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De Vleeschouwer, Martine; GUEUDET, Ghislaine

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SECONDARY-TERTIARY TRANSITION AND EVOLUTIONS OF DIDACTIC CONTRACT:

THE EXAMPLE OF DUALITY IN LINEAR ALGEBRA

Martine De Vleeschouwer, Unité de didactique des math., Université de Namur

Ghislaine Gueudet, *CREAD, IUFM de Bretagne UBO*

This contribution concerns the teaching and learning of duality in linear algebra. Combining an institutional, and a didactic contract perspective we argue that some of the novice students' difficulties can result from specific features of the university contract, at different levels. Analyzing university textbooks, we identify such features, in the case of duality. Drawing on these observations, we design an experimental teaching, aiming to support the students' entrance in the new contract, at different levels. We investigate the impact of this experimental teaching. Analyzing students' productions, we observe that they developed abilities specific to the university contract, concerning duality or more generally mathematics.

Keywords: Secondary-tertiary transition, Linear Algebra, Didactic contract, Institutions

Duality in linear algebra is recognized as an arduous topic for novice students. The general aim of our work is to understand the difficulties they meet in duality, and to propose a teaching of duality likely to overcome these difficulties. Duality can be considered as a content specific of university mathematics, far away from secondary school. It led us to situate our study within the wider issue of secondary-tertiary transition. In previous work (De Vleeschouwer 2010*a*), we studied the students' difficulties, and proposed categories of difficulties, using an institutional focus. The work we present here corresponds to two new directions of research. On a theoretical level, we propose to consider the change of didactic contract between secondary school and university, and to combine it with the institutional perspective. In our empirical work, we have designed and implemented an experimental teaching intervention, aiming to support the students' entrance in the new didactic contract. We investigate here its impact on the students' outputs.

In part 1, we outline our theoretical propositions, articulating the didactical contract and the institutional perspective. In part 2, we specify our analysis to the context of linear algebra, and especially to duality in linear algebra. We present in part 3 the experimental teaching; we analyse its impact, drawing on students answers to a questionnaire, in part 4.

DIDACTIC CONTRACT, INSTITUTIONS AND THE SECONDARY-TERTIARY TRANSITION

The notion of didactic contract was introduced by Brousseau (1997), to describe 'a system of rules, mostly implicit, associating the students and the teacher, for a given piece of knowledge" (Brousseau 1997). Another interpretation of the contract, which is especially relevant in our study, is formulated in terms of sharing responsibility towards knowledge, between the students and the teacher. It seems thereof straightforward to claim, like Artigue (2007), that when a student enters university "the didactic contract is no longer the same". Several authors retain this perspective to study novice students' difficulties (Bloch 2005, Grønbæk, Misfeldt & Winsløw 2009). Nevertheless, the contract features identified are often very general: the students must show more autonomy, they must be able to develop reasonings involving sevral frames (Douady 1987) etc. These features seem to characterize general institutional expectations and not a particular mathematical content.

Considering the work of Chevallard (2005) can enlighten this last issue. According to him, a subject encounters a given mathematical knowledge in an institution. The institution frames this knowledge as a mathematical organisation, or praxeology, entailing four components: a type of tasks, a technique to accomplish this type of tasks; a technology, which is a discourse justifying the technique, and a theory. Mathematical organisations exist at several levels, from specific to general.

Considering the didactic contract with this perspective leads to distinguish several levels of contract, in a given institution:

- a general contract, independent of the knowledge at stake (Sarrazy, 2005, terms it the pedagogic contract). For example, at university in some countries attending the courses is not compulsory; taking notes is under the students' responsibility etc.;

- a didactic contract for mathematics, concerning generally mathematics in the institution: for example, the requirement of rigorous proofs;

- a didactic contract for a given content, concerning particular mathematical notions.

With these distinctions, the main question studied in this article can be formulated as: is it possible to support the students' entrance in a new contract at different levels, and how? We address this issue in the context of duality in linear algebra. Firstly, we identify features of the didactic contract at university, correponding to different levels, for the teaching of duality.

INSTITUTIONAL DIDACTIC CONTRACT AND DUALITY IN LINEAR ALGEBRA

We do not consider here the general contract; we start with the level of the didactic contract for mathematics. Considering several research works about transition (Praslon 2000, Bloch 2005, Bosch *et al.* 2004, Winsløw 2008) we retain that the following difficulties of the students correspond to changes of the didactic contract for mathematics, between secondary school and university:

- difficulties with building examples;

- difficulties with working in different frames, with moving between different representations ;

- difficulties with working at the technology-theory level, which means in particular producing a discourse justifying a technique.

According to the authors mentioned above, at university the student is (at least sometimes) responsible for these issues, which were only under the responsibility of the teacher at secondary school.

On a more precise level, about linear algebra and duality, we infer rules of contract by analysing textbooks (De Vleeschouwer 2010*b*).

A central change is that several concepts, in linear algebra, can change status, according to the context. For example, a matrix can be considered as representing a linear function in given bases; it can also be considered as an element of a vector space. A function can be seen as process acting on given objects; it can also be an element of a vector space. This last example is crucial in duality, where the students will have to determine the dual of a given vector space: a set of linear forms. In Belgium where our study takes place, students also encounter matrices and functions at secondary school. But these matrices and functions are not considered as elements of sets. At university, the student must be able to switch between both statuses, which are moreover not explicitly presented.

In 2008-2009 we elaborated and tested a teaching of duality taking into account these features of the contract, both at the discipline level for mathematics and at the content level for duality.

SUPPORTING THE ENTRANCE IN A NEW CONTRACT: AN EXPERIMENT AT NAMUR UNIVERSITY

We present below the main choices made with regard to the experimental teaching. Its focus is on duality, but also develops some prerequisites (as a minimum repertoire of vector spaces). We first want to situate its context, both in terms of the students involved and of teaching organisation.

The University of Namur has set up a device called “springboard operation”, aiming to support novice students, entering university (De Vleeschouwer 2008). It consists in remedial sessions proposed to the students, lasting between 2 and 4 hours each week. The first author of this paper participated as a teacher in the springboard operation, for first year students seeking a Master's degree in mathematics at the university of Namur (26 students in this first year in 2008-2009). She implemented the experimental teaching mainly in the context of this springboard operation (the variety of vector spaces was developed in a group work). This choice is the result of institutional constraints: setting up an experimental teaching in the “normal” course would have been refused by the mathematicians responsible for this teaching. Usually, only some of the students attend the springboard sessions. For the experimental teaching, all the students were invited to participate; 20 of them finally followed the sessions. Our analysis concerns these 20 students.

Before the teaching of the duality, students had already seen, in the theoretical course and in the exercises concerning algebra, the vector spaces (algebraic structures, linear dependence and dimension, sub-vector spaces); the linear functions and the associated matrices.

The experimental teaching within the springboard operation starts before the teaching of duality with a mandatory group work, aiming to provide the students with a minimum repertoire of vector spaces. Duality itself was tackled one month after the start of the academic year, in October 2008, and the corresponding teaching lasted five weeks:

- during week 1, students received a theoretical course (1.5h) concerning linear forms and dual space. Then they participated in an activity, which purpose is to make students aware of the various statutes that a matrix may have in linear algebra: element of a group, a ring, a vector space or representing a linear function;

- during week 2, we proposed to the students an activity, "linear forms and dual", described below (1h). Moreover, a theoretical course was given (1.5h), concerning the bidual space, the reflexivity theorem and the transpose transformation;

- during week 3, illustrations of dual and bidual spaces are presented (1h);

- finally during weeks 4 and 5 students had sessions of exercises (2x2h) on duality.

We focus in this paper on the "linear forms and dual" activity. We now present this activity and the corresponding choices, described in detail in (De Vleeschouwer 2010*b*).

As mentioned above (§2) a function, and thus a linear form, can change status, according to the context. In the context of duality in linear algebra, different statuses of linear forms can appear in the same task.

A linear form φ_i belonging to a dual basis *X'* of a basis *X* of a vector space *E* combines indeed two statuses:

- the status of process, operating on the elements of a vector space *E*. This status appears in the relationship linking basis $X = \{x_1, \ldots, x_n\}$ of vector space *E* to its dual basis $X' = \{\varphi_1, \ldots, \varphi_n\} : \forall\ i, j = 1,\ldots,n : \varphi_i\ (x_j) = \delta_{ij}$, where δ_{ij} is Kronecker's delta;

- the status of element of a vector space: the dual space of *E* (denoted *E'*), as an element of a basis of *E'*.

This combination of statuses can be considered as an aspect of the institutional didactic contract, at the level of a specific content. "A linear form is a process, and an element of a vector space, and students should be able to switch between these two statuses" is a rule of this contract. It is certainly linked with the process/object dialectics (Dubinsky 1991), but we do not retain here a cognitive focus: we consider how the institution shapes the content. This rule remains implicit; and this change of status yields difficulties of the students. In the experimental teaching we organized, we have chosen to make this rule explicit to students.

We introduced for this purpose a specific vocabulary, presented to the students during the teaching in the springboard sessions. This vocabulary is thus not an analysis tool for our research; it can be seen as a meta-language proposed to students. From the researcher's point of view, it is directly connected with the levels introduced by Chevallard (2005); we can not develop the point here, details can be found in De Vleeschouwer (2010b).

We say that a linear form φ is considered at a *micro level* when it is seen as a process operating on the elements of a vector space E (on a field K). We explain to the students that this choice of vocabulary is a metaphor, indicating that φ is considered in detail, which permits to observe the transformation it operates on the vectors of E. At this *micro level,* we can consider its kernel, range, rank amongst others.

When a linear form is considered as an element of a vector space, we call it the *macro level*. In this case this linear form can be considered amongst other linear forms on the same space E, constituting thus a set. In this set, one can define addition and product laws; with these laws one obtains a vector space, the dual of E.

On both levels, the same object φ is considered, but under different statuses. We **explicitly** presented to the students these levels using the vocabulary "micro" and "macro" during teaching, and connected them by saying that the macro level is obtained by a zoom out, the micro level by a zoom in (see De Vleeschouwer 2010*b* for the figures associated with this metaphor).

In order to evaluate the precise impact of this experimental teaching, we proposed, four months after the experimental teaching, a questionnaire to the students who attended it, and analysed their answers. We present this work in the next section. More than their success or failure, we try to identify in the students' answers indices of their entrance, or non-entrance, in the new contract.

IMPACT OF THE EXPERIMENTAL TEACHING

We present here an extract of the questionnaire, and analyse the corresponding students' answers in terms of didactic contract.

34 Let $\mathcal{P}^3$ be the vector space of polynomials of degree less than or equal to 3, with real coefficients. Let, for $i = 1, \ldots, 4$, $p_i : \Re \rightarrow \Re$ such as $\forall x \in \Re$: $p_1(x) = x^3 + 2x^2 + 4$, $p_2(x) = 2x^3 - x + 2$, $p_3(x) = x^3 - 1$, $p_4(x) = 2x^3 + 3$.

Prove that $A = p_1, p_2, p_3, p_4$ is a basis of $\mathcal{P}^3$, and determine its dual basis.

35 Let f_1, f_2, f_3 such as:

$$f_1: \begin{array}{ccc} \Re^5 & \to & \Re \\ v=(a,b,c,d,e) & & f_1(v)=3a-2e \end{array} \qquad f_2: \begin{array}{ccc} \Re^5 & \to & \Re \\ v=(a,b,c,d,e) & & f_2(v)=a-b+2c \end{array}$$

$$f_3: \begin{array}{ccc} \Re^5 & \to & \Re \\ v=(a,b,c,d,e) & & f_3(v)=3b+6c-2e \end{array}$$

1 Give an example of vector space comprising f_1, f_2, f_3.

2 Does f_1, f_2, f_3 form a linearly independent set of vectors?

3 Give an example of linear form which does not belong to Span{ f_1, f_2, f_3 }.

36 Choose a vector space different from polynomials, $\Re^n$ or □^{n} ($\forall n \in$□), and give an example of linear form over this space.

Table 1 : Extract of the questionnaire proposed to the students

The methodology we employ here is based on the *a priori* analysis (Hejny *et al.* 1999) of the questionnaire. We identify, in the questionnaire, specific aspects of the university contract, and observe in the students' answers if these issues raised difficulties, or if they evidence on the opposite an entrance in this contract.

The first question is related with polynomials and refers explicitly to duality, since the students have to determine the dual basis of a given basis. The students must consider jointly the micro and the macro level of linear forms, which is typical of the new contract at the level of the 'linear forms' content. The second question concerns functions. The proposed functions f_1, f_2, f_3 are defined at a micro level. They are linear forms; nevertheless, this term is not used in the text in order to avoid that the students answer 'the dual space' to question 2.a). For their answers to question 2. b), the students must consider these functions as vectors, which means changing levels, to work at the macro level. They have to place their reasoning at a level different from the level of the text.

Question 3 requires that the students work in a frame different from the polynomials (question 1) or the algebraic frame (common in the courses). The change of frames is also typical of the university didactic contract at the discipline level. In the second part of question 3, a linear form must be proposed at a micro level, as a process acting on the elements of the chosen set.

Moreover, questions 2.a), 2.c) and 3 require to build an example. Such a task is typical of the university contract, at the discipline level; the same statement holds for the variety of frames, another feature of the questionnaire.

We now consider the students' answers, focusing on the issues identified in the *a priori* analysis.

Two main techniques have emerged from students' responses to determine if the polynomials in question 1 constitute a basis: working with polynomials or with *n*-tuples. Eight students (40%) prefer to work with 4-tuples instead of polynomials, but only two of them justify their reasoning (for instance a student cites the theorem asserting the existence of an isomorphism between $\mathcal{P}^3$ and $\Re^4$). We can consider that in doing so, these students comply with the didactical contract at the discipline's level which considers that the different steps of a mathematical reasoning should be justified. All students who responded to the questionnaire were able to determine that the given polynomials were linearly independent. It does not seem to be a problem for them to work in a non-usual frame, as often required at University.

70% of the students present the linear forms of the dual basis in a complete, detailed form (departure space, arrival space and image of any vector, see Figure 1). We interpret these kinds of answers as typical from the university contract, at two levels. At the discipline level, a mathematical answer has to be as complete as possible. At the content level, a function must be characterized by these three elements, whereas at secondary school generally only the expression "$f(x) = \ldots$" or the graph is given.

Twelve students (60%) describe analytically the four linear forms p_i' and, at the same time, consider them as elements of *a set* (the dual base): they write explicitly "$A' = p_1', p_2', p_3', p_4'$" (see Figure 1). In doing so, students consider the linear forms both at the macro level and at the micro level.

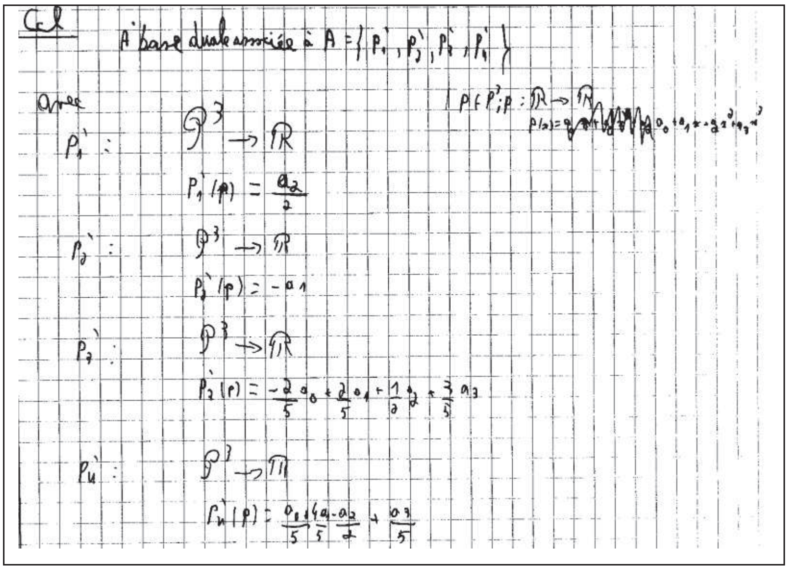

Figure 14: Conclusion of a student's answer to the second part of question 1

Concerning question 2.a) fifteen students (75%) succeed in giving a vector space comprising the given linear forms, and nine of them (45%) cite the dual space. It requires to consider at the macro level functions which have been described in the text as processes (micro level). This change of status does not seem to constitute a difficulty for a majority of students. The same statement holds for the sub-question 2.b): while the linear forms were given at the micro level, seventeen (85%) students succeed to consider them at the macro level, and answer correctly that they are linearly independent. Moreover six students convert the linear forms into 4-tuples before starting calculations (see Figure 2). We can interpret this fact as the conversion of a function from the mico level to the macro level. In Figure 2, we see that the student concludes question 2.b) by writing "the *vectors* are linearly indep." instead of "the *linear forms* are linearly indep."

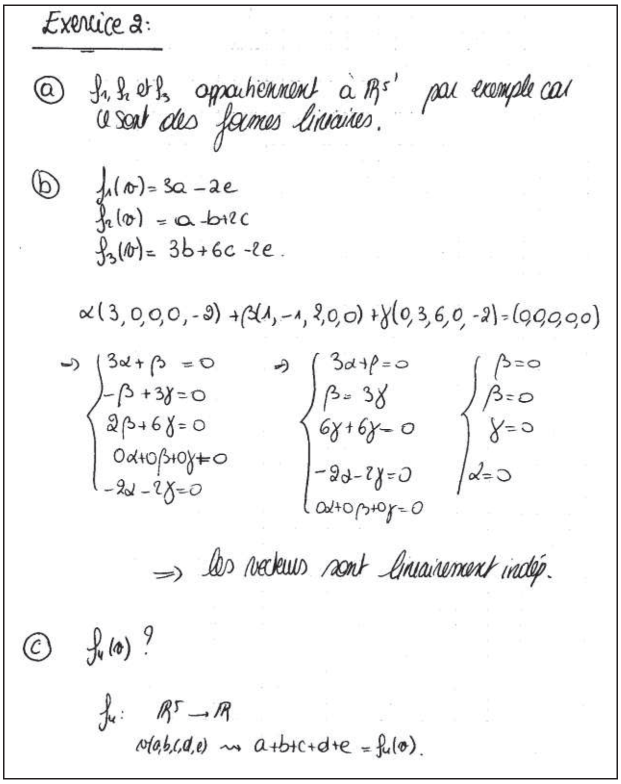

Figure 2 : Example of a student's answer to question 2

Analyzing students' answers to question 3 shows that they seem to have built a variety of vector spaces: amongst the vector spaces cited by the students, eleven are vector spaces of matrices (square matrices of size 2 or 3); two are vector spaces of functions (transformations of $\Re$ or of $\Re^2$), one can be considered as algebraic (□^{3} built over □). The unsuccessful attempts concern □^{2} or □^{3} (cited by three students) or □ (cited by two students); one student makes a non-relevant answer. Note that the module structure, which generalizes the vector space structure, was not presented to students. This perhaps explains the presence of proposals involving □^{2} or □^{3} in the responses of three of them. The variety of frames for linear algebra is typical of the new institutional contract at the discipline level. Moreover eight students *justify* the label 'linear form' given to their example although seven of them give only a partial explanation: "*arrival space is the field*". It seems that the part of justification in the didactic contract at the discipline level is not obvious.

CONCLUSION

In this work we attempted to determine rules of the didactic contract at university, in the case of duality. Using previous research works, and a textbooks analysis, we identified contract rules at different levels: some of these rules correspond to precise notions, like linear forms, while others concern more generally mathematics.

We designed a teaching intervention aiming to support the entrance of students in this new contract. The objective of such a teaching is not to change the contract, by reducing the students' responsibility. In the case we presented, we chose to make explicit a usually implicit rule, at the level of a specific content (linear forms), introducing a meta-language (*micro*/*macro*) for students' use. We have also proposed to the students exercises where they were required to change frames, to build examples, and more generally to comply with new university expectations, at the level of the discipline. We do not claim that all the rules should be made explicit. Some of the contract rules have to remain implicit, this well-known paradox is an essential condition for learning (Brousseau 1997). Introducing the micro-macro meta-language, we did not only unveil a rule about linear forms; we contributed to raise the awareness of the students about the different statuses of mathematical objects at university, and the possible need for change of status, according to the context. The meta-language makes this new responsibility explicit.

The analysis of the students' answers to our questionnaire evidence that they at least started to enter in the new contract. We did not carry out a comparison with other students (the conditions of our study did not offer such a possibility); but the first author of the paper, as a teacher, noticed that the students do not seem to meet the usual difficulties, and we consider that the experimental teaching significantly contributed to this progress. We now intend to extend our study to other topics: the precise rules of the didactic contract at university remain largely unknown.

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