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1 Introduction

The Koopman operator provides an alternative framework to describe the evolution of nonlinear systems in a purely linear fashion. Let us consider an autonomous dynamical system $x(t+1) = F(x(t))$ where $x \in \mathscr{X}$ is the state space and $F : \mathscr{X} \to \mathscr{X}$ is a nonlinear map. The Koopman operator is defined as the composition $\mathscr{K} f = f \circ F$ where $f : \mathscr{X} \to \mathbb{C}$ is an *observable* that belongs to a Banach space [1].

Since the Koopman operator is infinite-dimensional, it is natural and often necessary to compute its finite-dimensional approximation. This approximation is given by the *Koopman matrix*, which represents the projection of the operator onto a subspace spanned by the basis functions, also called *dictionary*. A finite-dimensional approximation of the Koopman operator can be obtained from data through the so-called Extended Dynamic Mode Decomposition (EDMD) algorithm [2].

Recently *dictionary learning* methods for EDMD have been proposed to provide a set of functions that yields the best representation of the Koopman operator. In particular, [3] has considered a feed-forward network for this purpose.

In this work, we propose to use a reservoir computer [4, 5] instead of a feed-forward network. This allows to train the dictionary with a dynamical network rather than with a static one. The reservoir computer is a discrete-time neural network which consists of three layers: the inputs, the reservoir, and the outputs (see [6] for a review). It is noticeable that the outputs are obtained through *linear* combinations of the states and *only* the output weights are trained. This is a key computational advantage of the reservoir computer that we exploit.

2 Two methods

We propose two methods to approximate the Koopman operator with a reservoir computer.

Our first method is to apply EDMD directly on the internal states of the reservoir. This method is efficient but provides a large Koopman matrix, since the number of internal states is typically large. This motivates our second method, which fully exploits the reservoir computer framework.

In order to reduce the size of the Koopman matrix, our a second method uses the outputs of the reservoirs as dictionary functions. These outputs are trained through the following optimization problem

$$\min_{W_{out},K} \sum_{t=1}^{T-1} \|W_{out}s(t+1) - KW_{out}s(t)\|^2 \tag{1}$$

where W_{out} is the output weights matrix, K the Koopman matrix and $s(t)$ is the vector of internal states of the reservoir at time t.

The optimization is solved with two alternating steps:

1. Computation of the Koopman matrix: fix W_{out} and optimize K using the least squares method ;
2. Dictionary learning: fix K and optimize W_{out} to minimize the norm of $W_{out}s(t+1) - KW_{out}s(t)$.

The second step of the method amounts at solving a Sylvester equation of the form $AX + XB = C$. We propose two different least squares techniques to solve the equation, which we will discuss.

It is noticeable that the second method can be seen as a compromise between our first method and linear DMD.

We will illustrate these two methods in the context of spectral analysis and prediction.

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