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F. M. CALLIER† and J. WINKIN†

σ_0 is a real number. We construct a transfer function algebra of fractions, viz. $\hat{\mathbf{F}}(\sigma_0)$, for modelling possibly unstable distributed systems such that (i) $\hat{f}$ in $\hat{\mathbf{F}}(\sigma_0)$ is holomorphic in $\mathrm{Re}\, s \geqq \sigma_0$, (i.e. is σ_0-stable), iff $\hat{f}$ is σ_0-exponentially stable, and (ii) we allow delay in the direct input–output transmission of the system. This algebra is (*a*) a restriction of the algebra $\hat{\mathbf{B}}(\sigma_0)$ developed by Callier and Desoer (1978, 1980 a), (*b*) an extension of the algebra of proper rational functions such that the exponential order properties of the latter transfer functions of lumped systems are maintained. The algebra $\hat{\mathbf{F}}(\sigma_0)$ can be used for modelling and feedback system design. It is shown that standard semigroup systems are better modelled by a transfer function in $\hat{\mathbf{F}}(\sigma_0)$ rather than $\hat{\mathbf{B}}(\sigma_0)$.

1. Introduction

In this paper we shall be involved with transfer functions of distributed systems, i.e. these with an infinite-dimensional state space. Our objective is to construct an algebra of transfer functions which fits well semigroup systems where (1) we allow delay in the direct transmission between input and output, and (ii) stability is exponential stability.

For the study of distributed systems essentially two approaches have been developed for studying open-loop systems, poles and zeros and problems of feedback system stability and design.

(*a*) *Semigroup state-space systems* (time-domain approach)

See for example Curtain and Pritchard (1978), Hille and Phillips (1957), Kato (1980), Pazy (1983), for the basic theory; Pandolfi (1984), Przyluski (1979), for poles and zeros, and Triggiani (1975), Pritchard and Zabczyk (1981), Curtain and Pritchard (1978), Balas (1978, 1982), Schumacher (1983), Curtain (1984), Pohjolainen (1982), for feedback.

(*b*) *Transfer function theory* (frequency-domain approach)

See for example Callier and Desoer (1978, 1980 a, b), Desoer *et al.* (1980), Nett *et al.* (1983), for the basic theory; Callier *et al.* (1981), for poles and zeros, and Callier and Desoer (1980 b), Desoer *et al.* (1980), Desoer and Vidyasagar (1975), Francis (1977), Francis and Vidyasagar (1983), Vidyasagar *et al.* (1982), Vidyasagar (1984), Ferreira and Callier (1982), Nett *et al.* (1983), Zames (1981), Saeks and Murray (1981), for feedback.

In this paper we shall be involved with subalgebras of the transfer function algebras $\hat{\mathbf{A}}_-(\sigma_0)$ and $\hat{\mathbf{B}}(\sigma_0)$: we proceed now to define the latter.

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Preliminaries (Callier and Desoer 1978, 1980 a)

In this paper RHS, LHS, SISO, MIMO, TF, I/O, a.e., are standard abbreviations meaning respectively right-hand side, left-hand side, single-input–single-output, multi-input–multi-output, transfer function, input–output, almost everywhere.

$\mathbb{R}$, ($\mathbb{C}$) denotes the field of real (or respectively complex), numbers. $\mathbb{R}_+ := [0, \infty)$, $\mathbb{C}_{\sigma+} := \{s : s \in \mathbb{C}$ such that $\operatorname{Re} s \geq \sigma\}$ and $\breve{\mathbb{C}}_{\sigma-} := \{s : s \in \mathbb{C}$ such that $\operatorname{Re} s < \sigma\}$. For $\sigma_0 \in \mathbb{R}$, $\mathbf{R}(\sigma_0)$ denotes the algebra of σ_0-exponentially stable rational TF's (i.e. proper and holomorphic in $\mathbb{C}_{\sigma_0^+}$) while $\mathbf{R}^\infty(\sigma_0)$ denotes its multiplicative subset of elements that are non-zero at infinity. $\mathbb{C}_p(s)$ is the algebra of proper rational functions in s with coefficients† in $\mathbb{C}$. It is well known that, e.g. in Callier and Desoer (1980 a), $\mathbb{C}_p(s)$ is an algebra of fractions of $\mathbf{R}(\sigma_0)$ with respect to $\mathbf{R}^\infty(\sigma_0)$, i.e. $\hat{f} \in \mathbb{C}_p(s)$ iff $\hat{f} = nd^{-1}$ for some $n \in \mathbf{R}(\sigma_0)$ and $d \in \mathbf{R}^\infty(\sigma_0)$ or equivalently $\mathbb{C}_p(s) = [\mathbf{R}(\sigma_0)][\mathbf{R}^\infty(\sigma_0)]^{-1}$. $L_p(\mathbb{R}_+)$, or for short L_p, denotes for $p \in [1, \infty)$, (for $p = \infty$), the Banach space of pth-power absolutely integrable, (or essentially bounded, respectively), functions with support on $\mathbb{R}_+$. For $\sigma \in \mathbb{R}$ and $p \in [1, \infty]$, $L_{p,\sigma} := \{f(.) : \exp(-\sigma.) f(.) \in L_p\}$. LTD_+ denotes the class of $\mathbb{C}$-valued Laplace transformable distributions with support on $\mathbb{R}_+$. The Laplace transform as well as corresponding sets of Laplace transforms will be denoted by a circumflex: e.g. $f \in \mathrm{LTD}_+$ iff $\hat{f} \in \widehat{\mathrm{LTD}}_+$ (sometimes we also use $\mathscr{L}(f) = \hat{f}$).

For $\sigma \in \mathbb{R}$, a distribution $f \in \mathrm{LTD}_+$ is said to belong to the convolution Banach algebra $\mathbf{A}(\sigma)$ iff, for $t < 0$, $f(t) = 0$, and, for $t \geq 0$, $f(t) = f_a(t) + f_{sa}(t)$, where the regular functional part $f_a(.) \in L_{1,\sigma}$ and the singular atomic part $f_{sa}(.) = \sum_{i=0}^{\infty} f_i \delta(. - t_i)$ such that $\delta(.)$ denotes the Dirac delta distribution, $t_0 = 0$, $t_i > 0$ for $i = 1, 2, \ldots$, and $f_i \in \mathbb{C}$ for $i = 0, 1, \ldots$ with $\sum_{i=0}^{\infty} |f_i| \exp(-\sigma t_i) < \infty$; (in Callier and Desoer (1978, 1980 a, b), the singular atomic part is denoted f_{ap} because $\hat{f}_{sa}$ is almost periodic in any vertical strip in $\mathbb{C}_{\sigma+}$, (Callier and Desoer (1978), p. 652)); the $\mathbf{A}(\sigma)$-norm of f is given by $\|f\|_{\mathbf{A}(\sigma)} := \int_0^\infty \exp(-\sigma t)|f_a(t)|\,dt + \sum_{i=0}^{\infty} \exp(-\sigma t_i)|f_i|$.

For $\sigma_0 \in \mathbb{R}$, the TF algebra $\hat{\mathbf{B}}(\sigma_0)$ is constructed as follows (Callier and Desoer 1978, 1980 a).

A distribution $f \in \mathrm{LTD}_+$ is said to belong to the convolution algebra $\mathbf{A}_-(\sigma_0)$ iff $f \in \mathbf{A}(\sigma)$ for some $\sigma < \sigma_0$; $\hat{\mathbf{A}}_-(\sigma_0)$ can be considered as a 'good class' of σ_0-stable TF's, (i.e. holomorphic in $\mathbb{C}_{\sigma_0^+}$), see e.g. Corollary 2.1. $\hat{\mathbf{A}}_-^\infty(\sigma_0)$ is the multiplicative subset of $\hat{\mathbf{A}}_-(\sigma_0)$ of elements that are bounded away from zero at infinity in $\mathbb{C}_{\sigma_0^+}$; see Callier and Desoer (1978), p. 652, and § 4. $\hat{\mathbf{B}}(\sigma_0)$ is the TF algebra of fractions of $\hat{\mathbf{A}}_-(\sigma_0)$ with respect to $\hat{\mathbf{A}}_-^\infty(\sigma_0)$, i.e. $\hat{\mathbf{B}}(\sigma_0) := [\hat{\mathbf{A}}_-(\sigma_0)][\hat{\mathbf{A}}_-^\infty(\sigma_0)]^{-1}$. It is known (Callier and Desoer 1978, 1980 a), that $\hat{\mathbf{B}}(\sigma_0) = [\hat{\mathbf{A}}_-(\sigma_0)][\mathbf{R}^\infty(\sigma_0)]^{-1}$ (the denominators may be chosen rational), and $\hat{\mathbf{B}}(\sigma_0)$ is an extension of $\mathbb{C}_p(s)$ for describing possibly unstable distributed systems. For further properties of $\hat{\mathbf{A}}_-(\sigma_0)$ and $\hat{\mathbf{B}}(\sigma_0)$, see Callier and Desoer (1978, 1980 a) and Callier and Desoer (1980 b), § 1.

Recently, Jacobson (1984 a, 1984 b), Nett *et al.* (1983) have clarified the relation between the semigroup system approach and the TF approach for distributed systems. There results the fact that, under conditions of exponential stabilizability and

† Almost all rational TF's have real coefficients. However, after partial fraction expansion complex coefficients may creep in and hence we consider $\mathbb{C}_p(s)$ rather than $\mathbb{R}_p(s)$.

detectability, any standard SISO semigroup system has a TF in $\hat{\mathbf{B}}(\sigma_0)$, (Jacobson 1984 b, Theorem 1). As is usual for semigroup systems, they used the notion of exponential stability, whence the TF of an exponentially stable semigroup system corresponds to an impulse response in $\mathbf{A}_-(0)$ with functional part bounded by a decreasing exponential. However, if $f = f_a + f_{sa}$ is in $\mathbf{A}_-(0)$ then f_a is not always exponentially stable.

Counterexample 1.1

Let $f: \mathbb{R} \to \mathbb{R}_+$, with support on $\mathbb{R}_+$, be such that for $n = 1, 2, \ldots, f(t) = t \exp(3n) + (\exp(n) - n\exp(3n))$ for $t \in [n - \exp(-2n), n], f(t) = -t\exp(3n) + (\exp(n) + n\exp(3n))$ for $t \in [n, n + \exp(-2n)]$, and $f(t) = 0$ elsewhere. Note that f has a graph $(t, f(t))$ that is a succession of triangles with vertices $(n - \exp(-2n), 0)$, $(n, \exp(n))$ and $(n + \exp(-2n), 0)$ for $n = 1, 2, \ldots$. Each triangle has a surface $\exp(-n)$, whence $f \in L_1$. Furthermore, for $\sigma < 0$ each transformed triangle of $\exp(-\sigma t) f(t)$ has a surface bounded by $K \exp(-(\sigma + 1)n)$ for n sufficiently large. Hence $f \in L_{1,\sigma}$ for all $\sigma > -1$ and $f \in \mathbf{A}_-(0)$. However, f is not exponentially stable because for $t \to \infty$ $f(t)$ does not tend to 0, $(f(n) = \exp(n))$.

On the other hand, the singular atomic part $f_{sa}(t) = \sum_{i=0}^{\infty} f_i \delta(t - t_i)$ of $f = f_a + f_{sa} \in \mathbf{A}_-(0)$ is appropriate for allowing delay in the direct I/O transmission of a system; in general it is not possible to restrict ourselves to a finite number of delays under feedback: e.g. with $\hat{f}(s) = 1 - \exp(-s)$, $(1 + \hat{f}(s))^{-1} = 2^{-1} \cdot \sum_{i=0}^{\infty} 2^{-i} \exp(-is)$. Therefore from our observations above it follows that for exponential stability a suitable restriction of $\mathbf{A}_-(0)$ is its subclass having exponentially stable functional parts: see $L_\varepsilon(0)$ (function subclass), $\mathbf{E}(0)$ (with singular part) and their generalizations $L_\varepsilon(\sigma_0)$, $\mathbf{E}(\sigma_0)$ in §§ 2 and 3 respectively. Their study in this paper leads, along the lines of Callier and Desoer (1978, 1980 a), to a TF algebra of fractions $\hat{\mathbf{F}}(\sigma_0) \subsetneqq \hat{\mathbf{B}}(\sigma_0)$. $\hat{\mathbf{F}}(\sigma_0)$ is of exponential order, suitable for feedback and, in § 6, it matches unstable semigroup systems better than $\hat{\mathbf{B}}(\sigma_0)$.

The rest of this paper is organized as follows. § 2: the ideal of σ_0-exponentially stable functions, [$L_\varepsilon(\sigma_0)$ is a proper ideal of $\mathbf{A}_-(\sigma_0)$, nice interaction of $L_\varepsilon(\sigma_0)$ with $\mathbf{A}_-(\sigma_0)$]. § 3: the algebra of σ_0-exponentially stable transfer functions, [$\hat{\mathbf{E}}(\sigma_0)$ is a rich subalgebra of $\hat{\mathbf{A}}_-(\sigma_0)$ closed under inversion]. § 4: boundedness from zero at infinity (the multiplicative subset $\hat{\mathbf{E}}^\infty(\sigma_0)$, product decomposition of $\hat{f} \in \hat{\mathbf{E}}(\sigma_0)$ and $\hat{f} \in \hat{\mathbf{E}}^\infty(\sigma_0)$), coprimeness (condition for satisfying the Bezout condition). § 5: fractions of σ_0-exponentially stable transfer functions, [$\hat{\mathbf{F}}(\sigma_0) = [\hat{\mathbf{E}}(\sigma_0)][\hat{\mathbf{E}}^\infty(\sigma_0)]^{-1}$, $\mathbb{C}_p(s) \subset \hat{\mathbf{F}}(\sigma_0) \subset \hat{\mathbf{B}}(\sigma_0)$, sum decomposition, exponential order]. § 6: Link with semigroup systems (TF in $\hat{\mathbf{F}}(\sigma_0)$, exponential stabilizability and detectability). § 7: conclusion.

2. The ideal of σ_0-exponentially stable functions

The following function subclass of $\mathbf{A}_-(\sigma_0)$ interacts well with that convolution algebra.

Definition 2.1

For $\sigma_0 \in \mathbb{R}$ the function $f: \mathbb{R} \to \mathbb{C}$ with support on $\mathbb{R}_+$ is said to be σ_0-*exponentially stable* iff there exists a $\sigma < \sigma_0$ such that $f \in L_{\infty,\sigma}$, or equivalently there exist $\sigma < \sigma_0$ and

$M > 0$ such that

$$|f(t)| \leqq M \exp(\sigma t) \quad \text{a.e. on } \mathbb{R}_+ \tag{2.1}$$

$L_\varepsilon(\sigma_0)$ denotes the set of σ_0-exponentially stable functions.

Fact 2.1

$L_\varepsilon(\sigma_0)$ has the following properties:

(*a*) If $f \in L_\varepsilon(\sigma_0)$, then

there exists $\sigma_1 < \sigma_0$ such that $f \in L_{p,\sigma_1}$ for all $p \in [1, \infty]$ (2.2)

(*b*) $L_\varepsilon(\sigma_0) \subset \mathbf{A}_-(\sigma_0)$ (2.3)

Proof

(*a*) By (2.1), for any $\sigma_1 \in (\sigma, \sigma_0)$, $|f(t)| \exp(-\sigma_1 t) \leqq M \exp((\sigma - \sigma_1)t)$ where $\sigma - \sigma_1 < 0$.

(*b*) From (2.2), $L_\varepsilon(\sigma_0) \subset U\{L_{1,\sigma_1} : \sigma_1 < \sigma_0\} \subset \mathbf{A}_-(\sigma_0)$ □

Comment 2.1

For $\sigma_0 \in \mathbb{R}$ a function $f \in L_\varepsilon(\sigma_0) \subset \mathbf{A}_-(\sigma_0)$ iff $\exp(-\sigma_0 t) f(t)$ is bounded† and tends to zero exponentially fast as $t \to \infty$.

Moreover, with $*$ denoting convolution, we have the following result.

Lemma 2.1 (regularization)

If $u \in L_\varepsilon(\sigma_0)$ and $f \in L_{1,\sigma}$ for some $\sigma < \sigma_0$, then $y := f * u \in L_\varepsilon(\sigma_0)$.

Proof

By (2.2) without loss of generality $u \in L_{1,\sigma}$ with $|u(t)| \leqq M \exp(\sigma t)$ for some $M > 0$. Hence $|(f*u)(t)| \leqq K \exp(\sigma t)$ on $\mathbf{R}_+$ with $0 < K := M \|f.\exp(-\sigma.)\|_{L_1} < \infty$. Hence, by Definition 2.1, $f*u \in L_\varepsilon(\sigma_0)$. □

Theorem 2.1 (ideal property)

For any $\sigma_0 \in \mathbb{R}$, $L_\varepsilon(\sigma_0)$ is a proper ideal of the convolution algebra $\mathbf{A}_-(\sigma_0)$, i.e.

(*a*) $L_\varepsilon(\sigma_0) \subsetneqq \mathbf{A}_-(\sigma_0)$, (2.4)

(*b*) $L_\varepsilon(\sigma_0)$ is a linear subspace of $\mathbf{A}_-(\sigma_0)$, (2.5)

(*c*) For all $u \in L_\varepsilon(\sigma_0)$ and $f \in \mathbf{A}_-(\sigma_0)$, $y = f*u \in L_\varepsilon(\sigma_0)$. (2.6)

Comments 2.2

(α) The ideal property (2.6) generalizes Lemma 2.1: for any convolution system $u \mapsto y$ with TF in $\hat{\mathbf{A}}_-(\sigma_0)$, if $\exp(-\sigma_0).u(.)$ is bounded and tends to zero exponentially fast as $t \to \infty$, then so does $\exp(-\sigma_0).y(.)$ (see Comment 2.1).

(β) $L_\varepsilon(\sigma_0)$ is also a subalgebra of $\mathbf{A}_-(\sigma_0)$ (having no multiplicative identity).

† We write 'bounded' instead of 'essentially bounded'.

Proof of Theorem 2.1

(*a*) and (*b*) are obvious from (2.3), Counterexample 1.1, and Definition 2.1.

(*c*) By assumption, $|u(t)| \leqq M\exp(\sigma t)$ for some $\sigma < \sigma_0$ and $M > 0$, and $f = f_a + \sum_{i=0}^{\infty} f_i \delta(. - t_i)$ is in $\mathbf{A}_-(\sigma_0)$ where without loss of generality $f_a \in L_{1,\sigma}$ and $\sum_{i=0}^{\infty} |f_i| \exp(-\sigma t_i) < \infty$ (by adapting σ eventually). Now,

$$(f*u)(t) = (f_a*u)(t) + \sum_{i=0}^{\infty} f_i u(t - t_i) \quad \text{on } \mathbb{R}_+ \tag{2.7}$$

where $u(t) := 0$ for $t < 0$. Hence, since $f_a \in L_{1,\sigma}$ for some $\sigma < \sigma_0$ and $u \in L_\varepsilon(\sigma_0)$, $f_a * u \in L_\varepsilon(\sigma_0)$ by Lemma 2.1. Moreover since $\sum_{i=0}^{\infty} |f_i| \exp(-\sigma t_i) =: K < \infty$, $\left|\sum_{i=0}^{\infty} f_i u(t - t_i)\right| \leqq MK\exp(\sigma t)$. Thus $\sum_{i=0}^{\infty} f_i u(. - t_i) \in L_\varepsilon(\sigma_0)$ and both terms on the right-hand side of (2.7) belong to $L_\varepsilon(\sigma_0)$. Hence (2.6) follows. □

The following result is a dynamic interpretation of Callier and Desoer (1978), Theorem 2.2 (see also the correction of that paper). We denote by $1(t)$ the unit step function.

Fact 2.2 (transmission of a σ_0-unstable exponential)

Let

$$\sigma_0 \in \mathbb{R} \quad \text{and} \quad f \in \mathbf{A}_-(\sigma_0) \tag{2.8 a}$$

Let

$$u(t) = \exp(zt).1(t) \tag{2.8 b}$$

where

$$z \in \mathbb{C}_{\sigma_0^+} \tag{2.8 c}$$

Let

$$y = f*u \tag{2.8 d}$$

Under these conditions

$$q(t) := y(t) - \hat{f}(z).\exp(zt).1(t) \quad \text{is in } L_\varepsilon(\sigma_0) \tag{2.9}$$

or equivalently, using the Laplace transform,

$$\hat{f}(s).(s - z)^{-1} = \hat{f}(z).(s - z)^{-1} + \hat{q}(s) \tag{2.10 a}$$

where

$$q \in L_\varepsilon(\sigma_0) \tag{2.10 b}$$

Comments 2.3

(α) By Callier and Desoer (1978), Theorem 2.2, q is in $L_{1,\sigma} \cap L_{\infty,\sigma}$ for $\sigma < \sigma_0$. Hence $q \in L_{\infty,\sigma} \subset L_\varepsilon(\sigma_0)$.

(β) A short proof is as follows: with (2.8), where especially $f = f_a + \sum_{i=0}^{\infty} f_i \delta(. - t_i)$, and e.g. Desoer and Vidyasagar (1975), p. 247, there exists a $\sigma < \sigma_0$ such that

$$-\exp(-\sigma t) q(t) = \int_t^{\infty} f_a(\tau) \exp(-\sigma\tau).\exp((z-\sigma)(t-\tau))\, d\tau$$
$$+ \sum_{t_i > t} f_i \exp(-\sigma t_i).\exp((z-\sigma)(t-t_i))$$

Hence, since Re $z > \sigma$, $\exp(-\sigma t)|q(t)| \leqq \|f.\exp(-\sigma .)\|_{A(0)} < \infty$.

(γ) By (2.9) and (2.1) we have

$$\exp(-\sigma_0 t)|y(t) - \hat{f}(z).\exp(zt).1(t)| \leqq M \exp((\sigma - \sigma_0)t)$$

for some $\sigma < \sigma_0$ and $M > 0$. Hence, as $t \to \infty$, $\exp(-\sigma_0 t).y(t)$ will be attracted exponentially fast, $(\sigma - \sigma_0 < 0)$, to the weighted stationary waveform $\exp(-\sigma_0 t).\hat{f}(z).\exp(zt).1(t)$. Moreover, as $t \to \infty$, for $\sigma_0 = 0$ and $\hat{f}(z) = 0$, the output will be attracted to zero exponentially fast; the blocking property (Kailath (1980), p. 449), of an unstable transmission zero of $\hat{f} \in \hat{\mathbf{A}}_-(0)$ is displayed in the same manner as for an exponentially stable rational TF in $\mathbf{R}(0)$. Nice consequences of this are visible in Callier *et al.* (1981).

The combination of Theorem 2.1, Fact 2.2 and (Desoer and Vidyasagar (1975), Exercise 2, p. 247) leads to the following transmission result; its proof is left as an exercise.

Corollary 2.1 (I/O properties of a transfer function† in $\hat{\mathbf{A}}_-(\sigma_0)$)

Let $\sigma_0 \in \mathbb{R}$, $f \in \mathbf{A}_-(\sigma_0)$, u be a $\mathbb{C}$-valued function with support on $\mathbb{R}_+$ and $y = f * u$. Under these conditions

(*a*) If $\exp(-\sigma_0 t).u$ is bounded on $\mathbb{R}_+$, then so is $\exp(-\sigma_0 t).y$.

(*b*) For all $p \in [1, \infty]$, if $\exp(-\sigma_0 t).u$ is in L_p, then so is $\exp(-\sigma_0 t).y$.

(*c*) If $\exp(-\sigma_0 t).u$ is bounded on $\mathbb{R}_+$ and tends to zero exponentially fast as $t \to \infty$, then so does $\exp(-\sigma_0 t).y$.

(*d*) For $z \in \mathbb{C}_{\sigma_0^+}$, if $\exp(-\sigma_0 t).u$ is bounded on $[0, T]$ for all $T > 0$ and tends to $\exp(-\sigma_0 t).u_\infty.\exp(zt).1(t)$ exponentially fast as $t \to \infty$, then $\exp(-\sigma_0 t).y$ is bounded on $[0, T]$ for all $T > 0$ and tends to $\exp(-\sigma_0 t).\hat{f}(z).u_\infty.\exp(zt).1(t)$ exponentially fast as $t \to \infty$.

Comment 2.4

For $\sigma_0 = 0$, the properties of $L_\varepsilon(0)$ are important to show that a TF in $\hat{\mathbf{A}}_-(0)$ has the I/O properties of an exponentially stable rational TF in $\mathbf{R}(0)$. The essential ideal property of $L_\varepsilon(\sigma_0)$ is also used below.

3. The algebra of σ_0-exponentially stable transfer functions

Definition 3.1

Let $\sigma_0 \in \mathbb{R}$ and let f be a distribution in $\mathbf{A}_-(\sigma_0)$, whence $f = f_a + f_{sa} = f_a$

† As is usual, we identify functions or distributions with their Laplace transforms.

$+\sum_{i=0}^{\infty} f_i\delta(.-t_i)$. f is said to be σ_0-exponentially stable iff $f_a \in L_\varepsilon(\sigma_0)$, i.e. the functional part f_a is σ_0-exponentially stable. We denote by $\mathbf{E}(\sigma_0)$ the subset of σ_0-exponentially stable distributions of $\mathbf{A}_-(\sigma_0)$, and we call $\hat{\mathbf{E}}(\sigma_0)$ the subset of σ_0-exponentially stable T F's of $\hat{\mathbf{A}}_-(\sigma_0)$.

Fact 3.1

For $\sigma_0 \in \mathbb{R}$ the following hold:

(*a*)
$$\hat{\mathbf{E}}(\sigma_0) \subset \hat{\mathbf{A}}_-(\sigma_0) \tag{3.1}$$

(*b*) With $f = f_a + f_{sa} = f_a + \sum_{i=0}^{\infty} f_i\delta(.-t_i)$,
$$\hat{f} \in \hat{\mathbf{E}}(\sigma_0) \tag{3.2}$$

iff there exist $\sigma < \sigma_0$ and $M > 0$ such that
$$|f_a(t)| \leqq M exp(\sigma t) \quad \text{a.e. on } \mathbb{R}_+ \tag{3.3 a}$$
and
$$\sum_{i=0}^{\infty} |f_i| \exp(-\sigma t_i) < \infty. \tag{3.3 b}$$

Proof

Obvious from Definitions 3.1 and 2.1, where σ has been eventually adapted. □

Comments 3.1

(α)
$$\mathbf{R}(\sigma_0) = \hat{\mathbf{E}}(\sigma_0) \cap \mathbb{C}_p(s) = \hat{\mathbf{A}}_-(\sigma_0) \cap \mathbb{C}_p(s) \tag{3.4}$$

(β) By (3.3 *a*) without loss of generality† $f_a \in L_{1,\sigma}$; hence by the Riemann–Lebesgue lemma (Desoer and Vidyasagar (1975), Theorem B.1.1.), $\hat{f}_a(s) \to 0$ as $|s| \to \infty$ in $\mathbb{C}_{\sigma+}$. By (3.3 *b*) and (Callier and Desoer (1978), p. 652), a non-zero $\hat{f}_{sa}$ is analytic almost periodic in any vertical strip of $\mathbb{C}_{\sigma+}$, and so is not zero at infinity.

(γ) In view of Comment (β), with $\sigma < \sigma_0$, a TF $\hat{f} \in \hat{\mathbf{E}}(\sigma_0)$ has a strictly proper part $\hat{f}_a$ such that exp $(-\sigma_0 t).f_a(t)$ is bounded by a decreasing exponential (see (3.3)): this property is characteristic for rational σ_0-exponentially stable TF's in $\mathbf{R}(\sigma_0)$, (e.g. in Callier and Desoer (1982), p. 127). Hence the decision to call elements of $\hat{\mathbf{E}}(\sigma_0)$ also σ_0-exponentially stable.

(δ) Counterexample 1.1 shows that $\hat{\mathbf{E}}(\sigma_0)$ is a proper subset of $\hat{\mathbf{A}}_-(\sigma_0)$. However, $\hat{\mathbf{E}}(\sigma_0)$ is a rich subset of $\hat{\mathbf{A}}_-(\sigma_0)$ as seen below.

Fact 3.2 (regular derivatives)

Let $\sigma_0 \in \mathbb{R}$ and let $\hat{f} \in \hat{\mathbf{D}}(\sigma_0)$, or equivalently:
$$f = f_a + f_{sa} \text{ is in } \mathbf{A}_-(\sigma_0) \tag{3.5 a}$$
such that, in the sense of distributions,
$$\dot{f}_a := g_a + \sum_{j=0}^{\infty} g_j\delta(.-\tau_j) \text{ is in } \mathbf{A}_-(\sigma_0) \tag{3.5 b}$$

† By increasing σ slightly to $\sigma_1 \in (\sigma, \sigma_0)$, if necessary.

where on $\mathbb{R}_+$

$$f_a(t) = \int_{0^-}^{t} \dot{f}_a(\tau)\, d\tau := \int_0^t g_a(\tau)\, d\tau + \sum_{\tau_j \le t} g_j . 1(t-\tau_j) \tag{3.5 c}$$

i.e. f_a is the primitive of $\dot{f}_a \in \mathbf{A}_-(\sigma_0)$ in the sense of distributions. Under these conditions

$$\hat{\mathbf{D}}(\sigma_0) \subset \hat{\mathbf{E}}(\sigma_0). \tag{3.6}$$

Comments 3.2

(α) Fact 3.2 is suggested by (Callier and Desoer (1980 b), Lemma 5.1., p. 36) and (Ferreira and Callier (1982), Theorem 2, p. 481).

(β) Conditions (3.5) guarantee that both f_a and $\dot{f}_a$ have a Laplace transform: a condition often encountered in dynamical systems.

(γ) In the proof below $f \in \mathbf{E}(\sigma_0)$ because for some $\sigma < \sigma_0$, $\exp(-\sigma .).f_a$ is in L_1 and 'of bounded variation on $\mathbb{R}_+$', (see (3.7) below).

Proof of Fact 3.2

Denote by $D(.)$ the derivative in the sense of distributions. Conditions (3.5) imply that, for some $\sigma < \sigma_0$,

$$\exp(-\sigma .) f_a \in L_1$$

and

$$D(\exp(-\sigma .) f_a) = \exp(-\sigma .)\dot{f}_a - \sigma \exp(-\sigma .) f_a \in \mathbf{A}(0) \tag{3.7}$$

such that on $\mathbb{R}_+$

$$\exp(-\sigma t) f_a(t) = \int_{0^-}^{t} \exp(-\sigma\tau)\dot{f}_a(\tau)\, d\tau - \sigma \int_0^t \exp(-\sigma\tau) f_a(\tau)\, d\tau$$

Hence on $\mathbb{R}_+$

$$\exp(-\sigma t)|f_a(t)| \leqq \|\exp(-\sigma .).\dot{f}_a\|_{\mathbf{A}(0)} + |\sigma|\, \|\exp(-\sigma .).f_a\|_{L_1} =: M < \infty$$

Therefore $f \in \mathbf{A}_-(\sigma_0)$ with $f_a \in L_\varepsilon(\sigma_0)$. □

We now investigate the algebraic properties of $\hat{\mathbf{E}}(\sigma_0)$.

Theorem 3.1 (subalgebra)

For any $\sigma_0 \in \mathbb{R}$, $\hat{\mathbf{E}}(\sigma_0)$ is a (commutative) subalgebra of $\hat{\mathbf{A}}_-(\sigma_0)$, i.e.

(*a*) $\hat{\mathbf{E}}(\sigma_0)$ is a linear subspace of $\hat{\mathbf{A}}_-(\sigma_0)$, (3.8)

(*b*) For all $\hat{f}$, $\hat{g}$ in $\hat{\mathbf{E}}(\sigma_0)$, $\hat{f}.\hat{g} \in \hat{\mathbf{E}}(\sigma_0)$. (3.9)

Comment 3.3

$\hat{\mathbf{E}}(\sigma_0)$ is not an ideal of $\hat{\mathbf{A}}_-(\sigma_0)$: if $\hat{g} = 1$, then for any $\hat{f} \in \hat{\mathbf{A}}_-(\sigma_0) \backslash \hat{\mathbf{E}}(\sigma_0)$, $\hat{f}.\hat{g} \in \hat{\mathbf{A}}_-(\sigma_0) \backslash \hat{\mathbf{E}}(\sigma_0)$.

Proof of Theorem 3.1

(*a*) follows by Definition 3.1 and (2.5).

(*b*) : with $f = f_a + f_{sa}$ and $g = g_a + g_{sa}$, $(f * g)_a = f_a * g_a + f_a * g_{sa} + f_{sa} * g_a$ with each term on the RHS in $L_\varepsilon(\sigma_0)$ because of the ideal property (2.6): note that such term has a factor in $L_\varepsilon(\sigma_0)$ and $\mathbf{A}_-(\sigma_0)$ respectively. Hence $f*g \in \mathbf{A}_-(\sigma_0)$ with $(f*g)_a \in L_\varepsilon(\sigma_0)$. □

We now study inversion.

Lemma 3.1 (unity feedback)

Let $\sigma_0 \in \mathbb{R}$. If $\hat{f} \in \hat{L}_\varepsilon(\sigma_0)$ and $(1+\hat{f})$ is invertible in $\hat{\mathbf{A}}_-(\sigma_0)$, then $\hat{h} = (1+\hat{f})^{-1} \in \hat{\mathbf{E}}(\sigma_0)$.

Proof

$h = \delta - (f*h)$, where $f \in L_\varepsilon(\sigma_0)$ and $h \in \mathbf{A}_-(\sigma_0)$. Hence h is in $\mathbf{E}(\sigma_0)$ because $h_a = -(f*h)$ is in $L_\varepsilon(\sigma_0)$ by the ideal property (2.6). □

Theorem 3.2 (Inversion)

Ley $\sigma_0 \in \mathbb{R}$ and $\hat{f} \in \hat{\mathbf{E}}(\sigma_0) \subset \hat{\mathbf{A}}_-(\sigma_0)$. Under these conditions

$$\hat{f} \text{ is invertible in } \hat{\mathbf{E}}(\sigma_0) \tag{3.10}$$

iff

$$\inf\{|\hat{f}(s)| : s \in \mathbb{C}_{\sigma_0}^+\} > 0. \tag{3.11}$$

Comments 3.4

(α) Invertibility condition (3.11) is also necessary and sufficient for $\hat{f}$ to be invertible in $\hat{\mathbf{A}}_-(\sigma_0)$ (Callier and Desoer (1980 a). Fact 2.3 (iii)). Hence $\hat{\mathbf{E}}(\sigma_0)$ as a subalgebra of $\hat{\mathbf{A}}_-(\sigma_0)$ is closed under inversion.

(β) The proof below shows that under (3.11) f_{sa} is invertible, whence Lemma 3.1 can be applied.

Proof of Theorem 3.2

Only if: (3.11) is necessary by (Callier and Desoer (1980 a), Fact 2.3 (iii)), because $\hat{\mathbf{E}}(\sigma_0) \subset \hat{\mathbf{A}}_-(\sigma_0)$.

If: By assumption $f = f_a + f_{sa} \in \mathbf{E}(\sigma_0) \subset \mathbf{A}_-(\sigma_0)$, whence the conclusions of Comment 3.1 (β) apply for some $\sigma < \sigma_0$. Hence condition (3.11) implies

$$\inf\{|\hat{f}_{sa}(s)| : s \in \mathbb{C}_{\sigma_0}^+\} > 0 \tag{3.12}$$

and

$$\inf\{|\hat{f}_a(s).\hat{f}_{sa}^{-1}(s) + 1| : s \in \mathbb{C}_{\sigma_0}^+\} > 0. \tag{3.13}$$

Condition (3.12) implies by Callier and Desoer (1980 a), Fact. 2.3 (iii)) that the singular atomic part f_{sa} is invertible in $\mathbf{A}_-(\sigma_0)$ with a singular atomic inverse (Hille and Phillips (1957), proof of Th. 4.18.6, p. 150). Therefore $\hat{f}_{sa}^{-1} \in \hat{\mathbf{E}}(\sigma_0) \subset \hat{\mathbf{A}}_-(\sigma_0)$ (if g denotes the inverse, then $g_a = 0$).

Now, by the ideal property (2.6), $\hat{f}_a.\hat{f}_{sa}^{-1}$ is in $L_\varepsilon(\sigma_0)$. Moreover by (3.13) $(1+\hat{f}_a\hat{f}_{sa}^{-1})$ is by (Callier and Desoer (1980 a), Fact. 2.3 (iii)) invertible in $\hat{\mathbf{A}}_-(\sigma_0)$. Hence by Lemma 3.1, $(1+\hat{f}_a\hat{f}_{sa}^{-1})^{-1}$ is in $\hat{\mathbf{E}}(\sigma_0)$. Finally, since $\hat{f}^{-1} = \hat{f}_{sa}^{-1}(1+\hat{f}_a\hat{f}_{sa}^{-1})^{-1}$, $\hat{f}^{-1}$ is in $\hat{\mathbf{E}}(\sigma_0)$ by Theorem 3.1 as a product of two factors in $\hat{\mathbf{E}}(\sigma_0)$. □

4. Boundedness from zero at infinity and coprimeness

Definition 4.1 (Callier and Desoer (1978), p. 652)

Let $\sigma_0 \in \mathbb{R}$, $\rho > 0$, $D(\sigma_0, \rho) := \{s \in \mathbb{C}_{\sigma_0^+} : |s - \sigma_0| \geqq \rho\}$ and $g \in \mathrm{LTD}_+ : g$ is said to be *bounded away from zero at infinity in* $\mathbb{C}_{\sigma_0^+}$ iff there exist $\eta > 0$ and $\rho > 0$ such that $|\hat{g}(s)| \geqq \eta$ for all $s \in D(\sigma_0, \rho)$.

Definition 4.2 (Zariski and Samuel (1958), p. 46)

Let M be a non-empty subset of a commutative ring $R : M$ is a *multiplicative system* of R iff (i) M does not contain either the zero nor divisors of zero of R, and (ii) M is closed under multiplication.

Notation 4.1.

For $\sigma_0 \in \mathbb{R}$, we shall denote by $\hat{\mathbf{E}}^\infty(\sigma_0)$ the subset of elements of $\hat{\mathbf{E}}(\sigma_0)$ that are bounded away from zero at infinity in $\mathbb{C}_{\sigma_0^+}$.

Recall now that the elements of $\mathbf{R}^\infty(\sigma_0)$ and $\hat{\mathbf{A}}_-^\infty(\sigma_0)$ are bounded away from zero at infinity in $\mathbb{C}_{\sigma_0^+}$ and form a multiplicative system of $\mathbf{R}(\sigma_0)$ or $\hat{\mathbf{A}}_-(\sigma_0)$ respectively. Moreover the latter has no divisors of zero (Callier and Desoer 1980 a, p. 321), and is by (3.1) a superset of $\hat{\mathbf{E}}(\sigma_0)$. Hence we have the following.

Fact 4.1

(*a*)
$$\hat{\mathbf{E}}^\infty(\sigma_0) \subset \hat{\mathbf{A}}_-^\infty(\sigma_0) \tag{4.1}$$

(*b*) $\hat{\mathbf{E}}^\infty(\sigma_0)$ is a multiplicative system of the commutative ring $\hat{\mathbf{E}}(\sigma_0)$. (4.2)

Comments 4.1

(α)
$$\mathbf{R}^\infty(\sigma_0) = \hat{\mathbf{E}}^\infty(\sigma_0) \cap \mathbb{C}_p(s) = \hat{\mathbf{A}}_-^\infty(\sigma_0) \cap \mathbb{C}_p(s) \tag{4.3}$$

(*β*) Recall (Callier and Desoer (1978); (1980 a), Fact 2.3): if $\hat{f} \in \hat{\mathbf{A}}_-(\sigma_0)$ then $\hat{f}$ is holomorphic in an open right half-plane strictly containing $\mathbb{C}_{\sigma_0^+}$, hence, counting multiplicities, (i) $\hat{f}$ has a finite number of zeros in any compact set of $\mathbb{C}_{\sigma_0^+}$, and (ii) if $\hat{f} \in \hat{\mathbf{A}}_-^\infty(\sigma_0)$ then $\hat{f}$ has a finite number of zeros in $\mathbb{C}_{\sigma_0^+}$. These properties apply also to $\hat{\mathbf{E}}(\sigma_0)$ and $\hat{\mathbf{E}}^\infty(\sigma_0)$ as subsets of $\hat{\mathbf{A}}_-(\sigma_0)$ or $\hat{\mathbf{A}}_-^\infty(\sigma_0)$ respectively. This observation is paramount for the results below inspired by Callier and Desoer (1978, Corollaries 2.2A, 2.2B, 2.2C; 1980 a, Corollary 2.2C).

Lemma 4.1 (division)

Let $\sigma_0 \in \mathbb{R}$ and $\hat{f} \in \hat{\mathbf{E}}(\sigma_0)$. Let $z \in \mathbb{C}_{\sigma_0^+}$ and $p \in \hat{\mathbb{C}}_{\sigma_0^-}$.

Under these conditions

$$(s - z)(s - p)^{-1} \in \mathbf{R}^\infty(\sigma_0) \subset \hat{\mathbf{E}}^\infty(\sigma_0) \tag{4.4}$$

and

$$\hat{f}(s).(s - p)(s - z)^{-1} = \hat{f}(z).(z - p)(s - z)^{-1} + \hat{q}(s) \tag{4.5 a}$$

where

$$\hat{q} \in \hat{\mathbf{E}}(\sigma_0). \tag{4.5 b}$$

Comments 4.2

Lemma 4.1 states that 'quotient' $\hat{q}$ of the division of $\hat{f} \in \hat{\mathbf{E}}(\sigma_0)$ by an elementary factor $(s-z)(s-p)^{-1}$ of $\mathbf{R}^\infty(\sigma_0)$ is in $\hat{\mathbf{E}}(\sigma_0)$, i.e. σ_0-exponentially stable. If $\hat{f}(z)=0$ and $\hat{f} \in \hat{\mathbf{E}}^\infty(\sigma_0)$, then $\hat{f}$ can be factored as $\hat{f}(s)=((s-z)(s-p)^{-1})\ \hat{q}(s)$ with two factors in $\hat{\mathbf{E}}^\infty(\sigma_0)$ and the first in $\mathbf{R}^\infty(\sigma_0)$. If, in addition, p is a pole of $\hat{f}$, then a pole-zero pair of $\hat{f}$ has been extracted.

Proof of Lemma 4.1

(4.4) is obvious. Concerning (4.5), note first that $\hat{f}(s).(s-p)(s-z)^{-1}=\hat{f}(s)(1+(z-p)(s-z)^{-1})$. Then by Fact 2.2, especially (2.10), $\hat{f}(s)(s-z)^{-1}=\hat{f}(z)(s-z)^{-1}+\hat{q}_1(s)$, where $\hat{q}_1 \in \hat{L}_\varepsilon(\sigma_0) \subset \hat{\mathbf{E}}(\sigma_0)$. Hence (4.5 *a*) follows where $\hat{q}(s)=\hat{f}(s)+(z-p)\hat{q}_1(s)$. Hence $\hat{q} \in \hat{\mathbf{E}}(\sigma_0)$ because $\hat{\mathbf{E}}(\sigma_0)$ is a linear space by (3.8). □

Repeated applications of Lemma 4.1 and Comment 4.1 (β) give the following result.

Theorem 4.1 (product decomposition)

Let $\sigma_0 \in \mathbb{R}$ and $\hat{f} \in \hat{\mathbf{E}}(\sigma_0)$. For $j=1,2,\dots,l$, let $\hat{f}$ have zeros $z_j \in \mathbb{C}_{\sigma_0^+}$ of order m_j such that $\sum_{j=1}^{l} m_j = n$. For $i=1,\dots,n$, let $p_i \in \mathring{\mathbb{C}}_{\sigma_0^-}$.

Under these conditions

$$(a) \qquad \hat{f}(s)=\hat{p}(s)\hat{q}(s) \tag{4.6 a}$$

where

$$\hat{p} \in \mathbf{R}^\infty(\sigma_0) \text{ and } \hat{q} \in \hat{\mathbf{E}}(\sigma_0) \tag{4.6 b}$$

$$\hat{p}(s)=\prod_{j=1}^{l}(s-z_j)^{m_j}.\prod_{i=1}^{n}(s-p_i)^{-1} \tag{4.6 c}$$

$$\hat{q}(z_j) \neq 0 \text{ for } j=1,\dots,l \tag{4.6 d}$$

(*b*) If, in addition, $\hat{f} \in \hat{\mathbf{E}}^\infty(\sigma_0)$ and $\hat{f}$ has no other zeros in $\mathbb{C}_{\sigma_0^+}$ than the z_j's of order m_j, then (4.6) holds and

$$\hat{q} \text{ is invertible in } \hat{\mathbf{E}}(\sigma_0). \tag{4.7}$$

Comments 4.3

(α) In Theorem 4.1(a) we remove from $\hat{f}$ the zeros $z_j \in \mathbb{C}_{\sigma_0^+}$ and stay in $\hat{\mathbf{E}}(\sigma_0)$.

(β) if $\hat{f} \in \hat{\mathbf{E}}^\infty(\sigma_0)$ then so does $\hat{q}$ in (4.6).

(γ) In Theorem 4.1 (b) we have removed all the zeros of $\hat{f}$ in $\mathbb{C}_{\sigma_0^+}$. Hence, since $\hat{q} \in \hat{\mathbf{E}}^\infty(\sigma_0)$, $\inf\{|\hat{q}(s)|: s \in \mathbb{C}_{\sigma_0^+}\} > 0$, and (4.7) follows by Theorem 3.2.

(δ) By Theorem 4.1 (b) $\hat{\mathbf{E}}^\infty(\sigma_0)$ and $\mathbf{R}^\infty(\sigma_0)$ are essentially identical in that their elements are the same modulo a factor invertible in $\hat{\mathbf{E}}(\sigma_0)$.

(ε) If in Theorem 4.1 (a) we remove from $\hat{f}$ only partially the zeros z_j in $\mathbb{C}_{\sigma_0^+}$, then (4.6 a)–(4.6 c) still hold with m_j the number of extracted zeros at z_j.

We now study coprimeness.

Definition 4.3

Let $\sigma_0 \in \mathbb{R}$ and $\hat{f}$ and $\hat{g}$ be in $\hat{\mathbf{E}}(\sigma_0)$. The pair $(\hat{f}, \hat{g})$ is said to be *$\varepsilon\sigma_0$-coprime*, (or $\hat{f}$ and $\hat{g}$ are said to be $\varepsilon\sigma_0$-coprime), iff

$$\text{there exist } \hat{u} \text{ and } \hat{v} \text{ in } \hat{\mathbf{E}}(\sigma_0) \text{ such that } \hat{u}\hat{f}+\hat{v}\hat{g}=1. \tag{4.8}$$

Comment 4.4.

Condition (4.8) is known as the Bezout condition. It is crucial for studying for instance algebras of fractions and feedback compensator design problems, e.g. in Callier and Desoer (1978, 1980 a) and Desoer et al. (1980).

A more operational characterization of coprimeness is the following.

Theorem 4.2 ($\varepsilon\sigma_0$-coprimeness)

Let $\sigma_0 \in \mathbb{R}$ and $\hat{f}$ and $\hat{g}$ be in $\hat{\mathbf{E}}(\sigma_0)$, then the pair $(\hat{f}, \hat{g})$ is $\varepsilon\sigma_0$-coprime iff

$$\inf \{|(\hat{f}(s), \hat{g}(s))| : s \in \mathbb{C}_{\sigma_0^+}\} > 0 \tag{4.9}$$

where $|.|$ is any $\mathbb{C}^2$-norm.

Comments 4.5

(α) Since $\hat{\mathbf{E}}(\sigma_0) \subset \hat{\mathbf{A}}_-(\sigma_0)$ it follows by (Callier and Desoer (1980 a), Facts 1 and 2) and (Callier and Desoer (1978), Theorem 2.1) that condition (4.9) is equivalent to $(\hat{f}, \hat{g})$ is σ_0-*coprime* or equivalently:

$$\text{there exist } \hat{u}_1 \text{ and } \hat{v}_1 \text{ in } \hat{\mathbf{A}}_-(\sigma_0) \text{ such that } \hat{u}_1\hat{f} + \hat{v}_1\hat{g} = 1 \tag{4.10}$$

i.e. the Bezout condition is satisfied with respect to $\hat{\mathbf{A}}_-(\sigma_0)$. The proof below shows that this condition also holds with respect to $\hat{\mathbf{E}}(\sigma_0)$ by a regularization procedure.

(β) Condition (4.9) is equivalent to (i) $(\hat{f}(s), \hat{g}(s)) \neq (0, 0)$ for all $s \in \mathbb{C}_{\sigma_0^+}$ and (ii) $(\hat{f}(s), \hat{g}(s))$ does not tend to $(0, 0)$ along any sequence in $\mathbb{C}_{\sigma_0^+}$ tending to infinity.

(γ) If, in addition, $\hat{g} \in \hat{\mathbf{E}}^\infty(\sigma_0)$, then condition (4.9) is equivalent to

$$(\hat{f}(s), \hat{g}(s)) \neq (0, 0) \text{ for all } s \in \mathbb{C}_{\sigma_0^+} \tag{4.11}$$

Moreover, in that case, if (4.11) is not satisfied, then, by applying Theorem 4.1 (a) under the conditions of Comment 4.3 (ε), $(\hat{f}, \hat{g})$ can be made to satisfy (4.11), (i.e. to be $\varepsilon\sigma_0$-coprime), by removing from $\hat{f}$ and $\hat{g}$ common factors $(s-z)(s-p)^{-1}$ in $\mathbf{R}^\infty(\sigma_0)$ associated with the common zeros z of $\hat{f}$ and $\hat{g}$ in $\mathbb{C}_{\sigma_0^+}$.

Proof of Theorem 4.2

Only if: if $(\hat{f}, \hat{g})$ is $\varepsilon\sigma_0$-coprime, then, by (4.8) with $\hat{\mathbf{E}}(\sigma_0) \subset \hat{\mathbf{A}}_-(\sigma_0)$, $(\hat{f}, \hat{g})$ is σ_0-coprime and (4.9) follows by Comment 4.5 (α).

If: By the equivalence of Comment 4.5(α), (4.10) holds with $\hat{u}_1 = \hat{u}_a + \hat{u}_{sa}$ and $\hat{v}_1 = \hat{v}_a + \hat{v}_{sa}$ both in $\hat{\mathbf{A}}_-(\sigma_0)$. Hence (4.10) can be rewritten as

$$\hat{u}_{sa}\hat{f} + \hat{v}_{sa}\hat{g} = 1 - (\hat{u}_a\hat{f} + \hat{v}_a\hat{g}) \tag{4.12}$$

Since they are singular atomic

$$\hat{u}_{sa} \text{ and } \hat{v}_{sa} \in \hat{\mathbf{E}}(\sigma_0) \tag{4.13}$$

and so do $\hat{f}$ and $\hat{g}$ (by assumption). Hence, by Theorem 3.1, the LHS of (4.12) is in $\hat{\mathbf{E}}(\sigma_0)$ and so does the RHS having the singular atomic part δ, $((\hat{u}_a\hat{f} + \hat{v}_a\hat{g})(s) \to 0$ as $|s| \to \infty$ in $\mathbb{C}_{\sigma_0^+}$). Therefore

$$1 - (\hat{u}_a\hat{f} + \hat{v}_a\hat{g}) \in \hat{\mathbf{E}}^\infty(\sigma_0) \tag{4.14}$$

with $\hat{u}_a\hat{f} + \hat{v}_a\hat{g}$ in $\hat{L}_\varepsilon(\sigma_0)$. Hence, by Theorem 4.1 (b) and (4.14)

$$1 - (\hat{u}_a\hat{f} + \hat{v}_a\hat{g}) = \hat{p}.\hat{q} \tag{4.15}$$

where

$$\hat{p} \in \mathbf{R}^{\infty}(\sigma_0),\ \hat{p}(\infty) = 1, \text{ and } 1 - \hat{p} \in \hat{L}_{\varepsilon}(\sigma_0) \tag{4.16}$$

and

$$\hat{q} \text{ is invertible in } \hat{\mathbf{E}}(\sigma_0) \tag{4.17}$$

Consequently (4.12) is equivalent to

$$\hat{q}^{-1}\hat{u}_{sa}\hat{f} + \hat{q}^{-1}\hat{v}_{sa}\hat{g} = \hat{p} \tag{4.18}$$

On the other hand, (4.10) implies

$$(1 - \hat{p})\hat{u}_1\hat{f} + (1 - \hat{p})\hat{v}_1\hat{g} = 1 - \hat{p} \tag{4.19}$$

Now by (4.13), (4.17) and Theorem 3.1, $\hat{q}^{-1}\hat{u}_{sa} \in \hat{\mathbf{E}}(\sigma_0)$; moreover, since $\hat{u}_1 \in \hat{\mathbf{A}}_-(\sigma_0)$, and by (4.16) and Theorem 2.1, $(1 - \hat{p})\hat{u}_1 \in \hat{L}_{\varepsilon}(\sigma_0) \subset \hat{\mathbf{E}}(\sigma_0)$. Hence, again by Theorem 3.1,

$$\hat{u} := \hat{q}^{-1}\hat{u}_{sa} + (1 - \hat{p})\hat{u}_1 \text{ is in } \hat{\mathbf{E}}(\sigma_0) \tag{4.20}$$

and for similar reasons

$$\hat{v} := \hat{q}^{-1}\hat{v}_{sa} + (1 - \hat{p})\hat{v}_1 \text{ is in } \hat{\mathbf{E}}(\sigma_0) \tag{4.21}$$

Thus, by (4.18)–(4.21), there exist $\hat{u}$ and $\hat{v}$ in $\hat{\mathbf{E}}(\sigma_0)$ such that $\hat{u}\hat{f} + \hat{v}\hat{g} = 1$, i.e., by Definition 4.3, $(\hat{f}, \hat{g})$ is $\varepsilon\sigma_0$-coprime. □

5. Fractions of σ_0-exponentially stable transfer functions

We follow the methods of Callier and Desoer (1980 a, § II) for modelling distributed systems.

Recall (Zariski and Samuel (1958), pp. 46–49) that if R is a commutative ring and M is a multiplicative system of R, then $F := RM^{-1}$ is a commutative ring of fractions of R with respect to M, i.e. $f \in F$ iff $f = nd^{-1}$ for some $n \in R$ and $d \in M$.

Now, (i) if, in addition, R is a commutative algebra, then $F = RM^{-1}$ is a commutative algebra of fractions, (ii) by Theorem 3.1 $\hat{\mathbf{E}}(\sigma_0)$ is a commutative algebra, and (iii) by Fact 4.1 $\hat{\mathbf{E}}^{\infty}(\sigma_0)$ is a multiplicative system of $\hat{\mathbf{E}}(\sigma_0)$. Hence the following definitions make sense.

Definitions 5.1

For $\sigma_0 \in \mathbb{R}$ we define $\hat{\mathbf{F}}(\sigma_0)$ as the commutative algebra of fractions of $\hat{\mathbf{E}}(\sigma_0)$ with respect to $\hat{\mathbf{E}}^{\infty}(\sigma_0)$, i.e.

$$\hat{\mathbf{F}}(\sigma_0) := [\hat{\mathbf{E}}(\sigma_0)][\hat{\mathbf{E}}^{\infty}(\sigma_0)]^{-1} \tag{5.1}$$

or equivalently $\hat{f} \in \hat{\mathbf{F}}(\sigma_0)$ iff

$$\hat{f} = \hat{n}\hat{d}^{-1} \text{ for some } \hat{n} \text{ in } \hat{\mathbf{E}}(\sigma_0) \text{ and } \hat{d} \text{ in } \hat{\mathbf{E}}^{\infty}(\sigma_0) \tag{5.2}$$

Any pair $(\hat{n}, \hat{d})$ such that (5.2) holds is called an *$\varepsilon\sigma_0$-fraction*† *of $\hat{f}$*; if, in addition, $(\hat{n}, \hat{d})$ is an $\varepsilon\sigma_0$-coprime pair, then $(\hat{n}, \hat{d})$ is called an *$\varepsilon\sigma_0$-coprime fraction*‡ *of $\hat{f}$*.

† In Desoer et al. (1980, p. 401), the term 'fractional representation' was used.

‡ In Callier and Desoer (1978, 1980 a, p. 321), a σ_0-coprime fraction of $\hat{f} \in \hat{\mathbf{B}}(\sigma_0)$ is called 'σ_0-admissible representation'.

Comments 5.1

(α) $\varepsilon\sigma_0$-coprime fractions exist by Comment 4.5 (γ) and Theorem 4.2. They are unique modulo a common factor invertible in $\hat{\mathbf{E}}(\sigma_0)$ (Desoer *et al.* 1980, p. 401). Hence, by Comment 4.1 (β), $\hat{f} \in \hat{\mathbf{F}}(\sigma_0)$ is meromorphic in an open right half-plane strictly containing $\mathbb{C}_{\sigma_0^+}$ and has a finite number of poles in $\mathbb{C}_{\sigma_0^+}$: see Theorem 5.1 below.

Fact 5.1

With $\mathbb{C}_p(s)$, $\mathbf{R}^\infty(\sigma_0)$ and $\hat{\mathbf{B}}(\sigma_0)$ given in the introduction, one has

(*a*) $$\hat{\mathbf{F}}(\sigma_0) = [\hat{\mathbf{E}}(\sigma_0)][\mathbf{R}^\infty(\sigma_0)]^{-1} \tag{5.3}$$

(*b*) $$\mathbb{C}_p(s) \subset \hat{\mathbf{F}}(\sigma_0) \subset \hat{\mathbf{B}}(\sigma_0) \tag{5.4}$$

where $\subset$ stands for 'subalgebra of'.

Proof

(*a*) follows from (5.1) and Comment 4.3(δ).

(*b*) follows because $\mathbb{C}_p(s) = [\mathbf{R}(\sigma_0)][\mathbf{R}^\infty(\sigma_0)]^{-1}$, $\hat{\mathbf{B}}(\sigma_0) = [\hat{\mathbf{A}}_-(\sigma_0)][\hat{\mathbf{A}}_-^\infty(\sigma_0)]^{-1} = [\hat{\mathbf{A}}_-(\sigma_0)][\mathbf{R}^\infty(\sigma_0)]^{-1}$, (Callier and Desoer 1980 a), and $\mathbf{R}(\sigma_0) \subset \hat{\mathbf{E}}(\sigma_0) \subset \hat{\mathbf{A}}_-(\sigma_0)$ where inclusion stands for 'subalgebra of'. Hence, using (5.3), (5.4) follows. □

The statements of Theorems 5.1 and 5.2 below are those of Callier and Desoer (1978, Theorem 3.3), or (1978, Theorem 3.7; 1980 a, Theorem 3.7) modulo interchanging $\hat{\mathbf{E}}(\sigma_0)$, $\hat{\mathbf{F}}(\sigma_0)$ for $\hat{\mathbf{A}}_-(\sigma_0)$, $\hat{\mathbf{B}}(\sigma_0)$ respectively. Their proofs are similar, and are omitted here. For Theorem 5.1 use Lemma 4.1 instead of Callier and Desoer (1978), Cor. 2.2A. Comments stress these results important for modelling and feedback.

Theorem 5.1 (sum decomposition)

Let $f \in \mathrm{LTD}_+$. Then

$$\hat{f} \in \hat{\mathbf{F}}(\sigma_0) \tag{5.5}$$

if and only if

$$\hat{f} = \hat{r} + \hat{g} \tag{5.6}$$

where

(*a*) $\hat{g} \in \hat{\mathbf{E}}(\sigma_0)$ (5.7)

(*b*) $\hat{r}$ is a strictly proper rational function which is zero iff $\hat{f} \in \hat{\mathbf{E}}(\sigma_0)$ (5.8)

(*c*) if $\hat{f} \notin \hat{\mathbf{E}}(\sigma_0)$, then

$$\hat{r} = p_n/p_d \tag{5.9 a}$$

is the sum of the principal parts of the Laurent expansions of $\hat{f}$ at its poles in $\mathbb{C}_{\sigma_0^+}$: in particular,

p_n and p_d are coprime polynomials (5.9 *b*)

p_d is monic (5.9 *c*)

$\deg(p_n) \leqq \deg(p_d) - 1$ (5.9 *d*)

$\hat{f}$ has an *m*th-order pole at $p \in \mathbb{C}_{\sigma_0^+}$ iff p_d has an *m*th-order zero at $p \in \mathbb{C}_{\sigma_0^+}$ (5.9 *e*)

$p_d(s) \neq 0$ for all $s \in \hat{\mathbb{C}}_{\sigma_0^-}$. (5.9 *f*)

Comments 5.2

(α) The analogue (Callier and Desoer (1978), Theorem 3.3) is crucial for showing that the class of infinite-dimensional semigroup systems considered in Nett *et al.* (1983, (3.1), (3.2)) have a transfer matrix with elements in $\hat{\mathbf{B}}(0)$: see the proof of Nett *et al.* (1983, Theorem 3.2). Our sharper Theorem 5.1 shows that the elements are in $\hat{\mathbf{F}}(0)$.

(β) Here, (5.9), $\hat{g} \in \hat{\mathbf{E}}(\sigma_0) \subsetneq \hat{\mathbf{A}}_-(\sigma_0)$: any TF in $\hat{\mathbf{F}}(\sigma_0)$ is the sum of two TF's, the first is strictly proper rational with poles in $\mathbb{C}_{\sigma_0^+}$, the second is σ_0-exponentially stable.

Corollary 5.1

The (transfer function) algebras $\hat{L}_\varepsilon(\sigma_0)$, $\hat{\mathbf{E}}(\sigma_0)$ and $\hat{\mathbf{F}}(\sigma_0)$ given in Definitions 2.1, 3.1 and 5.1 respectively are of exponential order in the sense that they are subalgebras of $U\{\hat{E}(\sigma_1): \sigma_1 \geqq \sigma_0\}$.

Proof

Obvious from the definitions, Theorem 5.1 and previous theory. □

Theorem 5.2 (invertibility)

Let $\hat{f} \in \hat{\mathbf{F}}(\sigma_0)$. Then $\hat{f}$ is invertible in $\hat{\mathbf{F}}(\sigma_0)$ iff $\hat{f}$ is bounded away from zero at infinity in $\mathbb{C}_{\sigma_0^+}$.

Comments 5.3.

(α) By Theorem 5.2, since $\hat{\mathbf{E}}^\infty(\sigma_0) \subset \hat{\mathbf{E}}(\sigma_0) \subset \mathbf{F}(\sigma_0)$, $I := \{\hat{f} \in \hat{\mathbf{E}}(\sigma_0): \hat{f}^{-1} \in \hat{\mathbf{F}}(\sigma_0)\} = \hat{\mathbf{E}}^\infty(\sigma_0)$. Hence, with $G = \hat{\mathbf{F}}(\sigma_0)$, $H = \hat{\mathbf{E}}(\sigma_0)$, $I = \hat{\mathbf{E}}^\infty(\sigma_0)$ and $J = \{\hat{f} \in \hat{\mathbf{E}}(\sigma_0): \hat{f}$ is invertible in $\hat{\mathbf{E}}(\sigma_0)\}$, every $\varepsilon\sigma_0$-fraction $\hat{n}\hat{d}^{-1} \in \hat{\mathbf{F}}(\sigma_0)$ is a fractional representation in $\{G, H, I, J\}$, (Desoer *et al.* (1980), p. 401). Therefore the feedback system design techniques of Desoer *et al.* (1980) apply over $\hat{\mathbf{E}}(\sigma_0)$.

(β) MIMO systems with transfer matrices having elements in $\hat{\mathbf{F}}(\sigma_0)$ can be handled according to the methods of Nett *et al.* (1983), Callier and Desoer (1980 b) and Desoer *et al.* (1980) for example.

6. Link with semigroup systems

We study recent feedback stabilizability results by Jacobson (1984 a, 1984 b), to which our algebras $L_\varepsilon(\sigma_0)$, $\mathbf{E}(\sigma_0)$ and $\mathbf{F}(\sigma_0)$ add precision. The main results of semigroup theory are available in for example Curtain and Pritchard (1978), Hille and Phillips (1957), Pazy (1983), Pritchard and Zabczyk (1981) and Triggiani (1975). Inspired by Jacobson (1984 a, 1984 b), we consider the following class of SISO semigroup state-space systems SGB with bounded sensing and control (generalization to the MIMO case is straightforward; for unbounded sensing and control see for example Curtain and Pritchard (1978), Ch. 8).

Definition 6.1

A dynamical system SGB is described by the equations

$$\dot{x}(t) = Ax(t) + Bu(t) \quad t \in \mathbb{R}_+,\ x(0) = x_0 \in D(A) \tag{6.1}$$

$$y(t) = Cx(t) + \sum_{i=0}^{\infty} d_i u(t - t_i) \quad t \in \mathbb{R}_+ \tag{6.2}$$

where

(a) $x(t) \in X$, a Hilbert space; $u(t)$, $y(t) \in \mathbb{R}$, (6.3)

(b) $A : D(A) \subset X \to X$ is the infinitesimal generator of a C_0-semigroup $(T_t)_{t \geq 0}$ of bounded linear operators in X; i.e. $T_t \in \mathbf{L}(X, X)$ (6.4)

(c) B and C are bounded linear operators, i.e. $B \in \mathbf{L}(\mathbb{R}, X)$ and $C \in \mathbf{L}(X, \mathbb{R})$, (6.5)

(d) $t_0 = 0$, $t_i > 0$ for all $i = 1, 2, \ldots$, and for all $i = 0, 1, \ldots$, $d_i \in \mathbb{R}$ such that

$$\sum_{i=0}^{\infty} |d_i| \exp(-\sigma t_i) < \infty \text{ for some } \sigma < \sigma_0. \tag{6.6}$$

Comments 6.1

(α) In (Jacobson 1984 a, b) X is a reflexive Banach space: generically X is a Hilbert space.

(β) The direct I/O transmission $\sum_{i=0}^{\infty} d_i u(t - t_i)$ is absent in Jacobson (1984 a, 1984 b) and most applications: it is added for obtaining a singular atomic part in the TF.

(γ) In (6.5) C is a bounded linear functional: hence by the Riesz representation theorem (Rudin (1974), p. 139)

$$Cx = \langle c, x \rangle \text{ for all } x \in X \tag{6.7}$$

where $c \in X$ and $\langle ., . \rangle$ is the inner product of X. Moreover, by (6.5)

$$Bu = bu \quad \text{for all } u \in \mathbb{R} \tag{6.8}$$

for some $b \in X$.

(δ) By standard analysis (Curtain and Pritchard 1978, Kato 1980, Pazy 1983), the (mild) solution $x(.)$ on $\mathbb{R}_+$ reads

$$x(t) = T_t x_0 + \int_0^t T_{t-\tau} Bu(\tau)\, d\tau \quad x_0 \in X \tag{6.9}$$

and the impulse response of system SGB is given by

$$f(t) = CT_t B + \sum_{i=0}^{\infty} d_i \delta(t - t_i) \tag{6.10}$$

Hence by the Laplace transform SGB has a TF

$$\hat{f}(s) = C(sI - A)^{-1} B + \sum_{i=0}^{\infty} d_i \exp(-st_i) \tag{6.11}$$

where $(sI - A)^{-1}$ is the resolvent of A (usually denoted by $R(s, A)$, i.e. the Laplace transform of $t \mapsto T_t$): it is a bounded linear operator in X for all s in an open right half-plane; as a function of s it is there a holomorphic operator-valued function.

Now, using (6.7) and (6.8), the TF reads more classically

$$\hat{f}(s) = \langle c, (sI - A)^{-1} b \rangle + \sum_{i=0}^{\infty} d_i \exp(-st_i) \tag{6.12}$$

where $\hat{f}$ is holomorphic in an open right half-plane. Hence, modulo b and c in X and $(t_i, d_i)_0^{\infty}$ as in (6.6), $\hat{f}$ will be specified by defining the state space $(X, \langle .,. \rangle)$ and the generator A with its domain $D(A)$: e.g. (i) if X is separable and A is self-adjoint with compact resolvent: see e.g. Curtain and Pritchard (1978), Example 2.40, and (ii) for differential delay equations on $X = M^2$: see e.g. Vinter (1978). Therefore we have a *systematic way* of writing TF's of systems SGB.

Similarly to Jacobson (1984 a, 1984 b) we show here that, modulo σ_0-exponential stabilizability and detectability, (i) $\hat{f}$, given by (6.11)–(6.12), is in $\hat{\mathbf{F}}(\sigma_0)$ rather than $\hat{\mathbf{B}}(\sigma_0)$, and (ii) internal σ_0-exponential stability is equivalent to external exponential stability in $\hat{\mathbf{E}}(\sigma_0)$, (rather than external stability in $\hat{\mathbf{A}}_-(\sigma_0)$). With $\|.\|$ denoting the uniform operator norm we need the following.

Definitions 6.2

Consider any system SGB.

(α) SGB is said to be *internally σ_0-exponentially stable* iff the semigroup $(T_t)_{t\geq 0}$ is *σ_0-exponentially stable*, i.e. there exist $\sigma < \sigma_0$ and $M > 0$ such that $\| T_t \| \leq M \exp(\sigma t)$ on $\mathbb{R}_+$.

(β) SGB is said to be *externally σ_0-exponentially stable* iff its TF $\hat{f}$ is in $\hat{\mathbf{E}}(\sigma_0)$.

(γ) The operator pair (A, B) is said to be *σ_0-exponentially stabilizable* iff there exists $K \in \mathbf{L}(X, \mathbb{R})$ such that the semigroup $(T_t^K)_{t\geq 0}$ generated by $A - BK$ is σ_0-exponentially stable.

(δ) The operator pair (C, A) is said to be *σ_0-exponentially detectable* iff there exists $F \in \mathbf{L}(\mathbb{R}, X)$ such that the semigroup $(T_t^F)_{t\geq 0}$, generated by $A - FC$, is σ_0-exponentially stable.

The following assumption is standard in semigroup theory (Curtain and Pritchard 1978, Pritchard and Zabczyk 1981, Triggiani 1975).

Definition 6.3

Consider any system SGB and let $A\colon D(A) \subset X \to X$ be the infinitesimal generator of the C_0-semigroup $(T_t)_{t\geq 0}$ with spectrum $\sigma(A) \subset \mathbb{C}$. A is said to satisfy the *spectrum decomposition assumption* (SDA) *at* $\sigma \in \mathbb{R}$ iff $\sigma_u(A) := \sigma(A) \cap \mathbb{C}_{\sigma+}$ is bounded and separated from $\sigma_s(A) := \sigma(A) \cap \mathring{\mathbb{C}}_{\sigma-}$ in such a way that a simply closed rectifiable oriented curve Γ can be drawn so as to enclose an open set containing $\sigma_u(A)$ in its interior and $\sigma_s(A)$ in its exterior.

Comment 6.2 (Curtain 1984, Curtain and Pritchard 1978, Triggiani 1975)

The SDA at σ induces a natural state-space decomposition $X = X^u \dotplus X^s$, where $X^u := \Pi(X)$ and $X^s := (I - \Pi)(X)$, with $\Pi := (2\pi j)^{-1} \int_\Gamma (sI - A)^{-1}\, ds$ a bounded projection in X. $A^u := A | X^u$ is bounded, $A^s := A | X^s$ is a generator; corresponding spectra are $\sigma(A^u) = \sigma_u(A)$, $\sigma(A^s) = \sigma_s(A)$. Furthermore Π reduces T_t, i.e. Π and $(I - \Pi)$ commute with A and T_t; $T_t^u = \Pi T_t$ is the semigroup generated by A^u, $(T_t^u = \exp(A^u t))$, and $T_t^s = (I - \Pi) T_t$ is the semigroup generated by A^s. Finally, the operators B and C are decomposed according to $B^u = \Pi B$, $B^s = (I - \Pi)B$ and $C^u = C\Pi$, $C^s = C(I - \Pi)$.

Hence, after decomposition, the impulse response, (6.10), and the transfer function, (6.11), read now

$$f(t) = C^u \exp(A^u t) B^u + C^s T_t^s B^s + \sum_{i=0}^{\infty} d_i \delta(t - t_i) \tag{6.13}$$

or

$$\hat{f}(s) = C^u(sI - A^u)^{-1}B^u + C^s(sI - A^s)^{-1}B^s + \sum_{i=0}^{\infty} d_i \exp(-st_i) \tag{6.14}$$

respectively.

We then have the following.

Fact 6.1 (Jacobson 1984 b, Theorem 1) (Exponential stabilizability and detectability)

Consider a system SGB of Definition 6.1. Then

$$(A, B) \text{ is } \sigma_0\text{-exponentially stabilizable and } (C, A) \text{ is } \sigma_0\text{-exponentially detectable} \tag{6.15}$$

if and only if

(*a*) the generator A satisfies the SDA at some $\sigma < \sigma_0$ (6.16)

(*b*) A^s generates a σ_0-exponentially stable semigroup $(T^s_t)_{t \geq 0}$ (6.17)

(*c*) X^u is finite-dimensional (6.18)

(*d*) (A^u, B^u) is controllable and (C^u, A^u) is observable. (6.19)

Comments 6.3

(α) In (6.19) A^u is a matrix and B^u, C^u are vectors.

(*β*) In (Jacobson 1984 b, Theorem 1), SGB does not contain a direct I/O transmission $\sum_{i=0}^{\infty} d_i u(t - t_i)$; however, this term is irrelevant in the proof whose sufficiency part is well known (Curtain and Pritchard (1978), Triggiani (1975)): the point is necessity: Jacobson (1984 a, Theorem 3.1.2, Theorem 3.2.1; 1984 b, Theorem 1).

Theorem 6.1

Consider a system SGB with transfer function $\hat{f}$ given by (6.11). Under these conditions, if

$$(A, B) \text{ is } \sigma_0\text{-exponentially stabilizable and } (C, A) \text{ is } \sigma_0\text{-exponentially detectable} \tag{6.15}$$

then

$$\hat{f} \in \hat{\mathcal{F}}(\sigma_0) \subsetneq \hat{\mathcal{B}}(\sigma_0). \tag{6.20}$$

Comment 6.4

An analog is the reasoning of Jacobson (1984 a, Theorem 3.1.1) whence $C(sI - A)^{-1}B \in \hat{\mathcal{B}}(\sigma_0)$.

Short proof

By fact 6.1 and (6.14)

$$\hat{f} = \hat{r} + \hat{g} \tag{6.21}$$

where

(i) $\hat{g}(s) := C^s(sI - A^s)^{-1}B^s + \sum_{i=0}^{\infty} d_i \exp(-st_i)$ is in $\hat{\mathbf{E}}(\sigma_0)$. Indeed $g(t) = C^s T_t^s B^s + \sum_{i=0}^{\infty} d_i \delta(t - t_i)$, where the last term is in $\mathbf{E}(\sigma_0)$ by (6.6) and $t \to C^s T_t^s B^s$ is in $L_\varepsilon(\sigma_0)$ because $|C^s T_t^s B^s| \leqq \|C^s\| M \|B^s\| \exp(\sigma t)$ for some $\sigma < \sigma_0$ and $M > 0$, (6.17).

(ii) $\hat{r}(s) := C^u(sI - A^u)^{-1}B^u = p_n(s)/p_d(s)$ is a rational function by Comment 6.3 (α) with $p_n(s) := C^u \mathrm{Adj}(sI - A^u)B^u$ and $p_d(s) := \det(sI - A^u)$ coprime polynomials satisfying the properties (5.9).

Hence $\hat{f} \in \hat{\mathbf{F}}(\sigma_0)$ by Theorem 5.1 and (6.21) with its properties. □

By Fact 6.1 and the reasoning of Jacobson (1984 b, Theorem A.1.1), we also have the following result.

Theorem 6.2

Consider a system SGB. SGB is internally σ_0-exponentially stable if and only if

(*a*) (A, B) is σ_0-exponentially stabilizable and (C, A) is σ_0-exponentially detectable (6.15)

(*b*) SGB is externally σ_0-exponentially stable. (6.22)

Comments 6.5

(α) In Jacobson (1984 b), Theorem A.1.1, (6.22) reads $\hat{f} \in \hat{\mathbf{A}}_-(\sigma_0)$; here $\hat{f} \in \hat{\mathbf{E}}(\sigma_0) \subsetneq \hat{\mathbf{A}}_-(\sigma_0)$.

(β) Fact 6.1 and Theorem 6.2 are the key results for output feedback stabilizability of a system SGB without a direct I/O transmission: see Jacobson (1984 b), Theorem 3). By Theorem 6.1 any such system necessarily has a TF in $\hat{\mathbf{F}}(\sigma_0) \subsetneq \hat{\mathbf{B}}(\sigma_0)$. If $\hat{f}$ has a direct I/O transmission term, then the same conclusion can be obtained from transfer function theory: see Nett (1984), Theorem 6.1 and its proof.

7. Conclusion

Four classes of distributed system transfer functions of exponential order have been defined:

(i) $\hat{L}_\varepsilon(\sigma_0)$, ($\sigma_0$-exponentially stable functions; Definition 2.1)

(ii) $\hat{\mathbf{E}}(\sigma_0)$, ($\sigma_0$-exponentially stable transfer functions; Definition 3.1)

(iii) $\hat{\mathbf{E}}^\infty(\sigma_0)$, ('biproper' elements of $\hat{\mathbf{E}}(\sigma_0)$; Notation 4.1)

(iv) $\hat{\mathbf{F}}(\sigma_0) = [\hat{\mathbf{E}}(\sigma_0)][\hat{\mathbf{E}}^\infty(\sigma_0)]^{-1}$ (fractions; Definition 5.1).

They are (a) restrictions of (i) $U\{L_{1\sigma} : \sigma < \sigma_0\}$, (ii) $\hat{\mathbf{A}}_-(\sigma_0)$, (iii) $\hat{\mathbf{A}}_-^\infty(\sigma_0)$, and (iv) $\hat{\mathcal{B}}(\sigma_0) = [\hat{\mathbf{A}}_-(\sigma_0)][\hat{\mathbf{A}}_-^\infty(\sigma_0)]^{-1}$ respectively, and (b) extensions of (i) the class of σ_0-exponentially stable 'exponential polynomials', (ii) $\mathbf{R}(\sigma_0)$, (iii) $\mathbf{R}^\infty(\sigma_0)$, and (iv) $\mathbb{C}_p(s) = [\mathbf{R}(\sigma_0)][\mathbf{R}^\infty(\sigma_0)]^{-1}$ *respectively such that they maintain the exponential order properties of the latter classes* associated with lumped systems; moreover, corresponding classes $A \subset B \subset C$ have the same algebraic properties (e.g. algebra or multiplicative system ...): see the results of §§ 2–5 of which two are *fundamental*, viz. (α) $\hat{L}_\varepsilon(\sigma_0)$ is a proper ideal of $\hat{\mathbf{A}}_-(\sigma_0)$ (Theorem 2.1), and (β) the transmission of a σ_0-unstable exponential by a system with transfer function in $\hat{\mathbf{A}}_-(\sigma_0)$ with an 'error' in $\hat{L}_\varepsilon(\sigma_0)$ (Fact 2.2).

For the transfer function algebra of fractions $\hat{\mathbf{F}}(\sigma_0)$ of § 5, we obtained that (i) $\hat{f} \in \hat{\mathbf{F}}(\sigma_0)$ is holomorphic in $\mathbb{C}_{\sigma_0^+}$ (i.e. σ_0-stable) iff $\hat{f} \in \hat{\mathbf{E}}(\sigma_0)$, (i.e. σ_0-exponentially stable), (ii) $\hat{f} \in \hat{\mathbf{F}}(\sigma_0)$ iff $\hat{f} = \hat{r} + \hat{g}$ (the sum decomposition of Theorem 5.1), and (iii) $\hat{\mathbf{F}}(\sigma_0)$ allows delay in the direct I/O transmission of the system. This resulted in § 6 in a nearly perfect fit for any standard semigroup system SGB: if (A, B) is σ_0-exponentially stabilizable and (C, A) is σ_0-exponentially detectable then (i) the transfer function $\hat{f}$ is in $\hat{\mathbf{F}}(\sigma_0)$, (rather than $\hat{\mathbf{B}}(\sigma_0)$), and (ii) internal σ_0-exponential stability is equivalent to external σ_0-exponential stability ($\hat{f}$ is in $\hat{\mathbf{E}}(\sigma_0)$ rather than $\hat{\mathbf{A}}_-(\sigma_0)$).

To conclude, we make three remarks.

(α) Numerically the sum decomposition of Theorem 5.1 involves computing poles and partial fractions of $\hat{f}$ in $\hat{\mathbf{F}}(\sigma_0)$: see e.g. Henrici (1974). In particular for semigroup systems SGB we must calculate eigenvalues and (generalized) eigenspaces of the generator A: see e.g. Chatelin (1983).

(β) A fractional representation of $\hat{f}$ in $\hat{\mathbf{F}}(\sigma_0)$ can be obtained by the methods of Nett *et al.* (1984, especially Remark 6) and Jacobson (1984 a, Theorem 3.1.5).

(γ) $\mathbf{E}(\sigma_0)$ together with the $\mathbf{A}(\sigma_0)$-norm is a *normed algebra* in which the *small gain property* holds, viz. if f is in $\mathbf{E}(\sigma_0)$ and $\|f\|_{\mathbf{A}(\sigma_0)} < 1$ then $(1+\hat{f})^{-1}$ is in $\hat{\mathbf{E}}(\sigma_0)$ (use Nett *et al.* (1983, Proof of Theorem 3.1) and Theorem 3.2). This property is paramount for robust feedback stability, e.g. Vidyasagar (1984), and sensitivity reduction, e.g. Zames (1981, p. 304).

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