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Spectral factorization and LQ-optimal regulation for multivariable distributed systems

FRANK M. CALLIER† and JOSEPH WINKIN†

A necessary and sufficient condition is proved for the existence of a bistable spectral factor (with entries in the distributed proper-stable transfer function algebra $\hat{\mathscr{A}}_-$) in the context of distributed multivariable convolution systems with no delays; a by-product is the existence of a normalized coprime fraction of the transfer function of such a possibly unstable system (with entries in the algebra $\hat{\mathscr{B}}$ of fractions over $\hat{\mathscr{A}}_-$). We next study semigroup state-space systems SGB with bounded sensing and control (having a transfer function with entries in $\hat{\mathscr{B}}$) and consider its standard LQ-optimal regulation problem having an optimal state feedback operator K_0. For a system SGB, a formula is given relating any spectral factor of a (transfer function) coprime fraction power spectral density to K_0; a by-product is the description of any normalized coprime fraction of the transfer function in terms of K_0. Finally, we describe an alternative way of finding the solution operator K_0 of the LQ-problem using spectral factorization and a diophantine equation: this is similar to Theorem 2 of Kucera (1981) for lumped systems.

Nomenclature

$\mathbb{R}$ (respectively $\mathbb{R}_-$, $\mathbb{R}_+$)	set of real (respectively non-positive real, non-negative real) numbers
$\mathbb{C}$	field of complex numbers
$\mathbb{C}_{\sigma+}$ (respectively $\mathbb{C}^0_{\sigma+}$)	$\{s \in \mathbb{C} : \mathrm{Re}(s) \geqslant \sigma$, (respectively $> \sigma$)$\}$ (σ is omitted if $\sigma = 0$)
S_σ, (respectively S^0_σ)	$\{s \in \mathbb{C} : \sigma \leqslant \mathrm{Re}(s) \leqslant -\sigma$, (respectively $\sigma < \mathrm{Re}(s) < -\sigma$)$\}$
LTD, (respectively LTD^-, LTD^+)	set of $\mathbb{C}$-valued Laplace transformable distributions with support on $\mathbb{R}$ (respectively $\mathbb{R}_-$, $\mathbb{R}_+$)
$\delta(\cdot)$	Dirac delta distribution (Dirac impulse)
$\hat{f}(\cdot)$	(two-sided) Laplace transform of $f \in$ LTD
$\hat{A}$	set of Laplace transforms of all $f \in A$
$L_{1\sigma}$	class of all functions f, with support on $\mathbb{R}_+$, such that $\int_0^\infty \lvert f(t)\rvert \exp(-\sigma t)\, dt < \infty$
Mat (A)	set of matrices having entries in A
M^*	adjoint of the operator M; hermitian transpose, when M is a matrix
$F_*(t)$	$F(-t)^*$, parahermitian transpose of $F \in$ Mat (LTD), equivalently $\hat{F}_*(s) = \hat{F}(-\bar{s})^*$ ($= \hat{F}(j\omega)^*$ for $s = j\omega$)
$M \geqslant 0$ (respectively > 0)	M is a positive semi-definite (respectively posi-

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	tive definite) matrix
$\mathscr{L}(X, Y)$	space of linear bounded (i.e. continuous) operators from X into Y (denoted by $\mathscr{L}(X)$ if $X = Y$)
RHS (LHS respectively)	right-hand side (left-hand side)
SGB	semigroup state-space system with bounded sensing and control
ORE	operator Riccati equation
$D(A)$	the domain of the operator A

Preliminary remark

To any distribution $f = f_a(\cdot) + f_0\delta(\cdot)$, (where f_a is a $\mathbb{C}$-valued function and $f_0 \in \mathbb{C}$), we associate $f^+ := f_a^+(\cdot) + 2^{-1}f_0\delta(\cdot) \in \mathrm{LTD}^+$ and $f^- := f_a^-(\cdot) + 2^{-1}f_0\delta(\cdot) \in \mathrm{LTD}^-$ such that $f_a^+ := f_a$ almost everywhere on $\mathbb{R}_+$, $f_a^- := f_a$ almost everywhere on $\mathbb{R}_-$ and $f = f^+ + f^-$.

1. Introduction

Spectral factorization for multivariable linear time-invariant systems can be viewed as a symmetric causal–anticausal factorization problem which plays a fundamental role in control. For example, it is useful in LQ- and H^∞-optimal control theory (e.g. Francis 1987). It is also used for solving certain types of integral equations (e.g. Krein 1962, Gohberg and Krein 1960). Here it is studied first in the context of multivariable distributed convolution systems with no delays and next, of state-space semigroup system realizations with bounded sensing and control. We are motivated by the possibility of applications to robust feedback stability (e.g. Vidyasagar 1985, Callier and Winkin 1987) and to LQ-optimal regulation (e.g. Kucera 1981).

In this theoretical paper we do not cover the most general cases. Our motivation is a typical one in theoretical applied science: to create relatively simple learning tools and models leading hopefully to insight, connections and generalization.

The paper is organised as follows. Section 2 describes our framework of distributed system transfer functions and state-space realizations. Section 3, on spectral factorization and the operator Riccati equation first gives a spectral factorization existence condition in our transfer function framework (Theorem 1); this leads to the existence of normalized coprime fractions of any possibly unstable transfer function (Corollary 1); next, in the context of our state-space realizations we consider the (transfer function) coprime fraction (power spectral density) spectral factorization problem (see (48) below): we discuss the structure of any spectral factor using Riccati-based LQ-optimal state-feedback (Theorem 2), which in turn determines normalized coprime fractions of the transfer function (Corollary 2). Finally in § 4 on spectral factorization based LQ-optimal regulation, we find an alternative way of solving the LQ-problem using spectral factorization and a diophantine equation (Theorem 3): the diophantine equation here determines the optimal state feedback operator (controller) given the spectral factor (closed-loop dynamics), playing its usual role in transfer function feedback system design: controller determination given the closed-loop dynamics (e.g. Kucera 1979, Chapter 3, Callier and Desoer 1982, equation (6.2.32), Vidyasagar 1985, equation (5.1.27)).

2. Transfer function and state-space system framework

The following classes of distributed system transfer functions are described by Callier and Desoer (1978, 1980 a, b):

Let $\sigma \leqslant 0$. An impulse response $f \in \text{LTD}^+$ is said to be in $\mathscr{A}(\sigma)$ if and only if for $t<0$, $f(t)=0$, and for $t \geqslant 0$, $f(t)=f_a(t)+f_{sa}(t)$ where the regular functional part $f_a \in L_{1\sigma}$ and the singular atomic part $f_{sa} := \Sigma_{i=0}^{\infty} f_i \delta(\cdot - t_i)$, where $t_0 = 0$, $t_i > 0$ for $i = 1, 2, \ldots$ and $f_i \in \mathbb{C}$ for $i = 0, 1, \ldots$ with $\Sigma_{i=0}^{\infty} |f_i| \exp(-\sigma t_i) < \infty$.

An impulse response f is said to be in $\mathscr{A}_-$ if and only if $f \in \mathscr{A}(\sigma)$ for some $\sigma < 0$. $\mathscr{A}(\sigma)$ and $\mathscr{A}_-$ are convolution algebras. $\hat{\mathscr{A}}_-$ (the class of Laplace transforms of elements in $\mathscr{A}_-$) is our selected class of distributed proper-stable transfer functions. It contains the multiplicative subset $\hat{\mathscr{A}}_-^{\infty}$, i.e. of transfer functions that are bounded away from zero at infinity in $\mathbb{C}_+$, i.e. that are biproper-stable. Possibly open-loop unstable transfer functions are selected to be in the algebra $\hat{\mathscr{B}}$, where $\hat{f} \in \hat{\mathscr{B}}$ if and only if $\hat{f} = \hat{n} \cdot \hat{d}^{-1}$ with $\hat{n} \in \hat{\mathscr{A}}_-$ and $\hat{d} \in \hat{\mathscr{A}}_-^{\infty}$. Note that by Theorem 3.3 of Callier and Desoer (1978), a transfer function is in $\hat{\mathscr{B}}$ if and only if it is the sum of a completely unstable strictly proper rational function and a stable function in $\hat{\mathscr{A}}_-$; hence $\hat{d}$ above can always be chosen biproper-stable *rational* (e.g. Vidyasagar 1985 Fact 20 p. 13). Multivariable plants have transfer matrices $\hat{P}$ in Mat ($\hat{\mathscr{B}}$) described by a right matrix fraction $\hat{P} = \hat{N}\hat{D}^{-1}$ where $\hat{N}$ and $\hat{D}$ are in Mat ($\hat{\mathscr{A}}_-$) and det $\hat{D}$ is in $\hat{\mathscr{A}}_-^{\infty}$; if this holds and there exist $\hat{U}$ and $\hat{V}$ in Mat ($\hat{\mathscr{A}}_-$) such that $\hat{U}\hat{N} + \hat{V}\hat{D} = I$ (the Bezout identity), (or equivalently $[\hat{N}(s)^{\mathrm{T}} \ \hat{D}(s)^{\mathrm{T}}]^{\mathrm{T}}$ has full column rank in $\mathbb{C}_+$), then we say that $\hat{P}$ in Mat ($\hat{\mathscr{B}}$) has a right coprime fraction (r.c.f.) $(\hat{N}, \hat{D})$ in Mat ($\hat{\mathscr{A}}_-$); r.c.f.s are unique up to multiplication on the right by a factor in Mat ($\hat{\mathscr{A}}_-$) together with its inverse; moreover $\hat{D}$ above can always be chosen biproper-stable *rational* such that $\hat{D}(\infty) = I$ (Callier and Desoer 1980 b proof of Theorem 2.1).

Warning

None of the impulse responses below has any *delayed impulses*. Hence it is important to consider the sub-class $L\Delta^+(\sigma)$ of $\mathscr{A}(\sigma)$ given by

$$L\Delta^+(\sigma) := \{f \in \mathscr{A}(\sigma) : f = f_a(\cdot) + f_0(\cdot),\ f_a(\cdot) \in L_{1\sigma} \text{ and } f_0 \in \mathbb{C}\} \tag{1}$$

where the following fact holds.

Fact 1: *Properties of* $L\Delta^+(\sigma) \subset \mathscr{A}(\sigma)$

Let $\sigma \leqslant 0$. Then

(*a*) $\mathscr{A}(\sigma)$ is a commutative convolution Banach algebra with unit element $\delta(\cdot)$ under the norm

$$\|f\|_{\mathscr{A}(\sigma)} := \int_0^{\infty} |f_a(t)| \exp(-\sigma t)\, dt + \sum_{i=0}^{\infty} |f_i| \exp(-\sigma t_i) \tag{2}$$

(*b*) $L\Delta^+(\sigma)$ given by (1) is a closed sub-algebra of $\mathscr{A}(\sigma)$

(*c*) If $f = f_a(\cdot) + f_0\delta(\cdot) \in L\Delta^+(\sigma)$, then

(i) $\hat{f}$ is uniformly continuous in $\mathbb{C}_{\sigma+}$ (3)

(ii) $\hat{f}$ is holomorphic in $\mathbb{C}_{\sigma+}^{0}$ (4)

(iii) $\hat{f}(s) \to f_0$ as $|s| \to \infty$ in $\mathbb{C}_{\sigma+}$ (5)

where in particular

$$|\hat{f}_a(\sigma_1 + j\omega)| \to 0 \text{ as } |\omega| \to \infty \text{ uniformly in } \sigma_1 \in [\sigma, \sigma'] \text{ for any } \sigma' \geq \sigma \text{ (Riemann–Lebesgue)} \tag{6}$$

(*d*) $F = F_a(\cdot) + F_0\delta(\cdot)$ is invertible in Mat $(L\Delta^+(\sigma)) \subset$ Mat $(\mathscr{A}(\sigma))$

if and only if

$$\inf\{|\det \hat{F}(s)| : s \in \mathbb{C}_{\sigma+}\} > 0 \tag{7 a}$$

or equivalently with $\det F_0 \neq 0$,

$$\det \hat{F}(s) \neq 0 \quad \forall s \in \mathbb{C}_{\sigma+} \tag{7 b}$$

(*e*) Let $F = F_a(\cdot) + F_0\delta(\cdot)$ be in Mat $(L\Delta^+(\sigma_1))$ for some $\sigma_1 < 0$ and hence in Mat $(\mathscr{A}_-)$. Then

$$\hat{F}^{-1} \in \text{Mat}\,(\hat{\mathscr{A}}_-) \text{ i.e. } \hat{F}^{-1} \in \text{Mat}\,(\widehat{L\Delta}^+(\sigma)) \text{ for some } \sigma < 0$$

if and only if

$$\inf\{|\det \hat{F}(s)| : s \in \mathbb{C}_+\} > 0 \tag{8 a}$$

or equivalently with $\det F_0 \neq 0$,

$$\det \hat{F}(s) \neq 0 \quad \forall s \in \mathbb{C}_+ \tag{8 b}$$

Comment 1

(*a*) The proof of Fact 1 follows by a straightforward adaptation of results due to Callier and Desoer (1978 p. 652, 1980 a). For Banach algebras, see, for example, Rudin (1970). For matrix algebras, see also Vidyasagar (1985 Fact 26 p. 393).

(*b*) $L\Delta^+(0)$ is denoted $\mathscr{L}$ by Vidyasagar and Anderson (1989 p. 96) where it is shown that an impulse response f in $\mathscr{A}(0)$ can be well-approximated by a lumped impulse response if and only if $f \in L\Delta^+(0)$. $\widehat{L\Delta}^+(0)$ is denoted $\mathscr{R}^+$ by Krein (1962 p. 173) and Gohberg and Krein (1960 p. 231) and has been used extensively.

(*c*) The impulse responses below have no delayed impulses, i.e. they have the form $f = f_a(\cdot) + f_0\delta(\cdot) \in$ Mat (LTD$^+$) where $f_a(\cdot)$ is a function and f_0 is a complex constant. Hence under this condition we have

(i) $\hat{f} \in \hat{\mathscr{A}}(\sigma)$ if and only if $\hat{f} \in \widehat{L\Delta}^+(\sigma)$;

(ii) $\hat{f} \in \hat{\mathscr{A}}_-$ ($\in \hat{\mathscr{A}}_-^\infty$ respectively) if and only if $\hat{f} \in \widehat{L\Delta}^+(\sigma)$ for some $\sigma < 0$ (and moreover $f_0 \neq 0$);

(iii) $\hat{f} \in \hat{\mathscr{B}}$ if and only if $f \in \hat{\mathscr{B}}_p$;

where $\hat{\mathscr{B}}_p$ is the sub-algebra of *constantly proper* elements of $\hat{\mathscr{B}}$, i.e.

$$\hat{\mathscr{B}}_p := \{f \in \hat{\mathscr{B}} : \hat{f}(s) \to \kappa \in \mathbb{C} \text{ as } |s| \to \infty \text{ in } \mathbb{C}_+\} \tag{9}$$

(*d*) Every *constantly proper* plant $\hat{P}$ in Mat $(\hat{\mathscr{B}}_p)$ has an r.c.f. $(\hat{N}, \hat{D})$ in Mat $(\widehat{L\Delta}^+(\sigma))$ for some $\sigma < 0$, where

$$N(t) = N_a(t) + N_0\delta(t) \tag{10 a}$$

$$D(t) = D_a(t) + I\delta(t) \tag{10 b}$$

with $N_a(\cdot)$ and $D_a(\cdot)$ in Mat $(L_{1\sigma})$ for some $\sigma < 0$. This structure is necessary as soon as one requires the denominator distribution $D(\cdot)$ to have no delayed impulses with a singular atomic part $D_{sa}(\cdot) = I\delta(\cdot)$. Hence by Vidyasagar and Anderson (1989 Theorems 4.2 and 5.1) such a $\hat{P}$ can be approximated and stabilized by lumped systems. Note finally that the power spectral density $\hat{F} = \hat{N}_*\hat{N} + \hat{D}_*\hat{D}$ originates from a correlation distribution $F \in \mathrm{Mat}\,(\mathrm{LTD})$ of the form $F(t) = F_a(t) + F_0\delta(t)$ where $F_a^+(\cdot)$ is in $\mathrm{Mat}\,(L_{1\sigma})$ for some $\sigma < 0$ and F_0 is a constant hermitian positive definite matrix. The spectral factorization of such an $\hat{F}$ is handled below.

The state-space systems below are of class SGB (i.e. semigroup systems with bounded sensing and control); see, for example, Curtain and Pritchard (1978) for the details. We shall denote the domain of the operator A by $D(A)$ and the uniform operator norm by $\|\cdot\|$.

Definition 1

An m-input p-output SGB state-space system is described by the equations

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) & \text{for } x(0) = x_0 \in D(A) \\ y(t) = Cx(t) \end{cases} \quad t \geqslant 0$$

where

(*a*) $x(t) \in X$, a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$; $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^p$;

(*b*) $A : D(A) \subset X \to X$ is the infinitesimal generator of a C_0-semigroup $(\exp(At))_{t \geqslant 0}$ of bounded linear operators in X, i.e. $\exp(At) \in \mathscr{L}(X)$ for all $t \geqslant 0$; and

(*c*) B and C are bounded linear operators, i.e. $B \in \mathscr{L}(\mathbb{R}^m, X)$ and $C \in \mathscr{L}(X, \mathbb{R}^p)$.

A SGB semigroup system has a transfer function

$$\hat{P}(s) = C(sI - A)^{-1}B \tag{11}$$

where $(sI - A)^{-1}$ is the resolvent of A (usually denoted by $R(s, A)$, i.e. the Laplace transform of $t \mapsto \exp(At)$): it is a bounded linear operator from X into $D(A) \subset X$ for all s in an open right half-plane; as a function of s it is there a holomorphic ($\mathscr{L}(X)$-operator valued) function. Hence $\hat{P}(s)$, given by (11), is a well-defined $p \times m$-matrix-valued transfer function holomorphic in an open right half-plane.

The following definitions are important to ensure that $\hat{P}$ is in $\mathrm{Mat}\,(\hat{\mathscr{B}})$ and for obtaining a well-defined LQ-optimal regulation problem.

Definition 2

Consider any SGB system.

(*a*) SGB is said to be *internally exponentially stable* if and only if the *semigroup* $(\exp(At))_{t \geqslant 0}$ is *exponentially stable*, i.e. there exist $\sigma < 0$ and $M > 0$ such that $\|\exp(At)\| \leqslant M \exp(\sigma t)$ on $t \geqslant 0$.

(*b*) The operator pair (A, B) is said to be *exponentially stabilizable* if and only if there exists a *stabilizing feedback* $K \in \mathscr{L}(X, \mathbb{R}^m)$, i.e. such that the semigroup $(\exp(A + BK)t)_{t \geqslant 0}$ is exponentially stable.

(*c*) The operator pair (C, A) is said to be *exponentially detectable* if and only if there exists a *stabilizing injection* $F \in \mathscr{L}(\mathbb{R}^p, X)$, i.e. such that the semigroup $(\exp(A + FC)t)_{t \geqslant 0}$ is exponentially stable.

The following obvious lemma plays a fundamental role.

Lemma 1

Consider any SGB system. Then if (A, B) is exponentially stabilizable with $K \in \mathscr{L}(X, \mathbb{R}^m)$ any stabilizable feedback, then

(*a*) $(K, A + BK)$ is exponentially detectable;

(*b*) $(A + BK, B)$ is exponentially stabilizable;

(*c*) (K, A) is exponentially detectable.

Lemma 2 below follows by a paramount characterization of joint stabilizability and detectability in infinite-dimensional space given by Jacobson and Nett (1988 Theorem 3.2) (see also Nefedov and Sholokhovich 1986); for parts (*c*) and (*d*) see Jacobson (1986 Theorem 2 pp. 25–30).

Lemma 2

Consider an SGB system with transfer function $\hat{P}$ given by (11). Assume that

$$(A, B) \text{ is exponentially stabilizable and } (C, A) \text{ is exponentially detectable} \tag{12}$$

Then

(*a*)
$$\hat{P} \in \text{Mat}(\hat{\mathscr{B}}) \tag{13}$$

(*b*) the SGB is internally exponentially stable if and only if the SGB is externally stable, i.e. $\hat{P} \in \text{Mat}(\hat{\mathscr{A}}_-)$;

(*c*) for any stabilizing feedback $K \in \mathscr{L}(X, \mathbb{R}^m)$, the pair $(\hat{N}, \hat{D})$ given by

$$(\hat{N}, \hat{D}) = (C(sI - A - BK)^{-1}B, I + K(sI - A - BK)^{-1}B) \tag{14 a}$$

is in $\text{Mat}(\hat{\mathscr{A}}_-)$ and is an r.c.f. of $\hat{P}$;

(*d*)
$$\hat{P}(s) \to 0 \text{ as } |s| \to \infty \text{ in } \mathbb{C}_+ \tag{15}$$

i.e. $\hat{P}(s)$ is strictly proper, or equivalently $P(t) \in \text{Mat}(\text{LTD}^+)$ contains no impulses.

Comment 2

(*a*) Recall the stable–unstable sum decomposition of Callier and Desoer (1980 b Proof of Theorem 2.1) or Curtain and Pritchard (1978 pp. 75 *et seq.*) with (A_u, B_u, C_u) a *matrix* minimal realization of the unstable rational part of $\hat{P}$ above: one can always reduce the feedback K in (14 *a*) to a finite-dimensional state feedback matrix K_u applied to this unstable part: (14 *a*) then reads

$$(\hat{N}, \hat{D}) = (\hat{P}\hat{D}, \hat{D}) \tag{14 b}$$

where

$$\hat{D}(s) = [I - K_u(sI - A_u)^{-1}B_u]^{-1} \tag{14 c}$$

is biproper-stable rational with $\hat{D}(\infty) = I$, $(\hat{D}(s)^{-1}$ is the control-loop return difference).

(*b*) The structure of $(\hat{N}, \hat{D})$ given by (14 *a*) is as in (10) where $N_0 = 0$ because $\hat{P}$ in $\text{Mat}(\hat{\mathscr{B}})$ is strictly proper.

3. Spectral factorization and the operator Riccati equation

In this section we first report the solution of a standard spectral factorization problem delivering a bistable spectral factor in Mat $(\hat{\mathscr{A}}_-)$ originating from a distribution on $t \geqslant 0$ with *no delayed impulses*, i.e. in Mat $(\widehat{L\Delta}^+(\sigma))$ for some $\sigma < 0$ (see (1) for a definition of $L\Delta^+(\sigma)$ and Fact 1 for properties). In order to consider the spectral density $\hat{F}$ on the imaginary axis, or equivalently the correlation distribution F, it is natural to introduce the following definitions and facts.

Definition 3

Let $\sigma \leqslant 0$ and $L\Delta^+(\sigma)$ be given by (1).

$$L\Delta^-(\sigma) := \{f \in \mathrm{LTD}^- : f(-\cdot) \in L\Delta^+(\sigma)\} \tag{16}$$

$$L\Delta(\sigma) := \left\{ f \in \mathrm{LTD} : f = f_a + f_0\delta(\cdot), \text{ with } f_0 \in \mathbb{C} \text{ and } f_a(\cdot) \text{ a } \mathbb{C}\text{-valued function such that } \int_{-\infty}^{\infty} |f_a(t)| \exp(-\sigma|t|)\, dt < \infty \right\} \tag{17}$$

We have

$$L\Delta(\sigma) = L\Delta^+(\sigma) + L\Delta^-(\sigma) \tag{18}$$

$$L\Delta(\sigma_1) \subset L\Delta(\sigma_2) \quad \forall\, \sigma_1 \leqslant \sigma_2 \leqslant 0 \tag{19}$$

We equip $L\Delta(\sigma)$ with a two-sided convolution product and a norm $\|\cdot\|_{1\sigma}$ as follows.

Let $f = f_a(\cdot) + f_0\delta(\cdot)$ and $g = g_a(\cdot) + g_0\delta(\cdot)$ be in $L\Delta(\sigma)$, then

$$(f * g)(t) := \int_{-\infty}^{\infty} f_a(t-s) g_a(s)\, ds + f_0 g_a(t) + f_a(t) g_0 + f_0 g_0 \delta(t)$$
$$=: \int_{-\infty}^{\infty} f(t-s) g(s)\, ds \quad \text{for } t \in \mathbb{R} \tag{20}$$

and

$$\|f\|_{1\sigma} := \int_{-\infty}^{\infty} |f_a(t)| \exp(-\sigma|t|)\, dt + |f_0| \tag{21}$$

Recall here the preliminary remark and the norm $\|\ \|_{\mathscr{A}(\sigma)}$ given in (2). We have for all f, g in $L\Delta(\sigma)$,

$$\|f\|_{1\sigma} = \|f^+(\cdot)\|_{\mathscr{A}(\sigma)} + \|f^{-1}(-\cdot)\|_{\mathscr{A}(\sigma)}$$

and

$$\|f * g\|_{1\sigma} \leqslant \|f\|_{1\sigma} \cdot \|g\|_{1\sigma}$$

Note that $L\Delta^+(0)$, $L\Delta^-(0)$, and $L\Delta(0)$, respectively, are denoted $\mathscr{R}^+$, $\mathscr{R}^-$ and $\mathscr{R}$ by Krein (1962 p. 173).

The observations above induce important properties: see Callier and Winkin (1987 Fact 3.1) for some details.

Fact 2: *Properties of distributions in* $L\Delta(\sigma)$

Let $\sigma \leqslant 0$. Then

(*a*) $L\Delta(\sigma)$ is a commutative convolution Banach algebra with unit element $\delta(\cdot)$ under the norm $\|\cdot\|_{1\sigma}$ given by (21);

(*b*) $L\Delta^+(\sigma)$ and $L\Delta^-(\sigma)$ given by (1) and (16) are closed sub-algebras of $L\Delta(\sigma)$;

(*c*) if $f = f_a(\cdot) + f_0\delta(\cdot) \in L\Delta(\sigma)$, then

(i) $$\hat{f} \text{ is uniformly continuous in } S_\sigma \tag{22}$$

(ii) $$\hat{f} \text{ is holomorphic in } S_\sigma^0 \tag{23}$$

(iii) $$\hat{f}(s) \to f_0 \text{ as } |s| \to \infty \text{ in } S_\sigma \tag{24}$$

where in particular

$$|\hat{f}_a(\sigma_1 + j\omega)| \to 0 \text{ as } |\omega| \to \infty \text{ uniformly in } \sigma_1 \in [\sigma, -\sigma] \tag{25}$$

(*d*) $F = F_a(\cdot) + F_0\delta(\cdot)$ is invertible in Mat $(L\Delta(\sigma))$ if and only if

$$\inf\{|\det \hat{F}(s)| : s \in S_\sigma\} > 0 \tag{26 a}$$

or, equivalently, with $\det F_0 \neq 0$,

$$\det \hat{F}(s) \neq 0 \quad \forall\, s \in S_\sigma \tag{26 b}$$

We are now ready for the spectral factorization inspired by Callier and Winkin (1987 Definition 3.1).

Definition 4

Let $F = F_a(\cdot) + F_0\delta(\cdot) \in$ Mat (LTD) where the following assumptions hold:

(*a*) $F_a(\cdot)$ is a function and F_0 is a constant hermitian positive definite matrix;

(*b*) F is parahermitian self-adjoint, i.e.

$$F(t) = F_*(t) = F(-t)^*$$

or equivalently

$$F^+ = (F^-)_* = F_a^+(\cdot) + 2^{-1}F_0\delta(\cdot)$$

(*c*) $\hat{F}^+ \in$ Mat $(\hat{\mathscr{A}}_-)$ or equivalently under (*b*)

$$F \in \text{Mat}\,(L\Delta(\sigma)) \text{ for some } \sigma < 0 \tag{27}$$

(*d*) $\hat{F}$ is positive semi-definite on the imaginary axis, i.e.

$$\hat{F}(j\omega)(= \hat{F}_*(j\omega) = \hat{F}(j\omega)^*) \geqslant 0 \quad \forall\, \omega \in \mathbb{R} \tag{28}$$

We say that a square matrix-valued function $\hat{R} = \hat{R}_a(\cdot) + R_0 \in \text{Mat}\,(\widehat{\text{LTD}}^+)$, (where $R_a(\cdot)$ is a function and R_0 is a constant non-singular matrix), is a (right) *spectral factor* of $\hat{F}$ (invertible in Mat $(\hat{\mathscr{A}}_-)$) if and only if

$$\hat{F}(j\omega) = \hat{R}_*(j\omega)\hat{R}(j\omega) \quad \forall\, \omega \in \mathbb{R} \tag{29 a}$$

where

$$\hat{R} \text{ and } \hat{R}^{-1} \text{ are in Mat}\,(\hat{\mathscr{A}}_-) \tag{29 b}$$

Comment 3

(i) In (*a*) above, $F_0 > 0$ is a well-posedness condition at $\omega = \pm\infty$ in (28).

(ii) In (*b*) $F(t) = F_*(t)$ reads more precisely $F_a(t) = F_{a*}(t)$ and $F_0 = F_0^*$.

(iii) The equivalence in (*c*) follows easily using (*b*). By Comment 1 (*c*) and (16), $\hat{F}^+ \in \text{Mat}(\hat{\mathscr{A}}_-)$ implies $F^{\pm} \in \text{Mat}(L\Delta^{\pm}(\sigma))$ for some $\sigma < 0$, whence $F = F^+ + F^- \in \text{Mat}(L\Delta(\sigma))$ for some $\sigma < 0$ (use (18)).

(iv) In the scalar case our concept of 'invertible spectral factor' is called the '0-outer spectral factor' by Callier and Winkin (1987).

(v) In Definition 4, '$\hat{R} = \hat{R}_a(\cdot) + R_0 \in \text{Mat}(\widehat{\text{LTD}}^+)$' means that R has no delayed impulses, whence R in $\text{Mat}(\mathscr{A}_-)$ has a singular atomic part $R_{sa} = R_0\delta(\cdot)$. That this is necessarily true follows by the fact that the invertibility of $\hat{R}$ in $\text{Mat}(\hat{\mathscr{A}}_-)$ results in the invertibility of $\hat{R}_{sa}$ in $\text{Mat}(\hat{\mathscr{A}}_-)$ (use, for example, Callier and Desoer 1978 Fact 2.2(i) applied to det $\hat{R}$). Hence the singular atomic part of (29 *a*), i.e. $F_0 = \hat{R}_{sa*}(j\omega)\hat{R}_{sa}(j\omega)$, leads to the identity

$$F_0^{-1}\hat{R}_{sa*}(j\omega) = \hat{R}_{sa}^{-1}(j\omega)$$

which can be extended to an entire bounded function. It follows by Hille (1959 Theorem 8.2.2) that $\hat{R}_{sa}$ is a constant matrix R_0, which is non-singular because $F_0 > 0$.

(vi) Since R has no delayed impulses we have:

a spectral factor invertible in $\text{Mat}(\hat{\mathscr{A}}_-)$ is invertible in $\text{Mat}(\widehat{L\Delta}^+(\sigma))$ for some $\sigma < 0$ (30)

(see Comment 1(*c*) and Fact 1 (*b*)).

(vii) Much insight is gained if in Definition 4 $F_0 = I$. See, for example, Gohberg and Krein (1960). This is easily obtained by considering in Definition 4 $(F_0^{-1/2})F(F_0^{-1/2})$ instead of F.

The following is important.

Theorem 1: *Existence of spectral factors*

Let $F = F_a(\cdot) + F_0\delta(\cdot) \in \text{Mat}(\text{LTD})$ satisfy assumptions (*a*)–(*d*) of Definition 4. Then,

(*a*) $\hat{F}$ has a spectral factor $\hat{R} = \hat{R}_a(\cdot) + R_0$ invertible in $\text{Mat}(\hat{\mathscr{A}}_-)$ if and only if

$$\det \hat{F}(j\omega) \neq 0 \quad \forall\, \omega \in \mathbb{R} \tag{31}$$

Moreover, if (30) holds, then

(*b*) all spectral factors of $\hat{F}$ are unique up to left multiplication by a constant unitary matrix U;

(*c*) if $F_0 = I$ then R_0 is a unitary matrix and $\hat{F}$ has a unique standard spectral factor; i.e. such that R_0 is the unit matrix.

Comment 4

(*a*) The proof of parts (*b*) and (*c*) above are left as an exercise. The main tool for part (*b*) is the analytic extension to a bounded entire function of the identity $\hat{R}_1\hat{R}_2^{-1} = \hat{R}_{1*}^{-1}\hat{R}_{2*}$, where $\hat{R}_1$ and $\hat{R}_2$ are two spectral factors, and (30) and the properties of Fact 1 (*c*) are used. For part (*c*) use also Fact 2 (*c*) as $s = j\omega \to \infty$ in (29 *a*).

(*b*) The crux of the proof of part (*a*) above is that under the assumptions of Definition 4,

(i) by Gohberg and Krein (1960 Theorem 8.2), condition (31) is necessary and sufficient for the existence of a spectral factor invertible in Mat $(\widehat{L\Delta}^+(0))$;

(ii) this spectral factor is actually invertible in Mat $(\widehat{L\Delta}^+(\sigma))$ for some $\sigma < 0$, and hence also in Mat $(\hat{\mathscr{A}}_-)$.

Proof of part (*a*) *of Theorem 1*

Necessity. If $\hat{R}$ is a spectral factor of $\hat{F}$, then by (29), (30) and $\hat{R}_*(j\omega) = \hat{R}(j\omega)^*$, we have, using Fact 1 (d), det $\hat{F}(j\omega) = |\det \hat{R}(j\omega)|^2 \neq 0$ for all $\omega \in \mathbb{R}$. Hence (31) holds.

Sufficiency. We proceed in two steps.

Step 1

Assumptions (*a*)–(*d*) from Definition 4 and condition (31) imply the existence of a square matrix-valued function $\hat{R} = \hat{R}_a(\cdot) + R_0 \in$ Mat $(\widehat{\text{LTD}}^+)$, with R_0 a constant non-singular matrix, such that

(*a*) $$\hat{F}(j\omega) = \hat{R}_*(j\omega)\hat{R}(j\omega) \quad \forall\, \omega \in \mathbb{R} \tag{32}$$

(*b*) $$\hat{R} \in \text{Mat}\,(\widehat{L\Delta}^+(0)) \text{ and } \det \hat{R}(s) \neq 0 \quad \forall\, s \in \mathbb{C}_+ \tag{33 a}$$

or equivalently

$$\hat{R} \text{ and } \hat{R}^{-1} \text{ are in Mat}\,(\widehat{L\Delta}^+(0)) \tag{33 b}$$

Indeed this follows (Gohberg and Krein 1960 Theorem 8.2) by assumptions (*a*)–(*d*) of Definition 4, where

(i) in (27) $F \in$ Mat $(L\Delta(0))$ by (19), and

(ii) in (28) $\hat{F}(j\omega) > 0$ for all $\omega \in \mathbb{R}$ by condition (31).

Note finally that by Fact 1 (*d*), (33 *a*) and (33 *b*) are equivalent since R_0 is non-singular.

Step 2

The square matrix-valued function $\hat{R}$ in (32)–(33) satisfies:

$$\hat{R} \text{ and } \hat{R}^{-1} \text{ are in Mat}\,(\hat{\mathscr{A}}_-) \tag{34}$$

Hence $\hat{F}$ has a spectral factor $\hat{R}$. □

Indeed denote by $S(\sigma_1, \sigma_2)$, ($S^0(\sigma_1, \sigma_2)$ respectively) the vertical strip $\{s \in \mathbb{C} : \sigma_1 \leqslant \text{Re}\,(s) \leqslant \sigma_2\}$, ($\{s \in \mathbb{C} : \sigma_1 < \text{Re}\,(s) < \sigma_2\}$ respectively). Rewrite (32) as

$$\hat{R} = \hat{R}_*^{-1}\hat{F} \tag{35}$$

Observe (33 *b*) where $\hat{R}_*^{-1}(s) = \hat{R}^{-1}(-\bar{s})^*$, and consider also (27). Then by (3)–(4) and (22)–(23),

(*a*) the RHS of (35) is holomorphic in $S^0(\sigma, 0)$ and continuous in $S(\sigma, 0)$,

(*b*) the LHS of (35) has the same properties w.r.t. $\mathbb{C}_+^0$ and $\mathbb{C}_+$, respectively.

Hence Theorem 7.7.1 of Hille (1959) may be applied to (35) such that by analytic extension (continuous up to the boundary), (35) holds in the strip $S(\sigma, 0)$. This implies

$$\hat{R}(\sigma + \cdot) = \hat{R}_*^{-1}(\sigma + \cdot)\hat{F}(\sigma + \cdot) \quad \text{on the } j\omega\text{-axis} \tag{36}$$

Note now that in (36),

$$\hat{F}(\sigma + \cdot) \in \mathrm{Mat}\,(\widehat{L\Delta}(0)) \tag{37}$$

or equivalently $\exp(-\sigma t)F \in \mathrm{Mat}\,(L\Delta(0))$, because of (27) and (17); similarly, using (16)

$$\hat{R}_*^{-1}(\sigma + \cdot) \in \mathrm{Mat}\,(\widehat{L\Delta}^-(0)) \tag{38}$$

or equivalently $\exp(\sigma t)R^{-1} \in \mathrm{Mat}\,(L\Delta^+(0))$ because of (33 *b*) and (1). Hence by (36)–(38) using the convolution (20) in $L\Delta(0)$, it follows by Fact 2 (*a*) that $\hat{R}(\sigma + \cdot) \in \mathrm{Mat}\,(\widehat{L\Delta}(0))$. Now note that $\exp(-\sigma t)R$ has its support on $t \geqslant 0$. Hence we get $\hat{R}(\sigma + \cdot) \in \mathrm{Mat}\,(\widehat{L\Delta}^+(0))$ or equivalently by (1)

$$\hat{R} \in \mathrm{Mat}\,(\widehat{L\Delta}^+(\sigma)) \quad \text{for some } \sigma < 0 \tag{39}$$

Finally observe that R_0 is non-singular and that by (33 *a*) $\det \hat{R}(s) \neq 0$ for all $s \in \mathbb{C}_+$: this together with (39) implies (34) by Fact 1 (*c*). □

Comment 5

(*a*) The analytic extension technique in Step 2 above is similar to that of Krein (1962 Proof of Theorem 3.1).

(*b*) Assumption (27) and condition (31) are necessary and sufficient for $\hat{F}$ to be invertible in $\mathrm{Mat}\,(\widehat{L\Delta}(\sigma))$ for some $\sigma < 0$ (use Facts 2 (*c*) and (*d*)). Hence (29 *a*) can be extended analytically to read

$$\hat{F} = \hat{R}_* \hat{R} \text{ and } \hat{F}^{-1} = \hat{R}^{-1}\hat{R}_*^{-1} \text{ on some strip } S_\sigma$$

(*c*) The philosophy of the proof above in Comment 4 (*b*) and §§ 6–8 of Gohberg and Krein (1960) show that, if $F_0 = I$ in Theorem 1, then the search for the standard spectral factor $\hat{R} = \hat{R}_a(\cdot) + I$ of $\hat{F}$ is equivalent to the following.

Let $\hat{G} := \hat{F}^{-1} =: \hat{G}_a(\cdot) + I$. Find the solution $R_a(\cdot)$ of the Wiener–Hopf integral equation

$$R_a(t) + \int_0^\infty R_a(s)G_a(t-s)\,ds = -G_a(t) \quad \text{on } t \geqslant 0 \tag{40}$$

such that $R_a(\cdot) \in \mathrm{Mat}\,(L_1)$ and $\det(I + \hat{R}_a(s)) \neq 0$ for all $s \in \mathbb{C}_+$. Approximate methods of solution of the Wiener–Hopf equation have been studied (see, for example, Stenger 1972). Other methods for solving vaguely related spectral factorization problems are also known (see, for example, Youla and Kazanjian 1978, Wilson 1978).

(*d*) Generalizations of Theorem 1 are known in some special cases. For the monovariable case of a correlation distribution giving a spectral factor invertible in $\hat{\mathscr{A}}_-$ with delayed impulses, see Callier and Winkin (1987 Theorem 3.1); for the multivariable case giving a spectral factor invertible in $\mathrm{Mat}\,(\hat{\mathscr{A}}_-)$ with equally spaced delays (e.g. transmission lines) see Winkin (1989 Theorem 3.1M). The multivariable case with delays that are not equally spaced is an open question (e.g. Gohberg and Fel'dman 1974 p. 252 Comments on Chapter VIII). For the case of bistable spectral factors of exponential order (Callier and Winkin 1986 Algebra $\hat{\mathscr{E}}$), see Winkin (1989 Theorem 3.2) for the multivariable case with equally spaced delays.

The following concept is important for defining the graph distance of two possibly unstable systems and for obtaining robustness estimates of feedback stability (see, for example, Vidyasagar 1985, Callier and Winkin 1987).

Definition 5

Let $\hat{P} \in \text{Mat}(\hat{\mathscr{B}})$ have an r.c.f. $(\hat{N}, \hat{D})$ in $\text{Mat}(\hat{\mathscr{A}}_-)$. We say that $(\hat{N}, \hat{D})$ is *normalized* if and only if

$$\hat{N}_*\hat{N} + \hat{D}_*\hat{D} = I \quad \text{for } s = j\omega \quad \forall\, \omega \in \mathbb{R} \tag{41}$$

We call the expression

$$\hat{F} = \hat{N}_*\hat{N} + \hat{D}_*\hat{D} \tag{42}$$

the *coprime fraction power spectral density.*

Comment 6

Normalized r.c.f.s of a plant $\hat{P} \in \text{Mat}(\hat{\mathscr{B}})$ are unique up to right multiplication by a constant unitary matrix.

By Theorem 1, we have the following corollary.

Corollary 1

Every *constantly proper* plant $\hat{P} \in \text{Mat}(\hat{\mathscr{B}}_p)$ has normalized r.c.f.s unique up to right multiplication by a unitary matrix.

Proof

Consider any r.c.f. $(\hat{N}, \hat{D})$ in $\text{Mat}(\widehat{L\Delta}^+(\sigma)) \subset \text{Mat}(\hat{\mathscr{A}}_-)$ as described by (10). The power spectral density $\hat{F}$ given by (42) satisfies assumptions (a)–(d) of Definition 4. Moreover coprimeness implies that (31) holds. By Theorem 1, $\hat{F}$ has a spectral factor $\hat{R}$ invertible in $\text{Mat}(\widehat{L\Delta}^+(\sigma)) \subset \text{Mat}(\hat{\mathscr{A}}_-)$. Hence $(\hat{N}\hat{R}^{-1}, \hat{D}\hat{R}^{-1})$ is a normalized r.c.f. of $\hat{P}$. □

Comment 7

The denominator $\hat{D}$ of any normalized r.c.f. of a plant $\hat{P} \in \text{Mat}(\hat{\mathscr{B}}_p)$ cannot be chosen rational unless $\hat{P}$ is rational: see (41).

Consider now for any SGB system of Lemma 2 the following problem.

Problem: *LQ-optimal regulation*

For any initial state $x_0 \in X$, find a square integrable control u_0 that minimizes the cost functional

$$J(x_0, u) = \int_0^\infty (\langle Cx(t), Cx(t)\rangle + \langle u(t), u(t)\rangle)\, dt \tag{43}$$

The solution of this problem is obtained by finding the positive semi-definite self-adjoint operator $Q \in \mathscr{L}(X)$ which solves the operator Riccati equation (ORE):

$$Q(D(A)) \subset D(A^*) \tag{44 a}$$

and

$$[A^*Q + QA + C^*C - QBB^*Q]x = 0 \quad \forall\, x \in D(A) \tag{44 b}$$

(see Zabczyk 1976, Curtain and Pritchard 1978 Chapter 4).

Lemma 3

For any SGB system with (A, B) exponentially stabilizable and (C, A) exponentially detectable, the (ORE) has a unique non-negative self-adjoint solution $Q_0 \in \mathscr{L}(X)$ and for any initial state $x_0 \in X$ the quadratic cost (43) is minimized by the unique control

$$u_0(t) = K_0 x(t) = -B^* Q_0 x(t) \quad \text{on } t \geqslant 0 \tag{45}$$

where the optimal state feedback operator

$$K_0 = -B^* Q_0 \in \mathscr{L}(X, \mathbb{R}^m) \tag{46}$$

is stabilizing, i.e. the feedback semigroup

$$(\exp(A + BK_0)t)_{t \geqslant 0} \text{ is exponentially stable} \tag{47}$$

For semigroup systems, we prove now the following theorem.

Theorem 2: *LQ-regulation dictated spectral factorization*

Let $\hat{P} \in \text{Mat}(\hat{\mathscr{B}}_p)$ be the strictly proper transfer function of an SGB system with (A, B) exponentially stabilizable and (C, A) exponentially detectable as in Lemma 2.

Let $(\hat{N}, \hat{D})$ be any r.c.f. in $\text{Mat}(\hat{\mathscr{A}}_-)$ of $\hat{P}$ described by (10) where $N_0 = 0$.

Consider the *coprime fraction spectral factorization problem*

$$\hat{F} := \hat{N}_* \hat{N} + \hat{D}_* \hat{D} = \hat{R}_* \hat{R} \tag{48}$$

where $\hat{R} = \hat{R}_a(\cdot) + R_0$ is a spectral factor of $\hat{F}$ invertible in $\text{Mat}(\hat{\mathscr{A}}_-)$.

Let $K_0 \in \mathscr{L}(X, \mathbb{R}^m)$ be the LQ-optimal feedback of Lemma 3 given by (46). Then

(*a*) $\hat{F}$ satisfies the assumptions and condition of Theorem 1 with $F_0 = I$, whence $\hat{F}$ has spectral factors with R_0 unitary, the standard spectral factor with $R_0 = I$ being unique.

(*b*) All spectral factors of $\hat{F}$ are related to K_0 by

$$\hat{R}(s) = U[I - K_0(sI - A)^{-1}B]\hat{D}(s) \tag{49 a}$$

$$= U[I + K_0(sI - A - BK_0)^{-1}B]^{-1}\hat{D}(s) \tag{49 b}$$

where U is a unitary constant matrix and $I - K_0(sI - A)^{-1}B$ in $\text{Mat}(\hat{\mathscr{B}}_p)$ is the LQ-optimal control-loop return difference. The standard spectral factor is obtained for $U = I$.

Comment 8

(*a*) Set $U = I$ in (49 *a*) and observe that

$$C(sI - A - BK_0)^{-1}B = \hat{N}\hat{R}^{-1} \in \text{Mat}(\hat{\mathscr{A}}_-)$$

is the optimal closed-loop transfer function. Hence as usual (49 *a*) describes the spectral factor $\hat{R}(s)$ (closed-loop dynamics) as the product of the (control-loop) return difference times the open-loop dynamics $(\hat{D}(s))$.

(*b*) Equation (49 *a*) suggests the identification of K_0 through a spectral factor $\hat{R}$, see Example 1 *et seq.* K_0 dictates also a spectral factor: an example involving a controlled delay differential equation is given by Winkin (1989 Example 4.2).

(*c*) The proof of (49) below follows closely the reasoning of Kucera (1981 Proof of Theorem 1 up to Eq. (14)).

(*d*) In (49) above $\hat{D}(s)$ can be chosen *rational*, see Comment 2 (*a*).

(*e*) In view of Lemma 2 (*c*) and (47) we can apply Theorem 2 in particular to the r.c.f.

$$(\hat{N}_{0p}, \hat{D}_{0p}) := (C(sI - A - BK_0)^{-1}B, I + K_0(sI - A - BK_0)^{-1}B) \tag{50}$$

(49 *b*) reads then $\hat{R}(s) = U$, such that in (48) $\hat{F} = I$. Hence by Definition 5, we have the following corollary.

Corollary 2: *Characterization of normalized r.c.f.'s*

Let $\hat{P}$ and K_0 be as in Theorem 2. Then any normalized r.c.f. of $\hat{P}$ reads

$$(\hat{N}(s), \hat{D}(s)) = (C(sI - A - BK_0)^{-1}B, I + K_0(sI - A - BK_0)^{-1}B)U \tag{51}$$

where U is an arbitrary unitary matrix.

Comment 9

(*a*) This result is a generalization of the lumped case (Meyer and Franklin 1987, Callier and Winkin 1987 Appendix C).

(*b*) More general state-space versions of (51) are known (Zhu 1988, Curtain 1988).

Proof of Theorem 2

Part (*a*) follows by the proof of Corollary 1. Note especially here that in (48) $F = F_a(\cdot) + I\delta(\cdot) \in \text{Mat}\,(L\Delta(\sigma))$ for some $\sigma < 0$, i.e. $F_0 = I$.

From part (*b*) observe that by Lemma 3, K_0 is stabilizing whence by Lemma 2 (*c*) the pair $(\hat{N}_{0p}, \hat{D}_{0p})$ in $\text{Mat}\,(\hat{\mathscr{A}}_-)$ given by (50) is an r.c.f. of $\hat{P}$.

Moreover, by Lemmas 1 and 2

$$\hat{D}_{0p}(s)^{-1} = I - K_0(sI - A)^{-1}B \in \text{Mat}\,(\hat{\mathscr{B}}_p)$$

and therefore has only a finite number of poles in a right half-plane containing the $j\omega$-axis in its interior (Callier and Desoer 1978, 1980 a). Furthermore, since $(sI - A)^{-1}Bu \in D(A)$ for all $u \in \mathbb{R}^m$, it follows by the ORE that

$$I + B^*(-j\omega I - A^*)^{-1}[-(-j\omega I - A^*)Q_0 - Q_0(j\omega I - A) + C^*C - Q_0BB^*Q_0] \times (j\omega I - A)^{-1}B = I$$

whence by (46)

$$I + \hat{P}_*\hat{P} = \hat{D}_{0p*}^{-1}\hat{D}_{0p}^{-1} \tag{52}$$

Consider now $\hat{F}$ in (48) where $(\hat{N}, \hat{D})$ is also an r.c.f. of $\hat{P}$. By the uniqueness of the r.c.f.'s of $\hat{P}$, $(\hat{N}, \hat{D}) = (\hat{N}_{0p}, \hat{D}_{0p})\hat{R}$ where $\hat{R}$ is invertible in $\text{Mat}\,(\hat{\mathscr{A}}_-)$ and $\hat{R} = \hat{D}_{0p}^{-1}\hat{D}$. Moreover, by (52) $\hat{F} = \hat{R}_*\hat{R}$. Hence $\hat{R}$ is a spectral factor. The conclusion (49) follows by the uniqueness of spectral factors. □

Example 1

Consider an exponentially stabilizable and detectable single input, single output SGB system as in Definition 1. In that case, C and K_0 are bounded linear functionals such that by the Riesz representation theorem (see e.g. Rudin 1970), $Cx = \langle c, x\rangle$ and $K_0x = \langle k_0, x\rangle$ for all $x \in X$ for some c and k_0 in X ($\langle\cdot,\cdot\rangle$ is the inner product of X).

Moreover $Bu = bu$ for all $u \in \mathbb{R}$ for some $b \in X$. The transfer function of such a system reads $\hat{P}(s) = \langle c, (sI - A)^{-1} b \rangle \in \hat{\mathscr{B}}_p$ and has the properties of Lemma 2. Assume in addition that X is separable (i.e. contains a countable dense subset), and that A is self-adjoint with compact resolvent. Then the spectrum of A consists only of real eigenvalues $\lambda_0 > \lambda_1 > \dots > \lambda_i > \dots$ (assumed to be of multiplicity one, for simplicity); and A has a complete orthonormal set of eigenvectors ϕ_i. Furthermore, the resolvent of A reads

$$(sI - A)^{-1} x = \sum_{i=0}^{\infty} (s - \lambda_i)^{-1} \langle x, \phi_i \rangle \phi_i \quad \forall\, x \in X$$

(see Curtain and Pritchard 1978 Example 2.40). If $\lambda_0 \geqslant 0$ is the only unstable eigenvalue of A, then an r.c.f. of $\hat{P}$ is, for example, $(\hat{N}, \hat{D}) = (\hat{P}(s)(s - \lambda_0)(s - \lambda)^{-1}, (s - \lambda_0)(s - \lambda)^{-1})$, where $\lambda < 0$. By Theorem 2, any spectral factor of $\hat{F}$ in (48) reads

$$\hat{R}(s) = U\left[1 - \sum_{i=0}^{\infty} \langle k_0, \phi_i \rangle \langle b, \phi_i \rangle (s - \lambda_i)^{-1}\right] \hat{D}(s) \tag{53}$$

where $|U| = 1$. This applies in particular to $A = \partial^2/\partial\xi^2$ operating on $X = L^2([0, 1])$, for modelling heat diffusion in a finite rod of unit length (Winkin 1989 Example 4.3), with

$$D(A) = \left\{ x \in L^2([0, 1]) : \frac{\partial^2 x}{\partial \xi^2} \in L^2([0, 1]);\ \ \frac{\partial x}{\partial \xi} = 0 \text{ at } \xi = 0, 1 \right\}$$

and

$$b(\xi) = \Delta_\omega(\xi) \text{ for } \xi \in [0, 1], \quad c(\xi) = b(1 - \xi) \text{ for } \xi \in [0, 1]$$

where for $\omega > 0$ small,

$$\Delta_\omega(\xi) := \begin{cases} \omega^{-1} & \text{for } \xi \in [0, \omega) \\ 0 & \text{elsewhere} \end{cases}$$

one has eigenvalues $\lambda_0 = 0$ and $\lambda_i = -i^2\pi^2$ for $i = 1, 2, \dots$ and corresponding eigenvectors $\phi_0(\xi) \equiv 1$ and $\phi_i(\xi) = \sqrt{2} \cos(i\pi\xi)$. In this case, (53) reads with $\hat{D}(s) = s(s - \lambda)^{-1}$,

$$\hat{R}(s) = U\left[1 - \frac{\langle k_0, \phi_0 \rangle}{s - \lambda} - \sqrt{2} \sum_{i=1}^{\infty} \frac{(\sin i\pi\omega) \langle k_0, \phi_i \rangle s}{(i\pi\omega)(s + (i\pi)^2)(s - \lambda)}\right]$$

4. Spectral factorization based LQ-optimal regulation

We prove an infinite-dimensional generalization of Kucera (1981 Theorem 2): viz. given a spectral factor (closed-loop dynamics), then the LQ-optimal feedback K_0, given by (46) (controller) can be obtained by a diophantine equation generated by a coprime fraction of the input-state transfer function, see below.

For technical reasons we need the following (Nakagiri and Yamamoto 1989).

Definition 6

Consider any SGB system with bounded linear *reachability operator* $G_{t_1} \in \mathscr{L}(L^2[0, t_1; \mathbb{R}^m], X)$ given by

$$G_{t_1} u = \int_0^{t_1} \exp(A(t_1 - s))\, Bu(s)\, ds \quad \forall\, t_1 > 0 \tag{54}$$

(*a*) We say that the operator pair (A, B) is *approximately reachable* (on $t > 0$) if and only if for all $x_1 \in X$ and for all $\varepsilon > 0$ there exists a time $t_1 > 0$ and a control $u \in L^2[0, t_1; \mathbb{R}^m]$ such that

$$\|x_1 - G_{t_1} u\|_X < \varepsilon$$

or equivalently

$$X = R(A, B) \tag{55 a}$$

where

$$R(A, B) := cl\left[\bigcup_{t_1 > 0} Ra[G_{t_1}]\right] \tag{55 b}$$

is the *(A, B)-approximately reachable subspace* (where $\|\cdot\|_X$ is the norm of X, $Ra[G_{t_1}] \subset X$ is the range of G_{t_1} and $cl[D]$ denotes the X-closure of a subset $D \subset X$).

(*b*) We say that the (dual) operator pair (B^*, A^*) is observable (on $t > 0$) if and only if

$$B^* \exp(A^* t)\, p = 0 \quad \text{on } t > 0 \text{ implies } p = 0$$

or equivalently,

$$NO(B^*, A^*) = \{0\} \tag{56 a}$$

where

$$NO(B^*, A^*) := \{p \in X : B^* \exp(A^* t)\, p = 0 \quad \text{on } t > 0\} \tag{56 b}$$

is the *(B^*, A^*)-unobservable subspace.*

The following result is a standard exercise (see Curtain and Pritchard 1978 Theorem 3.11 and Proof of Theorem 3.15).

Lemma 4

Consider any SGB system. Then

(*a*) (A, B) is approximately reachable if and only if (B^*, A^*) is observable, or equivalently (55)$\Leftrightarrow$(56).

(*b*) If in addition (A, B) is exponentially stabilizable and $K \in \mathscr{L}(X, \mathbb{R}^M)$ is any stabilizing feedback, then, the pair (A, B) is approximately reachable if and only if the pair $(A + BK, B)$ is approximately reachable, or equivalently

$$B^*[(A + BK)^*]^{-(l+1)} p = 0 \quad \forall\, l = 0, 1, 2, \ldots \text{ implies } p = 0 \tag{57}$$

Note

Observe also above that if $K \in \mathscr{L}(X, \mathbb{R}^m)$ is stabilizing then $[(A + BK)^*]^{-1}$ is a bounded operator since 0 is in the resolvent set of $(A + BK)^*$.

We now have the following theorem.

Theorem 3: *LQ-optimal regulation by spectral factorization*

Consider any system SGB such that

$$(A, B) \text{ is exponentially stabilizable and } (C, A) \text{ is exponentially detectable} \tag{12}$$

and assume moreover that

$$(A, B) \text{ is approximately reachable} \tag{58}$$

As in Lemma 2, let $K \in \mathscr{L}(X, \mathbb{R}^m)$ be any stabilizing feedback and let

$$(\hat{N}(s), \hat{D}(s)) := (C(sI - A - BK)^{-1} B, I + K(sI - A - BK)^{-1} B) \tag{14}$$

such that the pair $(\hat{N}, \hat{D})$ in Mat $(\hat{\mathscr{A}}_-)$ is an r.c.f. of the strictly proper transfer function $\hat{P}$ in Mat $(\hat{\mathscr{B}})$ of SGB. Define also

$$\hat{\mathscr{N}}(s) := (sI - A - BK)^{-1} B \tag{59}$$

Consider now as in Theorem 2 any solution of the spectral factorization problem

$$\hat{F} = \hat{N}_* \hat{N} + \hat{D}_* \hat{D} = \hat{\mathscr{N}}_* C^* C \hat{\mathscr{N}} + \hat{D}_* \hat{D} = \hat{R}_* \hat{R} \quad \text{on } s = j\omega \tag{60}$$

where $\hat{R}$ is a spectral factor invertible in Mat $(\hat{\mathscr{A}}_-)$. Consider finally the LQ-optimal feedback $K_0 \in \mathscr{L}(X, \mathbb{R}^m)$ given by (46) in Lemma 3.

Then, the LQ-optimal feedback $K_0 \in \mathscr{L}(X, \mathbb{R}^m)$ is also given by

$$K_0 = -\mathscr{U}^{-1} \mathscr{V} = -\mathscr{U}^* \mathscr{V} \tag{61}$$

where $(\mathscr{U}, \mathscr{V}) \in \mathscr{L}(\mathbb{R}^m) \times \mathscr{L}(X, \mathbb{R}^m)$, with $\mathscr{U}$ a unitary matrix, is the *unique constant* solution of the operator diophantine equation

$$\mathscr{U} \hat{D}(s) + \mathscr{V} \hat{\mathscr{N}}(s) = \hat{R}(s) \in \text{Mat}\,(\hat{\mathscr{A}}_-) \tag{62}$$

induced by the operator right coprime fraction over $\hat{\mathscr{A}}_-$ of the input-state transfer function

$$(sI - A)^{-1} B = \hat{\mathscr{N}}(s) \hat{D}(s)^{-1} \tag{63}$$

with operator Bezout identity over $\hat{\mathscr{A}}_-$ given by

$$\hat{D}(s) - K \hat{\mathscr{N}}(s) = I \tag{64}$$

The last statements are clarified in the proof. As noted in Comment 2 (*a*), $\hat{D}(s)$ in (14) can be chosen *rational.*

Proof of Theorem 3

The input-state transfer function $(sI - A)^{-1} B$ is an $\mathscr{L}(\mathbb{R}^m, X)$-valued meromorphic function in a half-plane containing $\mathbb{C}_+$ by the characterization of the stabilizability of (A, B) (see Nefedov and Sholokhovich 1986). In (59) $\hat{\mathscr{N}}(s)$ originates from $\mathscr{N}(t) = \exp((A + \text{BK})t)\, B$ where $K \in \mathscr{L}(X, \mathbb{R}^m)$ is stabilizing, hence

$$\int_0^\infty \exp(-\sigma t) \|\exp((A + BK)t)\, B\|_{\mathscr{L}(\mathbb{R}^m, X)}\, \mathrm{d}t$$

converges for some $\sigma < 0$. Hence $\hat{\mathscr{N}}(s)$ is an $\mathscr{L}(\mathbb{R}^m, x)$-valued $\hat{\mathscr{A}}_-$-function holomorphic in a half-plane containing $\mathbb{C}_+$. The latter property also holds for $\hat{D}(s) \in \text{Mat}\,(\hat{\mathscr{A}}_-)$ given by (14), where $\det \hat{D} \in \hat{\mathscr{A}}_-^\infty$ (it equals one at infinity in $\mathbb{C}_+$). Observe now that with $K \in \mathscr{L}(X, \mathbb{R}^m)$ stabilizing

$$(sI - A)(sI - A - BK)^{-1} B = [(sI - A - BK) + BK](sI - A - BK)^{-1} B$$

is justified because for all $x \in X$, $(sI - A - BK)^{-1} x \in D(A)$ for $\text{Re}\, s \geq \sigma$ for some $\sigma < 0$ (note that $D(A + BK) = D(A)$). Hence (14) and (59) lead to

$$(sI - A) \hat{\mathscr{N}}(s) = B \hat{D}(s) \tag{65}$$

in some half-plane containing $\mathbb{C}_+$, from which (63) follows and is valid similarly except at a finite number of poles. Equation (64) is an easy consequence of (14) and (59) and valid in a half-plane containing $\mathbb{C}_+$. Note finally that in (64) $K \in \mathscr{L}(X, \mathbb{R}^m)$ can be viewed as a constant operator valued $\hat{\mathscr{A}}_-$-function. Therefore (64) reads as a Bezout identity over $\hat{\mathscr{A}}_-$ and (63)–(64) can be seen as the expression of the fact that $(\hat{\mathscr{N}}, \hat{\mathscr{D}})$ is an operator r.c.f. of $(sI - A)^{-1}B$ over $\hat{\mathscr{A}}_-$.

Consider now the diophantine equation (62) in which the given data $\hat{R}$, $\hat{\mathscr{N}}$, $\hat{D}$ and the solution pair $(\mathscr{U}, \mathscr{V})$ are viewed as $\hat{\mathscr{A}}_-$-functions.

Observe now that by Theorem 2, i.e. (49 *a*), and (63), any spectral factor $\hat{R}$ of $\hat{F}$ in (60) is given by

$$\hat{R}(s) = U\hat{D}(s) - UK_0\hat{\mathscr{N}}(s) \tag{66}$$

where U is a unitary constant matrix. Hence $(\mathscr{U}, \mathscr{V}) = (U, -UK_0) \in \mathscr{L}(\mathbb{R}^m) \times \mathscr{L}(X, \mathbb{R}^m)$ is a constant solution of (62). We show now that this constant solution is unique. Thus assume that $(\mathscr{U}, \mathscr{V})$ is any constant solution of (62) and assume without loss of generality that $\hat{R}$ is the standard spectral factor of $\hat{F}$, whence $U = I$ (cf. Theorem 2 (*b*)). We must prove that $(\mathscr{U}, \mathscr{V}) = (I, -K_0)$.

By (62), (66) and (63) we have

$$\mathscr{U} + \mathscr{V}(sI - A)^{-1}B = I - K_0(sI - A)^{-1}B \tag{67}$$

where both sides are in Mat $(\hat{\mathscr{B}}_p)$ with the second terms strictly proper. This follows by Lemmas 1 and 2 for the RHS since by (46)–(47) K_0 is a stabilizing feedback. For the LHS, observe that by (63), $\mathscr{V}(sI - A)^{-1}B = \mathscr{V}\hat{\mathscr{N}}\hat{D}^{-1}$ where, using the structure of $\mathscr{V} \in \mathscr{L}(X, \mathbb{R}^m)$ and $\mathscr{N}$ above, $\mathscr{V}\mathscr{N} \in$ Mat $(L_{1\sigma})$ for some $\sigma < 0$, moreover $\hat{D} \in$ Mat $(\hat{\mathscr{A}}_-)$ with det $\hat{D}(j\infty) = 1$ by (14). Hence (67) is equivalent to

$$\mathscr{U} = I \tag{68}$$

and

$$(\mathscr{V} + K_0)(sI - A)^{-1}B = 0$$

where the last equation holds in a half-plane containing $\mathbb{C}_+$ except at a finite number of poles. Hence by the injectivity of the Laplace transform

$$(\mathscr{V} + K_0)\exp(At)\,B = O \quad \text{on } t \geqslant 0$$

or, equivalently, taking the adjoint and noting that $\mathscr{V}^*$ and $K_0^* \in \mathscr{L}(\mathbb{R}^m, X)$

$$B^* \exp(A^*t)(\mathscr{V}^* + K_0^*)v = 0 \quad \forall\, t \geqslant 0, \quad \forall\, v \in \mathbb{R}^m \tag{69}$$

Hence, by the reachability assumption (58), using Lemma 4 (*a*),

$$(\mathscr{V}^* + K_0^*)v = 0 \quad \forall\, v \in \mathbb{R}^m$$

i.e. $\mathscr{V}^* + K_0^* = O$, or, equivalently, $\mathscr{V} + K_0 = O$. This, together with (68), shows that necessarily with $U = I$ any constant solution of (62) reads $(\mathscr{U}, \mathscr{V}) = (I, -K_0)$, or, with U in (66) a fixed unitary matrix, $(\mathscr{U}, \mathscr{V}) = (U, -UK_0)$ is the unique constant solution of (62). Hence (61) holds. □

Comment 10

(*a*) The coprime fraction equations (63)–(64) and the diophantine equation (62) should be viewed as generalizations of Kucera (1981 Equations (16) and (19) respectively) with the same purpose: given a spectral factor $\hat{R}$ (closed-loop dynamics), determine the (LQ-optimal constant) controller (K_0). The same philosophy leads to the similar results of Kucera (1981 Theorem 2) and Theorem 3 modulo exchanging polynomial algebra for the 'distributed proper-stable algebra' $\hat{\mathscr{A}}_-$.

(*b*) The unique standard spectral factor of (60) is obtained for $\mathscr{U} = U = I$. Hence without loss of generality, (61)–(62) and (64) and (59) give

$$\hat{R}_a(s) = \hat{R}(s) - I = (K - K_0)(sI - A - BK)^{-1}B \in \mathrm{Mat}\,(\hat{\mathscr{A}}_-) \tag{70}$$

where (Callier and Desoer 1978 Fact 2.3), the LHS and RHS are holomorphic in a neighbourhood of $s = 0$. Hence follows the Taylor series matrix coefficient identification scheme

$$[l!]^{-1}\hat{R}_a^{(l)}(0) = (K - K_0)(A + BK)^{-(l+1)}B \quad \forall\, l = 0, 1, 2, \ldots \tag{71}$$

which determines $K_0 - K$ and hence K_0 uniquely by the reachability condition (58). This is an analogue of the usual polynomial matrix coefficient identification scheme used to define K_0 by Kucera (1981 Theorem 2).

(*c*) Equation (71) is probably of theoretical interest since, with $\mathscr{U} = U = I$, (49 *a*) gives rise to

$$\hat{R}(s)\hat{D}(s)^{-1} = I - K_0(sI - A)^{-1}B \in \mathrm{Mat}\,(\hat{\mathscr{B}}_p) \tag{72}$$

where in many cases both sides are meromorphic in $\mathbb{C}$. $K_0 \in \mathscr{L}(X, \mathbb{R}^m)$ is then recovered simply by residue calculus at the open-loop poles. Of course, this idea also applies at the 'stabilized open-loop poles' (eigenvalues of $(A + BK)$) in (70).

(*d*) In the particular case of Example 1 above, the LQ-optimal gain vector $(\langle k_0, \phi_i \rangle)_{i=0}^{\infty}$ can be obtained from any spectral factor provided that the pair (A, B) is approximately reachable or, equivalently, by Curtain and Pritchard (1978 Proof of Proposition 3.13)

$$\langle b, \phi_i \rangle \neq 0 \quad \forall\, i = 0, 1, 2, \ldots \tag{73}$$

Indeed, from (72) on comparing with (53) for all $i = 0, 1, 2, \ldots$,

$$\langle k_0, \phi_i \rangle = -\mathrm{Res}\,(\hat{R}(s)\hat{D}(s)^{-1}; \lambda_i)\langle b, \phi_i \rangle^{-1} \tag{74}$$

where $\mathrm{Res}\,(\hat{f}; \lambda)$ is the residue of the function $\hat{f}$ at λ and $\hat{R}$ is the standard spectral factor ($U = 1$).

(*e*) For the specific case at the end of Example 1, $\langle b, \phi_i \rangle$ in (73) reads

$$\langle b, \phi_i \rangle = \begin{cases} 1 & \text{for } i = 0 \\ \sqrt{2}\dfrac{\sin(i\pi\omega)}{(i\pi\omega)} & \text{for all } i = 1, 2, \ldots \end{cases} \tag{75}$$

where (73) is satisfied only if ω is irrational. Hence the approximate reachability condition is not robustly satisfied. Therefore an adaptation of Theorem 3 is in order. Preliminary analysis shows that an adaptation of Theorem 3 leads to the identification of the approximately reachable restriction of the LQ-optimal feedback operator K_0, which is still optimal for any approximately reachable initial state and stabilizing. Moreover, if $B^* = C$ and A is self-adjoint, as is practically the case in the specific example above, then this restricted operator K_0 is *the* optimal feedback operator owing to the fact that the (C, A)-unobservable subspace is in the null space of the solution of the ORE.

(*f*) One author (F. M. Callier) has studied the spectral factorization problem for the specific example above. It turns out that it is realizable by searching for the closed-loop spectrum (pole placement of a few dominant modes); the feedback operator determination by (74) is performed by almost finite-dimensional residue calculus.

5. Conclusion

The essential contribution of this paper on spectral factorization of multivariable distributed systems with no delays and LQ-optimal regulation of semigroup state-space SGB systems is Theorem 1 on the existence of a spectral factor. It turns out to be the regularization of a well-known result by Gohberg and Krein (1960 Theorem 8.2). Once this is done, Corollary 1 on the existence of normalized coprime fractions of transfer functions in Mat $(\hat{\mathscr{B}}_p)$ is rather straightforward.

Another important philosophical fact is the ability to generalize to our context Kucera's approach (Kucera 1981) to connect the structure of a (coprime fraction power spectral density) spectral factor with the LQ-optimal state feedback. This results in Theorem 2 on the structure of the considered spectral factor and Theorem 3 on the solution of an LQ-problem by spectral factorization and a diophantine equation. Furthermore one should observe that Corollary 2 on the determination of a normalized coprime fraction by an LQ-optimal state feedback is cheap once Theorem 2 is known.

The most urgent needs at this point are computational procedures for spectral factorization of physically motivated examples, showing connections between spectral factorization and the solution of an LQ-problem as, for example, in work by Davis and Barry (1977), indicating what the physics dictates as approximations. Another important viewpoint is the hamiltonian operator method (Kwakernaak and Sivan 1972 Theorem 3.8, Gibson 1983).

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