Stochastic Weighted Fractal Networks

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In this paper we introduce new models of complex weighted networks sharing several properties with fractal sets: the deterministic non-homogeneous weighted fractal networks and the stochastic weighted fractal networks. Networks of both classes can be completely analytically characterized in terms of the involved parameters. The proposed algorithms improve and extend the framework of weighted fractal networks recently proposed in [39].

PACS numbers: 89.75.Hc Complex networks, 05.45.Df Fractals, 05.40.-a Stochastic processes

I. INTRODUCTION

Fractal structures are ubiquitous in nature, coastlines [1], river networks [2, 3], snowflakes [4], growing colonies of bacteria [5–7], mammalian lungs [8–12], mammalian bloody vessels [12], just to mention few of them [50]. But also mankind artifacts can exhibit fractal features, for instance fractal antenna [15] or fluctuations in markets prices [16].

A distinction can be made between *mathematical* or deterministic fractals [17] for which a complete geometric description can be provided using simple tools such as homotheties, rotations and copying, and *random* or *pseudo* fractals [13] found in nature, being the latter characterized by exhibiting fractal properties, for instance self–similarity, only when statistical averages are computed, because unavoidable fluctuations and errors can alter the regular–geometric patterns. Moreover such scale invariance should be limited to a finite range of scale lengths because of physical constraints.

It is worth remarking that some of these physical fractals have functionalities, e.g. transportation of gases in mammalian lungs, or charges in fractal antenna, one can thus improve the geometrical description by including flows and growths constraints. Networks are therefore the most natural and useful tool to describe such growing complex structures with flows constraints. We thus hereby propose the *Stochastic Weighted Fractal Networks*, SWFN for short, a new class of complex networks whose construction is directly inspired by such physical fractal structures.

Starting from the pioneering works of Erdős and Rényi [18], network theory is nowadays a research field in its own [19, 20] and the scientific activity is mainly devoted to construct and characterize complex networks exhibiting some of the remarkable properties of real networks, scale–free [21], small–world [22], communities [23], weighted links [24–29], just to mention few of them.

In recent years we observed an increasing number of papers were authors proposed models of deterministic (pseudo) fractal networks [30–39] exhibiting scale-free and hierarchical structures. In a limited number of cases, models presented also a stochastic component [40–43].

The aim of the SWFN hereby introduced, is to provide a framework that could be used to (re)analyze flows on natural fractal structures using standard tools of transport theory on networks. Moreover SWFN share with physical fractals several interesting properties, for instance the self-similarity or the self-affinity, the presence of hierarchical structures and a stochastic growth process. Actually this allows us to generalize in a unifying scheme some of the above mentioned models existing in the literature.

The SWFN are constructed via a stochastic process and we are thus able to analytically characterize their topology as a function of the parameters involved in the construction, using *expectations* obtained constructing several replicas.

Let us conclude this introduction with two remarks. First of all we named our models "fractal"networks instead of "pseudo fractal", because some of the topological properties of SWFN depend on the fractal dimension of some underlying fractal set, whose value ranges all the positive real numbers, without any limitation. Second we rather prefer to talk about "stochastic"networks to emphasize the stochastic growth process instead of the randomness of some topological quantities; let us also stress that in the network theory "randomness"has a precise meaning that we cannot directly apply to this case.

The paper is organized as follows. In the next section we will introduce and study a deterministic model, that generalize the one proposed in [39], and that will serve as the basic building block to construct the SWFN in Section III. Then we conclude with some possible applications we sum up and draw our conclusions

II. DETERMINISTIC WEIGHTED FRACTAL NETWORK

According to Mandelbrot [17] "a fractal is by definition a set for which the Hausdorff dimension strictly exceeds the topological dimension". One of the most amazing and interesting feature of fractals is their *self-similarity*

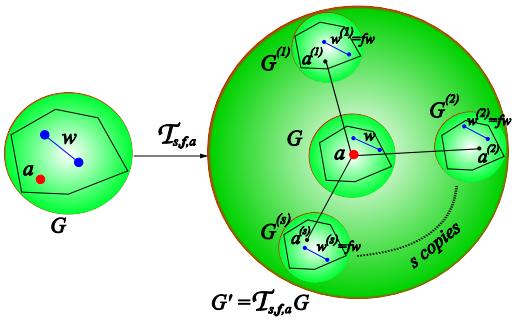

FIG. 1: *The map* $\mathcal{T}_{s,\mathbf{f},a}$. On the left a generic initial graph G with its attaching node a (red on-line) and a generic weighted edge $w \in G$ (blue on-line). On the right the new graph G' obtained as follows: Let $G^{(1)}, \ldots, G^{(s)}$ be s copies of G, whose weighted edges (blue on-line) have been scaled respectively by a factor $f_1, \ldots, f_s$, and let us denote by $a^{(i)}$, for $i = 1, \ldots, s$, the node in $G^{(i)}$ image of the labeled node $a \in G$, then link all those labeled nodes to $a \in G$ through edges of unitary weight. The connected network obtained in this way will be by definition the image of G through the map: $G' = \mathcal{T}_{s,\mathbf{f},a}(G)$.

or *self-affinity* [44, 45], namely looking at all scales we can find conformal or stretched copies of the whole set; this is actually the idea used to build up fractals as fixed point of *Iterated Function Systems* [46, 47], IFS for short. Such fractals have a Hausdorff dimension completely characterized by the number of copies and the scaling factors of the IFS. Let us observe that in this case this dimension coincides with the so called similarity dimension [47].

Recently, author proposed [39] a new general framework aiming to construct weighted networks with some a priori prescribed topology depending on the two main parameters: the number of copies and the scaling factors, hence on the fractal dimension of the "underlying"IFS fractal. The aim of this Section is to generalize such construction to obtain a larger class of networks; moreover exploiting the iterative construction we will be able to completely and analytically describe the network topology in terms of node strength distribution, average (weighted) shortest path and (weighted) clustering coefficient.

Let us fix a positive integer $s > 1$ and s real numbers $f_1, \ldots, f_s \in (0, 1)$ and let us consider a (possibly) weighted network G composed by N nodes, one of which has been labeled *attaching node* and denoted by a. We then introduce a map, $\mathcal{T}_{s,\mathbf{f},a}$, depending on the parameters s, $\mathbf{f} = (f_1, \ldots, f_s)$ and on the labeled node a, whose action on networks is described in Fig. 1.

So starting with a given initial network G_0 we can construct a family of weighted networks $(G_k)_{k\geq 0}$ iteratively applying the map $\mathcal{T}_{s,\mathbf{f},a}$: $G_k = \mathcal{T}_{s,\mathbf{f},a}(G_{k-1})$.

This construction improves the one recently proposed [39] by avoiding the introduction of an extra node, moreover it offers a unifying framework where several constructions presented in literature can be included and generalized, e.g. the model presented in [32] with $m = 3$ can be mapped into to the WFN with $s = 3$, $\mathbf{f} = (1, 1, 1)$, i.e. no weights, and $G_0 = \bullet$. Finally this deterministic construction will be the basic brick to develop the stochastic network introduced in the following Section III.

Given G_0 and the map $\mathcal{T}_{s,\mathbf{f},a}$ we are able to completely characterize the topology of each G_k for $k \geq 1$ and also of the limit network G_∞, defined as the fixed point of the map, $\mathcal{T}_{s,\mathbf{f},a}(G_\infty) = G_\infty$.

A. Results

The aim of this section is to describe the topology of the graphs G_k for all $k \geq 1$ and G_∞, by analytically studying their properties such as the average degree, the node strength distribution, the average (weighted) shortest path and the (weighted) clustering coefficient.

At each iteration step the graph G_k grows as the number of its nodes increases according to

$$N_k = (s+1)^k N_0 \,, \tag{1}$$

being N_0 the number of nodes in the initial graph, while the number of edges satisfies

$$E_k = (s+1)^k (E_0 + 1) - 1 \,, \tag{2}$$

being E_0 the number of edges in G_0. Hence in the limit of large k the average degree is asymptotically given by

$$\frac{E_k}{N_k} \underset{k\to\infty}{\longrightarrow} \frac{E_0+1}{N_0} \,. \tag{3}$$

Let us denote the weighted degree of node $i \in G_k$, also called *node strength* [25], by $\omega_i^{(k)} = \sum_j w_{ij}^{(k)}$, being $w_{ij}^{(k)}$ the weight of the edge $(ij) \in G_k$; then using the recursive construction, we can explicitly compute the total node strength, $W_k = \sum_i \omega_i^{(k)}$, and easily show that

$$W_k = \left[\frac{2s}{F} \left((F+1)^k - 1 \right) + (F+1)^k W_0 \right] \,, \tag{4}$$

being $F = \sum_{j=1}^{s} f_j$. Let us observe that using the hypothesis $f_j < 1$, it trivially follows that $F < s$, hence we can conclude that the average node strength goes to zero as k increases: $W_k/N_k \underset{k\to\infty}{\longrightarrow} 0$.

B. Node strength distribution.

Let $g_k(x)$ denote the number of nodes in G_k that have strength $\omega_i^{(k)} = x$ and let us assume g_0 to have values in some finite discrete subset of the positive reals, namely:

$$g_0(x) > 0 \text{ if and only if } x \in \{x_1, \ldots, x_m\} \,,$$

otherwise $g_0(x) = 0$. Using the property of the map $\mathcal{T}_{s,\mathbf{f},a}$ we get that after k steps of the construction the nodes strengths have been rescaled by a factor $f_1^{k_1} \ldots f_s^{k_s}$,

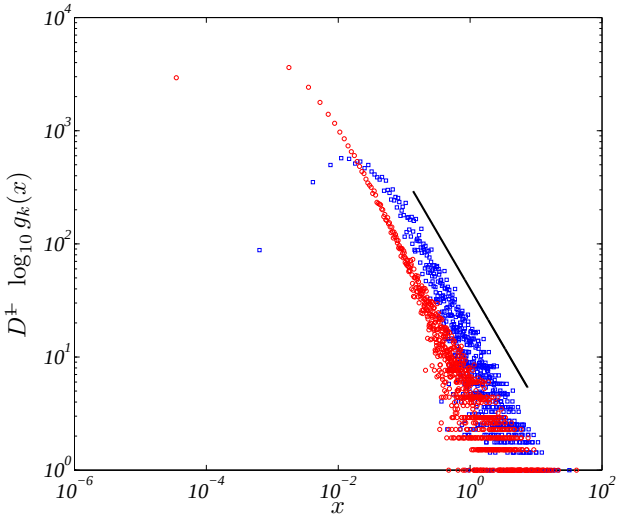

FIG. 2: *Node Strengths Distribution.* Plot of the renormalized node strengths distribution $D^{-1}\log_{10} g_k(x)$, where $D = -s\log s/\log(f_1 \dots f_s)$. Symbols refer to : □ the finite approximation G_{11} with 3145728 nodes of the WFN with $s = 3$, $\mathbf{f} = (1/\sqrt{2}, 1/\sqrt{3}, 1/\sqrt{5})$ and $G_0 =$  ; ◯ the finite approximation G_9 with 3359232 nodes of the WFN $s = 5$, $\mathbf{f} = (1/\sqrt{5}, 1/\sqrt{11}, 1/\sqrt{3}, 1/\sqrt{7}, 1/\sqrt{13})$ and $G_0 =$  . The reference line has slope -1, linear best fits (data not shown) provides a slope -1.037 ± 0.04 and $R^2 = 0.798$ for □ and -1.00 ± 0.03 and $R^2 = 0.8382$ for ◯.

where the non-negative integers k_i do satisfy $k_1 + \dots + k_s \leq k$. Because this can be done in $k!/(k_1! \dots k_s!)$ possible different ways, we get the following relation for the node strength distribution for the network G_k:

$$g_k(f_1^{k_1} \dots f_s^{k_s} x) = \frac{k!}{k_1! \dots k_s!} g_0(x) \quad \text{with } k_1 + \dots + k_s \leq k\,. \tag{5}$$

After sufficiently many steps and assuming that the main contribution arises from the choice $k_1 \sim \dots \sim k_s \sim k/s$, we can use Stirling formula to get the approximate distribution (see Fig. 2)

$$\log g_k(x) \sim \frac{s\log s}{\log(f_1 \dots f_s)} \log x\,, \tag{6}$$

so the nodes strength distribution follows a power law. Let us observe that in the case of homogeneous scaling, i.e. all f_j equal to some $f \in (0,1)$, one can prove [39] that Eq. (6) reduces to $\log g_k(x) \sim -d_{fract} \log x$ where $d_{fract} = -\log s/\log f$ is the fractal dimension of the underlying IFS fractal.

C. Average weighted shortest path.

By definition the average *weighted shortest path* [20] of the graph G_k is

$$\lambda_k = \frac{\Lambda_k}{N_k(N_k-1)}\,, \tag{7}$$

where

$$\Lambda_k = \sum_{ij \in G_k} p_{ij}^{(k)}\,, \tag{8}$$

being $p_{ij}^{(k)}$ the weighted shortest path linking nodes i and j in G_k. Taking advantage of the recursive construction and adapting the ideas used in [39], we get the following recursive relation for Λ_k

$$\Lambda_k = (F+1)\Lambda_{k-1} + 2s(F+1)N_{k-1}\Lambda_{k-1}^{(a_{k-1})} + 2s^2 N_{k-1}^2\,, \tag{9}$$

where we introduced $\Lambda_k^{(a_k)} = \sum_{i \in G_k} p_{ia_k}^{(k)}$, i.e. the sum of all weighted shortest paths ending at the attaching node, $a_k \in G_k$. We can prove that for large k the asymptotic behavior of $\Lambda_k^{(a_k)}$ is given by

$$\Lambda_k^{(a_k)} \underset{k\to\infty}{\sim} \frac{sN_0}{s-F}(s+1)^k\,, \tag{10}$$

and thus the recursive relation (9) can be explicitely solved to provide the following asymptotic behavior in the limit of large k (see Fig. 3)

$$\lambda_k \underset{k\to\infty}{\longrightarrow} \frac{2s^2(s+1)}{(s-F)[(1+s)^2-(1+F)]}\,. \tag{11}$$

One can explicitly compute the *average shortest path*, ℓ_k, formally obtained by setting $f_1 = \dots = f_s = 1$ in the previous formulas (7) and (8). Hence slightly modifying the results presented above we can prove that asymptotically we have (see Fig. 4)

$$\ell_k \underset{k\to\infty}{\sim} \frac{2s}{(1+s)\log(s+1)} \log\frac{N_k}{N_0}\,, \tag{12}$$

where growth law of N_k given by (1) has been used. Thus the network grows unbounded with the logarithm of the network size, while the weighted shortest distances stay bounded.

D. Clustering coefficient.

The clustering coefficient [20, 22] of the graph G_k is defined as the average over the whole set of nodes of the local clustering coefficient $c_i^{(k)}$, namely $< c_k >= C_k/N_k$, where $C_k = \sum_{i \in G_k} c_i^{(k)}$.

Because of the construction algorithm new triangles are created in the network "boundary" while their number

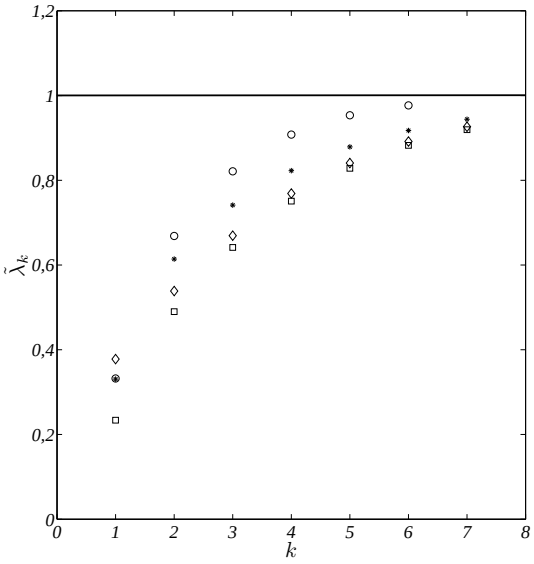

FIG. 3: *The average weighted shortest path.* Plot of the renormalized average weighted shortest path $\tilde{\lambda}_k$ versus the iteration number k, where $\tilde{\lambda}_k = \lambda_k \frac{(s-F)[(1+s)^2-(1+F)]}{2s^2(s+1)}$ and $F = f_1 + \cdots + f_s$. Symbols refer to : $\square$ the WFN $s = 3$, $\mathbf{f} = (1/\sqrt{2}, 1/\sqrt{3}, 1/\sqrt{5})$ and $G_0 =$  ; $\bigcirc$ the WFN $s =5$, $\mathbf{f} = (1/\sqrt{5}, 1/\sqrt{11}, 1/\sqrt{3}, 1/\sqrt{7}, 1/\sqrt{13})$ and $G_0 =$  ; $\Diamond$ the WFN $s = 2$, $\mathbf{f} = (1/\sqrt{3}, 1/\sqrt{5})$ and $G_0 =$  ; $*$ the WFN $s = 2$, $\mathbf{f} = (1/\sqrt{3}, 1/\sqrt{5})$ and $G_0 =$  .

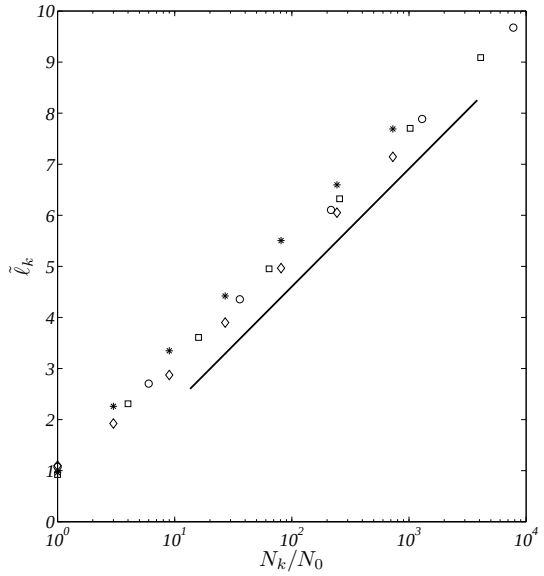

FIG. 4: *The average shortest path ℓ_k as a function of the network size* (semilog plot). Semilog plot of the renormalized average shortest path $\tilde{\ell}_k$ versus the network size N_k, where $\tilde{\ell}_k = \ell_k \frac{(s+1)\log(s+1)}{2s}$. Symbols are the same of Fig. 3. The reference line has slope 1. Linear best fits (data not shown) provides a slope 0.970 ± 0.017 and $R^2 = 0.9999$ for $\square$, 0.9654 ± 0.05 and $R^2 = 0.9997$ for $\bigcirc$, 0.97 ± 0.02 and $R^2 = 0.9998$ for $\Diamond$ and 1.01 ± 0.03 and $R^2 = 0.9993$ for $*$.

doesn't change in the inner core, hence the local clustering coefficient, at each step increases just by a factor $s+1$; thus after k–interactions we will have $C_k = (1+s)^k C_0$, being $C_0 = \sum_{i \in G_0} c_i^{(0)}$ the sum of local clustering coefficients in the initial graph. We can thus conclude that the clustering coefficient of the graph is asymptotically given by:

$$< c_k > \underset{k \to \infty}{\longrightarrow} \frac{C_0}{N_0} \,. \tag{13}$$

On the other hand, one can introduce the links values to weigh the clustering coefficient [48], generalizing the previous relation, we can easily prove that *weighted clustering coefficient* of the graph is asymptotically given by:

$$< \gamma_k > = \left(\frac{1+F}{1+s} \right)^k \frac{C_0}{N_0} \underset{k \to \infty}{\sim} \frac{1}{N_k^{1-d}} \,, \tag{14}$$

where $d = \frac{\log(1+F)}{\log(1+s)}$, that results smaller than one because of the assumption $f_j < 1$.

III. STOCHASTIC WEIGHTED FRACTAL NETWORKS

The aim of this section is to present a class of complex weighted networks that grow according to a *stochastic* process and exhibit self-similar or self-affine structures, hereby named *Stochastic Weighted Fractal Networks*, for short *SWFN*, whose construction is directly inspired by the stochastic growth phenomena present in nature. The idea is thus to mimic the growth of fractal structures in nature where "possible errors" could modify regular patterns.

So let us hypothesize that the growth process is the result of a stochastic process that selects the actual realization, i.e. the number of copies, between a number of different possibilities. Thus at each iteration the number of copies, s, is a stochastic variable distributed according to some probability distribution function $p(s)$. Once the numerical value for s has been set, s real numbers $f_1, \ldots, f_s$ are drawn according to some probability distribution function $q(f)$ with values in $(0, 1)$. Finally a new network is constructed by applying $\mathcal{T}_{s,(f_1,\ldots,f_s),a}$ to the actual network:

$$G \underset{p(s)}{\longmapsto} G^{(s)} = \mathcal{T}_{s,(f_1,\ldots,f_s),a} G \,. \tag{15}$$

Remark. *In the following we will assume the simpli-*

fying working hypothesis that $f_1 = \cdots = f_s = \alpha/s$, i.e. $q(f) = \delta(f - \alpha/s)$, for some given and fixed $\alpha \in (0,1)$, but of course the model applies to more general cases.

One can repeat the construction k times and thus obtain with probability $p(s_k)\dots p(s_1)$, starting from a network G_0, a new network, denoted by $G^{(s_k,\dots,s_1)}$:

$$G^{(s_k,\dots,s_1)} = \mathcal{T}_{s_k,(f_1^{(k)},\dots,f_{s_k}^{(k)}),a} \circ \cdots \circ \mathcal{T}_{s_1,(f_1^{(1)},\dots,f_{s_1}^{(1)}),a} G_0 \,. \tag{16}$$

The network growth results thus a stochastic process, hence we will describe the main topological network measures in terms of *expectations* obtained repeating several times the construction. Of course we could also consider and compute higher order momenta, but the computations become rapidly cumbersome, and thus we will non present these results except for some simple cases, such as the number of nodes.

A. Results: SWFN

At each step the number of nodes increases with respect the present ones, and the exact amount depends on the number of branches drawn. Starting from a network containing N_0 nodes we get a new network with $N^{(s_1)} = (1+s_1)N_0$ nodes with probability $p(s_1)$. Iterating the construction, after k steps we can obtain with probability $p(s_k)\dots p(s_1)$ a network with $N^{(s_k,\dots,s_1)} = (1+s_k)\dots(1+s_1)N_0$ nodes. Hence the expected value for the number of nodes in a network build after k iterations, is given by:

$$\begin{aligned} < N_k > &= \sum_{s_k,\dots,s_1} p(s_k)\dots p(s_1) N^{(s_k,\dots,s_1)} \qquad (17)\\ &= \sum_{s_k} p(s_k)(1+s_k) \sum_{s_{k-1},\dots,s_1} p(s_{k-1})\dots p(s_1) N^{(s_{k-1},\dots,s_1)} \\ &= (1+ < s >) < N_{k-1} > , \end{aligned}$$

where we denoted by $< s >= \sum_{s_k} p(s_k) s_k$ the average number of branches. We can thus conclude that the expected number of nodes increases exponentially

$$< N_k >= (1+ < s >)^k N_0 \,. \tag{18}$$

Using similar ideas one can prove that the variance of the number of nodes increases according to:

$$\sigma^2_{N_k} = N_0^2 \left[(1+ < s >)^2 + \sigma_s^2\right]^k - (1+ < s >)^{2k} N_0 \,, \tag{19}$$

where σ_s^2 is the variance of the distribution of number of branches.

On the other hand the number of edges can increase, with probability $p(s_k)$, in one iteration by $E^{(s_k,\dots,s_1)} = (1+s_k)E^{(s_{k-1},\dots,s_1)} + s_k$ and thus the expected number of edges do satisfy

$$< E_k >= (1+ < s >)^k (E_0 + 1) - 1 \,. \tag{20}$$

These findings are exact in the case of infinitely many replicas, nevertheless numerical simulations presented in Fig. 5 and in Fig. 6 show the good agreement also for finitely many repetitions.

Remark. *The numerical simulations presented in the following will be obtained assuming for the branch number a Poisson distribution translated by one, more precisely to avoid a non zero probability of drawing zero branches, we drawn with probability $p(k) = \lambda^k e^{-\lambda}/k!$ a non negative integer k, and then we set the number of branches to $s = k+1$, in this way we will get $< s >= \lambda + 1$, $\sigma^2 = \lambda$ and $s \geq 1$.*

Of course our findings are more general and do not rely on the particular choice for $p(s)$.

In a similar way we can compute the expected average degree after k steps, $< (E/N)_k >$, and the expected average node strength after k steps, $< (W/N)_k >$, where W is the total node strength for the given network realization, to get (see Fig. 6):

$$\left\langle \left(\frac{E}{N}\right)_k \right\rangle \underset{k\to\infty}{\to} \frac{E_0+1}{N_0} \quad \text{and} \quad \left\langle \left(\frac{W}{N}\right)_k \right\rangle \underset{k\to\infty}{\to} 0 \,. \tag{21}$$

As we did in the previous section, we are able to analytically study other relevant quantities such as the *expected* value for the *weighted shortest path* $< \lambda_k >$, defined for each network realization by (7). More precisely, starting from a network G_0 and applying iteratively the above construction we end up after k iterations with probability $p(s_k)\dots p(s_1)$ to a network $G^{(s_k,\dots,s_1)}$, we can thus define the weighted shortest path for the given network realization by $\lambda^{(s_k,\dots,s_1)} = \frac{\Lambda^{(s_k,\dots,s_1)}}{\left(N^{(s_k,\dots,s_1)}\right)^2}$. Then using the recursive construction we get:

$$\begin{aligned} \lambda^{(s_k,\dots,s_1)} &= \frac{(F_k+1)\Lambda^{s_{k-1},\dots,s_1} + 2s_k^2\left(N^{(s_{k-1},\dots,s_1)}\right)^2}{(1+s_k)^2\left(N^{(s_{k-1},\dots,s_1)}\right)^2} + \frac{2s_k(F_k+1)\Lambda_{a_k}^{s_{k-1},\dots,s_1} N^{(s_{k-1},\dots,s_1)}}{(1+s_k)^2\left(N^{(s_{k-1},\dots,s_1)}\right)^2} \\ &= \frac{F_k+1}{(1+s_k)^2}\lambda^{(s_{k-1},\dots,s_1)} + 2\left(\frac{s_k}{s_k+1}\right)^2 + \frac{2s_k(F_k+1)}{(1+s_k)^2}\hat{\lambda}^{(s_k,\dots,s_1)} \,, \end{aligned} \tag{22}$$

where we used the growth rate of the number of nodes and (9) and we introduced

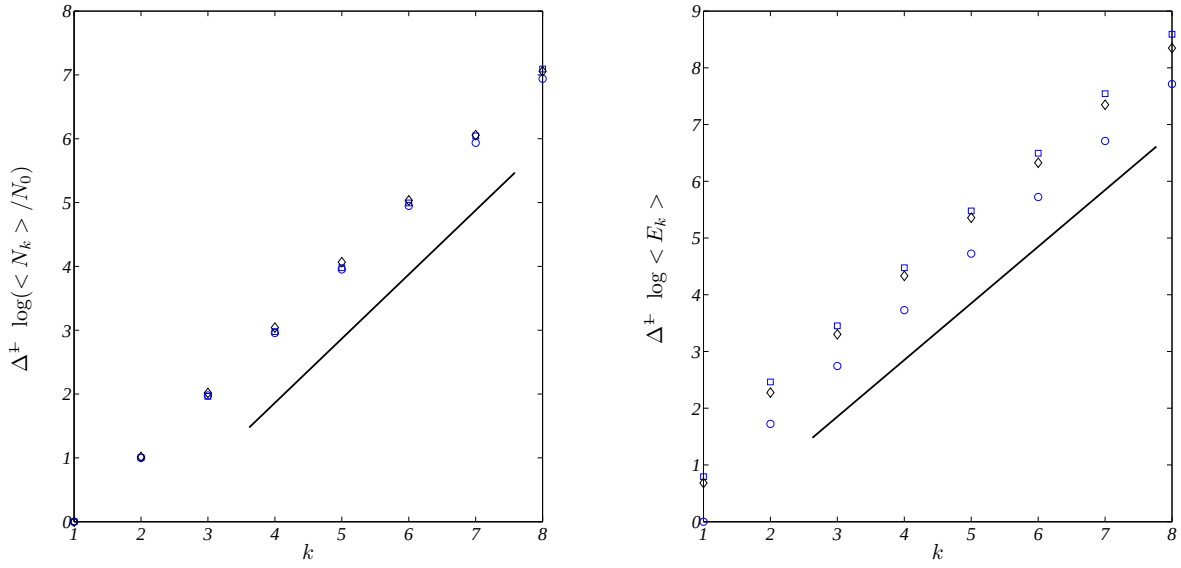

FIG. 5: *Expected values for number of Nodes and number of Edges.* Renormalized quantities : $\Delta^{-1}\log(< N_k > /N_0)$ and $\Delta^{-1}\log < E_k >$ where $\Delta = \log(1+ < s >)$. Symbols refer to : ◯ the SWFN with parameters $\lambda = 4$, $\alpha = 0.5$ and $G_0 =$; □ the SWFN with parameters $\lambda = 2$, $\alpha = 0.5$ and $G_0 =$  ; ◊ the SWFN with parameters $\lambda = 3$, $\alpha = 0.8$ and $G_0 =$  . Expectations are obtained over 100 replicas. Left panel, the reference line has slope 1, linear best fits (data not shown) give 0.9998 ± 0.03 $R^2 = 0.9991$ for ◯ and 1.017 ± 0.008 $R^2 = 0.9999$ for □, 1.008 ± 0.005 $R^2 = 1.000$ for ◊. Right panel, the reference line has slope 1, linear best fits (data not shown) give 0.9569 ± 0.06 $R^2 = 0.9988$ for ◯, 1.019 ± 0.03 $R^2 = 0.9997$ for □ and 1.06 ± 0.07 $R^2 = 0.9955$ for ◊.

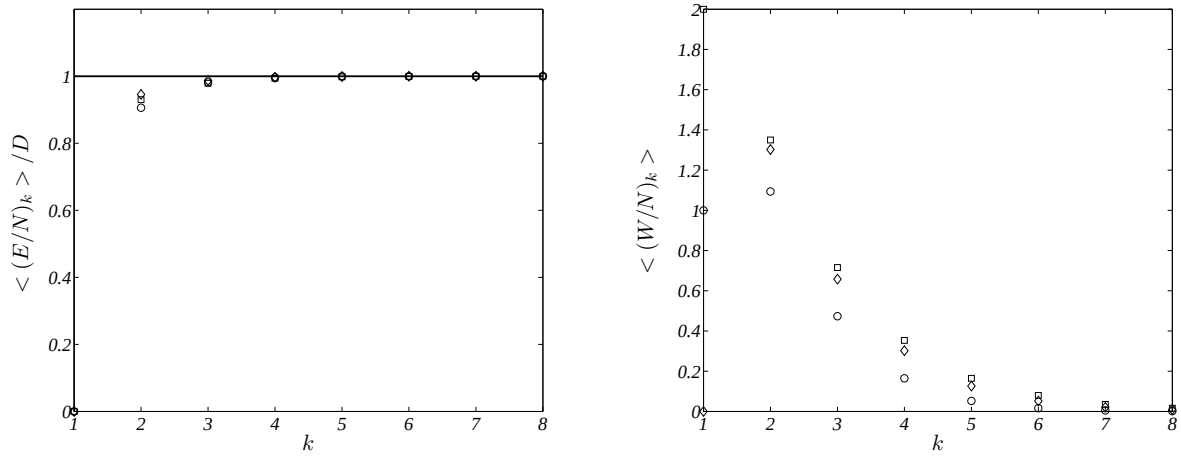

FIG. 6: *Expected values for the average degree and the average node strength.* Renormalized quantities : $< (E/N)_k > /D$ where $D = (E_0 + 1)/N_0$. Symbols are the same of Fig. 5. Expectations made over 100 replicas.

$\hat{\lambda}^{(s_k,\ldots,s_1)} = \Lambda_{a_k}^{(s_k,\ldots,s_1)}/N^{(s_k,\ldots,s_1)}$.

One can finally prove that the expected value for the average weighted shortest path satisfies the recurrence equation:

$$\langle \lambda_k \rangle = \langle \lambda_{k-1} \rangle \left\langle \frac{1+F}{(1+s)^2} \right\rangle + 2 \left\langle \left(\frac{s}{s+1} \right)^2 \right\rangle + 2 \left\langle \frac{s(1+F)}{(1+s)^2} \right\rangle \left\langle \hat{\lambda}_k \right\rangle , \tag{23}$$

where we defined

$$\left\langle \frac{1+F}{(1+s)^2} \right\rangle = \sum_k p(k) \frac{1+F_k}{(1+s_k)^2}\,,$$
$$\left\langle \left(\frac{s}{s+1}\right)^2 \right\rangle = \sum_k p(k) \left(\frac{s_k}{s_k+1}\right)^2 \quad \text{and}$$
$$\left\langle \frac{s(1+F)}{(1+s)^2} \right\rangle = \sum_k p(k) \frac{s_k(1+F_k)}{(1+s_k)^2}\,. \tag{24}$$

Under the simplifying assumption $f_1 = \cdots = f_{s_k} = \alpha/s_k$ we get $F_k = \alpha$ and thus we can simplify the previous equations and obtain (see Fig. 7):

$$< \lambda_k > \underset{k\to\infty}{\to} \left\langle \left(\frac{s}{s+1}\right)^2 \right\rangle \frac{2}{1-(\alpha+1)\left\langle 1/(1+s)^2 \right\rangle} + \left\langle \frac{s}{(1+s)^2} \right\rangle \left\langle \frac{s}{1+s} \right\rangle \frac{2(\alpha+1)}{1-(\alpha+1)\left\langle 1/(1+s)^2 \right\rangle} \frac{1}{1-(\alpha+1)\left\langle 1/(1+s) \right\rangle}\,. \tag{25}$$

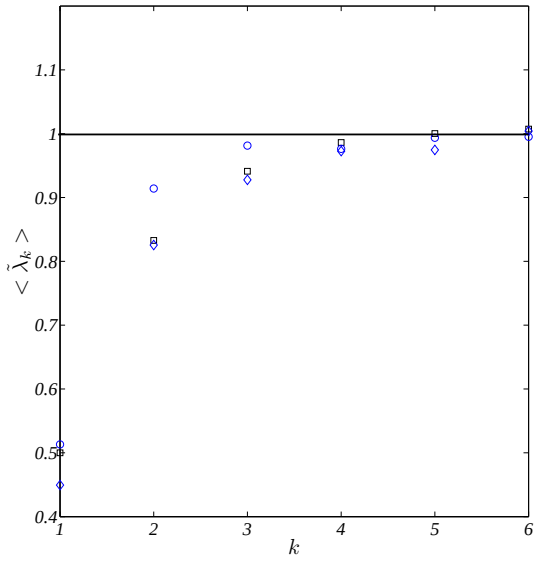

FIG. 7: *Expected values for the average weighted shortest path.* Renormalized quantities: $< \tilde{\lambda}_k >= L^{-1} < \lambda_k >$, where L is the right hand side of Eq. (25). Symbols are the same of Fig. 5. Expectations made over 20 replicas.

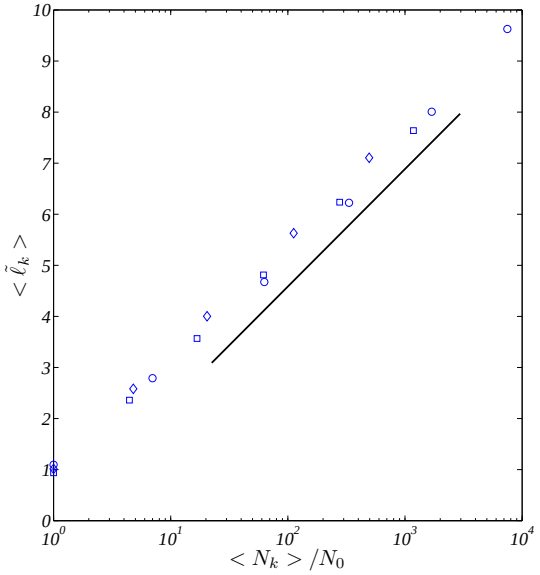

FIG. 8: *Expected values for the average shortest path as a function of the network size* (semilog plot). Semilog plot of the renormalized expected average shortest path $< \tilde{\ell}_k >=< \ell_k > /M$ versus the network size N_k, where M is the right hand side of Eq. (26). Symbols are the same of Fig. 5. Expectations made over 20 replicas. The reference line has slope 1, linear best fits (data not shown) provides 0.95 ± 0.01 with $R^2 = 0.9999$ for $\square$, 0.95 ± 0.07 $R^2 = 0.9968$ for $\bigcirc$ and 0.98 ± 0.01 $R^2 = 0.9999$ for $\Diamond$.

One can consider the *expected shortest path* by formally set all the scaling factors equal to 1 and similar technics allow to conclude that (see Fig. 8)

$$< \ell_k > \underset{k\to\infty}{\sim} \left\langle \left(\frac{s}{s+1}\right)^2 \right\rangle \frac{2}{1-\langle 1/(s+1)\rangle} \frac{1}{\log(1+< s >)} \log \frac{< N_k >}{N_0}\,. \tag{26}$$

Remark. *Let us observe that in the case where only one value of s is possible, i.e. the probability distribution of the number of branches reduces to a δ–distribution, $p(s) = \delta_{s,s'}$, then the above result coincide with the ones presented for the WFN in Section II.*

IV. CONCLUSIONS

In this paper we proposed a unifying general framework for complex weighted networks sharing several properties with fractal sets, the *Stochastic Weighted Fractal Networks*. This theory, that generalizes to networks the construction of physical fractals, allows us to build complex networks with a prescribed topology,

whose main quantities can be analytically predicted in terms of expectations and have been shown to depend on the fractal dimension of some underlying fractal; for instance the networks are scale-free, the exponent being the related to the fractal dimension of the underlying IFS. Moreover the SWFN share with fractals, the self-similar or self-affine structure.

These networks exhibit the small-world property. In fact the average shortest path increases logarithmically with the system size; hence it is as small as the average shortest path of a random network with the same number of nodes and same average degree. On the other hand the clustering coefficient is asymptotically constant, thus larger than the clustering coefficient of a random network that shrinks to zero as the system size increases.

As already observed [39] the self-similarity property of the SWFN make them suitable to model real problems involving some kind of diffusion over the network coupled with local looses of flow, here modeled via the parameters $f < 1$. Moreover the stochastic growth process allows us to introduce more realism in the construction and thus to extend the applicability domain of our framework to evolving structures.

[1] B.B. Mandelbrot, *How Long Is the Coast of Britain? Statistical Self-Similarity and Fractional Dimension*, Science, **156**, 3775, (1967), pp. 636-638.

[2] M. Cieplak et al., *Models of fractal river basins*, J. Stat. Phys., **91**, 1/2, (1998), pp. 1.

[3] I. Rodríguez-Iturbe and A. Rinaldo, *Fractal River Basins*, Cambridge University Press, (2001).

[4] J. Nittman and H.E. Stanley, *Non-deterministic approach to anisotropic growth patterns with continuously tunable morphology: the fractal properties of some real snowflakes*, J. Phys. A, **20**, (1987), pp. L1185.

[5] T. Matsuyama, M. Sogawa and Y. Nakagawa, *Fractal spreading growth of Serratia marcescens which produces surface active exolipids*, FEMS Microb. Lett., **61**, (1989), pp. 243.

[6] H. Fujikawa and M. Matsushita, *Fractal Growth of Bacillus subtilis on Agar Plates*, J. Phys. Soc. Jpn, **58**, (1989), pp. 387

[7] H. Fujikawa and M. Matsushita, *Bacterial Fractal Growth in the Concentration Field of Nutrient*, J. Phys. Soc. Jpn, **60**, (1991), pp. 88.

[8] B. Suki, A.-L. Barabási, Z. Hantos, F. Peták and H.E. Stanley *Avalanches and power–law behaviour in lung inflation*, Nature **368**, 615, (1994).

[9] A.-L. Barabási, S.V. Buldyrev, H.E. Stanley and B. Suki *Avalanches in lung: a statistical mechanical model* , Phys. Rev. Lett. **76**, (12), 2192, (1996).

[10] H. Kitaoka and B. Suki *Branching design of the bronchial tree based on diameter–flow relationship*, J. Appl. Physiol. **82**, 968, (1997).

[11] J.S. Andrade et al. *Asymmetric flow in symmetric branched structures*, Phys. Rev. Lett. **81**, (4), 926, (1998).

[12] A. Kamiya and T. Takahashi *Quantitative assessments of morphological and functional properties of biological trees based on their fractal nature*, J. Appl. Physiol. **102**, 2315, (2007).

[13] T. Vicsek, *Fractal growth phenomena*, World Scientific, (1992).

[14] D. Stauffer and H.E. Stanley, *From Newton to Mandelbrot*, Springer, (1996).

[15] R. Hohlfeld and N. Cohen *Self-similarity and the geometric requirements for frequency independence in Antennae*, Fractals, **7**, 1, (1999), pp. 7984.

[16] B.B. Mandelbrot, *The variation of certain speculative prices*, Journ. Business, **36**, (1963), pp. 394-419.

[17] B.B. Mandelbrot, *The Fractal Geometry of Nature*, W.H. Freeman and Company, New York (1982).

[18] P. Erdős and A. Rényi *On random graphs*, Publ. Math. Debrecen **6**, 290, (1959).

[19] R. Albert and A.-L. Barabási *Statistical mechanics of complex networks*, Rev. Mod. Phys. **74**, 47, (2002).

[20] S. Boccaletti, V. Latora, Y. Moreno, M. Chavez and D.-H. Hwang *Complex networks: Structure and dynamics*, Phys. Rep. **424**, 175, (2006).

[21] A.-L. Barabási and R. Albert *Emergence of Scaling in Random Networks*, Science **286**, 509, (1999).

[22] D.J. Watts and S.H. Strogatz *Collective dynamics of 'small-world' networks*, Nature **393**, 440, (1998).

[23] S. Fortunato *Community detection in graphs*, Physics Reports, **486**, 75, (2010).

[24] S.H. Yook, H. Jeong and A.-L. Barabási *Weighted evolving networks*, Phys. Rev. Lett. **86**, (25), 5835, (2001).

[25] A. Barrat, M. Barthélemy, R. Pastor-Satorras and A. Vespignani. *The architecture of complex weighted networks*, Proc. Nat. Acad. Sci. USA **101**, 3747, (2004).

[26] A. Barrat, M. Barthélemy and A. Vespignani *Modeling the evolution of weighted networks*, Phys. Rev. E **70**, 066149, (2004).

[27] D. Zheng, S. Trimper, B. Zheng and P.M. Hui, *Weighted scale–free networks with stochastic weight assigments*, Phys. Rev. E **67**, 040102, (2003).

[28] S.N. Dorogovtsev and J.F.F. Mendes *Minimal model of weighted scale–free networks*, preprint cond-mat/0408343v2, (2004).

[29] A. Barrat, M. Barthélemy and A. Vespignani *Weighted evolving networks: coupling topology and weight dynamics*, Phys. Rev. Lett. **92**, 22, (2004), pp. 066149.

[30] A.-L. Barabási, E. Ravasz and T. Vicsek *Deterministic scale–free networks*, Physica A, **299**, 559, (2001).

[31] S.N. Dorogovtsev, A.V. Goltsev and J.F.F. Mendes *Pseudofractal scale–free web*, Phys. Rev. E **65**, 066122, (2002).

[32] S. Jung, S. Kim and B. Kahng *Geometric fractal growth model for scale–free networks*, Phys. Rev. E **65**, 056101, (2002).

[33] E. Ravasz and A.-L. Barabási *Hierarchical organization in complex networks*, Phys. Rev. E **67**, 026112, (2003).

[34] Z. Zhang et al., *Incompatibily networks as models of scale–free small–world graphs*, Eur. Phys. J.B., **60**, (2007), pp. 259.

[35] Z. Zhang et al., *Recursive weighted treelike networks*, Eur. Phys. J.B., **59**, (2007), pp. 99.

[36] Z.Z. Zhang, S.G. Zhou, W.L. Xie, L.C. Chen, Y.Lin, and J.H. Guan, *Standard random walks and trapping on the Koch network with scale-free behavior and small-world effect*, Phys. Rev. E, **79**, (2009), pp. 061113.

[37] J. Guan, Y. Wu, Z. Zhang, S. Zhou, and Y. Wu *A unified model for Sierpinski networks with scale-free scaling and small-world effect*, Physica A **388**, 2571-2578, (2009).

[38] Y. Zhang, Z. Zhang, S. Zhou and J. Guan, *Deterministic weighted scale–free small–world networks*, arxiv:0910.1140v1 [cond-mat.stat-mech]

[39] T. Carletti and S. Righi, *Weighted Fractal Networks*, accepted Physica A, (2010).

[40] D.Y.C. Chan, B.D. Hughes, A.S. Leong and W.J. Reed, *Stochastically evolving networks*, Phys. Rev. E, **68**, (2003), pp.066124.

[41] L. Wang, F. Du, H.P. Dai and Y.X. Sun, *Random pseudofractal scale–free networks with small–world effect*, Eur. Phys. J.B, **53**, (2006), pp. 361.

[42] L. Wang, H.P. Dai and Y.X. Sun, *General random pseudofractal networks*, J. Phys. A, **40**, (2007), pp. 13279.

[43] Z. Zhang, S. Zhou, Z. Su, T. Zou and J. Guan *Random Sierpinski network with scale-free small-world and modular structure*, Eur. Phys. J. B. **65**, 141-147, (2008).

[44] B.B. Mandelbrot, *Self-affine fractals and fractal dimension*, Physica Scripta, **32**, (1985), pp. 257.

[45] B.B. Mandelbrot, in *Fractals in Physics*, edited by L. Pietronero and E. Tosatti (Elsevier, Amsterdam), (1986), pp. 3.

[46] M. Barnsley, *Fractals everywhere*, Academic Press London (1988)

[47] G.A. Edgar, *Measure, Topology and fractal geometry*, UTM, Springer–Verlag, New York (1990).

[48] J. Saramäki, M. Kivelä, J.-P. Onnela, K. Kaski and J. Kertész *Generalizations of the clustering coefficient to weighted complex networks*, Phys. Rev. E **75**, 027105, (2007).

[49] Mäkinen, V. Himmeli, a free software package for visualizing complex networks, available at http://www.artemis.kll.helsinki.fi/himmeli.

[50] The interested reader could find many more example in the beautiful books [13, 14].