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*Publication date:*  
1996

*Document Version*  
Peer reviewed version

[Link to publication](#)

*Citation for published version (HARVARD):*

Conn, A, Gould, N, Sartenaer, A & Toint, P 1996, *On iterated-subspace minimization methods for nonlinear optimization: Proceedings on Linear and Nonlinear Conjugate Gradient-Related Methods*.

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# On Iterated-Subspace Minimization Methods for Nonlinear Optimization

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February 16, 1996

**Abstract.** We consider a class of Iterated-Subspace Minimization (ISM) methods for solving large-scale unconstrained minimization problems. At each major iteration of such a method, a low-dimensional manifold, the iterated subspace, is constructed and an approximate minimizer of the objective function in this manifold then determined. The iterated subspace is chosen to contain vectors which ensure global convergence of the overall scheme and may also contain vectors which encourage fast asymptotic convergence. We demonstrate that this approach can sometimes be very advantageous and indicate the general performance on a collection of large problems. Moreover, comparisons with a limited memory approach and LANCELOT are made.

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**Keywords:** Unconstrained optimization, large-scale computation.

**Mathematics Subject Classifications:** 65K05, 90C30

**Running title:** Iterated-Subspace Minimization

This work was supported by the Belgian National Fund for Scientific Research. The research of Conn and Toint was supported in part by the Advanced Research Projects Agency of the Department of Defense and was monitored by the Air Force Office of Scientific Research under Contract No F49620-91-C-0079. The United States Government is authorized to reproduce and distribute reprints for governmental purposes notwithstanding any copyright notation hereon.

# On Iterated-Subspace Minimization Methods for Nonlinear Optimization

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## Abstract

We consider a class of Iterated-Subspace Minimization (ISM) methods for solving large-scale unconstrained minimization problems. At each major iteration of such a method, a low-dimensional manifold, the iterated subspace, is constructed and an approximate minimizer of the objective function in this manifold then determined. The iterated subspace is chosen to contain vectors which ensure global convergence of the overall scheme and may also contain vectors which encourage fast asymptotic convergence. We demonstrate that this approach can sometimes be very advantageous and indicate the general performance on a collection of large problems. Moreover, comparisons with a limited memory approach and LANCELOT are made.

## 1 Introduction

In this paper, we consider finding a local solution of the unconstrained minimization problem,

$$(1.1) \quad \begin{array}{l} \text{minimize } f(\boldsymbol{x}), \\ \boldsymbol{x} \in \mathbb{R}^n \end{array}$$

where we assume, for simplicity, that the objective function  $f \in C^2$ . We are particularly interested in the case where  $n$  is sufficiently large that methods appropriate for small problems — such as those which might maintain a dense factorization of a suitable approximation of the Hessian matrix, see, for example, Gill *et al.* (1981), Dennis and Schnabel (1983) and Fletcher (1987) — are impractical. We are especially interested in solving large problems for which exact second derivatives are explicitly available. At variance with a popular misconception, such problems frequently arise in practice, as is evident when one considers, amongst others, the large CUTE collection (see Bongartz *et al.*, 1995).

We are primarily concerned with the commonly occurring case in which the cost of evaluating the value of the objective function and its derivatives, at a given point  $\boldsymbol{x}$ , is less significant than the cost of solving, for instance, the Newton equations. Our experience with the large-scale nonlinear optimization package LANCELOT (see Conn *et al.*, 1992) has been that it is the linear-algebra cost which tends to dominate when solving a significant number of widely differing application problems (see, Conn *et al.*, 1993 and Conn *et al.*, 1996). Thus, it would appear desirable in these cases to attempt to reduce the linear-algebra costs, even if this results in an increase in the number of objective function evaluations.

The most common methods for unconstrained minimization either determine a search direction followed by a linesearch or use the trust-region approach (see, for example, Dennis and Schnabel, 1983). In the former case, a simple model of the underlying objective function

is constructed in order to determine the search direction. By contrast, in the latter case, an approximate minimizer of the model within a restricted domain (the trust region) is determined. This model minimizer is then used as a prediction of the actual minimizer of the true objective. In a trust-region method, success of this process is measured by comparing the model and true function values at the predicted minimizer. In linesearch methods, the true function is used to establish a step size. Thus, both of these approaches may be considered to perform their multi-dimensional work with respect to a model whilst probing the true function uni-dimensionally. Of course, the model does make use of the true function and perhaps its derivatives — maybe at more than a single point.

In this paper, we take the view that the above schemes are quite wasteful, given the amount of information that may have been accrued during the (approximate) minimization of the model. In particular, the model may have been sampled in a number of potentially interesting directions, of which only the aggregate direction is normally considered to be of significance.

We also believe that, provided function and derivative values are inexpensive to compute relative to the linear-algebra costs, an (approximate) low-dimensional minimization is a relatively simple calculation. Indeed, we feel that there is high-quality, robust, general-purpose software readily available for the small-scale unconstrained minimization problem, and that such software is normally capable of solving problems of modest dimensions - say up to a hundred variable problems - extremely fast on current workstations *provided that* function evaluation is cheap. Of course there are, and will continue to be, small-scale problems which are challenging, because they are so nonlinear that algorithms implemented in fixed, finite precision arithmetic are unsuccessful, but in our experience such examples occur rarely in practice.

Thus, in this paper, we propose methods which aim to investigate the *true* objective function in a space larger than the one-dimensional space which is normally associated with linesearch or trust-region methods. We do this knowing that, so long as the space is relatively modest, the approximate multi-dimensional minimization will still be a manageable calculation. Moreover, by carefully choosing the space that we investigate, we hope to reduce significantly the linear-algebra costs while still maintaining global, and fast asymptotic, convergence.

Previous work along the lines we are considering includes Cragg and Levy (1969) who propose minimizing in a  $k$ -dimensional subspace, which includes the steepest descent direction. The remaining  $k-1$  directions are the steps from the  $k-1$  previous outer iterations. See also, for example, Dennis and Turner (1987), Dixon *et al.* (1984), Miele and Cantrell (1969), and Vinsome (1976). The first minimizes a convex quadratic on a subspace by adding, at each iteration, an extra vector that is not dependent on the existing subspace. They thus provide a uniform framework for a wide class of conjugate directions. The second minimizes the objective function over a grid of 4096 points, generated from a four-dimensional subspace. This subspace is defined using the steepest descent direction, the Newton direction, and two others directions which are combinations of the two previous steps.

A particular form of these ideas has been given by Saad (1990) for the solution of nonlinear systems of equations. Here, a sequence of iterates are generated as least-squares solutions to the equations in suitable Krylov subspaces. The principal difference is that, in Saad's proposal, the entire Krylov subspace generated is used, while, as we shall see, this is in general quite unnecessary. Another example, that also is in the context of nonlinear equations is that of Kaporin and Axelsson (1995). Although developed independently, many

of the ideas are quite similar to those of this paper, but the emphasis is rather different.

Given an initial estimate of the solution to (1.1),  $\mathbf{x}^{(0)}$ , and an iteration count,  $k$ , set initially to zero, a prototype algorithm for the proposed method might be as follows:

1. Stop with the solution estimate  $\mathbf{x}^{(k)}$  if convergence tests are satisfied.
2. Determine a full-rank subspace matrix  $\mathbf{S}^{(k)} \in \mathfrak{R}^{n \times s^{(k)}}$ , where  $s^{(k)} \ll n$ .
3. Approximately solve the  $s^{(k)}$ -dimensional minimization problem

$$(1.2) \quad \underset{\mathbf{y} \in \mathfrak{R}^{s^{(k)}}}{\text{minimize}} \quad f(\mathbf{x}^{(k)} + \mathbf{S}^{(k)}\mathbf{y}),$$

set

$$(1.3) \quad \mathbf{x}^{(k+1)} = (\text{approximate}) \arg \min_{\mathbf{y} \in \mathfrak{R}^{s^{(k)}}} f(\mathbf{x}^{(k)} + \mathbf{S}^{(k)}\mathbf{y}),$$

replace  $k$  by  $k + 1$  and return to step 1.

We refer to such a method as *Iterated-Subspace Minimization* or ISM for short. This is, of course, a multi-dimensional subspace analog of the unidimensional-subspace linesearch method.

We are interested in the following issues:

- What is a good choice for  $s^{(k)}$ ?
- How do we determine the *Iterated-Subspace* matrix  $\mathbf{S}^{(k)}$ ?
- What do we mean by “approximate” in the problem (1.3)?
- Are there methods which are particularly appropriate for solving (1.2)?
- What can we say about the convergence of such a method?
- If we can establish convergence, what can we say about its asymptotic rate?

In this paper, we make preliminary attempts to answer all of these questions.

We will use the following notation. Bold lower and upper case Roman letters indicate vectors and matrices, respectively, while Greek and normal Roman letters denote scalars. Script style letters are index sets. A superscript  $(k)$  indicates a quantity which occurs at the  $k$ -th iteration or which is evaluated at  $\mathbf{x}^{(k)}$ .

We let  $\mathbf{g}(\mathbf{x})$  and  $\mathbf{H}(\mathbf{x})$ , respectively, indicate the gradient,  $\nabla_{\mathbf{x}} f(\mathbf{x})$ , and Hessian matrix,  $\nabla_{\mathbf{x}\mathbf{x}} f(\mathbf{x})$ , of the objective function. We define

$$(1.4) \quad f_s^{(k)}(\mathbf{y}) \stackrel{\text{def}}{=} f(\mathbf{x}^{(k)} + \mathbf{S}^{(k)}\mathbf{y}),$$

$\mathbf{g}_s^{(k)}(\mathbf{y}) \stackrel{\text{def}}{=} \nabla_{\mathbf{y}} f_s^{(k)}(\mathbf{y})$  and  $\mathbf{H}_s^{(k)}(\mathbf{y}) \stackrel{\text{def}}{=} \nabla_{\mathbf{y}\mathbf{y}} f_s^{(k)}(\mathbf{y})$ , and will make use of the derivative identities

$$(1.5) \quad \mathbf{g}_s^{(k)}(\mathbf{y}) = \mathbf{S}^{(k)T} \mathbf{g}(\mathbf{x}^{(k)} + \mathbf{S}^{(k)}\mathbf{y})$$

and

$$(1.6) \quad \mathbf{H}_s^{(k)}(\mathbf{y}) = \mathbf{S}^{(k)T} \mathbf{H}(\mathbf{x}^{(k)} + \mathbf{S}^{(k)}\mathbf{y}) \mathbf{S}^{(k)}.$$

The paper is organised as follows. In Section 2, we consider the convergence of the algorithm given in the introduction. We discuss a number of ISM methods in Section 3,

and we report on some preliminary numerical experience when solving some relatively large test examples, from the CUTE test suite (see Bongartz *et al.*, 1995), in Section 4. Possible extensions, to the cases where there are linear or nonlinear constraints present, are given in our concluding Section 5, where we also offer our perspectives of this and future work.

## 2 Convergence of a general algorithm

Global convergence of the above scheme can be guaranteed under fairly general assumptions. Suppose that we are able to pick consecutive iterates  $\mathbf{x}^{(k)}$  and  $\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} + \mathbf{S}^{(k)}\mathbf{y}^{(k)}$  for which the Goldstein (1964) conditions

$$(2.1) \quad f^{(k)} + \beta \mathbf{g}^{(k)T} \mathbf{S}^{(k)} \mathbf{y}^{(k)} \leq f^{(k+1)} \leq f^{(k)} + \alpha \mathbf{g}^{(k)T} \mathbf{S}^{(k)} \mathbf{y}^{(k)},$$

for some  $0 < \alpha \leq \beta < 1$ , are satisfied, where  $\mathbf{y}^{(k)}$  is the approximate solution of (1.2). Suppose, furthermore, that

$$(2.2) \quad \frac{-\mathbf{g}^{(k)T} \mathbf{S}^{(k)} \mathbf{y}^{(k)}}{\|\mathbf{S}^{(k)T} \mathbf{g}^{(k)}\|_2 \|\mathbf{y}^{(k)}\|_2} \geq \epsilon,$$

for some  $\epsilon > 0$ . Then the ISM algorithm from Section 1 is globally convergent to a stationary point for the problem (1.1) from any starting point so long as  $f$  is bounded from below and has a Lipschitz-continuous gradient (see, for example, Dennis and Schnabel, 1983, Theorem 6.3.3).

Using the definitions (1.4) and (1.5), we may write (2.1) as

$$(2.3) \quad f_s^{(k)}(\mathbf{0}) + \beta \mathbf{g}_s^{(k)}(\mathbf{0})^T \mathbf{y}^{(k)} \leq f_s^{(k)}(\mathbf{y}^{(k)}) \leq f_s^{(k)}(\mathbf{0}) + \alpha \mathbf{g}_s^{(k)}(\mathbf{0})^T \mathbf{y}^{(k)}$$

and ensure that (2.2) is satisfied by requiring that

$$(2.4) \quad \frac{-\mathbf{g}_s^{(k)}(\mathbf{0})^T \mathbf{y}^{(k)}}{\|\mathbf{g}_s^{(k)}(\mathbf{0})\|_2 \|\mathbf{y}^{(k)}\|_2} \geq \epsilon.$$

This is relevant as now the global convergence conditions may be verified in terms of the inner-minimization function  $f_s$  and its gradient. Similar global convergence results can be obtained if we replace condition (2.1) by the Armijo (1966) backtracking strategy (see Bertsekas, 1982, Section 1.3). It may, however, be difficult to design general algorithms which ensure that conditions (2.3) and (2.4) are satisfied on exit from the inner minimization. Thus, it may be preferable to impose extra conditions on the iterates generated during the inner minimization to ensure overall global convergence. With this in mind, suppose that we can find any point  $\mathbf{y}_s^{(k)}$  for which

$$(2.5) \quad f_s^{(k)}(\mathbf{0}) + \beta \mathbf{g}_s^{(k)}(\mathbf{0})^T \mathbf{y}_s^{(k)} \leq f_s^{(k)}(\mathbf{y}_s^{(k)}) \leq f_s^{(k)}(\mathbf{0}) + \alpha \mathbf{g}_s^{(k)}(\mathbf{0})^T \mathbf{y}_s^{(k)}$$

and

$$(2.6) \quad \frac{-\mathbf{g}_s^{(k)}(\mathbf{0})^T \mathbf{y}_s^{(k)}}{\|\mathbf{g}_s^{(k)}(\mathbf{0})\|_2 \|\mathbf{y}_s^{(k)}\|_2} \geq \epsilon$$

are satisfied. Suppose, furthermore, that we terminate the inner minimization at a point  $\mathbf{y}^{(k)}$  for which

$$(2.7) \quad f_s^{(k)}(\mathbf{0}) - f_s^{(k)}(\mathbf{y}^{(k)}) \geq \tau (f_s^{(k)}(\mathbf{0}) - f_s^{(k)}(\mathbf{y}_s^{(k)})),$$

where  $\tau > 0$ . Then it is easy to show that this scheme is globally convergent under the same conditions as stated above. The advantage here is that the tests (2.5) and (2.6) need

only be satisfied at an intermediate point to ensure convergence. Typically, the first inner-iterate provides such a point for carefully chosen subspaces and minimizers. For example, if the subspace contains the steepest-descent direction and the inner minimization starts by performing a linesearch in this direction, the resulting first inner-iterate satisfies (2.5) and (2.6). Similarly, if the subspace contains a (modified) truncated-Newton direction and the inner minimization starts by performing a linesearch in this direction, the same conclusion is true.

### 3 Computational Variants

We consider it important from a practical point of view to require that  $\mathbf{S}^{(k)}$  contains at least two components,

- a gradient-related direction, such as  $-\mathbf{g}^{(k)}$ , to encourage global convergence, and
- a Newton-related direction, such as might be computed by a truncated-Newton method, to encourage fast asymptotic convergence, with safeguards to account for indefiniteness.

Of course, these components play a key role in dog-leg trust-region methods (see, for example, Powell, 1970). Additional components have the advantage of enlarging the subspace searched, but the disadvantage of increasing the overheads in solving the  $s^{(k)}$ -dimensional subspace minimization problem. In this section, we consider various possible ways of choosing the enlarged iterated subspace by choosing suitable conjugate directions.

#### 3.1 Conjugate gradients

An appealing choice of  $\mathbf{S}^{(k)}$  may be obtained by picking the columns of  $\mathbf{S}^{(k)}$  as a set of  $\mathbf{H}^{(k)}$ -conjugate directions, especially if these directions are generated by a conjugate-gradient (CG) method. In practise one would typically precondition first.

Suppose that  $\mathbf{H}^{(k)}$  is positive definite. Let  $\phi^{(k)}(\mathbf{x}^{(k)} + \mathbf{p})$  be the quadratic model,

$$(3.1) \quad \phi^{(k)}(\mathbf{x}^{(k)} + \mathbf{p}) = f^{(k)} + \mathbf{p}^T \mathbf{g}^{(k)} + \frac{1}{2} \mathbf{p}^T \mathbf{H}^{(k)} \mathbf{p},$$

of  $f(\mathbf{x}^{(k)} + \mathbf{p})$  about  $\mathbf{x}^{(k)}$ . The preconditioned conjugate-gradient method (see, for example, Hestenes and Stiefel, 1952 and Golub and Loan, 1989, Section 10.3) is an iterative method which may be used to calculate the smallest value of (3.1). It is well known that the solution,  $\mathbf{p}_n$ , to this problem is the Newton direction.

A *preconditioner*  $\mathbf{P}^{(k)}$  is usually an easily invertible approximation to  $\mathbf{H}^{(k)}$ . We shall insist that  $\mathbf{P}^{(k)}$  has a uniformly bounded condition number. The  $j$ -th step of the preconditioned conjugate-gradient method determines the smallest value of (3.1) in the Krylov subspace spanned by the vectors  $\{-((\mathbf{P}^{(k)})^{-1} \mathbf{H}^{(k)})^i (\mathbf{P}^{(k)})^{-1} \mathbf{g}^{(k)}\}_{i=0}^j$ . Conjugacy properties ensure that each successive step may be accomplished by a univariate minimization of (3.1) in the direction  $\mathbf{s}_j$ ; the vectors  $\{\mathbf{s}_j\}$  are conjugate and are recurred from step to step. Significantly from our point of view, the first such vector,  $\mathbf{s}_0 = -(\mathbf{P}^{(k)})^{-1} \mathbf{g}^{(k)}$ . In exact arithmetic the method would terminate with the Newton direction,  $\mathbf{p}_n$ , after at most  $n$  steps, but numerical rounding errors ensure that the method behaves more like an infinite iteration (see Reid, 1971). Moreover, for the large-scale case, we would be unwilling to consider anywhere close to  $n$  iterations. Nonetheless, the method is an effective technique for calculating approximations to the Newton direction, especially if a good preconditioner is used or if low accuracy solutions may be tolerated (see Toint, 1981, Dembo *et al.*, 1982, and Dembo and Steihaug, 1983).

In truncated-Newton methods, (see Dembo *et al.*, 1982), the method of conjugate gradients is used to generate approximations to the Newton direction. The resulting search direction,  $\mathbf{p}_{tn}$ , is employed within a linesearch framework for solving unconstrained optimization problems. Highly accurate approximations to the Newton correction are only needed to accelerate the convergence of the iteration in the neighbourhood of a limit point, and crude improvements upon the steepest-descent direction suffice elsewhere. Furthermore, by monitoring the gradient of the model at each step of the conjugate-gradient method, we can decide when to terminate the iteration.

While such an approach has undoubtedly proved successful in practice, we note that a considerable amount of work is invested, in such a scheme, in calculating an “average” direction and that much of the information gleaned on the way is subsequently ignored. We take the point of view that directions generated by the conjugate-gradient method are of interest for the quadratic model, but might also be locally of interest for the true objective function. We thus propose to construct our iterated subspace from the subspace investigated by the conjugate gradient method.

We intend to include the following:

- The preconditioned steepest-descent direction,  $\mathbf{s}_0 = -(\mathbf{P}^{(k)})^{-1}\mathbf{g}^{(k)}$ ;
- A number of other conjugate directions,  $\mathbf{s}_j$ , determined by the preconditioned conjugate-gradient method; and
- The overall truncated-Newton direction,  $\mathbf{p}_{tn}$ .

We note that the first of these components is designed to encourage global convergence, while the last will ensure that convergence occurs at a fast asymptotic rate. Thus, although we could include only the steepest descent direction and not Newton or some other combination, the choice above seems most prudent.

In the next two subsections, we will discuss the choice of the conjugate directions and the size of the iterated subspace.

### 3.2 Choice of conjugate directions

Suppose that, in addition to the (preconditioned) steepest-descent and (truncated) Newton directions, we wish to include  $q$  directions that are  $\mathbf{H}^{(k)}$ -conjugate in the subspace. Although there are other possibilities, we propose two specific ways to choose the conjugate directions. The simplest choice is just to take the first  $q$  generated (excluding, of course, the steepest-descent direction). Alternatively one might propose choosing the  $q$  which gave the largest decrease in the model. Experiments suggest that this is rarely more successful than the simpler scheme.

Another possibility is to include approximations from the extreme eigenspaces, that is the set of eigenvectors which correspond to the smallest and largest eigenvalues (recall  $\mathbf{H}^{(k)}$  is assumed positive definite). In the unpreconditioned case, eigenvectors corresponding to large eigenvalues correspond to the directions along which the function and CG model change most rapidly. In the preconditioned case, they still have the same effect for the transformed model.

Similarly, those associated with small eigenvalues reflect the (transformed) space in which the Newton direction is likely to be sensitive and contributions from this space are necessary if rapid progress is to be made. Thus both sets of vectors are reasonable candidates for subspace directions.

Clearly, the calculation of these spaces is generally prohibitively expensive, but they may be approximated by directions generated during the CG process (see, eg, Parlett, 1980, Chapter 13). One possibility is to monitor the Rayleigh quotients

$$(3.2) \quad \frac{\mathbf{s}_j^T \mathbf{H}^{(k)} \mathbf{s}_j}{\mathbf{s}_j^T \mathbf{s}_j}$$

and include the  $\mathbf{s}_j$  which give rise to the most extreme Rayleigh quotients. Of course, these vectors are not eigenvectors of  $\mathbf{H}^{(k)}$ , but they usually contain significant contributions in the extreme eigenspaces.

### 3.3 Choice of subspace dimension

The choice of subspace dimension is clearly important. The simplest choice is to fix an upper bound  $s$  on this dimension before the computation proceeds (perhaps  $s = 10$ , see Section 4), and to select  $s^{(k)}$  to be the smaller of  $s$  and the total number of CG directions sampled during the  $k$ -th CG iteration — recall that the CG process may be truncated and thus fewer than  $s$  directions may have been computed.

A more sophisticated approach is to try to dynamically select the size of subspace based upon the needs of the  $k$ -th iteration. For instance, if the Hessian is relatively well-conditioned, it is reasonable that a subspace made up from the steepest-descent and (truncated) Newton directions will suffice. If, on the other hand, the Hessian is ill-conditioned, further subspace directions to account for the extreme eigenvalues are likely to prove beneficial.

A simple heuristic would be to monitor the Rayleigh quotient as the CG iteration proceeds. Typically the first search direction,  $\mathbf{s}_0 = -(\mathbf{P}^{(k)})^{-1} \mathbf{g}^{(k)}$ , for the CG iteration will contain components of all eigenvectors and hence some components of those corresponding to the largest eigenvalues. The influence of the eigenvectors corresponding to the large eigenvalues is reduced in the subsequent directions  $\mathbf{s}_1, \dots$ , and this is reflected in a reduction in the Rayleigh quotient during these iterations. This effect is reversed after a number of iterations, when the influence of the larger eigenvalues reappears. It would thus seem sensible to record the iteration number,  $i^{(k)}$ , at which the Rayleigh quotient first starts to increase after its initial sequence of decreases. As we know that we then have sampled eigenvectors corresponding to both “large” and “small” eigenvalues, it is appropriate to set  $s^{(k)} = i^{(k)}$ .

### 3.4 The inner minimization

As we have stated, we believe that there are a number of highly effective algorithms for the unconstrained minimization of a function of several variables. Indeed, it would not be unreasonable to say that the problem has effectively been solved so long as derivatives are available. Among the most successful methods are the Newton-like second-derivative methods and the finite-difference and secant methods which require only gradients (see, for example, Dennis and Schnabel, 1983, Gill *et al.*, 1981 or Fletcher, 1987).

When considering the minimization of (1.4), we note that the calculation of derivatives of  $f_s^{(k)}$  requires those of  $f$ . We see that the calculation of the second derivatives (1.6) requires significantly more products involving  $\mathbf{S}^{(k)}$  than do the first derivatives (1.5). Thus, we would prefer to use methods which either only require relatively few Hessian-vector products, such as (preconditioned and truncated) conjugate-gradient methods, or secant methods, which

build up approximations to the second derivatives from gradients in the  $s^{(k)}$ -dimensional subspace as they proceed.

The most widely used secant methods are those in the convex Broyden class of positive-definite approximations, of which the BFGS method has the best reputation (again see, for example, Dennis and Schnabel, 1983). While there is some controversy as to whether there are better nonconvex secant updates (see for example, Conn *et al.*, 1991, Khalfan *et al.*, 1993, and Byrd *et al.*, 1996), we feel that convex secant methods are most natural in a linesearch, as opposed to a trust-region, context. Such methods start with a positive-definite second-derivative approximation and generate a sequence of matrices which mimic the curvature in the space of directions searched. Traditionally, the Cholesky factors of the sequence are updated as the iteration proceeds. We now show that building a good starting matrix in our case is easy.

Firstly, suppose that  $\mathbf{S}^{(k)}$  is made up purely of  $\mathbf{H}^{(k)}$ -conjugate directions. Then the exact second-derivative matrix  $\mathbf{H}_s^{(k)}$  is diagonal because of the conjugacy and moreover its diagonal entries will have been calculated during the conjugate-gradient process. This matrix, then, is a good starting approximation; its Cholesky factors are trivial to determine. Furthermore, this choice ensures that the first quasi-Newton search direction is identical to that generated by minimizing the quadratic model (3.1) in the manifold  $\mathbf{x}^{(k)} + \mathbf{S}^{(k)}\mathbf{y}$  (see, Gill *et al.*, 1981, section 4.8.3.1).

Now suppose that  $\mathbf{S}^{(k)}$  is made by augmenting a set of  $\mathbf{H}^{(k)}$ -conjugate directions by the overall truncated-Newton direction  $\mathbf{p}_{tn}$ . Then, the exact second-derivative matrix has an arrowhead structure with the leading  $s^{(k)} - 1$  by  $s^{(k)} - 1$  submatrix being diagonal and the remaining row and column easy to obtain. To be precise, if we denote the residual  $\mathbf{H}^{(k)}\mathbf{p}_{tn} + \mathbf{g}^{(k)}$  following the truncated conjugate-gradient process by  $\mathbf{r}^{(k)}$ , the last column of the required second-derivative matrix is  $\mathbf{S}^{(k)T}(\mathbf{r}^{(k)} - \mathbf{g}^{(k)})$ . Thus, once again this matrix provides a good starting approximation in that its Cholesky factors are extremely cheap to compute. Moreover, as before, the first quasi-Newton direction gives the minimum of the model (3.1) in the manifold  $\mathbf{x}^{(k)} + \mathbf{S}^{(k)}\mathbf{y}$ . Significantly, as the truncated-Newton direction is in the subspace, this first quasi-Newton direction is thus the same as the truncated-Newton direction.

## 4 Numerical Experiments

We start this section by investigating, from a numerical point of view, the impact of different subspace sizes on the convergence of the method. As the problem FMINSURF<sup>1</sup> is particularly efficiently solved using ISM in comparison with the default version of LANCELOT, we examined this problem in detail. We ran ISM, without preconditioner, but constructing the subspace from a combination of the steepest-descent, truncated-Newton and extreme directions as described in Section 3.2, using a variety of subspace dimensions and illustrate, in Figure 1, the effects of the choice of this dimension on the CPU time<sup>2</sup> required to solve the problem.

We observe that the CPU time for small subspace dimension is large, but that, for subspaces of dimension between nine and forty, the time is relatively constant, being within ten percent of the least ( $s = 11$ ) time. Thus, it appears that for this problem, adding information above the steepest-descent and truncated-Newton directions is beneficial but

<sup>1</sup>From the CUTE collection, see Bongartz *et al.* (1995).

<sup>2</sup>This experiment was performed on an IBM RISC/6000 320h workstation, using optimized (-O) Fortran 77 code and IBM-supplied BLAS.

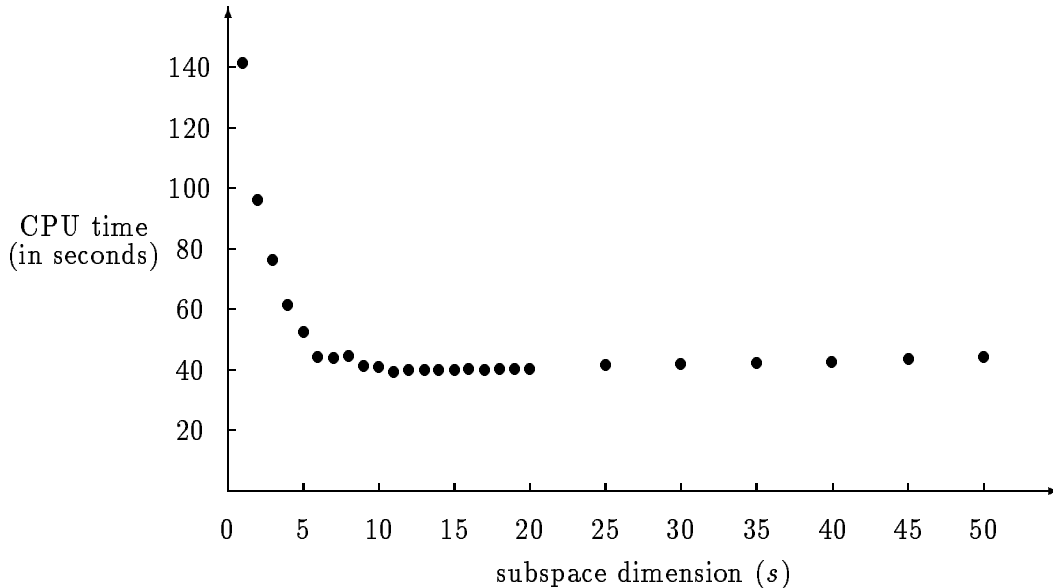


FIG. 1. *The impact of varying the subspace dimension on FMINSURF (using an unpreconditioned ISM in which the subspace is chosen from the the steepest-descent, truncated-Newton and extreme CG directions).*

there is little extra payoff from using more than eleven directions. When we monitored the Rayleigh quotients for this problem, we observed that the quotient decreases for on average ten CG iterations before increasing for the first time. Thus, unless we insist on at least ten CG iterations, it is possible that we may not have sampled the complete eigenspace. Similar runs on different problems indicate that this behaviour is not exceptional. In other words it appears to be not exceptional that searching in a subspace that uses eight directions corresponding to the extreme Rayleigh quotients with the addition of the steepest descent and the truncated Newton directions captures the bulk of the possible improvements at a given iteration. Of course, this cannot be the case for all iterations or all problems, but the overhead associated with choosing substantially more directions, and the loss of descent incurred by using less directions (recall  $s$  is fixed), does not payoff in general.

We next report the results of running a number of variants of our Iterated-Subspace Minimization code, based on the suggestions made in Section 3, on some large or difficult test problems. The simplest variant has the following features.

- No preconditioning is used.
- $s^{(k)}$  is the smaller of the *fixed* bound  $s = 10$  (as suggested by the above example) and the number of inner iterations required to determine the truncated-Newton direction.
- The subspace is constructed from the *first*  $s^{(k)} - 1$  conjugate directions plus the truncated-Newton direction. The truncation is performed when the residual (gradient) of the model is smaller than  $\|\mathbf{g}^{(k)}\|_2 \min(0.1, \|\mathbf{g}^{(k)}\|_2^{0.1})$  or when more than  $n$  conjugate-gradient iterations were performed.

- The model is modified, if necessary, to ensure that it is strictly convex. The modification is carried out as the conjugate-gradient iteration proceeds using the method of Arioli *et al.* (1993).
- The iteration is deemed to have converged when  $\|\mathbf{g}^{(k)}\|_2$  is smaller than  $10^{-5}$ .
- A BFGS linesearch method is used to solve the inner-minimization problem. An Armijo backtracking linesearch is used, starting with a unit step and dividing the step by two until the Armijo sufficient decrease condition is satisfied. If a step of one proves acceptable, but the model has been modified to ensure that it is strictly convex, the step is doubled until an unacceptable stepsize is determined, when the last-found acceptable step is chosen. A maximum of  $2s^{(k)}$  BFGS iterations are permitted and the iteration is stopped if  $\|\mathbf{g}_s^{(k)}\|_2$  is smaller than  $10^{-6}$ .

Note that one Hessian evaluation is made for each major iteration and one gradient evaluation for each inner iteration. We denote this method by the symbol  $\text{ISM}(n,f,f)$ , where  $n$  means that *no* preconditioning is used, the first  $f$  that the subspace is constructed from the *first* CG directions, and the second  $f$  that the maximal subspace dimension  $s$  is *fixed* (to ten).

We also consider the  $\text{ISM}(p,f,f)$  method, which is identical to  $\text{ISM}(n,f,f)$ , except that an 11-band modified Cholesky factorization *preconditioner* is used in the conjugate-gradient calculation. The factorization takes the elements of  $\mathbf{H}^{(k)}$  within a band of semi-bandwidth five of the diagonal, replacing any other elements by zeros. The resulting band matrix is factorized, modifications being made according to the method of Schnabel and Eskow (1991) to ensure that the preconditioner is positive definite with bounded condition number.

We define the methods  $\text{ISM}(n,e,f)$  and  $\text{ISM}(p,e,f)$  as the modifications of  $\text{ISM}(n,f,f)$  and  $\text{ISM}(p,f,f)$ , respectively, where we construct the subspace from a set of *extreme*  $s^{(k)} - 2$  conjugate directions plus the steepest-descent and truncated-Newton directions. We pick half of the extreme directions to be those whose Rayleigh quotient is largest, while the remainder correspond to the smallest Rayleigh quotients.

We next consider the possibility of generating the subspace dimension *automatically*, as discussed in Section 3.3. Once the subspace dimension has been determined, we construct the subspace from the set of first or extreme  $s^{(k)} - 2$  conjugate directions plus the steepest-descent and truncated-Newton direction, just as before. This results in four additional variants, namely  $\text{ISM}(n,f,a)$ ,  $\text{ISM}(p,f,a)$ ,  $\text{ISM}(n,e,a)$  and  $\text{ISM}(p,e,a)$ .

As a yard-stick, we compare the above methods with three other algorithms. The first is the default version of the LANCELOT A nonlinear optimization package (see Conn *et al.*, 1992) (denoted  $\text{LAN}(p)$ ) in which an 11-band preconditioner is used, together with the same unpreconditioned version (denoted  $\text{LAN}(n)$ ). LANCELOT is a trust-region method in which a gradient and Hessian evaluation are made on every successful iteration. The trust region subproblem is solved using a truncated conjugate gradient method. The second algorithm included in our comparison is a truncated-Newton method (see Dembo and Steihaug (1983)). The truncated-Newton search direction is obtained by an inexact minimization of the Newton model using unpreconditioned or preconditioned conjugate gradients. We denote the resulting methods by  $\text{TN}(n)$  and  $\text{TN}(p)$ , respectively. These two variants are obtained from our ISM algorithms by restricting the subspace minimization to a single linesearch along the truncated-Newton direction. Here, the truncation is performed when the residual (gradient) of the model is smaller than  $\|\mathbf{g}^{(k)}\|_2 \min(0.1, \|\mathbf{g}^{(k)}\|_2^{0.5})$  or when

more than  $n$  conjugate-gradient iterations are performed<sup>3</sup>. Finally, we also compare our ISM algorithms with the limited memory algorithm L-BFGS-B of Zhu *et al.* (1996) (see also Byrd *et al.*, 1995). In the unconstrained case, Algorithm L-BFGS-B approximately minimizes a quadratic model, whose Hessian is a limited memory BFGS approximation of the Hessian of the objective function, to compute a search direction, and performs a linesearch along this search direction. We set parameter `isbmin` to three in the code, meaning that the conjugate-gradient method is used to compute the search direction. As no preconditioning is considered in this method (denoted LM( $n$ )), we compare it with the unpreconditioned variants of the ISM algorithms. We note that all the algorithms considered in this comparison, except the limited memory one of course, use exact first and second derivatives.

We selected our 34 test examples as the majority of large and/or difficult unconstrained test examples in the CUTE (see Bongartz *et al.*, 1995) test set. Only problems which took excessive CPU time (more than 30 minutes), or which were variations on the reported problems, were excluded. All experiments were made on a DEC 3000 workstation, using optimized (-O) Fortran 77 code.

TABLE 1

*Cumulative statistics on the performance of all methods on 32 problems (unpreconditioned case)*

| Method         | Details  | #f    | Time |
|----------------|----------|-------|------|
| LAN( $n$ )     | Table 5  | 5585  | 486  |
| TN( $n$ )      | Table 7  | 19443 | 649  |
| LM( $n$ )      | Table 9  | 15046 | 1175 |
| ISM( $n$ ,f,f) | Table 10 | 21743 | 695  |
| ISM( $n$ ,e,f) | Table 12 | 22168 | 720  |
| ISM( $n$ ,f,a) | Table 14 | 14753 | 532  |
| ISM( $n$ ,e,a) | Table 16 | 15172 | 533  |

TABLE 2

*Cumulative statistics on the performance of all except the limited memory method on 32 problems (preconditioned case)*

| Method         | Details  | #f    | Time |
|----------------|----------|-------|------|
| LAN( $p$ )     | Table 6  | 4197  | 456  |
| TN( $p$ )      | Table 8  | 7852  | 471  |
| ISM( $p$ ,f,f) | Table 11 | 22096 | 649  |
| ISM( $p$ ,e,f) | Table 13 | 22193 | 652  |
| ISM( $p$ ,f,a) | Table 15 | 12616 | 448  |
| ISM( $p$ ,e,a) | Table 17 | 12774 | 458  |

Tables 1 and 2 first report cumulative statistics on the performance of the considered algorithms on 32 problems for the unpreconditioned case (the limited memory algorithm failed to solve problems DIXON3DQ and VARDIM), and on 32 problems for the preconditioned case (different approximate local optima have been reached for problems LIARWH and

<sup>3</sup>The exponent differs from the 0.1 used for the ISM methods because otherwise the results for the truncated-Newton method deteriorate considerably.

NONDIA). In these tables and the following ones,  $\#f$  indicates the total number of function evaluations and *time* the total CPU time (in seconds). The second column indicates which of the tables in the appendix gives the complete and detailed results for the considered method.

These tables show some interesting results. In particular, they indicate that the automatic choice of the subspace dimension is advantageous on average. On the other hand, there is little difference between the average performance of ISM methods using the first CG directions to define the subspace and those using the extreme ones. One also sees that, on average, the preconditioned ISM variants are all better in CPU time than their unpreconditioned counterparts. When comparing with the other methods, the unpreconditioned ISM variants with automatic choice of the subspace dimension require, on average, more function evaluations than LANCELOT, but approximately the same amount as limited memory and less than truncated-Newton. In the preconditioned case, the total number of function evaluations is less for LANCELOT and truncated-Newton. The overall CPU time for the unpreconditioned case indicates a better performance, on average, for LANCELOT, directly followed (within less than ten per cent) by ISM variants with automatic choice of the subspace dimension. The limited memory method did not perform as well on these examples. This is more than likely because it uses less derivative information. When preconditioning is used, the total CPU time for the ISM methods with automatic choice of the subspace dimension is comparable with (and even slightly better than) the total CPU time for LANCELOT and for truncated-Newton.

We next consider a more disaggregate presentation of what happens problem by problem to refine our analysis. We present in Tables 3 and 4 the number of times that each of the considered method ranks first, second, third, etc. for the two criteria used above. In these rankings, two CPU times are reputed identical if they differ by less than five percent or by less than half a second.

TABLE 3  
*Rankings (unpreconditioned case)*

| Method         | Function evaluations ( $\#f$ ) |     |     |     |     |     |     | Time |     |     |     |     |     |     |
|----------------|--------------------------------|-----|-----|-----|-----|-----|-----|------|-----|-----|-----|-----|-----|-----|
|                | 1st                            | 2nd | 3rd | 4th | 5th | 6th | 7th | 1st  | 2nd | 3rd | 4th | 5th | 6th | 7th |
| LAN( $n$ )     | 26                             | 2   | 1   | 1   | 0   | 2   | 0   | 12   | 3   | 1   | 1   | 3   | 6   | 6   |
| TN( $n$ )      | 3                              | 9   | 4   | 3   | 0   | 0   | 13  | 18   | 0   | 0   | 1   | 1   | 0   | 12  |
| LM( $n$ )      | 1                              | 10  | 1   | 2   | 4   | 5   | 9   | 14   | 1   | 1   | 0   | 5   | 5   | 6   |
| ISM( $n,f,f$ ) | 1                              | 6   | 6   | 3   | 5   | 9   | 2   | 19   | 0   | 1   | 3   | 4   | 2   | 3   |
| ISM( $n,e,f$ ) | 1                              | 6   | 5   | 3   | 5   | 8   | 4   | 19   | 1   | 1   | 1   | 2   | 7   | 1   |
| ISM( $n,f,a$ ) | 3                              | 9   | 7   | 9   | 4   | 0   | 0   | 22   | 1   | 2   | 4   | 1   | 2   | 0   |
| ISM( $n,e,a$ ) | 3                              | 6   | 9   | 8   | 4   | 2   | 0   | 21   | 0   | 4   | 2   | 4   | 1   | 0   |

These tables strengthen the indications drawn above from average measures, except that they indicate that the supremacy of LANCELOT in CPU time for the unpreconditioned ISM variants is misleading. It appears indeed that all the ISM methods (with or without preconditioner) are very competitive in CPU time. We further observe that truncated-Newton is sometimes very good and sometimes rather poor for the unpreconditioned case (as may be expected), while this dichotomy is less marked in the preconditioned case.

We thus conclude that in general ISM methods are efficient from the CPU time

TABLE 4  
*Rankings (preconditioned case)*

| Method     | Function evaluations (#f) |     |     |     |     |     | Time |     |     |     |     |     |
|------------|---------------------------|-----|-----|-----|-----|-----|------|-----|-----|-----|-----|-----|
|            | 1st                       | 2nd | 3rd | 4th | 5th | 6th | 1st  | 2nd | 3rd | 4th | 5th | 6th |
| LAN(p)     | 31                        | 0   | 0   | 0   | 0   | 1   | 19   | 2   | 0   | 4   | 0   | 7   |
| TN(p)      | 5                         | 13  | 0   | 10  | 0   | 4   | 20   | 0   | 0   | 2   | 3   | 7   |
| ISM(p,f,f) | 5                         | 2   | 3   | 5   | 16  | 1   | 17   | 2   | 1   | 3   | 7   | 2   |
| ISM(p,e,f) | 5                         | 2   | 4   | 3   | 15  | 3   | 15   | 0   | 3   | 3   | 9   | 2   |
| ISM(p,f,a) | 5                         | 13  | 9   | 1   | 3   | 1   | 19   | 4   | 6   | 2   | 0   | 1   |
| ISM(p,e,a) | 6                         | 8   | 11  | 2   | 5   | 0   | 19   | 2   | 4   | 4   | 3   | 0   |

point of view, even though they may require substantially more function evaluations than LANCELOT (but not more than truncated-Newton and limited memory). Further, the automatic determination of the subspace dimension is often quite helpful, both in the unpreconditioned and the preconditioned cases. On the other hand, there is little difference in performance between ISM variants that build the subspace from the first CG directions and variants that use the extreme ones. As the former is easier to implement, one might prefer it in practice. In fact ISM(p,f,a) is the best, slightly<sup>4</sup>.

## 5 Perspectives and Conclusions

In this paper, we have shown that it is possible to solve large-scale nonlinear optimization problems using methods designed for small-scale problems. These methods may be regarded as a generalization of linesearch-type methods more usually used to solve unconstrained minimization problems. We have indicated that the convergence of our methods depends upon using robust algorithms for the small-dimensional subproblems, and have suggested a number of ways of selecting promising subspaces in which to search. Furthermore, we feel that there are a number of important areas for future investigation.

- It is possible to extend the iterated-subspace minimization idea to treat linearly constrained optimization problems. For example, if we suppose the original problem is of the form

$$(5.1) \quad \underset{\mathbf{x} \in \mathbb{R}^n}{\text{minimize}} \quad f(\mathbf{x}) \quad \text{subject to} \quad \mathbf{l} \leq \mathbf{A}\mathbf{x} \leq \mathbf{u},$$

where  $\mathbf{A}$  is an  $m$  by  $n$  matrix,  $\mathbf{l}$  and  $\mathbf{u}$  are  $m$ -vectors and  $\mathbf{x}^{(k)}$  satisfies  $\mathbf{l} \leq \mathbf{A}\mathbf{x}^{(k)} \leq \mathbf{u}$ , we may apply the following linearly-constrained ISM algorithm:

1. Stop if convergence tests are satisfied.
2. Determine a full-rank subspace matrix  $\mathbf{S}^{(k)} \in \mathbb{R}^{n \times s^{(k)}}$ , where  $s^{(k)} \ll n$ .
3. Approximately solve the  $s^{(k)}$ -dimensional minimization problem

$$(5.2) \quad \underset{\mathbf{y} \in \mathbb{R}^{s^{(k)}}}{\text{minimize}} \quad f(\mathbf{x}^{(k)} + \mathbf{S}^{(k)}\mathbf{y}) \quad \text{subject to} \quad \mathbf{l}^{(k)} \leq \mathbf{A}^{(k)}\mathbf{y} \leq \mathbf{u}^{(k)},$$

---

<sup>4</sup>One should also bear in mind, at this point, that LANCELOT is a much more sophisticated code than our ISM variants, because it is designed for solving generally constrained in addition to unconstrained problems, has been extensively tested and refined, and contains a number of safeguards that are not included in the simpler ISM codes.

where  $\mathbf{A}^{(k)} = \mathbf{A}\mathbf{S}^{(k)}$ ,  $\mathbf{l}^{(k)} = \mathbf{l} - \mathbf{A}\mathbf{x}^{(k)}$  and  $\mathbf{u}^{(k)} = \mathbf{u} - \mathbf{A}\mathbf{x}^{(k)}$ , and set

$$(5.3) \quad \mathbf{x}^{(k+1)} = (\text{approximate}) \arg \min_{\mathbf{y} \in \mathbb{R}^{s^{(k)}}} f(\mathbf{x}^{(k)} + \mathbf{S}^{(k)}\mathbf{y})$$

subject to  $\mathbf{l}^{(k)} \leq \mathbf{A}^{(k)}\mathbf{y} \leq \mathbf{u}^{(k)}$ .

The central issues remain those discussed in Section 1. However, extra care must be exercised when picking the subspace matrix, as it is now desirable for a *constrained* steepest-descent and (truncated) Newton directions to lie in the subspace. The real difficulty is in adding appropriate additional directions since, for example, a straightforward analogue of the unconstrained case would require that we have conjugate directions in the constrained (reduced) space. One instance where this supposition is reasonable is when the constraints are just simple bounds. We also now need to use efficient methods for solving *small linearly-constrained minimization* problems when determining  $\mathbf{x}^{(k+1)}$ , but fortunately the state-of-the-art here is as advanced as it is for unconstrained minimization.

The ISM philosophy does not obviously extend to handle nonlinearly-constrained minimization problems except that, of course, any unconstrained or linearly-constrained subproblems may be treated by existing ISM methods. This may be important for nonlinearly-constrained minimization methods which are based on the sequential minimization of penalty or barrier functions, or their augmented or shifted counterparts.

- In our investigations, we found it convenient to use a linesearch (BFGS) method to solve the inner-iteration subproblems. One might, of course, alternatively use a trust-region method to solve the subproblem. However, as the performance of such methods depends upon building a good model within an adequate trust region, and as our ISM method will solve a sequence of subproblems, it may be that a good trust-region radius for one subproblem is poor for the next, and inefficiencies may occur. Thus care may be needed in determining interactions between successive models.

Another important issue is how to pick stopping rules, analogous to (2.5)–(2.7), which are appropriate for trust-region based methods. The main difficulty here is that the initial trust-region radius may interfere with a condition like (2.5).

- While we have suggested a number of methods for computing a good iterated subspace, more work clearly needs to be performed. We believe that we have identified some of the ingredients of a good subspace, but our understanding is far from complete.

## 6 Acknowledgement

Nick Gould would like to thank CERFACS for the facilities which made some of this research possible.

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## A Detailed numerical results

In this appendix, we give comprehensive details of the performance of each of the methods discussed in the main body of the paper on the complete set of test examples. We include these results so that others may, in future, compare new proposals with ours.

TABLE 5  
*Results for LAN(n) (unpreconditioned LANCELOT) on large or hard problems.*

| problem  | n    | #f   | #it  | #cg   | time   | f          |
|----------|------|------|------|-------|--------|------------|
| ARWHEAD  | 1000 | 6    | 5    | 2     | 0.95   | 1.6903D-10 |
| BDQRTIC  | 1000 | 13   | 12   | 66    | 2.01   | 3.9838D+03 |
| BROWNAL  | 100  | 3    | 2    | 3     | 0.17   | 1.1263D-13 |
| BRYBND   | 1000 | 14   | 13   | 177   | 3.53   | 2.8797D-13 |
| CRAGGLVY | 1000 | 14   | 13   | 166   | 2.40   | 3.3642D+02 |
| DIXMAANA | 1500 | 7    | 6    | 10    | 1.40   | 1.0000D+00 |
| DIXON3DQ | 1000 | 6    | 5    | 1427  | 5.04   | 7.7032D-09 |
| DQDRTIC  | 1000 | 3    | 2    | 0     | 0.73   | 1.6602D-23 |
| DQRTIC   | 1000 | 36   | 35   | 223   | 2.87   | 3.7687D-06 |
| EDENSCH  | 1000 | 15   | 14   | 35    | 1.58   | 6.0033D+03 |
| EIGENALS | 110  | 16   | 15   | 142   | 0.76   | 2.7564D-11 |
| ENGVAL1  | 1000 | 8    | 7    | 20    | 1.08   | 1.1082D+03 |
| FLETCHCR | 1000 | 3529 | 3528 | 20619 | 219.85 | 5.0581D-14 |
| FMINSURF | 1024 | 190  | 189  | 527   | 15.69  | 1.0000D+00 |
| FREUROTH | 1000 | 21   | 20   | 40    | 1.76   | 1.2147D+05 |
| GENROSE  | 1000 | 1214 | 1213 | 6640  | 77.43  | 1.0000D+00 |
| LIARWHD  | 1000 | 14   | 13   | 14    | 1.27   | 2.3011D-16 |
| MANCINO  | 100  | 12   | 11   | 13    | 11.28  | 4.3367D-17 |
| MOREBV   | 1000 | 3    | 2    | 265   | 2.01   | 2.0186D-09 |
| NCB20B   | 1000 | 28   | 27   | 1274  | 96.53  | 1.6760D+03 |
| NONDIA   | 1000 | 30   | 29   | 29    | 1.99   | 4.3483D-16 |
| NONDQUAR | 1000 | 92   | 91   | 853   | 6.43   | 9.5606D-06 |
| PENALTY1 | 1000 | 56   | 55   | 44    | 8.15   | 9.6862D-03 |
| POWELLSG | 1000 | 16   | 15   | 60    | 1.35   | 1.1167D-06 |
| POWER    | 1000 | 29   | 28   | 613   | 3.73   | 5.8399D-09 |
| QUARTC   | 1000 | 36   | 35   | 223   | 3.07   | 3.7687D-06 |
| SINQUAD  | 1000 | 70   | 69   | 126   | 5.54   | 6.3238D-05 |
| SROSENBR | 1000 | 11   | 10   | 20    | 1.06   | 6.6723D-12 |
| TOINTGSS | 1000 | 9    | 8    | 15    | 0.94   | 1.0010D+01 |
| TQUARTIC | 1000 | 14   | 13   | 14    | 1.26   | 1.0168D-16 |
| TRIDIA   | 1000 | 5    | 4    | 91    | 1.14   | 8.5487D-14 |
| VARDIM   | 1000 | 37   | 36   | 0     | 1.87   | 1.1203D-20 |
| VAREIGVL | 1000 | 13   | 12   | 313   | 4.32   | 3.0691D-08 |
| WOODS    | 1000 | 58   | 57   | 194   | 3.42   | 3.5419D-16 |

Key:  $n$  = number of variables,  $\#f$  = number of function evaluations,  $\#it$  = number of iterations,  $\#cg$  = total number of CG iterations,  $time$  = total CPU time in seconds,  $f$  = smallest function value obtained.

TABLE 6

*Results for LAN(p) (default preconditioned LANCELOT) on large or hard problems.*

| problem  | n    | #f   | #it  | #cg  | time   | f          |
|----------|------|------|------|------|--------|------------|
| ARWHEAD  | 1000 | 6    | 5    | 1    | 0.72   | 1.6903D-10 |
| BDQRTIC  | 1000 | 12   | 11   | 13   | 1.56   | 3.9838D+03 |
| BROWNAL  | 100  | 4    | 3    | 40   | 0.97   | 6.3470D-11 |
| BRYBND   | 1000 | 17   | 16   | 30   | 2.53   | 1.8775D-12 |
| CRAGGLVY | 1000 | 15   | 14   | 11   | 1.51   | 3.3642D+02 |
| DIXMAANA | 1500 | 8    | 7    | 10   | 1.14   | 1.0000D+00 |
| DIXON3DQ | 1000 | 3    | 2    | 2    | 0.40   | 0.0000D+00 |
| DQDRTIC  | 1000 | 3    | 2    | 0    | 0.40   | 1.6602D-23 |
| DQRTIC   | 1000 | 36   | 35   | 27   | 2.31   | 3.6952D-06 |
| EDENSCH  | 1000 | 13   | 12   | 9    | 1.24   | 6.0033D+03 |
| EIGENALS | 110  | 25   | 24   | 56   | 1.18   | 1.4887D-12 |
| ENGVAL1  | 1000 | 8    | 7    | 7    | 0.81   | 1.1082D+03 |
| FLETCHCR | 1000 | 2138 | 2137 | 2136 | 119.01 | 2.8160D-12 |
| FMINSURF | 1024 | 316  | 315  | 436  | 106.32 | 1.0000D+00 |
| FREUROTH | 1000 | 11   | 10   | 7    | 1.06   | 1.2147D+05 |
| GENROSE  | 1000 | 1095 | 1094 | 1158 | 63.37  | 1.0000D+00 |
| LIARWHD  | 1000 | 15   | 14   | 21   | 1.29   | 2.0274D-20 |
| MANCINO  | 100  | 16   | 15   | 8    | 17.39  | 9.2449D-18 |
| MOREBV   | 1000 | 2    | 1    | 1    | 0.39   | 7.3289D-13 |
| NCB20B   | 1000 | 23   | 22   | 568  | 50.35  | 1.6760D+03 |
| NONDIA   | 1000 | 30   | 29   | 47   | 2.18   | 6.9910D-16 |
| NONDQUAR | 1000 | 18   | 17   | 19   | 1.21   | 1.3932D-09 |
| PENALTY1 | 1000 | 64   | 63   | 432  | 23.16  | 9.6862D-03 |
| POWELLSG | 1000 | 16   | 15   | 15   | 1.06   | 2.2324D-06 |
| POWER    | 1000 | 28   | 27   | 53   | 7.60   | 1.8888D-08 |
| QUARTC   | 1000 | 36   | 35   | 27   | 2.32   | 3.6952D-06 |
| SINQUAD  | 1000 | 132  | 131  | 341  | 14.82  | 9.0365D-07 |
| SROSENBR | 1000 | 11   | 10   | 10   | 0.82   | 5.8309D-13 |
| TOINTGSS | 1000 | 3    | 2    | 2    | 0.44   | 1.0000D+01 |
| TQUARTIC | 1000 | 13   | 12   | 24   | 1.19   | 2.9487D-11 |
| TRIDIA   | 1000 | 3    | 2    | 1    | 0.49   | 1.3827D-32 |
| VARDIM   | 1000 | 37   | 36   | 0    | 19.03  | 1.1203D-20 |
| VAREIGVL | 1000 | 13   | 12   | 199  | 7.06   | 4.1621D-08 |
| WOODS    | 1000 | 72   | 71   | 69   | 4.18   | 2.4231D-11 |

Key:  $n$  = number of variables,  $\#f$  = number of function evaluations,  $\#it$  = number of iterations,  $\#cg$  = total number of CG iterations,  $time$  = total CPU time in seconds,  $f$  = smallest function value obtained.

TABLE 7

*Results for the unpreconditioned truncated-Newton method TN(n) on large or hard problems.*

| problem  | n    | #f   | #its | #cg   | time   | f        |
|----------|------|------|------|-------|--------|----------|
| ARWHEAD  | 1000 | 1545 | 389  | 395   | 26.63  | 0.00D+00 |
| BDQRTIC  | 1000 | 43   | 17   | 97    | 1.68   | 3.98D+03 |
| BROWNAL  | 100  | 5    | 3    | 4     | 0.04   | 2.45D-08 |
| BRYBND   | 1000 | 37   | 14   | 48    | 1.49   | 1.22D-12 |
| CRAGGLVY | 1000 | 45   | 16   | 129   | 1.98   | 3.36D+02 |
| DIXMAANA | 1500 | 189  | 64   | 73    | 7.10   | 1.00D+00 |
| DIXON3DQ | 1000 | 7    | 4    | 1748  | 3.91   | 1.25D-11 |
| DQDRTIC  | 1000 | 10   | 5    | 12    | 0.23   | 1.91D-14 |
| DQRTIC   | 1000 | 48   | 17   | 70    | 0.38   | 5.14D-08 |
| EDENSCH  | 1000 | 46   | 17   | 44    | 1.35   | 6.00D+03 |
| EIGENALS | 110  | 69   | 22   | 179   | 0.98   | 8.43D-12 |
| ENGVAL1  | 1000 | 31   | 13   | 28    | 0.81   | 1.11D+03 |
| FLETCHCR | 1000 | 3046 | 1184 | 13315 | 100.03 | 3.04D-15 |
| FMINSURF | 1024 | 156  | 26   | 5559  | 46.24  | 1.00D+00 |
| FREUROTH | 1000 | 92   | 19   | 110   | 2.42   | 1.21D+05 |
| GENROSE  | 1000 | 2896 | 540  | 9838  | 71.28  | 1.00D+00 |
| LIARWHD  | 1000 | 78   | 28   | 32    | 1.42   | 1.43D-16 |
| MANCINO  | 100  | 72   | 21   | 29    | 68.27  | 1.87D-21 |
| MOREBV   | 1000 | 3    | 2    | 549   | 2.14   | 1.34D-09 |
| NCB20B   | 1000 | 74   | 19   | 1053  | 81.98  | 1.68D+03 |
| NONDIA   | 1000 | 156  | 55   | 58    | 3.46   | 5.93D-21 |
| NONDQUAR | 1000 | 382  | 94   | 26963 | 67.04  | 3.76D-08 |
| PENALTY1 | 1000 | 124  | 52   | 78    | 1.73   | 9.69D-03 |
| POWELLSG | 1000 | 837  | 281  | 575   | 6.04   | 5.31D-08 |
| POWER    | 1000 | 5636 | 1414 | 1419  | 29.24  | 4.19D-09 |
| QUARTC   | 1000 | 48   | 17   | 70    | 0.38   | 5.14D-08 |
| SINQUAD  | 1000 | 285  | 119  | 216   | 9.06   | 1.33D-06 |
| SROSENBR | 1000 | 26   | 12   | 15    | 0.29   | 6.11D-20 |
| TOINTGSS | 1000 | 298  | 101  | 202   | 8.52   | 1.00D+01 |
| TQUARTIC | 1000 | 57   | 20   | 27    | 0.74   | 1.24D-13 |
| TRIDIA   | 1000 | 16   | 9    | 576   | 1.40   | 1.31D-14 |
| VARDIM   | 1000 | 60   | 15   | 15    | 0.33   | 2.28D-21 |
| VAREIGVL | 1000 | 3018 | 1007 | 1010  | 103.70 | 6.02D-11 |
| WOODS    | 1000 | 75   | 24   | 63    | 0.96   | 6.63D-18 |

Key:  $n$  = number of variables,  $\#f$  = number of function evaluations,  $\#its$  = total number of iterations,  $\#cg$  = total number of CG iterations,  $time$  = total CPU time in seconds,  $f$  = smallest function value obtained.

TABLE 8

Results for the preconditioned truncated-Newton method TN( $p$ ) on large or hard problems.

| problem  | n    | #f   | #its | #cg  | time  | f        |
|----------|------|------|------|------|-------|----------|
| ARWHEAD  | 1000 | 20   | 8    | 10   | 0.68  | 0.00D+00 |
| BDQRTIC  | 1000 | 30   | 11   | 15   | 1.17  | 3.98D+03 |
| BROWNAL  | 100  | 74   | 26   | 213  | 7.10  | 1.21D-11 |
| BRYBND   | 1000 | 72   | 15   | 41   | 2.65  | 1.91D-16 |
| CRAGGLVY | 1000 | 122  | 42   | 42   | 4.91  | 3.36D+02 |
| DIXMAANA | 1500 | 91   | 31   | 35   | 4.56  | 1.00D+00 |
| DIXON3DQ | 1000 | 3    | 1    | 1    | 0.05  | 0.00D+00 |
| DQDRTIC  | 1000 | 3    | 1    | 1    | 0.09  | 0.00D+00 |
| DQRTIC   | 1000 | 43   | 15   | 15   | 0.45  | 7.86D-08 |
| EDENSCH  | 1000 | 42   | 12   | 13   | 1.30  | 6.00D+03 |
| EIGENALS | 110  | 140  | 36   | 162  | 2.19  | 8.15D-12 |
| ENGVAL1  | 1000 | 26   | 10   | 10   | 0.82  | 1.11D+03 |
| FLETCHCR | 1000 | 2962 | 1169 | 1172 | 79.57 | 4.78D-14 |
| FMINSURF | 1024 | 248  | 28   | 1438 | 25.48 | 1.00D+00 |
| FREUROTH | 1000 | 47   | 9    | 1008 | 19.38 | 1.21D+05 |
| GENROSE  | 1000 | 2290 | 748  | 1548 | 58.31 | 1.00D+00 |
| LIARWHD  | 1000 | 48   | 16   | 27   | 1.24  | 3.51D-15 |
| MANCINO  | 100  | 105  | 23   | 55   | 88.31 | 3.43D-21 |
| MOREBV   | 1000 | 2    | 1    | 1    | 0.08  | 7.33D-13 |
| NCB20B   | 1000 | 47   | 16   | 565  | 49.92 | 1.68D+03 |
| NONDIA   | 1000 | 50   | 19   | 35   | 1.57  | 8.47D-20 |
| NONDQUAR | 1000 | 20   | 9    | 17   | 0.39  | 2.45D-09 |
| PENALTY1 | 1000 | 106  | 32   | 223  | 9.71  | 9.69D-03 |
| POWELLSG | 1000 | 21   | 10   | 16   | 0.37  | 5.55D-08 |
| POWER    | 1000 | 43   | 15   | 15   | 3.81  | 1.00D-10 |
| QUARTC   | 1000 | 43   | 15   | 15   | 0.45  | 7.86D-08 |
| SINQUAD  | 1000 | 784  | 307  | 861  | 37.85 | 2.35D-10 |
| SROSENBR | 1000 | 36   | 12   | 13   | 0.53  | 8.11D-25 |
| TOINTGSS | 1000 | 3    | 1    | 1    | 0.14  | 1.00D+01 |
| TQUARTIC | 1000 | 25   | 9    | 16   | 0.55  | 6.46D-21 |
| TRIDIA   | 1000 | 3    | 1    | 1    | 0.05  | 1.97D-26 |
| VARDIM   | 1000 | 78   | 28   | 3241 | 29.14 | 9.73D-16 |
| VAREIGVL | 1000 | 179  | 49   | 1932 | 38.61 | 2.35D-14 |
| WOODS    | 1000 | 144  | 35   | 48   | 2.11  | 3.78D-15 |

Key:  $n$  = number of variables,  $\#f$  = number of function evaluations,  $\#its$  = total number of iterations,  $\#cg$  = total number of CG iterations,  $time$  = total CPU time in seconds,  $f$  = smallest function value obtained.

TABLE 9

Results for the (unpreconditioned) limited memory method LM( $n$ ) on large or hard problems.

| problem  | $n$  | $\#f$ | $\#its$ | $\#cg$ | time   | $f$       |
|----------|------|-------|---------|--------|--------|-----------|
| ARWHEAD  | 1000 | 22    | 13      | 20     | 0.70   | 0.000D+00 |
| BDQRTIC  | 1000 | 294   | 196     | 1104   | 11.84  | 3.984D+03 |
| BROWNAL  | 100  | 22    | 9       | 18     | 0.12   | 2.449D-08 |
| BRYBND   | 1000 | 31    | 24      | 80     | 1.39   | 2.152D-11 |
| CRAGGLVY | 1000 | 94    | 81      | 467    | 5.50   | 3.364D+02 |
| DIXMAANA | 1500 | 10    | 7       | 10     | 0.58   | 1.000D+00 |
| DIXON3DQ | 1000 | 9999  | 9668    |        | 343.62 | 5.065D-05 |
| DQDRTIC  | 1000 | 18    | 11      | 31     | 0.52   | 9.088D-14 |
| DQRTIC   | 1000 | 48    | 38      | 73     | 0.86   | 8.112D-06 |
| EDENSCH  | 1000 | 31    | 23      | 77     | 1.29   | 6.003D+03 |
| EIGENALS | 110  | 444   | 408     | 2565   | 6.30   | 1.609D-09 |
| ENGVAL1  | 1000 | 24    | 17      | 52     | 0.87   | 1.108D+03 |
| FLETCHCR | 1000 | 5795  | 5108    | 35941  | 255.57 | 7.962D-12 |
| FMINSURF | 1024 | 186   | 175     | 1219   | 11.29  | 1.000D+00 |
| FREUROTH | 1000 | 114   | 29      | 91     | 4.25   | 1.215D+05 |
| GENROSE  | 1000 | 2460  | 2163    | 14874  | 111.52 | 1.000D+00 |
| LIARWHD  | 1000 | 28    | 22      | 40     | 0.88   | 2.268D-16 |
| MANCINO  | 100  | 11    | 9       | 11     | 15.46  | 8.786D-20 |
| MOREBV   | 1000 | 75    | 71      | 526    | 3.32   | 9.964D-09 |
| NCB20B   | 1000 | 2581  | 2393    | 18491  | 651.93 | 1.676D+03 |
| NONDIA   | 1000 | 28    | 17      | 30     | 0.99   | 7.125D-21 |
| NONDQUAR | 1000 | 1073  | 961     | 7250   | 35.67  | 1.521D-05 |
| PENALTY1 | 1000 | 121   | 85      | 166    | 2.76   | 9.687D-03 |
| POWELLSG | 1000 | 48    | 37      | 116    | 0.99   | 6.583D-06 |
| POWER    | 1000 | 149   | 131     | 773    | 4.11   | 1.218D-08 |
| QUARTC   | 1000 | 48    | 38      | 73     | 0.84   | 8.112D-06 |
| SINQUAD  | 1000 | 141   | 104     | 385    | 6.41   | 1.129D-04 |
| SROSENBR | 1000 | 21    | 15      | 24     | 0.41   | 1.037D-09 |
| TOINTGSS | 1000 | 8     | 5       | 9      | 0.36   | 1.001D+01 |
| TQUARTIC | 1000 | 25    | 19      | 34     | 0.62   | 4.907D-17 |
| TRIDIA   | 1000 | 912   | 871     | 5992   | 29.15  | 1.606D-11 |
| VARDIM   | 1000 | 21    | 1       | 0      | 0.17   | 1.242D+22 |
| VAREIGVL | 1000 | 107   | 99      | 646    | 6.97   | 9.973D-09 |
| WOODS    | 1000 | 77    | 52      | 178    | 2.01   | 1.287D-11 |

Key:  $n$  = number of variables,  $\#f$  = number of function evaluations,  $\#its$  = total number of iterations,  $\#cg$  = total number of CG iterations,  $time$  = total CPU time in seconds,  $f$  = smallest function value obtained.

TABLE 10

Results for ISM( $n, f, f$ ) on large or hard problems, in which no preconditioning is used and the subspace is chosen from the first 10 CG directions.

| problem  | n    | #f    | #min | #its  | #cg   | $s^{(k)}$ | time   | f        |
|----------|------|-------|------|-------|-------|-----------|--------|----------|
| ARWHEAD  | 1000 | 11    | 2    | 6     | 3     | 1.50      | 0.25   | 0.00D+00 |
| BDQRTIC  | 1000 | 85    | 10   | 47    | 63    | 5.40      | 2.12   | 3.98D+03 |
| BROWNAL  | 100  | 5     | 2    | 3     | 3     | 1.50      | 0.04   | 2.45D-08 |
| BRYBND   | 1000 | 105   | 10   | 80    | 60    | 5.90      | 3.62   | 8.79D-14 |
| CRAGGLVY | 1000 | 108   | 12   | 90    | 126   | 7.58      | 4.46   | 3.36D+02 |
| DIXMAANA | 1500 | 24    | 7    | 15    | 7     | 1.00      | 1.09   | 1.00D+00 |
| DIXON3DQ | 1000 | 7     | 4    | 4     | 1748  | 9.00      | 3.97   | 1.25D-11 |
| DQDRTIC  | 1000 | 10    | 5    | 5     | 12    | 2.40      | 0.24   | 1.91D-14 |
| DQRTIC   | 1000 | 188   | 8    | 145   | 21    | 2.63      | 1.18   | 9.10D-09 |
| EDENSCH  | 1000 | 65    | 7    | 44    | 25    | 3.57      | 1.76   | 6.00D+03 |
| EIGENALS | 110  | 101   | 8    | 64    | 86    | 9.00      | 1.16   | 1.90D-10 |
| ENGVAL1  | 1000 | 52    | 7    | 34    | 22    | 3.14      | 1.19   | 1.11D+03 |
| FLETCHCR | 1000 | 13014 | 1049 | 10640 | 13187 | 9.92      | 316.24 | 3.60D-14 |
| FMINSURF | 1024 | 189   | 11   | 103   | 2309  | 10.00     | 21.61  | 1.00D+00 |
| FREUROTH | 1000 | 261   | 11   | 69    | 47    | 4.09      | 4.92   | 1.21D+05 |
| GENROSE  | 1000 | 5985  | 411  | 4327  | 6827  | 9.90      | 138.24 | 1.00D+00 |
| LIARWHD  | 1000 | 29    | 3    | 24    | 4     | 1.33      | 0.57   | 1.46D-14 |
| MANCINO  | 100  | 68    | 12   | 37    | 17    | 1.42      | 67.72  | 1.34D-18 |
| MOREBV   | 1000 | 4     | 3    | 3     | 519   | 10.00     | 1.98   | 1.37D-09 |
| NCB20B   | 1000 | 174   | 17   | 101   | 1034  | 9.88      | 99.47  | 1.68D+03 |
| NONDIA   | 1000 | 14    | 3    | 10    | 4     | 1.33      | 0.38   | 1.48D-15 |
| NONDQUAR | 1000 | 466   | 40   | 368   | 2643  | 9.05      | 10.84  | 2.11D-06 |
| PENALTY1 | 1000 | 72    | 3    | 58    | 4     | 1.33      | 0.85   | 9.69D-03 |
| POWELLSG | 1000 | 41    | 3    | 37    | 9     | 3.00      | 0.35   | 4.92D-09 |
| POWER    | 1000 | 105   | 10   | 80    | 150   | 7.20      | 1.18   | 1.43D-10 |
| QUARTC   | 1000 | 188   | 8    | 145   | 21    | 2.63      | 1.18   | 9.10D-09 |
| SINQUAD  | 1000 | 171   | 11   | 129   | 25    | 2.27      | 4.48   | 9.92D-07 |
| SROSENBR | 1000 | 16    | 2    | 12    | 3     | 1.50      | 0.17   | 2.45D-16 |
| TOINTGSS | 1000 | 7     | 1    | 4     | 1     | 1.00      | 0.22   | 1.00D+01 |
| TQUARTIC | 1000 | 27    | 2    | 19    | 3     | 1.50      | 0.34   | 4.05D-15 |
| TRIDIA   | 1000 | 17    | 10   | 10    | 643   | 9.80      | 1.63   | 7.32D-15 |
| VARDIM   | 1000 | 56    | 3    | 48    | 3     | 1.00      | 0.41   | 3.77D-23 |
| VAREIGVL | 1000 | 66    | 9    | 57    | 179   | 6.89      | 4.22   | 7.52D-11 |
| WOODS    | 1000 | 75    | 5    | 42    | 14    | 2.80      | 0.83   | 9.56D-15 |

Key:  $n$  = number of variables,  $\#f$  = number of function evaluations,  $\#min$  = number of minimizations,  $\#its$  = total number of iterations,  $\#cg$  = total number of CG iterations,  $s^{(k)}$  = average subspace dimension,  $time$  = total CPU time in seconds,  $f$  = smallest function value obtained.

TABLE 11

Results for ISM(p,f,f) on large or hard problems, in which preconditioning is used and the subspace is chosen from the first 10 CG directions.

| problem  | n    | #f    | #min | #its  | #cg  | $s^{(k)}$ | time   | f        |
|----------|------|-------|------|-------|------|-----------|--------|----------|
| ARWHEAD  | 1000 | 23    | 4    | 17    | 5    | 1.25      | 0.67   | 0.00D+00 |
| BDQRTIC  | 1000 | 62    | 7    | 40    | 7    | 1.00      | 1.55   | 3.98D+03 |
| BROWNAL  | 100  | 125   | 10   | 108   | 134  | 7.70      | 3.31   | 1.05D-11 |
| BRYBND   | 1000 | 159   | 11   | 97    | 26   | 2.36      | 4.58   | 7.78D-14 |
| CRAGGLVY | 1000 | 93    | 9    | 48    | 9    | 1.00      | 2.76   | 3.36D+02 |
| DIXMAANA | 1500 | 21    | 1    | 18    | 3    | 3.00      | 0.97   | 1.00D+00 |
| DIXON3DQ | 1000 | 3     | 1    | 1     | 1    | 1.00      | 0.05   | 0.00D+00 |
| DQDRTIC  | 1000 | 3     | 1    | 1     | 1    | 1.00      | 0.09   | 1.24D-21 |
| DQRTIC   | 1000 | 46    | 2    | 40    | 2    | 1.00      | 0.33   | 5.20D-08 |
| EDENSCH  | 1000 | 73    | 6    | 37    | 6    | 1.00      | 1.73   | 6.00D+03 |
| EIGENALS | 110  | 213   | 13   | 118   | 58   | 4.46      | 2.24   | 1.25D-11 |
| ENGVAL1  | 1000 | 32    | 5    | 21    | 5    | 1.00      | 0.83   | 1.11D+03 |
| FLETCHCR | 1000 | 12749 | 1212 | 10320 | 1212 | 1.00      | 266.31 | 4.00D-19 |
| FMINSURF | 1024 | 248   | 9    | 89    | 459  | 9.44      | 11.15  | 1.00D+00 |
| FREUROTH | 1000 | 113   | 6    | 40    | 15   | 2.50      | 2.55   | 1.21D+05 |
| GENROSE  | 1000 | 6586  | 598  | 4955  | 985  | 1.65      | 135.03 | 1.00D+00 |
| LIARWHD  | 1000 | 30    | 2    | 24    | 4    | 2.00      | 0.63   | 1.92D-14 |
| MANCINO  | 100  | 111   | 13   | 46    | 14   | 1.08      | 97.22  | 3.03D-18 |
| MOREBV   | 1000 | 2     | 1    | 1     | 1    | 1.00      | 0.08   | 7.33D-13 |
| NCB20B   | 1000 | 110   | 15   | 76    | 656  | 6.67      | 69.90  | 1.68D+03 |
| NONDIA   | 1000 | 116   | 7    | 84    | 17   | 2.43      | 2.74   | 9.90D-01 |
| NONDQUAR | 1000 | 86    | 11   | 73    | 18   | 1.64      | 0.98   | 2.18D-09 |
| PENALTY1 | 1000 | 145   | 7    | 114   | 64   | 5.00      | 3.76   | 9.69D-03 |
| POWELLSG | 1000 | 92    | 14   | 71    | 16   | 1.14      | 0.94   | 6.14D-11 |
| POWER    | 1000 | 61    | 4    | 51    | 4    | 1.00      | 1.37   | 3.44D-10 |
| QUARTC   | 1000 | 46    | 2    | 40    | 2    | 1.00      | 0.33   | 5.20D-08 |
| SINQUAD  | 1000 | 184   | 13   | 147   | 41   | 3.15      | 5.71   | 2.93D-10 |
| SROSENBR | 1000 | 36    | 7    | 26    | 7    | 1.00      | 0.50   | 6.14D-25 |
| TOINTGSS | 1000 | 3     | 1    | 1     | 1    | 1.00      | 0.14   | 1.00D+01 |
| TQUARTIC | 1000 | 21    | 2    | 18    | 3    | 1.50      | 0.36   | 2.01D-19 |
| TRIDIA   | 1000 | 3     | 1    | 1     | 1    | 1.00      | 0.05   | 1.01D-25 |
| VARDIM   | 1000 | 91    | 14   | 77    | 2660 | 6.86      | 20.02  | 5.72D-12 |
| VAREIGVL | 1000 | 70    | 9    | 56    | 154  | 7.33      | 6.98   | 1.25D-09 |
| WOODS    | 1000 | 486   | 35   | 321   | 45   | 1.29      | 6.27   | 6.79D-16 |

Key:  $n$  = number of variables,  $\#f$  = number of function evaluations,  $\#min$  = number of minimizations,  $\#its$  = total number of iterations,  $\#cg$  = total number of CG iterations,  $s^{(k)}$  = average subspace dimension,  $time$  = total CPU time in seconds,  $f$  = smallest function value obtained.

TABLE 12

Results for ISM( $n, e, f$ ) on large or hard problems, in which no preconditioning is used and the subspace is chosen from 10 extreme CG directions.

| problem  | n    | #f    | #min | #its  | #cg   | $s^{(k)}$ | time   | f        |
|----------|------|-------|------|-------|-------|-----------|--------|----------|
| ARWHEAD  | 1000 | 11    | 2    | 6     | 3     | 1.50      | 0.25   | 0.00D+00 |
| BDQRTIC  | 1000 | 63    | 10   | 46    | 56    | 5.20      | 1.89   | 3.98D+03 |
| BROWNAL  | 100  | 5     | 2    | 3     | 3     | 1.50      | 0.04   | 2.45D-08 |
| BRYBND   | 1000 | 106   | 9    | 81    | 58    | 6.33      | 3.57   | 1.01D-12 |
| CRAGGLVY | 1000 | 109   | 12   | 91    | 113   | 7.58      | 4.42   | 3.36D+02 |
| DIXMAANA | 1500 | 24    | 7    | 15    | 7     | 1.00      | 1.08   | 1.00D+00 |
| DIXON3DQ | 1000 | 7     | 4    | 4     | 1748  | 9.00      | 4.03   | 1.25D-11 |
| DQDRTIC  | 1000 | 10    | 5    | 5     | 12    | 2.40      | 0.24   | 1.91D-14 |
| DQRTIC   | 1000 | 188   | 8    | 145   | 21    | 2.63      | 1.18   | 9.10D-09 |
| EDENSCH  | 1000 | 65    | 7    | 44    | 25    | 3.57      | 1.76   | 6.00D+03 |
| EIGENALS | 110  | 108   | 10   | 70    | 102   | 8.40      | 1.28   | 1.08D-10 |
| ENGVAL1  | 1000 | 52    | 7    | 34    | 22    | 3.14      | 1.19   | 1.11D+03 |
| FLETCHCR | 1000 | 12762 | 1075 | 10268 | 13810 | 9.93      | 314.35 | 8.11D-14 |
| FMINSURF | 1024 | 226   | 12   | 107   | 2140  | 10.00     | 20.82  | 1.00D+00 |
| FREUROTH | 1000 | 223   | 9    | 68    | 45    | 4.78      | 4.44   | 1.21D+05 |
| GENROSE  | 1000 | 6642  | 431  | 4689  | 7088  | 9.90      | 150.00 | 1.00D+00 |
| LIARWHD  | 1000 | 29    | 3    | 24    | 4     | 1.33      | 0.58   | 1.46D-14 |
| MANCINO  | 100  | 68    | 12   | 37    | 17    | 1.42      | 67.98  | 1.34D-18 |
| MOREBV   | 1000 | 4     | 3    | 3     | 519   | 10.00     | 1.99   | 1.37D-09 |
| NCB20B   | 1000 | 284   | 18   | 131   | 1135  | 9.78      | 119.17 | 1.68D+03 |
| NONDIA   | 1000 | 14    | 3    | 10    | 4     | 1.33      | 0.38   | 1.48D-15 |
| NONDQUAR | 1000 | 390   | 39   | 303   | 1636  | 8.62      | 7.79   | 2.67D-06 |
| PENALTY1 | 1000 | 72    | 3    | 58    | 4     | 1.33      | 0.85   | 9.69D-03 |
| POWELLSG | 1000 | 41    | 3    | 37    | 9     | 3.00      | 0.36   | 4.92D-09 |
| POWER    | 1000 | 105   | 10   | 80    | 150   | 7.20      | 1.20   | 1.43D-10 |
| QUARTC   | 1000 | 188   | 8    | 145   | 21    | 2.63      | 1.19   | 9.10D-09 |
| SINQUAD  | 1000 | 171   | 11   | 129   | 25    | 2.27      | 4.47   | 9.92D-07 |
| SROSENBR | 1000 | 16    | 2    | 12    | 3     | 1.50      | 0.17   | 2.45D-16 |
| TOINTGSS | 1000 | 7     | 1    | 4     | 1     | 1.00      | 0.22   | 1.00D+01 |
| TQUARTIC | 1000 | 27    | 2    | 19    | 3     | 1.50      | 0.35   | 4.05D-15 |
| TRIDIA   | 1000 | 17    | 10   | 10    | 644   | 9.80      | 1.67   | 5.18D-15 |
| VARDIM   | 1000 | 56    | 3    | 48    | 3     | 1.00      | 0.41   | 3.77D-23 |
| VAREIGVL | 1000 | 66    | 9    | 57    | 161   | 6.89      | 4.09   | 1.50D-10 |
| WOODS    | 1000 | 75    | 5    | 42    | 14    | 2.80      | 0.83   | 9.56D-15 |

Key:  $n$  = number of variables,  $\#f$  = number of function evaluations,  $\#min$  = number of minimizations,  $\#its$  = total number of iterations,  $\#cg$  = total number of CG iterations,  $s^{(k)}$  = average subspace dimension,  $time$  = total CPU time in seconds,  $f$  = smallest function value obtained.

TABLE 13

Results for ISM(p,e,f) on large or hard problems, in which preconditioning is used and the subspace is chosen from 10 extreme CG directions.

| problem  | n    | #f    | #min | #its  | #cg  | $s^{(k)}$ | time   | f        |
|----------|------|-------|------|-------|------|-----------|--------|----------|
| ARWHEAD  | 1000 | 23    | 4    | 17    | 5    | 1.25      | 0.67   | 0.00D+00 |
| BDQRTIC  | 1000 | 62    | 7    | 40    | 7    | 1.00      | 1.57   | 3.98D+03 |
| BROWNAL  | 100  | 122   | 12   | 107   | 140  | 7.08      | 3.82   | 6.64D-13 |
| BRYBND   | 1000 | 159   | 11   | 97    | 26   | 2.36      | 4.58   | 7.78D-14 |
| CRAGGLVY | 1000 | 93    | 9    | 48    | 9    | 1.00      | 2.73   | 3.36D+02 |
| DIXMAANA | 1500 | 21    | 1    | 18    | 3    | 3.00      | 0.97   | 1.00D+00 |
| DIXON3DQ | 1000 | 3     | 1    | 1     | 1    | 1.00      | 0.05   | 0.00D+00 |
| DQDRTIC  | 1000 | 3     | 1    | 1     | 1    | 1.00      | 0.09   | 1.24D-21 |
| DQRTIC   | 1000 | 46    | 2    | 40    | 2    | 1.00      | 0.33   | 5.20D-08 |
| EDENSCH  | 1000 | 73    | 6    | 37    | 6    | 1.00      | 1.73   | 6.00D+03 |
| EIGENALS | 110  | 213   | 13   | 118   | 58   | 4.46      | 2.24   | 1.25D-11 |
| ENGVAL1  | 1000 | 32    | 5    | 21    | 5    | 1.00      | 0.82   | 1.11D+03 |
| FLETCHCR | 1000 | 12749 | 1212 | 10320 | 1212 | 1.00      | 265.25 | 4.00D-19 |
| FMINSURF | 1024 | 214   | 11   | 99    | 487  | 9.55      | 12.12  | 1.00D+00 |
| FREUROTH | 1000 | 113   | 6    | 40    | 15   | 2.50      | 2.55   | 1.21D+05 |
| GENROSE  | 1000 | 6586  | 598  | 4955  | 985  | 1.65      | 135.23 | 1.00D+00 |
| LIARWHD  | 1000 | 30    | 2    | 24    | 4    | 2.00      | 0.63   | 1.92D-14 |
| MANCINO  | 100  | 111   | 13   | 46    | 14   | 1.08      | 97.85  | 3.03D-18 |
| MOREBV   | 1000 | 2     | 1    | 1     | 1    | 1.00      | 0.08   | 7.33D-13 |
| NCB20B   | 1000 | 117   | 15   | 78    | 520  | 6.67      | 62.12  | 1.68D+03 |
| NONDIA   | 1000 | 116   | 7    | 84    | 17   | 2.43      | 2.83   | 9.90D-01 |
| NONDQUAR | 1000 | 86    | 11   | 73    | 18   | 1.64      | 1.00   | 2.18D-09 |
| PENALTY1 | 1000 | 244   | 16   | 187   | 155  | 5.25      | 7.70   | 9.69D-03 |
| POWELLSG | 1000 | 92    | 14   | 71    | 16   | 1.14      | 0.95   | 6.14D-11 |
| POWER    | 1000 | 61    | 4    | 51    | 4    | 1.00      | 1.38   | 3.44D-10 |
| QUARTC   | 1000 | 46    | 2    | 40    | 2    | 1.00      | 0.34   | 5.20D-08 |
| SINQUAD  | 1000 | 184   | 13   | 147   | 41   | 3.15      | 5.76   | 2.93D-10 |
| SROSENBR | 1000 | 36    | 7    | 26    | 7    | 1.00      | 0.50   | 6.14D-25 |
| TOINTGSS | 1000 | 3     | 1    | 1     | 1    | 1.00      | 0.14   | 1.00D+01 |
| TQUARTIC | 1000 | 21    | 2    | 18    | 3    | 1.50      | 0.36   | 2.01D-19 |
| TRIDIA   | 1000 | 3     | 1    | 1     | 1    | 1.00      | 0.05   | 1.01D-25 |
| VARDIM   | 1000 | 105   | 19   | 89    | 2842 | 7.21      | 23.54  | 4.61D-13 |
| VAREIGVL | 1000 | 84    | 10   | 61    | 269  | 7.60      | 8.84   | 3.95D-10 |
| WOODS    | 1000 | 486   | 35   | 321   | 45   | 1.29      | 6.27   | 6.79D-16 |

Key:  $n$  = number of variables,  $\#f$  = number of function evaluations,  $\#min$  = number of minimizations,  $\#its$  = total number of iterations,  $\#cg$  = total number of CG iterations,  $s^{(k)}$  = average subspace dimension,  $time$  = total CPU time in seconds,  $f$  = smallest function value obtained.

TABLE 14

Results for ISM(n,f,a) on large or hard problems, in which no preconditioning is used and the subspace dimension is chosen automatically from the first CG directions.

| problem  | n    | #f   | #min | #its | #cg   | $s^{(k)}$ | time   | f        |
|----------|------|------|------|------|-------|-----------|--------|----------|
| ARWHEAD  | 1000 | 11   | 2    | 6    | 3     | 2.00      | 0.25   | 0.00D+00 |
| BDQRTIC  | 1000 | 52   | 11   | 40   | 62    | 2.18      | 1.69   | 3.98D+03 |
| BROWNAL  | 100  | 5    | 2    | 3    | 3     | 2.00      | 0.04   | 2.45D-08 |
| BRYBND   | 1000 | 55   | 10   | 37   | 74    | 2.40      | 2.27   | 8.21D-14 |
| CRAGGLVY | 1000 | 64   | 12   | 52   | 122   | 3.17      | 2.85   | 3.36D+02 |
| DIXMAANA | 1500 | 22   | 7    | 14   | 7     | 2.00      | 1.00   | 1.00D+00 |
| DIXON3DQ | 1000 | 7    | 4    | 4    | 1748  | 23.00     | 3.98   | 1.25D-11 |
| DQDRTIC  | 1000 | 10   | 5    | 5    | 12    | 2.40      | 0.24   | 1.91D-14 |
| DQRTIC   | 1000 | 58   | 9    | 45   | 18    | 2.00      | 0.38   | 1.75D-08 |
| EDENSCH  | 1000 | 60   | 8    | 33   | 29    | 2.50      | 1.47   | 6.00D+03 |
| EIGENALS | 110  | 96   | 15   | 74   | 138   | 3.87      | 1.37   | 1.23D-11 |
| ENGVAL1  | 1000 | 33   | 8    | 27   | 23    | 2.25      | 0.91   | 1.11D+03 |
| FLETCHCR | 1000 | 7720 | 1258 | 6300 | 14645 | 2.04      | 194.48 | 1.07D-12 |
| FMINSURF | 1024 | 190  | 11   | 106  | 2346  | 12.09     | 23.10  | 1.00D+00 |
| FREUROTH | 1000 | 59   | 9    | 32   | 25    | 2.44      | 1.75   | 1.21D+05 |
| GENROSE  | 1000 | 5170 | 497  | 3674 | 7780  | 4.91      | 128.63 | 1.00D+00 |
| LIARWHD  | 1000 | 25   | 5    | 19   | 8     | 2.00      | 0.54   | 7.19D-14 |
| MANCINO  | 100  | 31   | 11   | 20   | 13    | 2.00      | 39.91  | 1.01D-16 |
| MOREBV   | 1000 | 4    | 3    | 3    | 519   | 11.00     | 2.13   | 1.37D-09 |
| NCB20B   | 1000 | 147  | 21   | 89   | 1095  | 2.71      | 103.56 | 1.68D+03 |
| NONDIA   | 1000 | 13   | 3    | 10   | 4     | 2.00      | 0.37   | 1.48D-15 |
| NONDQUAR | 1000 | 280  | 47   | 231  | 2491  | 2.28      | 8.75   | 9.81D-07 |
| PENALTY1 | 1000 | 79   | 14   | 58   | 19    | 2.00      | 1.03   | 9.69D-03 |
| POWELLSG | 1000 | 58   | 11   | 49   | 32    | 2.27      | 0.56   | 5.24D-11 |
| POWER    | 1000 | 94   | 10   | 72   | 140   | 5.20      | 1.13   | 6.64D-10 |
| QUARTC   | 1000 | 58   | 9    | 45   | 18    | 2.00      | 0.39   | 1.75D-08 |
| SINQUAD  | 1000 | 186  | 34   | 139  | 67    | 2.21      | 5.91   | 2.02D-06 |
| SROSENBR | 1000 | 16   | 5    | 13   | 7     | 2.00      | 0.21   | 1.88D-15 |
| TOINTGSS | 1000 | 7    | 1    | 4    | 1     | 2.00      | 0.23   | 1.00D+01 |
| TQUARTIC | 1000 | 44   | 9    | 36   | 13    | 2.00      | 0.69   | 3.27D-16 |
| TRIDIA   | 1000 | 17   | 10   | 10   | 591   | 9.70      | 1.51   | 7.14D-14 |
| VARDIM   | 1000 | 58   | 9    | 40   | 9     | 2.00      | 0.40   | 1.14D-19 |
| VAREIGVL | 1000 | 52   | 9    | 44   | 162   | 3.78      | 3.66   | 1.77D-10 |
| WOODS    | 1000 | 37   | 7    | 29   | 17    | 2.57      | 0.60   | 2.76D-11 |

Key:  $n$  = number of variables,  $\#f$  = number of function evaluations,  $\#min$  = number of minimizations,  $\#its$  = total number of iterations,  $\#cg$  = total number of CG iterations,  $s^{(k)}$  = average subspace dimension,  $time$  = total CPU time in seconds,  $f$  = smallest function value obtained.

TABLE 15

Results for ISM(p,f,a) on large or hard problems, in which preconditioning is used and the subspace dimension is chosen automatically from the first CG directions.

| problem  | n    | #f   | #min | #its | #cg  | $s^{(k)}$ | time   | f        |
|----------|------|------|------|------|------|-----------|--------|----------|
| ARWHEAD  | 1000 | 22   | 4    | 15   | 5    | 2.00      | 0.63   | 7.22D-14 |
| BDQRTIC  | 1000 | 38   | 7    | 29   | 8    | 2.00      | 1.27   | 3.98D+03 |
| BROWNAL  | 100  | 55   | 10   | 46   | 126  | 2.80      | 3.08   | 1.75D-11 |
| BRYBND   | 1000 | 61   | 10   | 36   | 25   | 2.40      | 2.33   | 6.05D-14 |
| CRAGGLVY | 1000 | 54   | 10   | 42   | 10   | 2.00      | 2.06   | 3.36D+02 |
| DIXMAANA | 1500 | 20   | 6    | 15   | 8    | 2.17      | 1.23   | 1.00D+00 |
| DIXON3DQ | 1000 | 3    | 1    | 1    | 1    | 2.00      | 0.05   | 0.00D+00 |
| DQDRTIC  | 1000 | 3    | 1    | 1    | 1    | 2.00      | 0.09   | 1.24D-21 |
| DQRTIC   | 1000 | 48   | 8    | 40   | 8    | 2.00      | 0.41   | 8.23D-09 |
| EDENSCH  | 1000 | 33   | 6    | 26   | 6    | 2.00      | 1.09   | 6.00D+03 |
| EIGENALS | 110  | 122  | 16   | 79   | 55   | 2.56      | 1.62   | 4.21D-11 |
| ENGVAL1  | 1000 | 26   | 5    | 17   | 5    | 2.00      | 0.70   | 1.11D+03 |
| FLETCHCR | 1000 | 7053 | 1246 | 6220 | 1246 | 2.00      | 161.54 | 3.32D-16 |
| FMINSURF | 1024 | 199  | 11   | 77   | 540  | 5.91      | 11.94  | 1.00D+00 |
| FREUROTH | 1000 | 65   | 7    | 29   | 8    | 2.00      | 1.68   | 1.21D+05 |
| GENROSE  | 1000 | 3682 | 593  | 2950 | 957  | 2.00      | 81.75  | 1.00D+00 |
| LIARWHD  | 1000 | 124  | 15   | 65   | 31   | 2.07      | 2.26   | 1.11D+01 |
| MANCINO  | 100  | 53   | 14   | 32   | 16   | 2.00      | 62.23  | 1.09D-18 |
| MOREBV   | 1000 | 2    | 1    | 1    | 1    | 2.00      | 0.08   | 7.33D-13 |
| NCB20B   | 1000 | 94   | 15   | 69   | 534  | 5.20      | 59.05  | 1.68D+03 |
| NONDIA   | 1000 | 142  | 19   | 99   | 44   | 2.53      | 3.69   | 9.90D-01 |
| NONDQUAR | 1000 | 124  | 18   | 102  | 57   | 2.39      | 1.53   | 1.07D-07 |
| PENALTY1 | 1000 | 77   | 13   | 60   | 60   | 2.08      | 4.30   | 9.69D-03 |
| POWELLSG | 1000 | 73   | 14   | 63   | 16   | 2.07      | 0.81   | 1.17D-09 |
| POWER    | 1000 | 48   | 8    | 40   | 8    | 2.00      | 2.22   | 1.05D-11 |
| QUARTC   | 1000 | 48   | 8    | 40   | 8    | 2.00      | 0.41   | 8.23D-09 |
| SINQUAD  | 1000 | 283  | 56   | 215  | 177  | 2.29      | 11.50  | 6.90D-08 |
| SROSENBR | 1000 | 30   | 7    | 23   | 7    | 2.00      | 0.44   | 9.58D-26 |
| TOINTGSS | 1000 | 3    | 1    | 1    | 1    | 2.00      | 0.14   | 1.00D+01 |
| TQUARTIC | 1000 | 26   | 5    | 20   | 8    | 2.00      | 0.49   | 3.50D-16 |
| TRIDIA   | 1000 | 3    | 1    | 1    | 1    | 2.00      | 0.05   | 1.01D-25 |
| VARDIM   | 1000 | 78   | 22   | 57   | 2887 | 2.27      | 24.91  | 5.87D-13 |
| VAREIGVL | 1000 | 47   | 9    | 40   | 147  | 3.67      | 6.06   | 1.57D-10 |
| WOODS    | 1000 | 143  | 23   | 105  | 25   | 2.00      | 2.19   | 1.21D-26 |

Key:  $n$  = number of variables,  $\#f$  = number of function evaluations,  $\#min$  = number of minimizations,  $\#its$  = total number of iterations,  $\#cg$  = total number of CG iterations,  $s^{(k)}$  = average subspace dimension,  $time$  = total CPU time in seconds,  $f$  = smallest function value obtained.

TABLE 16

Results for ISM(n,e,a) on large or hard problems, in which no preconditioning is used and the subspace dimension is chosen automatically from extreme CG directions.

| problem  | n    | #f   | #min | #its | #cg   | $s^{(k)}$ | time   | f        |
|----------|------|------|------|------|-------|-----------|--------|----------|
| ARWHEAD  | 1000 | 11   | 2    | 6    | 3     | 2.00      | 0.25   | 0.00D+00 |
| BDQRTIC  | 1000 | 52   | 11   | 40   | 62    | 2.18      | 1.69   | 3.98D+03 |
| BROWNAL  | 100  | 5    | 2    | 3    | 3     | 2.00      | 0.04   | 2.45D-08 |
| BRYBND   | 1000 | 53   | 10   | 36   | 72    | 2.40      | 2.21   | 9.91D-14 |
| CRAGGLVY | 1000 | 66   | 12   | 55   | 126   | 3.58      | 3.00   | 3.36D+02 |
| DIXMAANA | 1500 | 22   | 7    | 14   | 7     | 2.00      | 1.00   | 1.00D+00 |
| DIXON3DQ | 1000 | 7    | 4    | 4    | 1748  | 23.00     | 4.19   | 1.25D-11 |
| DQDRTIC  | 1000 | 10   | 5    | 5    | 12    | 2.40      | 0.24   | 1.91D-14 |
| DQRTIC   | 1000 | 58   | 9    | 45   | 18    | 2.00      | 0.38   | 1.75D-08 |
| EDENSCH  | 1000 | 60   | 8    | 33   | 29    | 2.50      | 1.47   | 6.00D+03 |
| EIGENALS | 110  | 94   | 17   | 73   | 127   | 3.47      | 1.35   | 5.34D-11 |
| ENGVAL1  | 1000 | 33   | 8    | 27   | 23    | 2.25      | 0.93   | 1.11D+03 |
| FLETCHCR | 1000 | 7722 | 1262 | 6300 | 14691 | 2.03      | 195.36 | 3.98D-15 |
| FMINSURF | 1024 | 216  | 12   | 100  | 2259  | 8.33      | 22.14  | 1.00D+00 |
| FREUROTH | 1000 | 80   | 9    | 32   | 28    | 2.56      | 1.92   | 1.21D+05 |
| GENROSE  | 1000 | 5445 | 475  | 3895 | 7394  | 5.71      | 126.73 | 1.00D+00 |
| LIARWHD  | 1000 | 25   | 5    | 19   | 8     | 2.00      | 0.49   | 7.19D-14 |
| MANCINO  | 100  | 31   | 11   | 20   | 13    | 2.00      | 38.06  | 1.01D-16 |
| MOREBV   | 1000 | 4    | 3    | 3    | 519   | 11.00     | 1.97   | 1.37D-09 |
| NCB20B   | 1000 | 159  | 21   | 95   | 1139  | 3.62      | 105.14 | 1.68D+03 |
| NONDIA   | 1000 | 13   | 3    | 10   | 4     | 2.00      | 0.35   | 1.48D-15 |
| NONDQUAR | 1000 | 341  | 53   | 257  | 3507  | 2.42      | 11.35  | 1.64D-06 |
| PENALTY1 | 1000 | 79   | 14   | 58   | 19    | 2.00      | 0.96   | 9.69D-03 |
| POWELLSG | 1000 | 58   | 11   | 49   | 32    | 2.27      | 0.52   | 5.24D-11 |
| POWER    | 1000 | 94   | 10   | 72   | 140   | 5.20      | 1.06   | 6.64D-10 |
| QUARTC   | 1000 | 58   | 9    | 45   | 18    | 2.00      | 0.39   | 1.75D-08 |
| SINQUAD  | 1000 | 208  | 41   | 156  | 88    | 2.20      | 6.43   | 1.71D-07 |
| SROSENBR | 1000 | 16   | 5    | 13   | 7     | 2.00      | 0.19   | 1.88D-15 |
| TOINTGSS | 1000 | 7    | 1    | 4    | 1     | 2.00      | 0.21   | 1.00D+01 |
| TQUARTIC | 1000 | 44   | 9    | 36   | 13    | 2.00      | 0.65   | 3.27D-16 |
| TRIDIA   | 1000 | 17   | 10   | 10   | 591   | 9.40      | 1.57   | 8.20D-14 |
| VARDIM   | 1000 | 58   | 9    | 40   | 9     | 2.00      | 0.38   | 1.14D-19 |
| VAREIGVL | 1000 | 54   | 10   | 46   | 228   | 4.40      | 4.18   | 2.07D-11 |
| WOODS    | 1000 | 37   | 7    | 29   | 17    | 2.57      | 0.54   | 2.76D-11 |

Key:  $n$  = number of variables,  $\#f$  = number of function evaluations,  $\#min$  = number of minimizations,  $\#its$  = total number of iterations,  $\#cg$  = total number of CG iterations,  $s^{(k)}$  = average subspace dimension,  $time$  = total CPU time in seconds,  $f$  = smallest function value obtained.

TABLE 17

Results for ISM(p,e,a) on large or hard problems, in which preconditioning is used and the subspace dimension is chosen automatically from extreme CG directions.

| problem  | n    | #f   | #min | #its | #cg  | $s^{(k)}$ | time   | f        |
|----------|------|------|------|------|------|-----------|--------|----------|
| ARWHEAD  | 1000 | 22   | 4    | 15   | 5    | 2.00      | 0.59   | 7.22D-14 |
| BDQRTIC  | 1000 | 38   | 7    | 29   | 8    | 2.00      | 1.17   | 3.98D+03 |
| BROWNAL  | 100  | 56   | 11   | 48   | 140  | 3.27      | 3.28   | 3.25D-13 |
| BRYBND   | 1000 | 66   | 11   | 40   | 26   | 2.27      | 2.58   | 2.06D-14 |
| CRAGGLVY | 1000 | 54   | 10   | 42   | 10   | 2.00      | 2.04   | 3.36D+02 |
| DIXMAANA | 1500 | 20   | 6    | 15   | 8    | 2.17      | 1.20   | 1.00D+00 |
| DIXON3DQ | 1000 | 3    | 1    | 1    | 1    | 2.00      | 0.05   | 0.00D+00 |
| DQDRTIC  | 1000 | 3    | 1    | 1    | 1    | 2.00      | 0.09   | 1.24D-21 |
| DQRTIC   | 1000 | 48   | 8    | 40   | 8    | 2.00      | 0.42   | 8.23D-09 |
| EDENSCH  | 1000 | 33   | 6    | 26   | 6    | 2.00      | 1.08   | 6.00D+03 |
| EIGENALS | 110  | 112  | 16   | 70   | 56   | 2.44      | 1.53   | 2.50D-13 |
| ENGVAL1  | 1000 | 26   | 5    | 17   | 5    | 2.00      | 0.70   | 1.11D+03 |
| FLETCHCR | 1000 | 7053 | 1246 | 6220 | 1246 | 2.00      | 162.80 | 3.32D-16 |
| FMINSURF | 1024 | 190  | 13   | 86   | 549  | 6.54      | 12.77  | 1.00D+00 |
| FREUROTH | 1000 | 65   | 7    | 29   | 8    | 2.00      | 1.67   | 1.21D+05 |
| GENROSE  | 1000 | 3682 | 593  | 2950 | 957  | 2.00      | 81.89  | 1.00D+00 |
| LIARWHD  | 1000 | 124  | 15   | 65   | 31   | 2.07      | 2.24   | 1.11D+01 |
| MANCINO  | 100  | 53   | 14   | 32   | 16   | 2.00      | 64.59  | 1.09D-18 |
| MOREBV   | 1000 | 2    | 1    | 1    | 1    | 2.00      | 0.08   | 7.33D-13 |
| NCB20B   | 1000 | 101  | 15   | 70   | 505  | 4.00      | 57.74  | 1.68D+03 |
| NONDIA   | 1000 | 142  | 19   | 99   | 44   | 2.53      | 3.66   | 9.90D-01 |
| NONDQUAR | 1000 | 122  | 17   | 96   | 46   | 2.47      | 1.46   | 9.13D-08 |
| PENALTY1 | 1000 | 80   | 14   | 64   | 62   | 2.07      | 4.60   | 9.69D-03 |
| POWELLSG | 1000 | 73   | 14   | 63   | 16   | 2.07      | 0.80   | 1.17D-09 |
| POWER    | 1000 | 48   | 8    | 40   | 8    | 2.00      | 2.24   | 1.05D-11 |
| QUARTC   | 1000 | 48   | 8    | 40   | 8    | 2.00      | 0.41   | 8.23D-09 |
| SINQUAD  | 1000 | 440  | 93   | 301  | 281  | 2.15      | 17.97  | 1.11D-08 |
| SROSENBR | 1000 | 30   | 7    | 23   | 7    | 2.00      | 0.44   | 9.58D-26 |
| TOINTGSS | 1000 | 3    | 1    | 1    | 1    | 2.00      | 0.14   | 1.00D+01 |
| TQUARTIC | 1000 | 26   | 5    | 20   | 8    | 2.00      | 0.49   | 3.50D-16 |
| TRIDIA   | 1000 | 3    | 1    | 1    | 1    | 2.00      | 0.05   | 1.01D-25 |
| VARDIM   | 1000 | 82   | 22   | 59   | 2804 | 2.32      | 24.92  | 1.24D-12 |
| VAREIGVL | 1000 | 49   | 9    | 42   | 134  | 3.67      | 6.03   | 3.26D-10 |
| WOODS    | 1000 | 143  | 23   | 105  | 25   | 2.00      | 2.20   | 1.21D-26 |

Key:  $n$  = number of variables,  $\#f$  = number of function evaluations,  $\#min$  = number of minimizations,  $\#its$  = total number of iterations,  $\#cg$  = total number of CG iterations,  $s^{(k)}$  = average subspace dimension,  $time$  = total CPU time in seconds,  $f$  = smallest function value obtained.