evolutionary dead-end or, at worst, altogether inconsistent. To

[1]Systems of quantities encompassing infinitesimal ones were used by Leibniz, Bernoulli, Euler, and others. Our choice of the term is explained in entry 2.5. It *encompasses* modern non-Archimedean systems.

[2]Similar views were expressed by M. Parmentier in (Leibniz 1989 [83, p. 36, note 92]).

say that infinitesimals provoked passions would be an understatement. Parkhurst and Kingsland, writing in *The Monist*, proposed applying a *saline solution* (if we may be allowed a pun) to the problem of the infinitesimal:

> [S]ince these two words [infinity and infinitesimal] have sown nearly as much faulty logic in the fields of mathematics and metaphysics as all other fields put together, they should be rooted out of both the fields which they have contaminated. And not only should they be rooted out, lest more errors be propagated by them: *a due amount of salt* should be ploughed under the infected territory, that the damage be mitigated as well as arrested (Parkhurst and Kingsland 1925 [96, pp. 633-634]) [emphasis added–the authors].

Writes P. Vickers:

> So entrenched is the understanding that the early calculus was inconsistent that many authors don't provide a reference to support the claim, and don't present the set of inconsistent propositions they have in mind. (Vickers 2013 [114, section 6.1])

Such an *assumption* of inconsistency can influence one's appreciation of historical mathematics, make a scholar myopic to certain significant developments due to their automatic placement in an "evolutionary dead-end" track, and inhibit potential fruitful applications in numerous fields ranging from physics to economics. One example is the visionary work of Enriques exploiting infinitesimals, recently analyzed in an article by David Mumford, who wrote:

> In my own education, I had assumed [that Enriques and the Italians] were irrevocably stuck. ... As I see it now, Enriques must be credited with a nearly complete geometric proof using, as did Grothendieck, higher order infinitesimal deformations. ... Let's be careful: he certainly had the correct ideas about infinitesimal geometry, though he had no idea at all how to make precise definitions (Mumford 2011 [93]).

Another example is important work by Cauchy (see entry 2.8 below) on singular integrals and Fourier series using infinitesimals and infinitesimally defined "Dirac" delta functions (these precede Dirac by a century), which was forgotten for a number of decades because of shifting foundational biases. The presence of Dirac delta functions in

Cauchy's oeuvre was noted in (Freudenthal 1971 [40]) and analyzed by Laugwitz (1989 [78]), (1992a [79]); see also (Katz & Tall 2012 [72]) and (Tall & Katz 2013 [112]).

Recent papers on Leibniz (Katz & Sherry [71], [70]; Sherry & Katz [105]) argue that, contrary to widespread perceptions, Leibniz's system for infinitesimal calculus was *not* inconsistent (see entry 4.5 on *mathematical rigor* for a discussion of the term). The significance and coherence of Berkeley's critique of infinitesimal calculus have been routinely exaggerated. Berkeley's sarcastic tirades against infinitesimals fit well with the ontological limitations imposed by the A-approach favored by many historians, even though Berkeley's opposition, on empiricist grounds, to an infinitely divisible continuum is profoundly at odds with the A-approach.

A recent study of Fermat (Katz, Schaps & Shnider 2013 [69]) shows how the nature of his contribution to the calculus was distorted in recent Fermat scholarship, similarly due to an "evolutionary dead-end" bias (see entry 3.7).

The Marburg school of Hermann Cohen, Cassirer, Natorp, and others explored the philosophical foundations of the infinitesimal method underpinning the mathematized natural sciences. Their versatile, and insufficiently known, contribution is analyzed in (Mormann & Katz 2013 [92]).

A number of recent articles have pioneered a re-evaluation of the history and philosophy of mathematics, analyzing the shortcomings of received views, and shedding new light on the deleterious effect of the latter on the philosophy, the practice, and the applications of mathematics. Some of the conclusions of such a re-evaluation are presented below.

2. Adequality to Chimeras

Some topics from the history of infinitesimals illustrating our approach appear below in alphabetical order.

2.1. **Adequality.**

Adequality is a technique used by Fermat to solve problems of tangents, problems of maxima and minima, and other variational problems. The term *adequality* derives from the $\pi\alpha\rho\iota\sigma\acute{o}\tau\eta\varsigma$ of Diophantus (see entry 3.2). The technique involves an element of approximation and "smallness", represented by a small variation E, as in the familiar difference $f(A+E)-f(A)$. Fermat used adequality in particular to find the tangents of transcendental curves such as the cycloid, that were considered to be "mechanical" curves off-limits to geometry, by Descartes. Fermat also used it to solve the variational problem of the refraction of light so as to obtain Snell's law. Adequality incorporated a procedure of discarding higher-order terms in E (*without* setting them equal to zero). Such a heuristic procedure was ultimately formalized mathematically in terms of the *standard part principle* (entry 5.3) in Robinson's theory of infinitesimals starting with (Robinson 1961 [99]). Fermat's adequality is comparable to Leibniz's *transcendental law of homogeneity* (entry 4.4).

2.2. **Archimedean axiom.** What is known today as the Archimedean axiom first appears in Euclid's *Elements*, Book V, as Definition 4 (Euclid [34], definition V.4). It is exploited in (Euclid [34], Proposition V.8). We include bracketed symbolic notation so as to clarify the definition:

> Magnitudes [a, b] are said to have a ratio with respect to one another which, being multiplied [na] are capable of exceeding one another [$na > b$].

It can be formalized as follows:[3]

$$(\forall a, b)(\exists n \in \mathbb{N})\,[na > b], \quad \text{where} \quad na = \underbrace{a + ... + a}_{n-times}. \tag{2.1}$$

Next, it appears in the papers of Archimedes as the following lemma (see Archimedes [2], I, Lamb. 5):

> Of unequal lines, unequal surfaces, and unequal solids [a, b, c], the greater exceeds the lesser [$a < b$] by such a magnitude [$b - a$] as, when added to itself [$n(b - a)$], can be made to exceed any assigned magnitude [c] among those which are comparable with one another (Heath 1897 [50, p. 4]).

This can be formalized as follows:

$$(\forall a, b, c)(\exists n \in \mathbb{N})\,[a < b \rightarrow n(b - a) > c]. \tag{2.2}$$

Note that Euclid's definition V.4 and the lemma of Archimedes are *not* logically equivalent (see entry 3.3, footnote 11).

[3]See e.g., the version of the Archimedean axiom in (Hilbert 1899 [54, p. 19]). Note that we have avoided using "0" in formula (2.1), as in "$\forall a > 0$", since 0 was not part of the conceptual framework of the Greeks. The term "multiplied" in the English translation of Euclid's definition V.4 corresponds to the Greek term $\pi o\lambda\lambda\alpha\pi\lambda\alpha\sigma\iota\alpha\zeta \acute{o}\mu\varepsilon\nu\alpha$. A common formalisation of the noun "multiple", $\pi o\lambda\lambda\alpha\pi\lambda \acute{\alpha}\sigma\iota o\nu$, is $na = a + ... + a$.

The Archimedean axiom plays no role in the plane geometry as developed in Books I-IV of *The Elements*.[4] Interpreting geometry in ordered fields, or in geometry over fields in short, one knows that $\mathbb{F}^2$ is a model of Euclid's plane, where $(\mathbb{F}, +, \cdot, 0, 1, <)$ is a Euclidean field, i.e., an ordered field closed under the square root operation. Consequently, $\mathbb{R}^* \times \mathbb{R}^*$ (where $\mathbb{R}^*$ is a hyperreal field) is a model of Euclid's plane, as well (see entry 4.6 on *modern implementations*). Euclid's definition V.4 is discussed in more detail in entry 3.3.

Otto Stolz rediscovered the Archimedean axiom for mathematicians, making it one of his axioms for magnitudes and giving it the following form: if $a > b$, then there is a multiple of b such that $nb > a$ (Stolz 1885 [111, p. 69]).[5] At the same time, in his development of the integers Stolz implicitly used the Archimedean axiom. Stolz's visionary realisation of the importance of the Archimedean axiom, and his work on non-Archimedean systems, stand in sharp contrast with Cantor's remarks on infinitesimals (see entry 4.5 on *mathematical rigor*).

In modern mathematics, the theory of ordered fields employs the following form of the Archimedean axiom (see e.g., Hilbert 1899 [54, p. 27]):

$$(\forall x > 0)\ (\forall \epsilon > 0)\ (\exists n \in \mathbb{N})\ [n\epsilon > x],$$

or equivalently

$$(\forall \epsilon > 0)\ (\exists n \in \mathbb{N})\ \ [n\epsilon > 1]. \tag{2.3}$$

A number system satisfying (2.3) will be referred to as an *Archimedean continuum*. In the contrary case, there is an element $\epsilon > 0$ called an infinitesimal such that no finite sum $\epsilon + \epsilon + \ldots + \epsilon$ will ever reach 1; in other words,

$$(\exists \epsilon > 0)\ (\forall n \in \mathbb{N})\ \ \left[\epsilon \leq \tfrac{1}{n}\right]. \tag{2.4}$$

A number system satisfying (2.4) is referred to as a *Bernoullian continuum* (i.e., a non-Archimedean continuum); see entry 2.5.

2.3. **Berkeley, George.** George Berkeley (1685-1753) was a cleric whose *empiricist* (i.e., based on sensations, or *sensationalist*) metaphysics tolerated no conceptual innovations, like infinitesimals, without an *empirical* counterpart or referent. Berkeley was similarly opposed, on metaphysical grounds, to infinite divisibility of the continuum (which he referred to as *extension*), an idea widely taken for granted

[4]With the exception of Proposition III.16 where so called *horn angles* appear, that could be considered as non-Archimedean magnitudes relative to rectilinear angles.

[5]See Ehrlich [32] for additional historical details concerning Stolz's account of the Archimedean Axiom.

today. In addition to his outdated *metaphysical criticism* of the infinitesimal calculus of Newton and Leibniz, Berkeley also formulated a *logical criticism*.[6] Berkeley claimed to have detected a logical fallacy at the basis of the method. In terms of Fermat's E occuring in his *adequality* (entry 2.1), Berkeley's objection can be formulated as follows:

> The increment E is assumed to be nonzero at the beginning of the calculation, but zero at its conclusion, an apparent logical fallacy.

However, E is not *assumed to be zero* at the end of the calculation, but rather is *discarded* at the end of the calculation (see entry 2.4 for more details). Such a technique was the content of Fermat's *adequality* (see entry 2.1) and Leibniz's *transcendental law of homogeneity* (see entry 4.4), where the relation of equality has to be suitably interpreted (see entry 5.2 on relation $\sqcap$). The technique is equivalent to taking the *limit* (of a typical expression such as $\frac{f(A+E)-f(A)}{E}$ for example) in Weierstrass's approach, and to taking the *standard part* (see entry 5.3) in Robinson's approach.

Meanwhile, Berkeley's own attempt to explain the calculation of the derivative of x^2 in *The Analyst* contains a logical circularity. Namely, Berkeley's argument relies on the determination of the tangents of a parabola by Apollonius (which is eqivalent to the calculation of the derivative). This circularity in Berkeley's argument was analyzed in (Andersen 2011 [1]).

2.4. **Berkeley's logical criticism.** Berkeley's *logical criticism* of the calculus amounts to the contention that the evanescent increment is first assumed to be non-zero to set up an algebraic expression, and then *treated as zero* in *discarding* the terms that contained that increment when the increment is said to vanish. In modern terms, Berkeley was claiming that the calculus was based on an inconsistency of type

$$(dx \neq 0) \wedge (dx = 0).$$

The criticism, however, involves a misunderstanding of Leibniz's method. The rebuttal of Berkeley's criticism is that the evanescent increment need *not* be "treated as zero", but, rather, is merely *discarded* through an application of the *transcendental law of homogeneity* by Leibniz, as illustrated in entry 5.1 in the case of the product rule.

While consistent (in the sense of entry 4.5, level (2)), Leibniz's system unquestionably relied on *heuristic* principles such as the laws of

[6]Berkeley's criticism was dissected into its logical and metaphysical components in (Sherry 1987 [103]).

continuity and homogeneity, and thus fell short of a standard of rigor *if* measured by today's criteria (see entry 4.5 on *mathematical rigor*). On the other hand, the consistency and resilience of Leibniz's system is confirmed through the development of *modern implementations* of Leibniz's heuristic principles (see entry 4.6).

2.5. **Bernoulli, Johann.** Johann Bernoulli (1667-1748) was a disciple of Leibniz's who, having learned an infinitesimal methodology for the calculus from the master, never wavered from it. This is in contrast to Leibniz himself, who, throughout his career, used both

(A) an Archimedean methodology (proof by exhaustion), and
(B) an infinitesimal methodology,

in a symbiotic fashion. Thus, Leibniz relied on the A-methodology to underwrite and justify the B-methodology, and he exploited the B-methodology to shorten the path to discovery (*Ars Inveniendi*). Historians often name Bernoulli as the first mathematician to have adhered systematically to the infinitesimal approach as the basis for the calculus. We refer to an infinitesimal-enriched number system as a *B-continuum*, as opposed to an Archimedean *A-continuum*, i.e., a continuum satisfying the *Archimedean axiom* (see entry 2.2).

2.6. **Bishop, Errett.** Errett Bishop (1928-1983) was a mathematical constructivist who, unlike his fellow intuitionist[7] *Arend Heyting* (see entry 3.8), held a dim view of classical mathematics in general and Robinson's infinitesimals in particular. Discouraged by the apparent non-constructivity of his early work in functional analysis under P. Halmos, he believed to have found the culprit in the law of excluded middle (LEM), the key logical ingredient in every proof by contradiction. He spent the remaining 18 years of his life in an effort to expunge the reliance on LEM (which he dubbed "the principle of omniscience" in [11]) from analysis, and sought to define *meaning* itself in mathematics in terms of such LEM-extirpation.

Accordingly, he described classical mathematics as both a *debasement of meaning* (Bishop 1973 [13, p. 1]) and *sawdust* (Bishop 1973 [13, p. 14]), and did not hesitate to speak of both *crisis* (Bishop 1975

[7]Bishop was not an Intuitionist in the narrow sense of the term, in that he never worked with Brouwer's continuum or "free choice sequences". We are using the term "Intuitionism" in a broader sense (i.e., mathematics based on intuitionistic logic) that incorporates constructivism, as used for example by Abraham Robinson in the comment quoted at the end of this entry.

[11]) and *schizophrenia* (Bishop 1973 [13]) in contemporary mathematics, predicting an imminent demise of classical mathematics in the following terms:

> Very possibly classical mathematics will cease to exist as an independent discipline (Bishop 1968 [10, p. 54]).

His attack in (Bishop 1977 [12]) on calculus pedagogy based on Robinson's infinitesimals was a natural outgrowth of his general opposition to the logical underpinnings of classical mathematics, as analyzed in (Katz & Katz 2011 [66]). Robinson formulated a brief but penetrating appraisal of Bishop's ventures into the history and philosophy of mathematics, when he noted that

> The sections of [Bishop's] book that attempt to describe the philosophical and historical background of [the] remarkable endeavor [of Intuitionism] are more vigorous than accurate and tend to belittle or ignore the efforts of others who have worked in the same general direction (Robinson 1968 [100, p. 921]).

See entry 2.9 for a related criticism by Alain Connes.

2.7. **Cantor, Georg.** Georg Cantor (1845-1918) is familiar to the modern reader as the *underappreciated* creator of the "Cantorian paradise" which David Hilbert would not be expelled out of, as well as the tragic hero, allegedly *persecuted* by Kronecker, who ended his days in a lunatic asylum. Cantor historian J. Dauben notes, however, an *underappreciated* aspect of Cantor's scientific activity, namely his principled *persecution* of infinitesimalists:

> Cantor devoted some of his most vituperative correspondence, as well as a portion of the *Beiträge*, to attacking what he described at one point as the 'infinitesimal Cholera bacillus of mathematics', which had spread from Germany through the work of Thomae, du Bois Reymond and Stolz, to infect Italian mathematics ... Any acceptance of infinitesimals necessarily meant that his own theory of number was incomplete. Thus to accept the work of Thomae, du Bois-Reymond, Stolz and Veronese was to deny the perfection of Cantor's own creation. Understandably, Cantor launched a thorough campaign to discredit Veronese's work in every way possible (Dauben 1980 [27, pp. 216-217]).

A discussion of Cantor's flawed investigation of the *Archimedean axiom* (see entry 2.2) may be found in entry 4.5 on *mathematical rigor*.[8]

2.8. **Cauchy, Augustin-Louis.** Augustin-Louis Cauchy (1789-1857) is often viewed in the history of mathematics literature as a precursor of Weierstrass. Note, however, that contrary to a common misconception, Cauchy never gave an ϵ, δ definition of either limit or continuity (see entry 5.4 on *variable quantity* for Cauchy's definition of limit). Rather, his approach to continuity was via what is known today as *microcontinuity* (see entry 3.1). Several recent articles (Błaszczyk et al. [14]; Borovik & Katz [16]; Bråting [20]; Katz & Katz [65], [67]; Katz & Tall [72]), have argued that a proto-Weierstrassian view of Cauchy is one-sided and obscures Cauchy's important contributions, including not only his infinitesimal definition of continuity but also such innovations as his infinitesimally defined ("Dirac") delta function, with applications in Fourier analysis and evaluation of singular integrals, and his study of orders of growth of infinitesimals that anticipated the work of Paul du Bois-Reymond, Borel, Hardy, and ultimately Skolem ([107], [108], [109]) and Robinson.

To elaborate on Cauchy's "Dirac" delta function, note the following formula from (Cauchy 1827 [23, p. 188]) in terms of an infinitesimal α:

$$\frac{1}{2}\int_{a-\epsilon}^{a+\epsilon} F(\mu)\frac{\alpha\, d\mu}{\alpha^2+(\mu-a)^2} = \frac{\pi}{2}F(a). \tag{2.5}$$

Replacing Cauchy's expression $\frac{\alpha}{\alpha^2+(\mu-a)^2}$ by $\delta_a(\mu)$, one obtains Dirac's formula up to trivial modifications (see Dirac [30, p. 59]):

$$\int_{-\infty}^{\infty} f(x)\delta(x) = f(0)$$

Cauchy's 1853 paper on a notion closely related to uniform convergence was recently examined in (Katz & Katz 2011 [65]) and (Błaszczyk et al. 2012 [14]). Cauchy handles the said notion using infinitesimals, including one generated by the null sequence $(\frac{1}{n})$.

Meanwhile, Núñez et al. (1999 [95, p. 54]) coined the term 'Cauchy–Weierstrass definition of continuity'. Since Cauchy gave an infinitesimal definition and Weierstrass, an ϵ, δ one, such a coinage is an oxymoron. J. Gray (2008a [45, p. 62]) lists continuity among concepts Cauchy allegedly defined using 'limiting arguments', but Gray unfortunately confuses the term 'limit' as *bound* with 'limit' as in *variable tending to a quantity*, since the term 'limits' appear in Cauchy's definition only

[8]Cantor's dubious claim that the infinitesimal leads to contradictions was endorsed by no less an authority than B. Russell; see footnote 15 in entry 4.5.

in the sense of endpoints (bounds) of an interval. Not to be outdone, Kline (1980 [76, p. 273]) claims that "Cauchy's work not only banished [infinitesimals] but disposed of any need for them." Hawking (2007 [48, p. 639]) does reproduce Cauchy's infinitesimal definition, yet on the same page 639 claims that Cauchy "was particularly concerned to banish infinitesimals," apparently unaware of a comical non-sequitur he committed.

2.9. **Chimeras.** Alain Connes (1947–) formulated criticisms of Robinson's infinitesimals between the years 1995 and 2007, on at least seven separate occasions (see Kanovei et al. 2012 [60], Section 3.1, Table 1). These range from pejorative epithets such as "inadequate", "disappointing", "chimera", and "irremediable defect", to "the end of the rope for being 'explicit'".

Connes sought to exploit the Solovay model $\mathcal{S}$ (Solovay 1970 [110]) as ammunition against non-standard analysis, but the model tends to boomerang, undercutting Connes' own earlier work in functional analysis. Connes described the hyperreals as both a "virtual theory" and a "chimera", yet acknowledged that his argument relies on the transfer principle (see entry 4.6). In $\mathcal{S}$, all definable sets of reals are Lebesgue measurable, suggesting that Connes views a theory as being "virtual" if it is not *definable* in a suitable model of ZFC. If so, Connes' claim that a theory of the hyperreals is "virtual" is refuted by the existence of a definable model of the hyperreal field (Kanovei & Shelah [62]). Free ultrafilters aren't definable, yet Connes exploited such ultrafilters both in his own earlier work on the classification of factors in the 1970s and 80s, and in his magnum opus *Noncommutative Geometry* (Connes 1994 [26, ch. V, sect. 6.δ, Def. 11]), raising the question whether the latter may not be vulnerable to Connes' criticism of virtuality. The article [60] analyzed the philosophical underpinnings of Connes' argument based on Gödel's incompleteness theorem, and detected an apparent circularity in Connes' logic. The article [60] also documented the reliance on non-constructive foundational material, and specifically on the Dixmier trace $⨍$ (featured on the front cover of Connes' *magnum opus*) and the Hahn–Banach theorem, in Connes' own framework; see also [68].

See entry 2.6 for a related criticism by Errett Bishop.

3. Continuity to indivisibles

3.1. **Continuity.** Of the two main definitions of continuity of a function, Definition A (see below) is operative in either a B-continuum or an A-continuum (satisfying the Archimedean axiom; see entry 2.2), while Definition B only works in a B-continuum (i.e., an infinitesimal-enriched or Bernoullian continuum; see entry 2.5).

- *Definition A* (ϵ, δ approach): A real function f is continuous at a real point x if and only if
$$(\forall \epsilon > 0)\ (\exists \delta > 0)\ (\forall x')\ \big[\, |x - x'| < \delta \rightarrow |f(x) - f(x')| < \epsilon \,\big].$$
- *Definition B* (microcontinuity): A real function f is continuous at a real point x if and only if
$$(\forall x')\ \big[\, x' \sqcap x \ \rightarrow\ f(x') \sqcap f(x) \,\big]. \tag{3.1}$$

In formula (3.1), the natural extension of f is still denoted f, and the symbol "$\sqcap$" stands for the relation of being infinitely close; thus, $x' \sqcap x$ if and only if $x' - x$ is infinitesimal (see entry 5.2 on relation "$\sqcap$").

3.2. **Diophantus.** Diophantus of Alexandria (who lived about 1800 years ago) contributed indirectly to the development of infinitesimal calculus through the technique called $\pi\alpha\rho\iota\sigma\acute{o}\tau\eta\varsigma$, developed in his work *Arithmetica*, Book Five, problems 12, 14, and 17. The term $\pi\alpha\rho\iota\sigma\acute{o}\tau\eta\varsigma$ can be literally translated as "approximate equality". This was rendered as *adaequalitas* in Bachet's Latin translation [4], and *adégalité* in French (see entry 2.1 on *adequality*). The term was used by Fermat to describe the comparison of values of an algebraic expression, or what would today be called a function f, at nearby points A and $A + E$, and to seek extrema by a technique equivalent to the vanishing of $\frac{f(A+E)-f(A)}{E}$ after discarding the remaining E-terms; see (Katz, Schaps & Shnider 2013 [69]).

3.3. **Euclid's definition V.4.** Euclid's definition V.4 was already discussed in entry 2.2. In addition to book V, it appears in Books X and XII and is used in the method of exhaustion (see Euclid [34], Propositions X.1, XII.2). The method of exhaustion was exploited intensively by both Archimedes and Leibniz (see entry 4.2 on Leibniz's work *De Quadratura*). It was revived in the 19th century in the theory of the Riemann integral.

Euclid's Book V sets the basis for the theory of similar figures developed in Book VI. Great mathematicians of the 17th century like Descartes, Leibniz, and Newton exploited Euclid's theory of similar figures of Book VI while paying no attention to its axiomatic background.[9] Over time Euclid's Book V became a subject of interest for historians and editors alone.

[9]Leibniz and Newton apparently applied Euclid's conclusions in a context where the said conclusions did not technically speaking apply, namely to infinitesimal figures such as the characteristic triangle, i.e., triangle with sides dx, dy, and ds.

To formalize Definition V.4, one needed a formula for Euclid's notion of "multiple" and an idea of total order. Some progress in this direction was made by Robert Simson in 1762.[10] In 1876, Hermann Hankel provided a modern reconstruction of Book V. Combining his own historical studies with an idea of order compatible with addition developed by Hermann Grassmann (1861 [44]), he gave a formula that to this day is accepted as a formalisation of Euclid's definition of proportion in V.5 (Hankel 1876 [49, pp. 389-398]). Euclid's proportion is a relation among four "magnitudes", such as

$$A : B :: C : D.$$

It was interpreted by Hankel as the relation

$$(\forall m, n) \left[(nA >_1 mB \to nC >_2 mD) \wedge \right.$$

$$\left. \wedge (nA = mB \to nC = mD) \wedge (nA <_1 mB \to nC <_2 mD) \right],$$

where n, m are natural numbers. The indices on the inequalities emphasize the fact that the "magnitudes" A, B have to be of "the same kind", e.g., line segments, whereas C, D could be of another kind, e.g., triangles.

In 1880, J. L. Heiberg in his edition of Archimedes' *Opera omnia*, in a comment on a lemma of Archimedes, cites Euclid's definition V.4, noting that these two are the same axioms (Heiberg 1880 [52, p. 11]).[11] This is the reason why Euclid's definition V.4 is commonly known as the Archimedean axiom. Today we formalize Euclid's definition V.4 as in (2.1), while the Archimedean lemma is rendered by formula (2.2).

3.4. **Euler, Leonhard.** Euler's *Introductio in Analysin Infinitorum* (1748 [35]) contains remarkable calculations carried out in an extended number system in which the basic algebraic operations are applied to infinitely small and infinitely large quantities. Thus, in Chapter 7, "Exponentials and Logarithms Expressed through Series", we find a derivation of the power series for a^z starting from the formula $a^\omega = 1 + k\omega$, for ω infinitely small and then raising the equation to the

[10] See Simson's axioms that supplement the definitions of Book V as elaborated in (Simson 1762 [106, p. 114-115]).

[11] In point of fact, Euclid's axiom V.4 and Archimedes' lemma are not equivalent from the logical viewpoint. Thus, the additive semigroup of positive appreciable limited hyperreals satisfies V.4 but not Archimedes' lemma.

infinitely great power[12] $j = \frac{z}{\omega}$ for a finite (appreciable) z to give

$$a^z = a^{j\omega} = (1 + k\omega)^j$$

and finally expanding the right hand side as a power series by means of the binomial formula. In the chapters following Euler finds infinite product expansions factoring the power series expansion for transcendental functions (see entry 3.5 for his infinite product formula for sine). By Chapter 10, he has the tools to sum the series for $\zeta(2)$ where $\zeta(s) = \sum_n n^{-s}$. He explicitly calculates $\zeta(2k)$ for $k = 1, \ldots, 13$ as well as many other related infinite series.

In Chapter 3 of his *Institutiones Calculi Differentialis* (1755 [37]), Euler deals with the methodology of the calculus, such as the nature of infinitesimal and infinitely large quantities. We will cite the English translation [38] of the Latin original [37]. Here Euler writes that

> [e]ven if someone *denies* that infinite numbers really exist in this world, still in mathematical speculations there arise questions to which answers cannot be given unless we admit an infinite number (ibid., § 82) [emphasis added–the authors].

Euler's approach, countenancing the possibility of *denying* that "infinite numbers really exist", is consonant with a Leibnizian view of infinitesimal and infinite quantities as "useful fictions" (see Katz & Sherry [71]; Sherry & Katz [105]). Euler then notes that "an infinitely small quantity is nothing but a vanishing quantity, and so it is really equal to 0" (ibid., § 83).

Similarly, Leibniz combined a view of infinitesimals as "useful fictions" and *inassignable quantities*, with a generalized notion of "equality" which was an equality up to an incomparably negligible term. Leibniz sought to codify this idea in terms of his *transcendental law of homogeneity* (TLH); see entry 4.4. Thus, Euler's formulas such as

$$a + dx = a \tag{3.2}$$

(where a "is any finite quantity"; ibid., §§ 86, 87) are consonant with a Leibnizian tradition (cf. formula (4.1) in entry 4.4). To explain formulas such as (3.2), Euler elaborated two distinct ways (arithmetic and geometric) of comparing quantities in the following terms:

> Since we are going to show that an infinitely small quantity is really zero, we must meet the objection of why

[12] Euler used the symbol i for the infinite power. Blanton replaced this by j in the English translation so as to avoid a notational clash with the standard symbol for $\sqrt{-1}$.

> we do not always use the same symbol 0 for infinitely small quantities, rather than some special ones… [S]ince we have two ways to compare them, either *arithmetic* or *geometric*, let us look at the quotients of quantities to be compared in order to see the difference.
>
> If we accept the notation used in the analysis of the infinite, then dx indicates a quantity that is infinitely small, so that both $dx = 0$ and $a\,dx = 0$, where a is any finite quantity. Despite this, the *geometric* ratio $a\,dx : dx$ is finite, namely $a : 1$. For this reason, these two infinitely small quantities, dx and $a\,dx$, both being equal to 0, cannot be confused when we consider their ratio. In a similar way, we will deal with infinitely small quantities dx and dy (ibid., § 86, p. 51-52) [emphasis added–the authors].

Euler proceeds to clarify the difference between the arithmetic and geometric comparisons as follows:

> Let a be a finite quantity and let dx be infinitely small. The arithmetic ratio of equals is clear: Since $ndx = 0$, we have
>
> $$a \pm ndx - a = 0.$$
>
> On the other hand, the geometric ratio is clearly of equals, since
>
> $$\frac{a \pm ndx}{a} = 1. \tag{3.3}$$
>
> From this we obtain the well-known rule that *the infinitely small vanishes in comparison with the finite and hence can be neglected* (Euler 1755 [38, §87]) [emphasis in the original–the authors].

Like Leibniz, Euler considers more than one way of comparing quantities. Euler's formula (3.3) indicates that his geometric comparison is procedurally identical with the Leibnizian TLH. Namely, Euler's geometric comparision of a pair of quantities amounts to their ratio being infinitely close to 1; the same is true for TLH. Thus, one has $a + dx = a$ in this sense for an appreciable $a \neq 0$, but *not* $dx = 0$ (which is true only *arithmetically* in Euler's sense). Euler's "geometric" comparison was dubbed "the principle of cancellation" in (Ferraro 2004 [39, p. 47]).

Euler proceeds to present the usual rules of infinitesimal calculus, which go back to Leibniz, L'Hôpital, and the Bernoullis, such as

$$a\,dx^m + b\,dx^n = a\,dx^m \tag{3.4}$$

provided $m < n$ "since dx^n vanishes compared with dx^m" (ibid., § 89), relying on his "geometric" equality. Euler introduces a distinction between infinitesimals of different order, and directly *computes*[13] a ratio of the form

$$\frac{dx \pm dx^2}{dx} = 1 \pm dx = 1$$

of two particular infinitesimals, assigning the value 1 to it (ibid., § 88). Euler concludes:

> Although all of them [infinitely small quantities] are equal to 0, still they must be carefully distinguished one from the other if we are to pay attention to their mutual relationships, which has been explained through a geometric ratio (ibid., § 89).

The Eulerian hierarchy of orders of infinitesimals harks back to Leibniz's work (see entry 4.7 on Nieuwentijt for a historical dissenting view). The remarkable lucidity of Euler's procedures for dealing with infinitesimals has unfortunately not been appreciated by all commentators. Thus, J. Gray interrupts his biography of Euler by suddenly declaring: "At some point it should be admitted that Euler's attempts at explaining the foundations of calculus in terms of differentials, which are and are not zero, are dreadfully weak" (Gray 2008b [46, p. 6]) but provides no evidence whatsoever for his dubious claim.

3.5. **Euler's infinite product formula for sine.** The fruitfulness of Euler's infinitesimal approach can be illustrated by some of the remarkable applications he obtained. Thus, Euler derived an infinite product decomposition for the sine and sinh functions of the following form:

$$\sinh x = x\left(1+\frac{x^2}{\pi^2}\right)\left(1+\frac{x^2}{4\pi^2}\right)\left(1+\frac{x^2}{9\pi^2}\right)\left(1+\frac{x^2}{16\pi^2}\right)\ldots \quad (3.5)$$

$$\sin x = x\left(1-\frac{x^2}{\pi^2}\right)\left(1-\frac{x^2}{4\pi^2}\right)\left(1-\frac{x^2}{9\pi^2}\right)\left(1-\frac{x^2}{16\pi^2}\right)\ldots \quad (3.6)$$

Decomposition (3.6) generalizes an infinite product formula for $\frac{\pi}{2}$ due to Wallis [115]. Euler also summed the inverse square series: $1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\ldots=\frac{\pi^2}{6}$ (see [90]) and obtained additional identities. A common feature of these formulas is that Euler's computations involve not only infinitesimals but also infinitely large natural numbers, which Euler

[13]Note that Euler does not "prove that the expression is equal to 1"; such *indirect* proofs are a trademark of the ϵ, δ approach. Rather, Euler *directly* computes (what would today be formalized as the standard part of) the expression, illustrating one of the advantages of the B-methodology over the A-methodology.

sometimes treats as if they were ordinary natural numbers.[14] Similarly, Euler treats infinite series as polynomials of a specific infinite degree.

The derivation of (3.5) and (3.6) in (Euler 1748 [35, § 156]) can be broken up into seven steps as follows.

Step 1. Euler observes that

$$2\sinh x \quad = \quad e^x - e^{-x} = \left(1+\frac{x}{j}\right)^j - \left(1-\frac{x}{j}\right)^j, \tag{3.7}$$

where j (or "i" in Euler [35]) is an infinitely large natural number. To motivate the next step, note that the expression $x^j - 1 = (x-1)(1+x+x^2+\ldots+x^{j-1})$ can be factored further as $\prod_{k=0}^{j-1}(x-\zeta^k)$, where $\zeta = e^{2\pi i/j}$; conjugate factors can then be combined to yield a decomposition into real quadratic terms.

Step 2. Euler uses the fact that $a^j - b^j$ is the product of the factors

$$a^2 + b^2 - 2ab\cos\frac{2k\pi}{j}, \quad \text{where} \quad k \geq 1, \tag{3.8}$$

together with the factor $a-b$ and, if j is an even number, the factor $a+b$, as well.

Step 3. Setting $a = 1 + \frac{x}{j}$ and $b = 1 - \frac{x}{j}$ in (3.7), Euler transforms expression (3.8) into the form

$$2 + 2\frac{x^2}{j^2} - 2\left(1 - \frac{x^2}{j^2}\right)\cos\frac{2k\pi}{j}. \tag{3.9}$$

Step 4. Euler then replaces (3.9) by the expression

$$\frac{4k^2\pi^2}{j^2}\left(1 + \frac{x^2}{k^2\pi^2} - \frac{x^2}{j^2}\right), \tag{3.10}$$

justifying this step by means of the formula

$$\cos\frac{2k\pi}{j} = 1 - \frac{2k^2\pi^2}{j^2}. \tag{3.11}$$

Step 5. Next, Euler argues that the difference $e^x - e^{-x}$ is divisible by the expression

$$1 + \frac{x^2}{k^2\pi^2} - \frac{x^2}{j^2}$$

from (3.10), where "we omit the term $\frac{x^2}{j^2}$ since even when multiplied by j, it remains infinitely small" (English translation from [36]).

[14] Euler's procedure is therefore consonant with the Leibnizian *law of continuity* (see entry 4.3) though apparently Euler does not refer explicitly to the latter.

Step 6. As there is still a factor of $a - b = 2x/j$, Euler obtains the final equality (3.5), arguing that then "the resulting first term will be x" (in order to conform to the Maclaurin series for $\sinh x$).

Step 7. Finally, formula (3.6) is obtained from (3.5) by means of the substitution $x \mapsto ix$. □

We will discuss modern formalisations of Euler's argument in entry 3.6.

3.6. **Euler's sine factorisation formalized.** Euler's argument in favor of (3.5) and (3.6) was formalized in terms of a "nonstandard" proof in (Luxemburg 1973 [86]). However, the formalisation in [86] deviates from Euler's argument beginning with steps 3 and 4, and thus circumvents the most problematic steps 5 and 6.

A proof in the framework of modern nonstandard analysis, formalizing Euler's argument step-by-step throughout, appeared in (Kanovei 1988 [59]); see also (McKinzie & Tuckey 1997 [90]) and (Kanovei & Reeken 2004 [61, Section 2.4a]). This formalisation interprets problematic details of Euler's argument on the basis of general principles of modern nonstandard analysis, as well as general analytic facts that were known in Euler's time. Such principles and facts behind some early proofs in infinitesimal calculus are sometimes referred to as "hidden lemmas" in this context; see (Laugwitz [77], [78]), (McKinzie & Tuckey 1997 [90]). For instance, the "hidden lemma" behind Step 4 above is the fact that for a fixed x, the terms of the Maclaurin expansion of $\cos x$ tend to 0 faster than a convergent geometric series, allowing one to infer that the effect of the transformation of step 4 on the product of the factors (3.9) is infinitesimal. Some "hidden lemmas" of a different kind, related to basic principles of nonstandard analysis, are discussed in [90, pp 43ff.].

What clearly stands out from Euler's argument is his explicit use of infinitesimal expressions such as (3.9) and (3.10), as well as the approximate formula (3.11) which holds "up to" an infinitesimal of higher order. Thus, Euler used infinitesimals *par excellence*, rather than merely ratios thereof, in a routine fashion in some of his best work.

Euler's use of infinite integers and their associated infinite products (such as the decomposition of the sine function) were interpreted in Robinson's framework in terms of hyperfinite sets. Thus, Euler's product of j-infinitely many factors in (3.6) is interpreted as a hyperfinite product in [61, formula (9), p. 74]. A hyperfinite formalisation of Euler's argument involving infinite integers and their associated products illustrates the successful remodeling of the *arguments* (and not merely the *results*) of classical infinitesimal mathematics, as discussed in entry 4.5.

3.7. **Fermat, Pierre.** Pierre de Fermat (1601-1665) developed a pioneering technique known as *adequality* (see entry 2.1) for finding tangents to curves and for solving problems of maxima and minima. (Katz, Schaps & Shnider 2013 [69]) analyze some of the main approaches in the literature to the method of adequality, as well as its source in the $\pi\alpha\rho\iota\sigma\acute{o}\tau\eta\varsigma$ of Diophantus (see entry 3.2). At least some of the manifestations of adequality, such as Fermat's treatment of transcendental curves and Snell's law, amount to variational techniques exploiting a small (alternatively, infinitesimal) variation E. Fermat's treatment of geometric and physical applications suggests that an aspect of approximation is inherent in adequality, as well as an aspect of smallness on the part of E.

Fermat's use of the term *adequality* relied on Bachet's rendering of Diophantus. Diophantus coined the term *parisotes* for mathematical purposes. Bachet performed a semantic calque in passing from *parisoō* to *ad-aequo*. A historically significant parallel is found in the similar role of, respectively, adequality and the *transcendental law of homogeneity* (see entry 4.4) in the work of, respectively, Fermat and Leibniz on the problems of maxima and minima.

Breger (1994 [21]) denies that the idea of "smallness" was relied upon by Fermat. However, a detailed analysis (see [69]) of Fermat's treatment of the cycloid reveals that Fermat did rely on issues of "smallness" in his treatment of the cycloid, and reveals that Breger's interpretation thereof contains both mathematical errors and errors of textual analysis. Similarly, Fermat's proof of Snell's law, a variational principle, unmistakably relies on ideas of "smallness".

Cifoletti (1990 [25]) finds similarities between Fermat's adequality and some procedures used in smooth infinitesimal analysis of Lawvere and others. Meanwhile, (J. Bell 2009 [9]) seeks the historical sources of Lawvere's infinitesimals mainly in Nieuwentijt (see entry 4.7).

3.8. **Heyting, Arend.** Arend Heyting (1898-1980) was a mathematical Intuitionist whose lasting contribution was the formalisation of the Intuitionistic logic underpinning the Intuitionism of his teacher Brouwer. While Heyting never worked on any theory of infinitesimals, he had several opportunities to present an expert opinion on Robinson's theory. Thus, in 1961, Robinson made public his new idea of non-standard models for analysis, and "communicated this almost immediately to . . . Heyting" (see Dauben [28, p. 259]). Robinson's first paper on the subject was subsequently published in *Proceedings of the Netherlands Royal Academy of Sciences* [99]. Heyting praised non-standard analysis as "a standard model of important mathematical research" (Heyting 1973 [53, p. 136]). Addressing Robinson, he declared:

> you connected this extremely abstract part of model theory with a theory apparently so far apart as the elementary calculus. In doing so you threw new light on the history of the calculus by giving a clear sense to Leibniz's notion of infinitesimals (ibid).

Intuitionist Heyting's admiration for the application of Robinson's infinitesimals to calculus pedagogy is in stark contrast with the views of his fellow constructivist E. Bishop (entry 2.6).

3.9. **Indivisibles versus Infinitesimals.** Commentators use the term *infinitesimal* to refer to a variety of conceptions of the infinitely small, but the variety is not always acknowledged. It is important to distinguish the infinitesimal methods of Archimedes and Cavalieri from those employed by Leibniz and his successors. To emphasize this distinction, we will say that tradition prior to Leibniz employed *indivisibles.* For example, in his heuristic proof that the area of a parabolic segment is 4/3 the area of the inscribed triangle with the same base and vertex, Archimedes imagines both figures to consist of perpendiculars of various heights erected on the base. The perpendiculars are indivisibles in the sense that they are limits of division and so one dimension less than the area. In the same sense, the indivisibles of which a line consists are points, and the indivisibles of which a solid consists are planes.

Leibniz's infinitesimals are not indivisibles, for they have the same dimension as the figures that comprise them. Thus, he treats curves as composed of infinitesimal line intervals rather than indivisible points. The strategy of treating infinitesimals as dimensionally homogeneous with the objects they compose seems to have originated with Roberval or Torricelli, Cavalieri's student, and to have been explicitly arithmetized in (Wallis 1656 [115]).

Zeno's *paradox of extension* admits of resolution in the framework of Leibnizian infinitesimals (see entry 5.5). Furthermore, only with the dimensionality retained is it possible to make sense of the fundamental theorem of calculus, where one must think about the rate of change of the *area* under a curve, another reason why indivisibles had to be abandoned in favor of infinitesimals so as to enable the development of the calculus (see Ely 2012 [33]).

4. Leibniz to Nieuwentijt

4.1. **Leibniz, Gottfried.** Gottfried Wilhelm Leibniz (1646-1716), the co-inventor of infinitesimal calculus, is a key player in the parallel infinitesimal track referred to by Felix Klein [75, p. 214] (see Section 1).

Leibniz's *law of continuity* (see entry 4.3) together with his *transcendental law of homogeneity* (which he already discussed in his response to Nieuwentijt in 1695 as noted by M. Parmentier [83, p. 38], and later in greater detail in a 1710 article [82] cited in the seminal study of Leibnizian methodology by H. Bos [17]) form a basis for implementing the calculus in the context of a B-continuum.

Many historians of the calculus deny significant continuity between infinitesimal calculus of the 17th century and 20th century developments such as Robinson's theory (see further discussion in Katz & Sherry [71]). Robinson's hyperreals require the resources of modern logic; thus many commentators are comfortable denying a historical continuity. A notable exception is Robinson himself, whose identification with the Leibnizian tradition inspired Lakatos, Laugwitz, and others to consider the history of the infinitesimal in a more favorable light. Many historians have overestimated the force of Berkeley's criticisms (see entry 2.3), by underestimating the mathematical and philosophical resources available to Leibniz.

Leibniz's infinitesimals are fictions, not logical fictions, as (Ishiguro 1990 [57]) proposed, but rather pure fictions, like imaginaries, which are not eliminable by some syncategorematic paraphrase; see (Sherry & Katz 2012 [105]) and entry 4.2 below.

In fact, Leibniz's defense of infinitesimals is more firmly grounded than Berkeley's criticism thereof. Moreover, Leibniz's system for differential calculus was free of logical fallacies (see entry 2.4). This strengthens the conception of modern infinitesimals as a formalisation of Leibniz's strategy of relating inassignable to assignable quantities by means of his *transcendental law of homogeneity* (see entry 4.4).

4.2. **Leibniz's *De Quadratura*.** In 1675 Leibniz wrote a treatise on his infinitesimal methods, *On the Arithmetical Quadrature of the Circle, the Ellipse, and the Hyperbola*, or *De Quadratura*, as it is widely known. However, the treatise appeared in print only in 1993 in a text edited by Knobloch (Leibniz [84]).

De Quadratura was interpreted by R. Arthur [3] and others as supporting the thesis that Leibniz's infinitesimals are mere shortcuts, eliminable by long-winded paraphrase. This so-called syncategorematic interpretation of Leibniz's calculus has gained a number of adherents.

We believe this interpretation to be a mistake. In the first place, Leibniz wrote the treatise at a time when infinitesimals were despised by the French Academy, a society whose approval and acceptance he eagerly sought. More importantly, as (Jesseph 2013 [58]) has shown, *De Quadratura* depends on infinitesimal resources in order to construct an approximation to a given curvilinear area less than any previously specified error. This problem is reminiscent of the difficulty that led to infinitesimal methods in the first place. Archimedes' method of exhaustion required one to determine a value for the quadrature *in advance* of showing, by *reductio* argument, that any departure from that value entails a contradiction. Archimedes possessed a heuristic, indivisible method for finding such values, and the results were justified by exhaustion, but only after the fact. By the same token, the use of infinitesimals is 'just' a shortcut only if it is entirely eliminable from quadratures, tangent constructions, etc. Jesseph's insight is that this is not the case.

Finally, the syncategorematic interpretation misrepresents a crucial aspect of Leibniz's mathematical philosophy. His conception of mathematical fiction includes imaginary numbers, and he often sought approbation for his infinitesimals by comparing them to imaginaries, which were largely uncontroversial. There is no suggestion by Leibniz that imaginaries are eliminable by long-winded paraphrase. Rather, he praises imaginaries for their capacity to achieve universal harmony by the greatest possible systematisation, and this characteristic is more central to Leibniz's conception of infinitesimals than the idea that they are mere shorthand. Just as imaginary roots both unified and extended the method for solving cubics, likewise infinitesimals unified and extended the method for quadrature so that, e.g., quadratures of general parabolas and hyperbolas, could be found by the same method used for quadratures of less difficult curves. See also (Tho 2012 [113]).

4.3. ***Lex continuitatis.*** A heuristic principle called *The law of continuity* (LC) was formulated by Leibniz and is a key to appreciating Leibniz's vision of infinitesimal calculus. The LC asserts that whatever succeeds in the finite, succeeds also in the infinite. This form of the principle appeared in a letter to Varignon (Leibniz 1702 [81]). A more detailed form of LC in terms of the concept of *terminus* appeared in his text *Cum Prodiisset*:

> In any supposed continuous transition, ending in any terminus, it is permissible to institute a general reasoning, in which the final terminus may also be included (Leibniz 1701 [80, p. 40])

$$\left\{\begin{matrix} \textit{assignable} \\ \textit{quantities} \end{matrix}\right\} \overset{\text{LC}}{\rightsquigarrow} \left\{\begin{matrix} \textit{inassignable} \\ \textit{quantities} \end{matrix}\right\} \overset{\text{TLH}}{\rightsquigarrow} \left\{\begin{matrix} \textit{assignable} \\ \textit{quantities} \end{matrix}\right\}$$

FIGURE 2. Leibniz's law of continuity (LC) takes one from assignable to inassignable quantities, while his transcendental law of homogeneity (TLH; entry 4.4) returns one to assignable quantities.

To elaborate, the LC postulates that whatever properties are satisfied by ordinary or assignable quantities, should also be satisfied by inassignable quantities (see entry 5.4) such as infinitesimals (see Figure 2). Thus, the trigonometric formula $\sin^2 x + \cos^2 x = 1$ should be satisfied for an inassignable (e.g., infinitesimal) input x, as well. In the 20th century this heuristic principle was formalized as the *transfer principle* (see entry 4.6) of Łoś–Robinson.

The significance of LC can be illustated by the fact that a failure to take note of the law of continuity often led scholars astray. Thus, Nieuwentijt (see entry 4.7) was led into something of a dead-end with his nilpotent infinitesimals (ruled out by LC) of the form $\frac{1}{\infty}$. J. Bell's view of Nieuwentijt's approach as a precursor of nilsquare infinitesimals of Lawvere (see Bell 2009 [9]) is plausible, though it could be noted that Lawvere's nilsquare infinitesimals cannot be of the form $\frac{1}{\infty}$.

4.4. ***Lex homogeneorum transcendentalis.*** Leibniz's *transcendental law of homogeneity*, or *lex homogeneorum transcendentalis* in the original Latin (Leibniz 1710 [82]), governs equations involving differentials. Leibniz historian H. Bos interprets it as follows:

> A quantity which is infinitely small with respect to another quantity can be neglected if compared with that quantity. Thus all terms in an equation except those of the highest order of infinity, or the lowest order of infinite smallness, can be discarded. For instance,
>
> $$a + dx = a \tag{4.1}$$
>
> $$dx + ddy = dx$$
>
> etc. The resulting equations satisfy this . . . requirement of homogeneity (Bos 1974 [17, p. 33]).

For an interpretation of the equality sign in the formulas above, see entry 5.2 on the relation $\sqcap$.

The TLH associates to an inassignable quantity (such as $a + dx$), an assignable one (such as a); see Figure 2 for a relation between LC and TLH.

4.5. **Mathematical rigor.** There is a certain lack of clarity in the historical literature with regard to issues of fruitfulness, consistency, and rigorousness of mathematical writing. As a rough guide, and so as to be able to formulate useful distinctions when it comes to evaluating mathematical writing from centuries past, we would like to consider three levels of judging mathematical writing:

(1) potentially fruitful but (logically) inconsistent;
(2) (potentially) consistent but informal;
(3) formally consistent and fully rigorous according to currently prevailing standards.

As an example of level (1) we would cite the work of Nieuwentijt (entry 4.7; see there for a discussion of the inconsistency). Our prime example of level (2) is provided by the Leibnizian *laws of continuity and homogeneity* (entries 4.3 and 4.4), which found rigorous implementation at level (3) only centuries later (see entry 4.6 on *modern implementations*).

A foundation rock of the received history of mathematical analysis is the belief that mathematical rigor emerged starting in the 1870s through the efforts of Cantor, Dedekind, Weierstrass, and others, thereby replacing formerly unrigorous work of infinitesimalists from Leibniz onward. The philosophical underpinnings of such a belief were analyzed in (Katz & Katz 2012a [67]) where it was pointed out that in mathematics, as in other sciences, former errors are eliminated through a process of improved conceptual understanding, evolving over time, of the key issues involved in that science.

Thus, no scientific development can be claimed to have attained perfect clarity or rigor merely on the grounds of having eliminated earlier errors. Moreover, no such claim for a single scientific development is made either by the practitioners or by the historians of the natural sciences. It was further pointed out in [67] that the term *mathematical rigor* itself is ambiguous, its meaning varying according to context. Four possible meanings for the term were proposed in [67]:

(1) it is a shibboleth that identifies the speaker as belonging to a clan of professional mathematicians;
(2) it represents the idea that as a scientific field develops, its practitioners attain greater and more conceptual understanding of key issues, and are less prone to error;

(3) it represents the idea that a search for greater correctness in analysis *inevitably* led Weierstrass specifically to epsilontics (i.e., the A-approach) in the 1870s;
(4) it refers to the establishment of what are perceived to be the ultimate foundations for mathematics by Cantor, eventually explicitly expressed in axiomatic form by Zermelo and Fraenkel.

Item (1) may be pursued by a fashionable academic in the social sciences, but does not get to the bottom of the issue. Meanwhile, item (2) would be agreed upon by historians of the other sciences.

In this context, it is interesting to compare the investigation of the Archimedean property as performed by the would-be rigorist Cantor, on the one hand, and the infinitesimalist Stolz, on the other. Cantor sought to derive the Archimedean property as a consequence of those of a linear continuum. Cantor's work in this area was not only unrigorous but actually erroneous, whereas Stolz's work was fully rigorous and even visionary. Namely, Cantor's arguments "proving" the inconsistency of infinitesimals were based on an implicit assumption of what is known today as the Kerry-Cantor axiom (see Proietti 2008 [97]). Meanwhile, Stolz was the first modern mathematician to realize the importance of the *Archimedean axiom* (see entry 2.2) as a separate axiom in its own right (see Ehrlich 2006 [32]), and moreover developed some non-Archimedean systems (Stolz 1885 [111]).

In his *Grundlagen der Geometrie* (Hilbert 1899 [54]), Hilbert did not develop a new geometry, but rather remodeled Euclid's geometry. More specifically Hilbert brought rigor into Euclid's geometry, in the sense of formalizing both Euclid's *propositions* and Euclid's style of *procedures and style of reasoning.*

Note that Hilbert's system works for geometries built over a non-Archimedean field, as Hilbert was fully aware. Hilbert (1900 [55, p. 207]) cites Dehn's counterexamples to Legendre's theorem in the absence of the Archimedean axiom. Dehn planes built over a non-Archimedean field were used to prove certain cases of the independence of Hilbert's axioms (see Cerroni 2007 [24]).[15]

Robinson's theory similarly formalized 17th and 18th century analysis by remodeling both its *propositions* and its *procedures and reasoning.* Using Weierstrassian ϵ, δ techniques, one can recover only the

[15]It is a melancholy comment to note that, fully three years later, the philosopher-mathematician Bertrand Russell was still claiming, on Cantor's authority, that the infinitesimal "leads to contradictions" (Russell 2003 [102, p. 345]). This set the stage for several decades of anti-infinitesimal vitriol, including the *saline solution* of Parkhurst and Kingsland (see Section 1.)

propositions but not the proof procedures. Thus, Euler's *result* giving an infinite product formula for sine (entry 3.5) admits of numerous proofs in a Weierstrassian context, but Robinson's framework provides a suitable context in which Euler's *proof*, relying on infinite integers, can also be recovered. This is the crux of the historical debate concerning ϵ, δ *versus* infinitesimals. In short, Robinson did for Leibniz what Hilbert did for Euclid. Meanwhile, epsilontists failed to do for Leibniz what Robinson did for Leibniz, namely formalizing the procedures and reasoning of the historical infinitesimal calculus. This theme is pursued further in terms of the internal/external distinction in entry 5.4 on *variable quantity*.

4.6. **Modern implementations.** In the 1940s, Hewitt [51] developed a modern implementation of an infinitesimal-enriched continuum extending $\mathbb{R}$, by means of a technique referred to today as the ultrapower construction. We will denote such an infinitesimal-enriched continuum by the new symbol $\mathbb{I}\mathbb{R}$ ("thick-R").[16] In the next decade, (Łoś 1955 [85]) proved his celebrated theorem on ultraproducts, implying in particular that elementary (more generally, first-order) statements over $\mathbb{R}$ are true if and only if they are true over $\mathbb{I}\mathbb{R}$, yielding a modern implementation of the Leibnizian *law of continuity* (entry 4.3). Such a result is equivalent to what is known in the literature as the *transfer principle*; see Keisler [73]. Every finite element of $\mathbb{I}\mathbb{R}$ is infinitely close to a unique real number; see entry 5.3 on the *standard part principle*. Such a principle is a mathematical implementation of Fermat's *adequality* (entry 2.1); of Leibniz's *transcendental law of homogeneity* (see entry 4.4); and of Euler's *principle of cancellation* (see discussion between formulas (3.2) and (3.4) in entry 3.4).

4.7. **Nieuwentijt, Bernard.** (1654-1718).[17] In Nieuwentijt's *Analysis Infinitorum* (1695), the Dutch philosopher proposed a system containing an infinite number, as well as infinitesimal quantities formed by dividing finite numbers by this infinite one. Nieuwentijt postulated that the product of two infinitesimals should be exactly equal to zero. In particular, an infinitesimal quantity is nilpotent. In an exchange of publications with Nieuwentijt on infinitesimals (see Mancosu 1996 [88, p. 161]), Leibniz and Hermann claimed that this system is consistent only if all infinitesimals are equal, rendering differential calculus useless. Leibniz instead advocated a system in which the product of two infinitesimals is incomparably smaller than either infinitesimal.

[16] A more traditional symbol is $^*\mathbb{R}$ or $\mathbb{R}^*$.

[17] Alternative spellings are Nieuwentijdt or Nieuwentyt.

Nieuwentijt's objections compelled Leibniz in 1696 to elaborate on the hierarchy of infinite and infinitesimal numbers entailed in a robust infinitesimal system.

Nieuwentijt's nilpotent infinitesimals of the form $\frac{1}{\infty}$ are ruled out by Leibniz's *law of continuity* (entry 4.3). J. Bell's view of Nieuwentijt's approach as a precursor of nilsquare infinitesimals of Lawvere (see Bell 2009 [9]) is plausible, though it could be noted that Lawvere's nilsquare infinitesimals cannot be of the form $\frac{1}{\infty}$.

5. Product rule to Zeno

5.1. **Product rule.** In the area of Leibniz scholarship, the received view is that Leibniz's infinitesimal system was logically faulty and contained internal contradictions allegedly exposed by the cleric *George Berkeley* (entry 2.3). Such a view is fully compatible with the A-track-dominated outlook, bestowing supremacy upon the reconstruction of analysis accomplished through the efforts of Cantor, Dedekind, Weierstrass, and their rigorous followers (see entry 4.5 on *mathematical rigor*). Does such a view represent an accurate appraisal of Leibniz's system?

The articles (Katz & Sherry 2012 [70]; 2013 [71]; Sherry & Katz [105]) building on the earlier work (Sherry 1987 [103]), argued that Leibniz's system was in fact consistent (in the sense of level (2) of entry 4.5),[18] and featured resilient heuristic principles such as the *law of continuity* (entry 4.3) and the *transcendental law of homogeneity* (TLH) (entry 4.4), which were implemented in the fullness of time as precise mathematical principles guiding the behavior of modern infinitesimals.

How did Leibniz exploit the TLH in developing the calculus? We will now illustrate an application of the TLH in the particular example of the derivation of the product rule. The issue is the justification of the last step in the following calculation:

$$\begin{aligned} d(uv) &= (u+du)(v+dv) - uv = udv + vdu + du\,dv \\ &= udv + vdu. \end{aligned} \tag{5.1}$$

The last step in the calculation (5.1), namely

$$udv + vdu + du\,dv = udv + vdu$$

[18]Concerning the status of Leibniz's system for differential calculus, it may be more accurate to assert that it was *not inconsistent*, in the sense that the contradictions alleged by Berkeley and others turn out not to have been there in the first place, once one takes into account Leibniz's generalized notion of equality and his transcendental law of homogeneity.

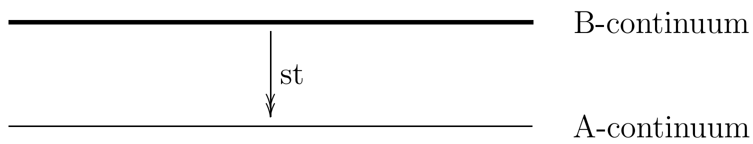

Figure 3. Thick-to-thin: applying the law of homogeneity, or taking standard part (the thickness of the top line is merely conventional)

is an application of the TLH.[19]

In his 1701 text *Cum Prodiisset* [80, p. 46-47], Leibniz presents an alternative justification of the product rule (see Bos [17, p. 58]). Here he divides by dx and argues with differential quotients rather than differentials. Adjusting Leibniz's notation to fit with (5.1), we obtain an equivalent calculation[20]

$$\begin{aligned} \frac{d(uv)}{dx} &= \frac{(u+du)(v+dv)-uv}{dx} = \frac{udv+vdu+du\,dv}{dx} \\ &= \frac{udv+vdu}{dx} + \frac{du\,dv}{dx} = \frac{udv+vdu}{dx} \end{aligned}$$

Under suitable conditions the term $\left(\frac{du\,dv}{dx}\right)$ is infinitesimal, and therefore the last step

$$\frac{udv+vdu}{dx} + \frac{du\,dv}{dx} = u\,\frac{dv}{dx} + v\,\frac{du}{dx} \tag{5.2}$$

is legitimized as a special case of the TLH. The TLH interprets the equality sign in (5.2) and (4.1) as the relation of being infinitely close, i.e., an equality up to infinitesimal error.

5.2. **Relation $\sqcap$.** Leibniz did not use our equality symbol but rather the symbol "$\sqcap$" (see McClenon 1923 [89, p. 371]). Using such a symbol to denote the relation of being infinitely close, one could write the

[19]Leibniz had two laws of homogeneity, one for dimension and the other for the order of infinitesimalness. Bos notes that they 'disappeared from later developments' [17, p. 35], referring to Euler and Lagrange. Note, however, the similarity to Euler's *principle of cancellation* (see Bair et al. [5]).

[20]The special case treated by Leibniz is $u(x) = x$. This limitation does not affect the conceptual structure of the argument.

calculation of the derivative of $y = f(x)$ where $f(x) = x^2$ as follows:

$$\begin{aligned} f'(x) &\sqcap \frac{dy}{dx} \\ &= \frac{(x+dx)^2 - x^2}{dx} \\ &= \frac{(x+dx+x)(x+dx-x)}{dx} \\ &= 2x + dx \\ &\sqcap 2x. \end{aligned}$$

Such a relation is formalized by the *standard part function*; see entry 5.3 and Figure 3.

5.3. **Standard part principle.** In any totally ordered field extension E of $\mathbb{R}$, every finite element $x \in E$ is infinitely close to a suitable unique element $x_0 \in \mathbb{R}$. Indeed, via the total order, the element x defines a Dedekind cut on $\mathbb{R}$, and the cut specifies a real number $x_0 \in \mathbb{R} \subset F$. The number x_0 is infinitely close to $x \in E$. The subring $E_f \subset E$ consisting of the finite elements of E therefore admits a map

$$\text{st} : E_f \to \mathbb{R}, \; x \mapsto x_0,$$

called the *standard part function*.

The standard part function is illustrated in Figure 3. A more detailed graphic representation may be found in Figure 4.[21]

The key remark, due to Robinson, is that the limit in the A-approach and the standard part function in the B-approach are essentially equivalent tools. More specifically, the limit of a sequence (u_n) can be expressed, in the context of a hyperreal enlargement of the number system, as the standard part of the value u_H of the natural extension of the sequence at an infinite hypernatural index $n = H$. Thus,

$$\lim_{n\to\infty} u_n = \text{st}(u_H). \tag{5.3}$$

Here the standard part function "st" associates to each finite hyperreal, the unique finite real infinitely close to it (i.e., the difference between them is infinitesimal). This formalizes the natural intuition that for "very large" values of the index, the terms in the sequence are "very close" to the limit value of the sequence. Conversely, the standard part

[21] For a recent study of optical diagrams in non-standard analysis, see (Dossena & Magnani [31], [87]) and (Bair & Henry [8]).

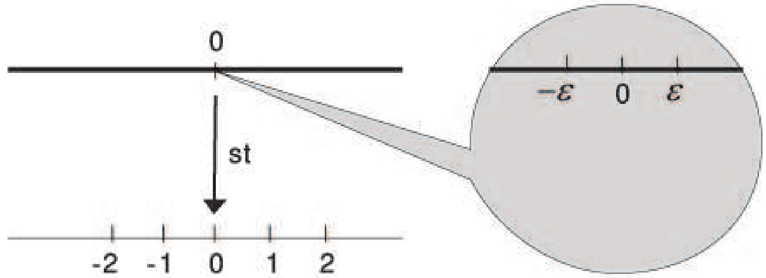

FIGURE 4. Zooming in on infinitesimal ε (here $\mathrm{st}(\pm\varepsilon) = 0$). The standard part function associates to every finite hyperreal, the unique real number infinitely close to it. The bottom line represents the "thin" real continuum. The line at top represents the "thick" hyperreal continuum. The "infinitesimal microscope" is used to view an infinitesimal neighborhood of 0. The derivative $f'(x)$ of $f(x)$ is then defined by the relation $f'(x) \approx \frac{f(x+\varepsilon)-f(x)}{\varepsilon}$.

of a hyperreal $u = [u_n]$ represented in the ultrapower construction by a Cauchy sequence (u_n), is simply the limit of that sequence:

$$\mathrm{st}(u) = \lim_{n\to\infty} u_n. \tag{5.4}$$

Formulas (5.3) and (5.4) express limit and standard part in terms of each other. In this sense, the procedures of taking the limit and taking the standard part are logically equivalent.

5.4. **Variable quantity.** The mathematical term $\mu\acute{\varepsilon}\gamma\varepsilon\vartheta o\varsigma$ in ancient Greek has been translated into Latin as *quantitas*. In modern languages it has two competing counterparts: in English – *quantity*, *magnitude*;[22] in French – *quantité*, *grandeur*; in German – *Quantität*, *Grösse*. The term *grandeur* with the meaning *real number* is still in use in (Bourbaki 1947 [19]). *Variable quantity* was a primitive notion in analysis as presented by Leibniz, l'Hôpital, and later Carnot and Cauchy. Other key notions of analysis were defined in terms of variable quantities. Thus, in Cauchy's terminology, a variable quantity *becomes* an *infinitesimal* if it eventually drops below any *given* (i.e., constant) quantity (see Borovik & Katz [16] for a fuller discussion). Cauchy notes that

[22]The term "magnitude" is etymologically related to $\mu\acute{\varepsilon}\gamma\varepsilon\vartheta o\varsigma$. Thus, $\mu\acute{\varepsilon}\gamma\varepsilon\vartheta o\varsigma$ in Greek and *magnitudo* in Latin both mean "bigness"; "big" being *mega* ($\mu\acute{\varepsilon}\gamma\alpha$) in Greek and *magnum* in Latin.

the *limit* of such a quantity is zero. The notion of *limit* itself is defined as follows:

> Lorsque les valeurs successivement attribuées à une même variable s'approchent indéfiniment d'une valeur fixe, de manière à finir par en différer aussi peu que l'on voudra, cette dernière est appelée la limite de toutes les autres (Cauchy, *Cours d'Analyse* [22]).

Thus, Cauchy defined both infinitesimals and limits in terms of the primitive notion of a variable quantity. In Cauchy, any variable quantity q that does not tend to infinity is expected to decompose as the sum of a given quantity c and an infinitesimal α:

$$q = c + \alpha. \tag{5.5}$$

In his 1821 text [22], Cauchy worked with a hierarchy of infinitesimals defined by polynomials in a base infinitesimal α. Each such infinitesimal decomposes as

$$\alpha^n(c + \varepsilon) \tag{5.6}$$

for a suitable integer n and infinitesimal ε. Cauchy's expression (5.6) can be viewed as a generalisation of (5.5).

In Leibniz's terminology, c is an *assignable* quantity while α and ε are *inassignable*. Leibniz's *transcendental law of homogeneity* (see entry 4.4) authorized the replacement of the inassignable $q = c + \alpha$ by the assignable c since α is negligible compared to c:

$$q \sqcap c \tag{5.7}$$

(see entry 5.2 on relation "$\sqcap$"). Leibniz emphasized that he worked with a generalized notion of equality where expressions were declared "equal" if they differed by a negligible term. Leibniz's procedure was formalized in Robinson's B-approach by the *standard part function* (see entry 5.3), which assigns to each finite hyperreal number, the unique real number to which it is infinitely close. As such, the standard part allows one to work "internally" (not in the technical NSA sense but) in the sense of exploiting concepts already available in the toolkit of the historical infinitesimal calculus, such as Fermat's *adequality* (entry 2.1), Leibniz's *transcendental law of homogeneity* (entry 4.4), and Euler's *principle of cancellation* (see Bair et al. [5]). Meanwhile, in the A-approach as formalized by Weierstrass one is forced to work with "external" concepts such as the multiple-quantifier ϵ, δ *definitions* (see entry 3.1) which have no counterpart in the historical infinitesimal calculus of Leibniz and Cauchy.

Thus, the notions of *standard part* and *epsilontic limit*, while logically equivalent (see entry 5.3), have the following difference between them:

the standard part principle corresponds to an "internal" development of the historical infinitesimal calculus, whereas the epsilontic limit is "external" to it.

5.5. **Zeno's paradox of extension.** Zeno of Elea (who lived about 2500 years ago) raised a puzzle (the *paradox of extension*, which is distinct from his better known paradoxes of motion) in connection with treating any continuous magnitude as though it consists of infinitely many indivisibles; see (Sherry 1988 [104]); (Kirk et al. 1983 [74]). If the indivisibles have no magnitude, then an *extension* (such as space or time) composed of them has no magnitude; but if the indivisibles have some (finite) magnitude, then an extension composed of them will be infinite. There is a further puzzle: If a magnitude is composed of indivisibles, then we ought to be able to add or concatenate them in order to produce or increase a magnitude. But indivisibles are not next to one another; as limits or boundaries, any pair of indivisibles is separated by what they limit. Thus, the concepts of addition or concatenation seem not to apply to indivisibles.

The paradox need not apply to infinitesimals in Leibniz's sense, however (see entry 3.9 on *indivisibles and infinitesimals*). For, having neither zero nor finite magnitude, infinitely many of them may be just what is needed to produce a finite magnitude. And in any case, the addition or concatenation of infinitesimals (of the same dimension) is no more difficult to conceive of than adding or concatenating finite magnitudes. This is especially important, because it allows one to apply arithmetic operations to infinitesimals (see entry 4.3 on the *law of continuity*). See also (Reeder 2012 [98]).

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