October the 2nd, 2024, Gent, Belgium

Timoteo Carletti

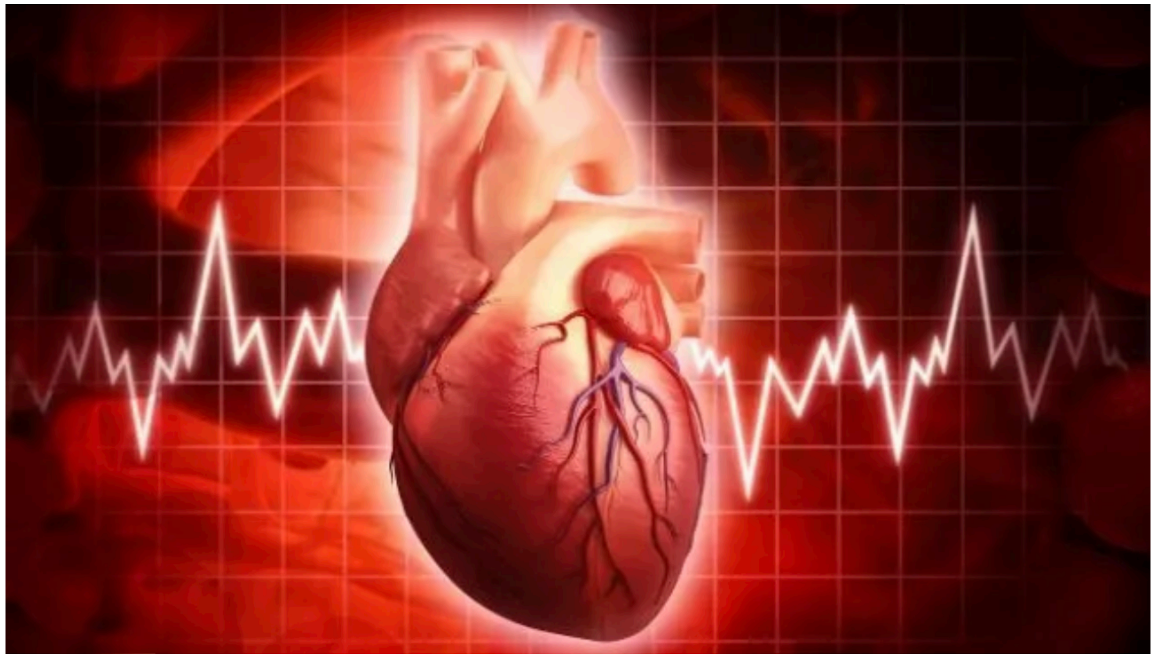
Global synchronization on networks and beyond

naxys
Namur Institute for Complex Systems

UNIVERSITÉ DE NAMUR

timoteo.carletti@unamur.be

Synchronisation



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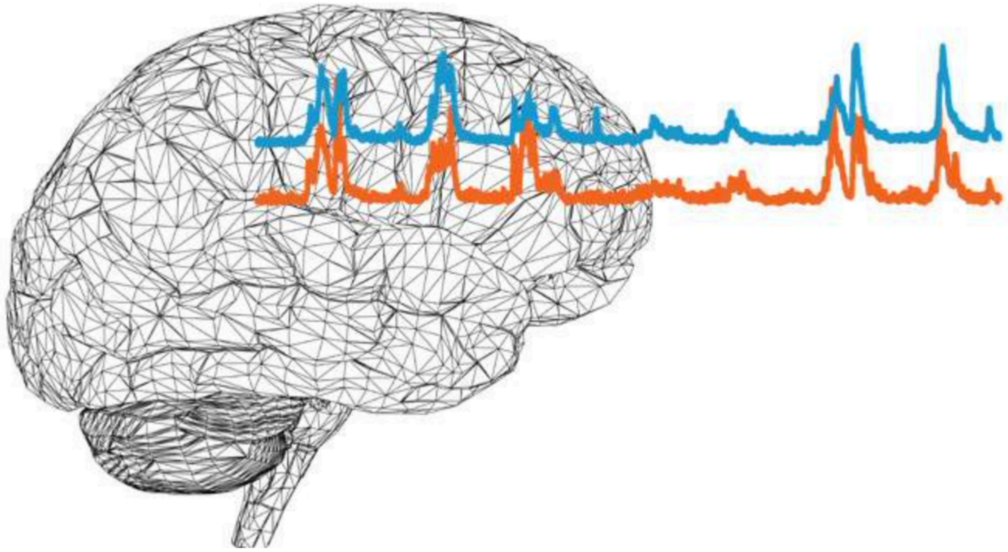
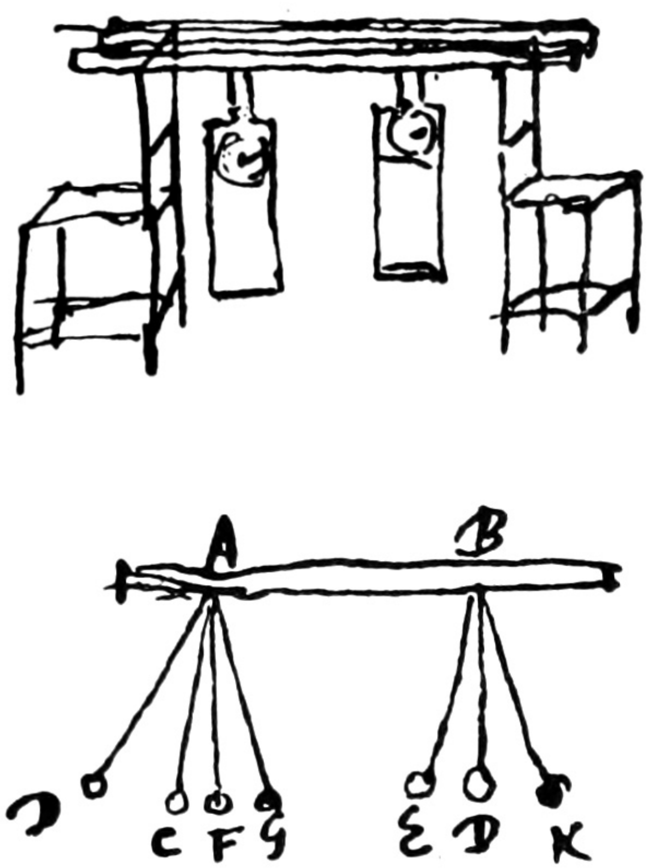


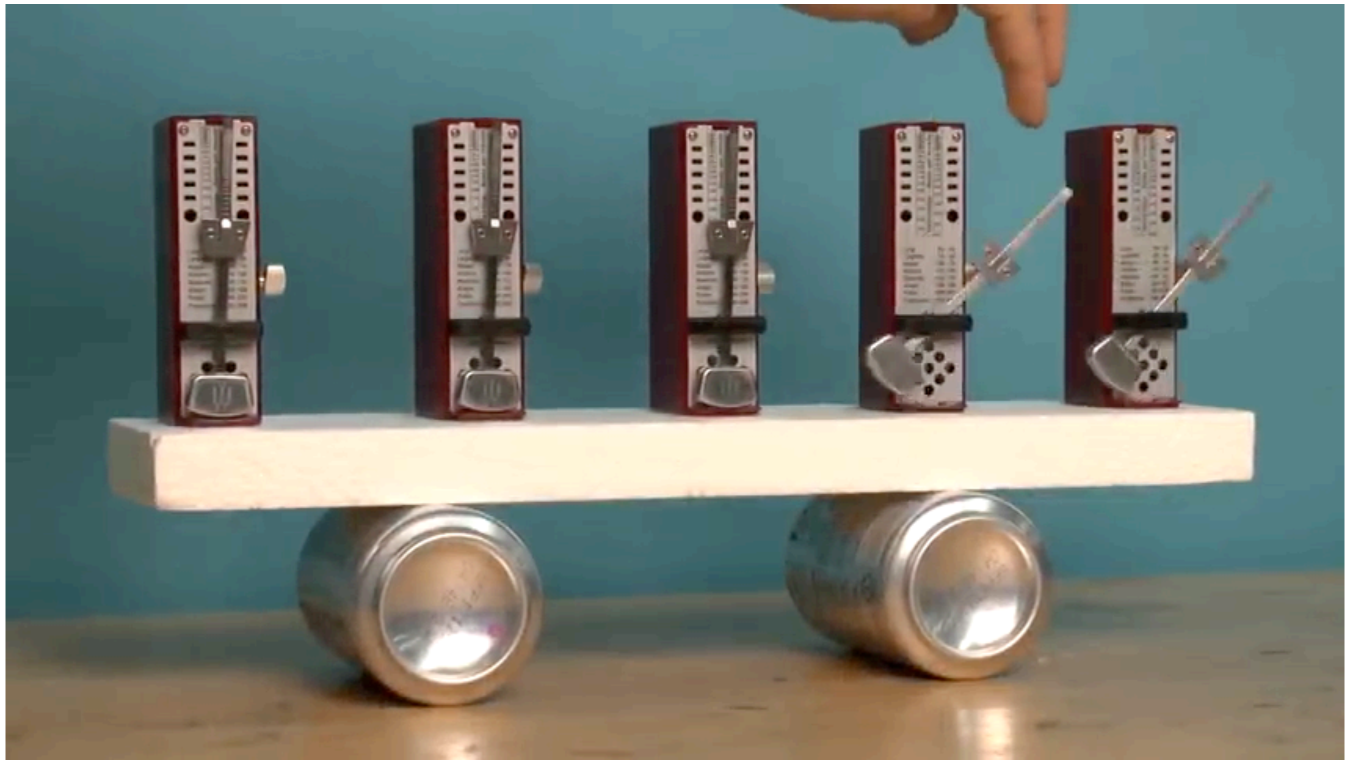
www.quantamagazine.org

Global Synchronisation



Huygen

"An odd kind of sympathy"



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Kuramoto model

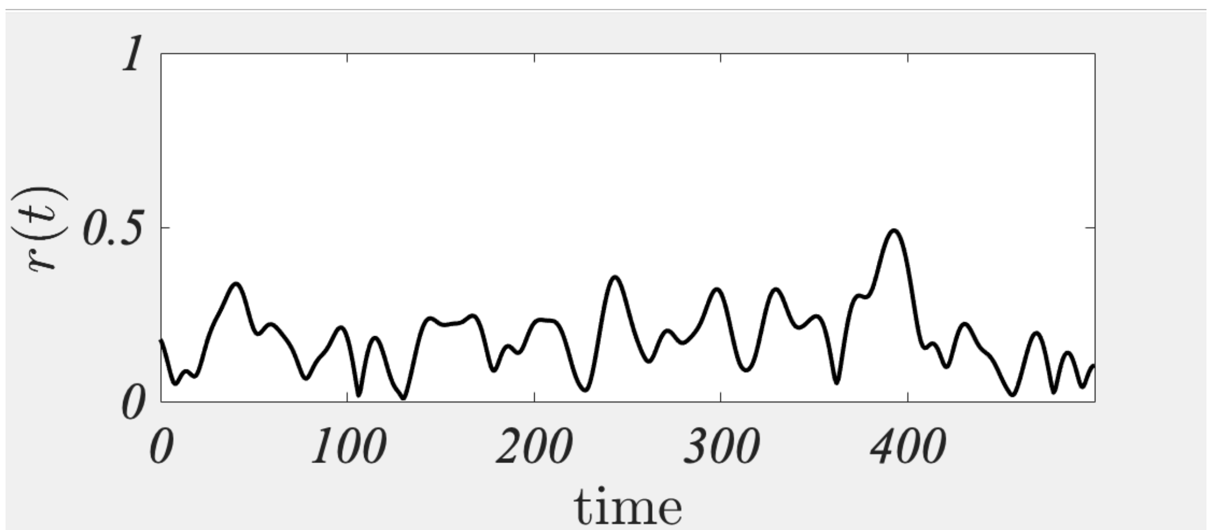
order parameter

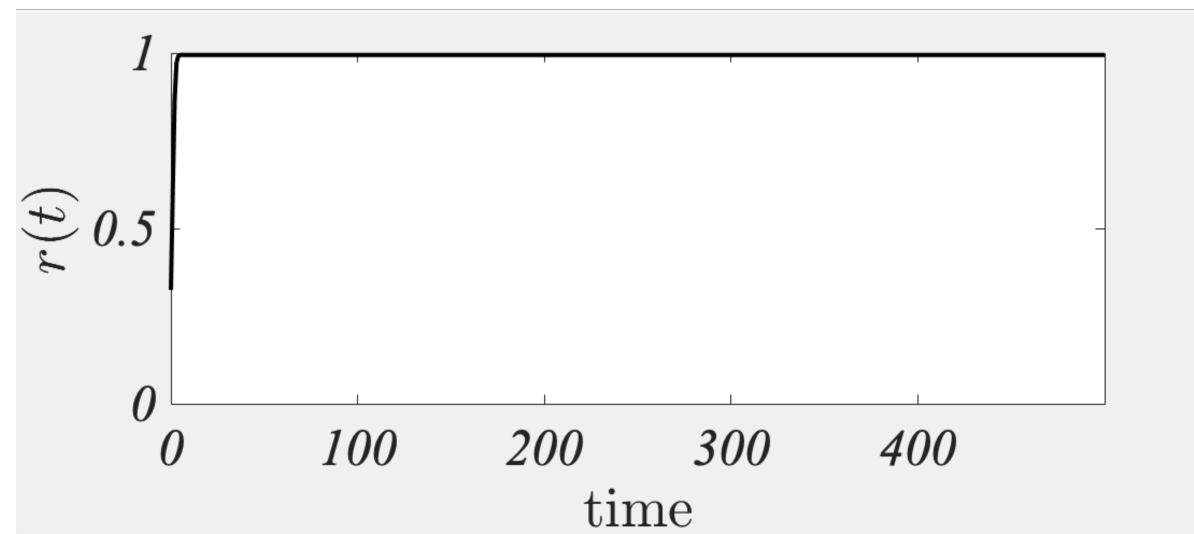
$$\frac{d\theta_i}{dt} = \omega_i + \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i)$$

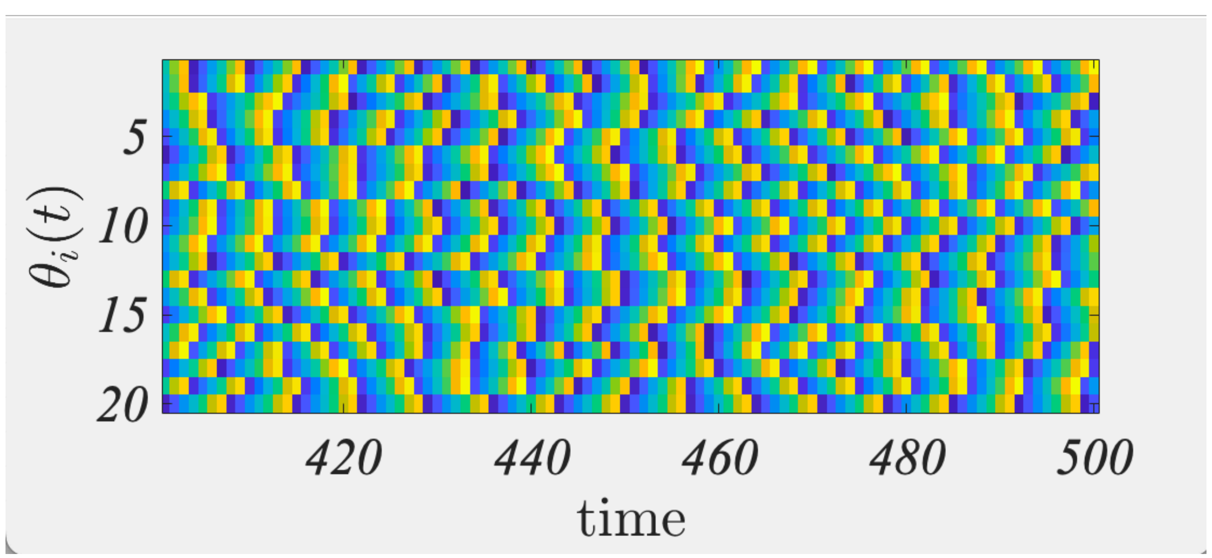
$$re^{i\psi} = \frac{1}{N}\sum_{j=1}^{N} e^{i\theta_j}$$

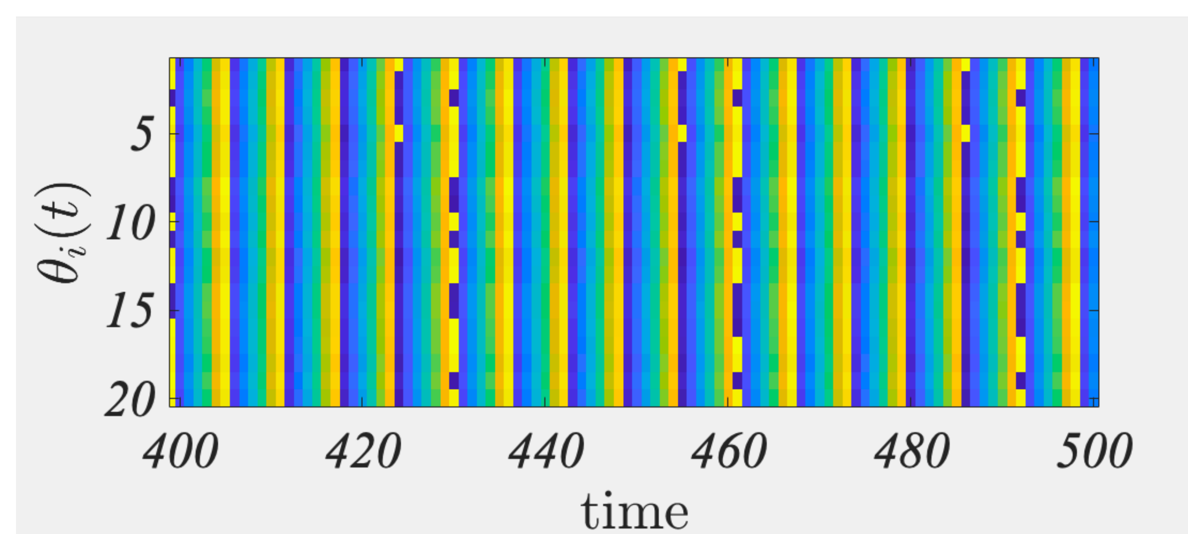
$K = 0.01$

$K = 1.0$









Global Synchronisation

Progress of Theoretical Physics, Vol. 69, No. 1, January 1983

Stability Theory of Synchronized Motion in Coupled-Oscillator Systems

Hirokazu FUJISAKA and Tomoji YAMADA*

Department of Physics, Kagoshima University, Kagoshima 890
**Department of Physics, Kyushu Institute of Technology Kitakyushu 804*

VOLUME 64, NUMBER 8 PHYSICAL REVIEW LETTERS 19 FEBRUARY 1990

Synchronization in Chaotic Systems

Louis M. Pecora and Thomas L. Carroll
Code 6341, Naval Research Laboratory, Washington, D.C. 20375
(Received 20 December 1989)

VOLUME 80, NUMBER 10 PHYSICAL REVIEW LETTERS 9 MARCH 1998

Master Stability Functions for Synchronized Coupled Systems

Louis M. Pecora and Thomas L. Carroll
Code 6343, Naval Research Laboratory, Washington, D.C. 20375
(Received 7 July 1997)

PHYSICAL REVIEW E **80**, 036204 (2009)

Generic behavior of master-stability functions in coupled nonlinear dynamical systems

Liang Huang,[1] Qingfei Chen,[1] Ying-Cheng Lai,[1,2] and Louis M. Pecora[3]
[1]*School of Electrical, Computer and Energy Engineering, Arizona State University, Tempe, Arizona 85287, USA*
[2]*Department of Physics, Arizona State University, Tempe, Arizona 85287, USA*
[3]*Code 6362, Naval Research Laboratory, Washington, DC 20375, USA*
(Received 9 June 2009; published 15 September 2009)

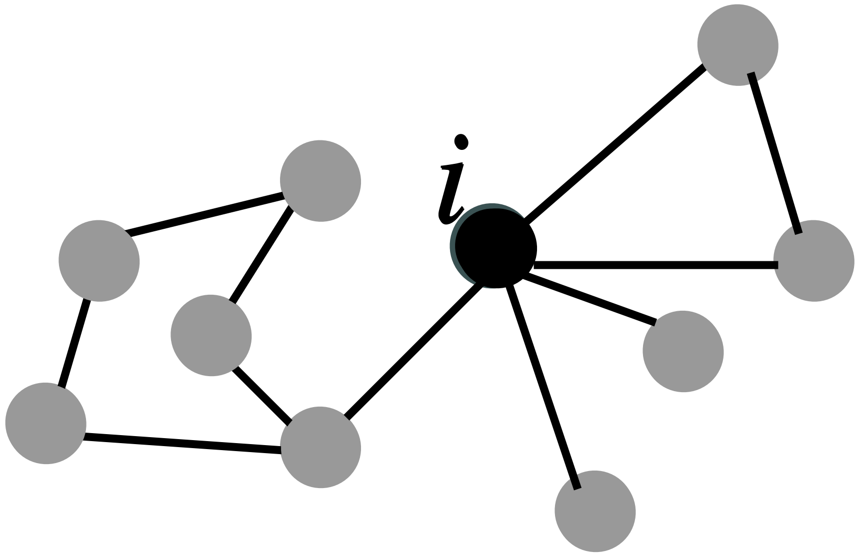
Global Synchronisation on networks

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}) \quad \mathbf{x} \in \mathbb{R}^d$$

1

$$\frac{d\mathbf{x}^{(i)}}{dt} = \mathbf{f}(\mathbf{x}^{(i)}) \quad \mathbf{x}^{(i)} \in \mathbb{R}^d$$

$$i = 1, \dots, n$$



2

$$\frac{d\mathbf{x}^{(i)}}{dt} = \mathbf{f}(\mathbf{x}^{(i)}) + \sigma \sum_{j=1}^{n} A_{ij} \mathbf{g}(\mathbf{x}^{(i)}, \mathbf{x}^{(j)})$$

$$\mathbf{g}(\mathbf{x}^{(i)}, \mathbf{x}^{(j)}) = \mathbf{h}(\mathbf{x}^{(j)}) - \mathbf{h}(\mathbf{x}^{(i)})$$

$$= \mathbf{f}(\mathbf{x}^{(i)}) + \sigma \sum_{j=1}^{n} L_{ij} \mathbf{h}(\mathbf{x}^{(j)})$$

Diffusive-like coupling

$$L_{ij} = A_{ij} - \delta_{ij} k_i$$

Global Synchronisation on networks

Reference orbit $\mathbf{s}(t)$ solution of $$\frac{d\mathbf{s}}{dt} = \mathbf{f}(\mathbf{s})$$

Global Synchronisation : $\mathbf{x}^{(i)}(t) = \mathbf{s}(t) \quad \forall i = 1, \dots, n$

Does the whole system admit such (spatially) homogeneous solution?

$$\frac{d\mathbf{x}^{(i)}}{dt}\Big|_{\mathbf{x}^{(i)}=\mathbf{s}} = \mathbf{f}(\mathbf{x}^{(i)}) + \sigma \sum_{j=1}^{n} L_{ij}\mathbf{h}(\mathbf{x}^{(j)})\Big|_{\mathbf{x}^{(i)}=\mathbf{s}} = 0$$

$$\mathbf{L}\mathbf{u} = 0 \qquad \mathbf{u} = (1, \dots, 1)^\top$$

Laplace matrix

Global Synchronisation on networks

Is $\mathbf{x}^{(i)}(t) = \mathbf{s}(t) \quad \forall i = 1, \dots, n$ stable?

- $\delta\mathbf{x}^{(i)}(t) = \mathbf{x}^{(i)}(t) - \mathbf{s}(t) \quad \forall i = 1, \dots, n$

- $$\frac{d\delta\mathbf{x}^{(i)}}{dt} = \mathbf{J_f}(\mathbf{s}(t))\delta\mathbf{x}^{(i)} + \sigma\sum_{j=1}^{n} L_{ij}\mathbf{J_h}(\mathbf{s}(t))\delta\mathbf{x}^{(j)}$$

Time dependent linear system

Global Synchronisation on networks

- $$\mathbf{L}\phi^{(\alpha)} = \Lambda^{(\alpha)}\phi^{(\alpha)} \qquad \phi^{(\alpha)} \cdot \phi^{(\beta)} = \delta_{\alpha\beta} \qquad \Lambda^{(1)} = 0 \quad \Lambda^{(\alpha)} < 0$$
- $$\delta\mathbf{x}^{(i)} = \sum_{\alpha} \delta\mathbf{x}_{\alpha}\phi_i^{(\alpha)}$$
- $$\frac{d\delta\mathbf{x}_{\alpha}}{dt} = \mathbf{J}_{\mathbf{f}}(\mathbf{s}(t))\delta\mathbf{x}_{\alpha} + \sigma\Lambda^{(\alpha)}\mathbf{J}_{\mathbf{h}}(\mathbf{s}(t))\delta\mathbf{x}_{\alpha} := \mathbf{J}_{\alpha}(\mathbf{s}(t))\delta\mathbf{x}_{\alpha}$$

$\lambda(\Lambda^{(\alpha)})$ Master Stability Function = largest Lyapunov exponent of $\mathbf{J}_{\alpha}(\mathbf{s}(t))$

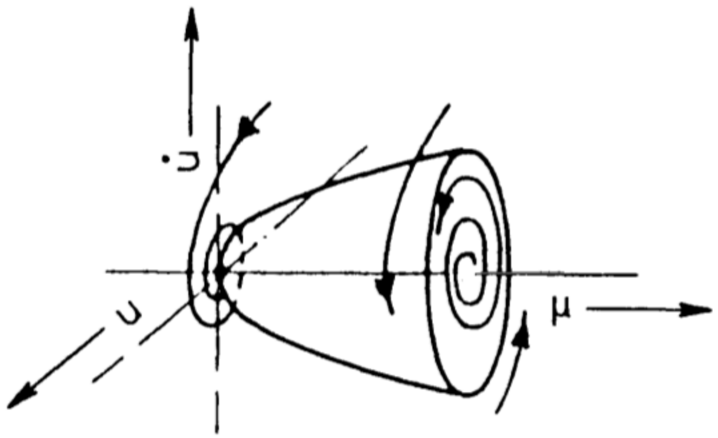
(function of $\Lambda^{(\alpha)}$ eigenvalue of L)

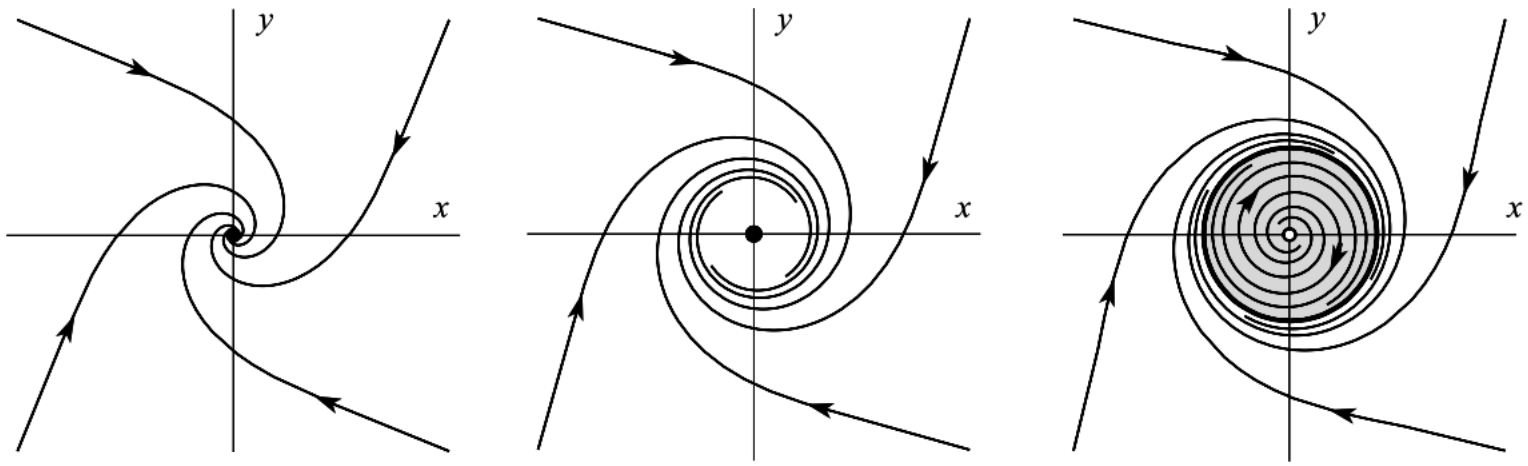
timoteo.carletti@unamur.be

Stuart - Landau oscillator

$$\frac{dz}{dt} = z(a + ib - |z|^2) \qquad z = x + iy \in \mathbb{C} \quad a \in \mathbb{R} \quad b \in \mathbb{R}_+$$

Hopf Bifurcation



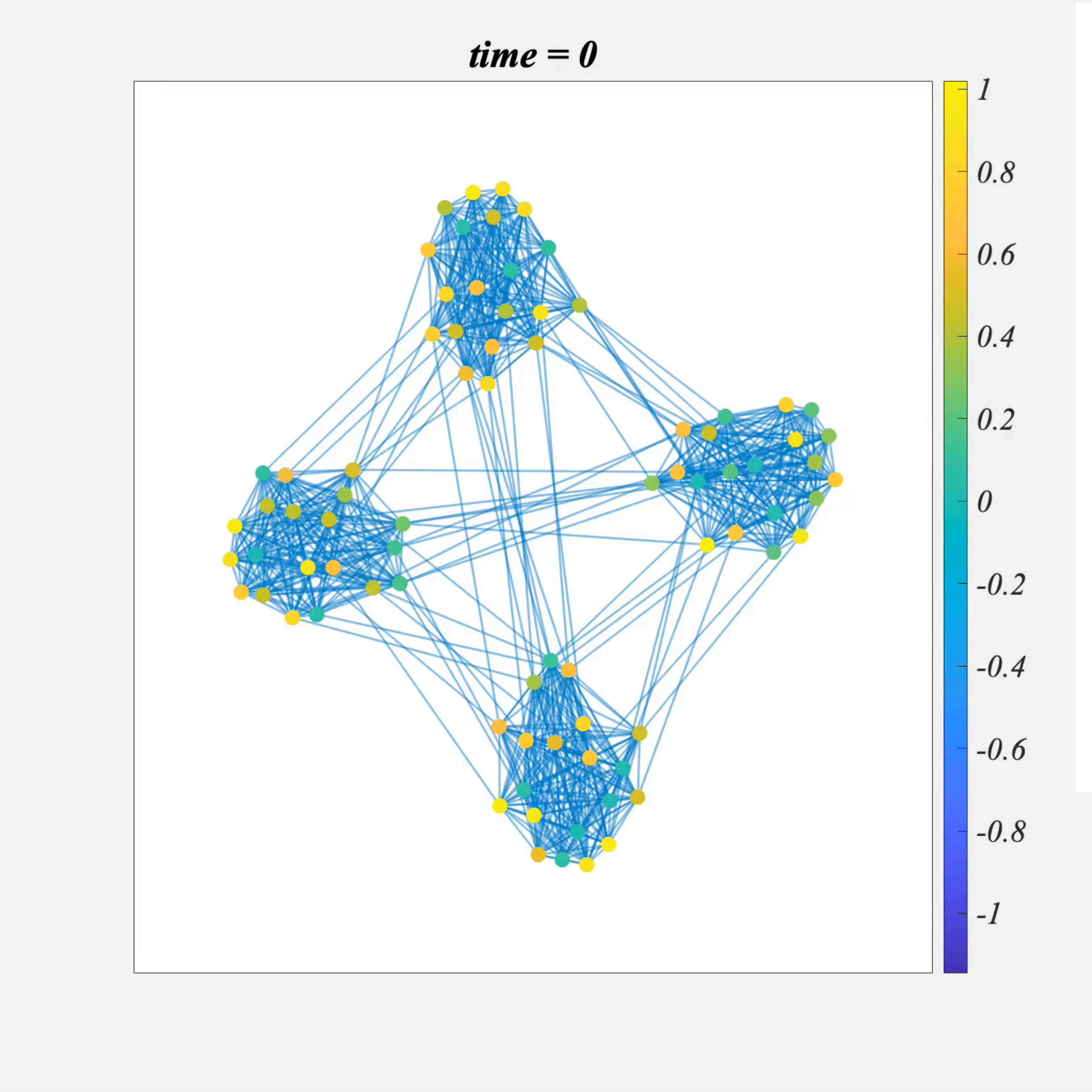


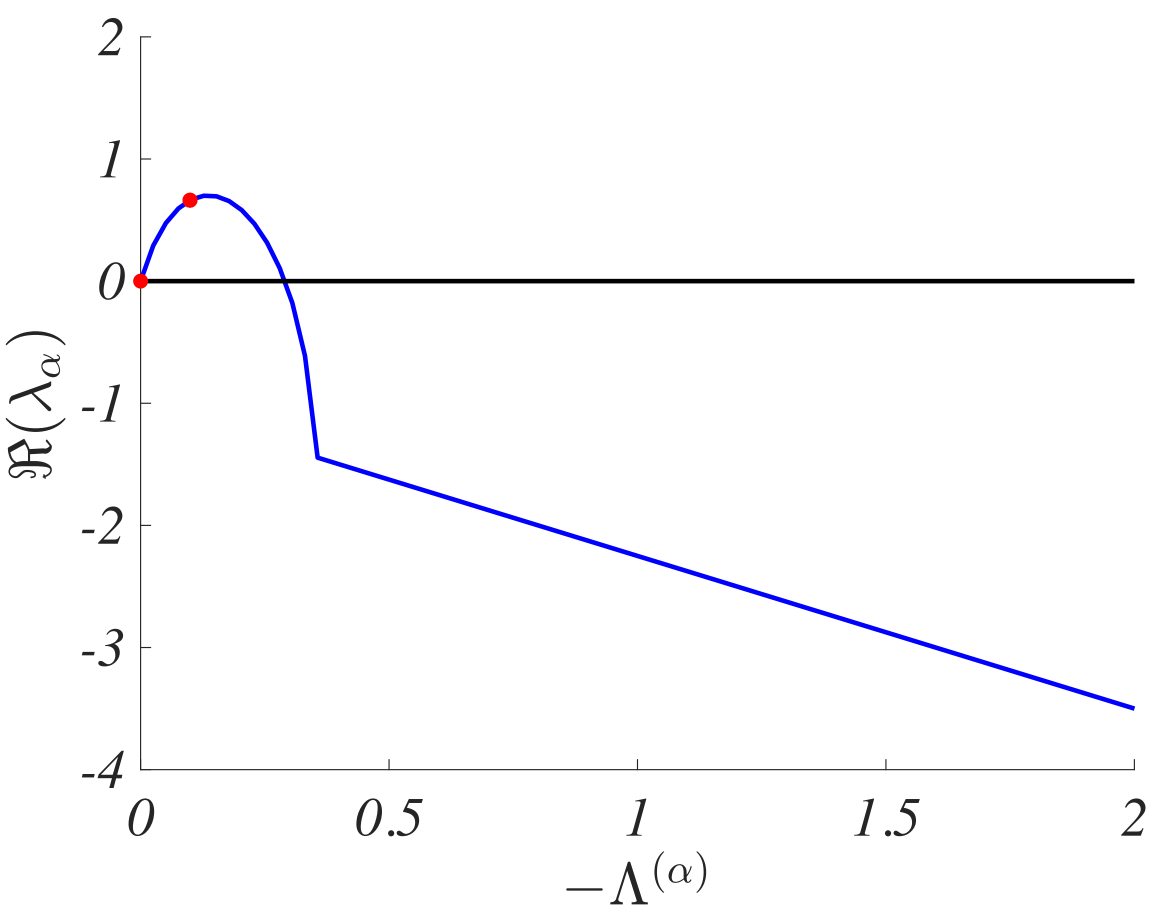
$a < 0$ $\quad a = 0$ $\quad a > 0$

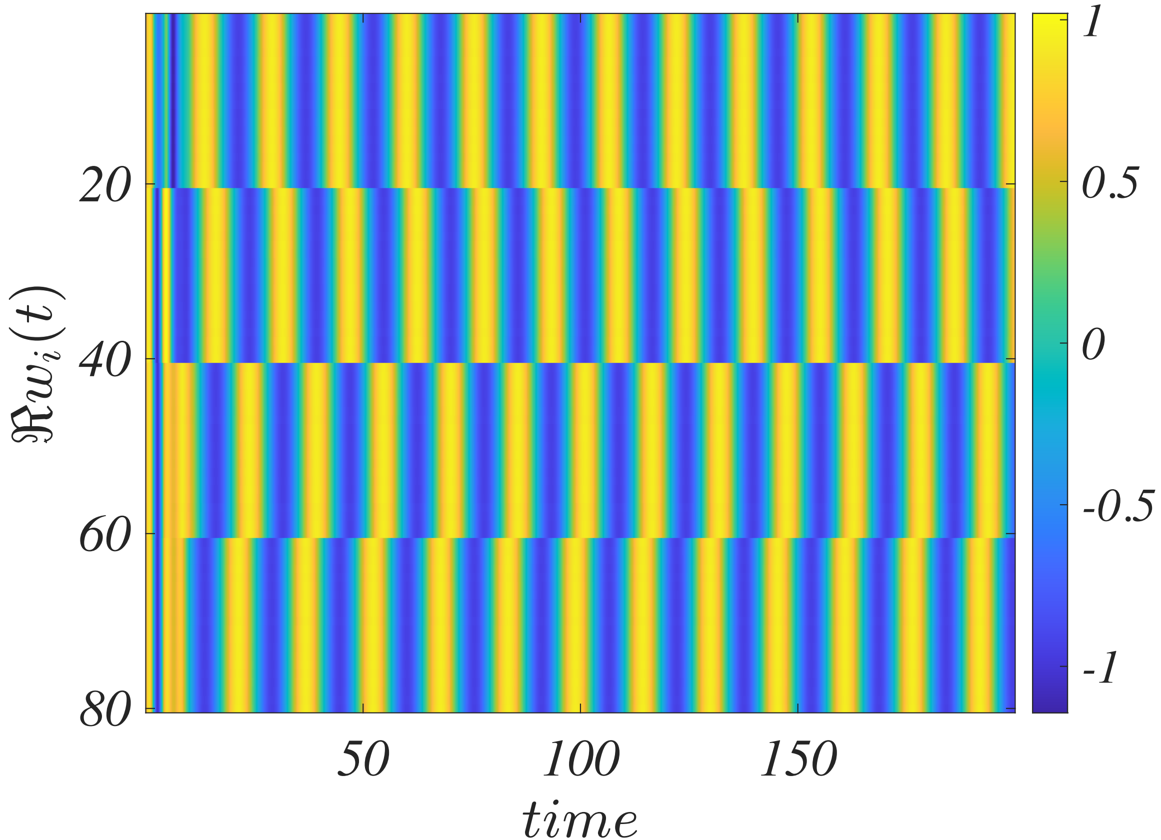
$$\frac{dz_j}{dt} = z_j(a + ib - |z_j|^2) + \mu \sum_{j=1}^{n} A_{j\ell} \left[h(z_\ell) - h(z_j)\right]$$

$$h(z) = |z|^{m-1} z$$

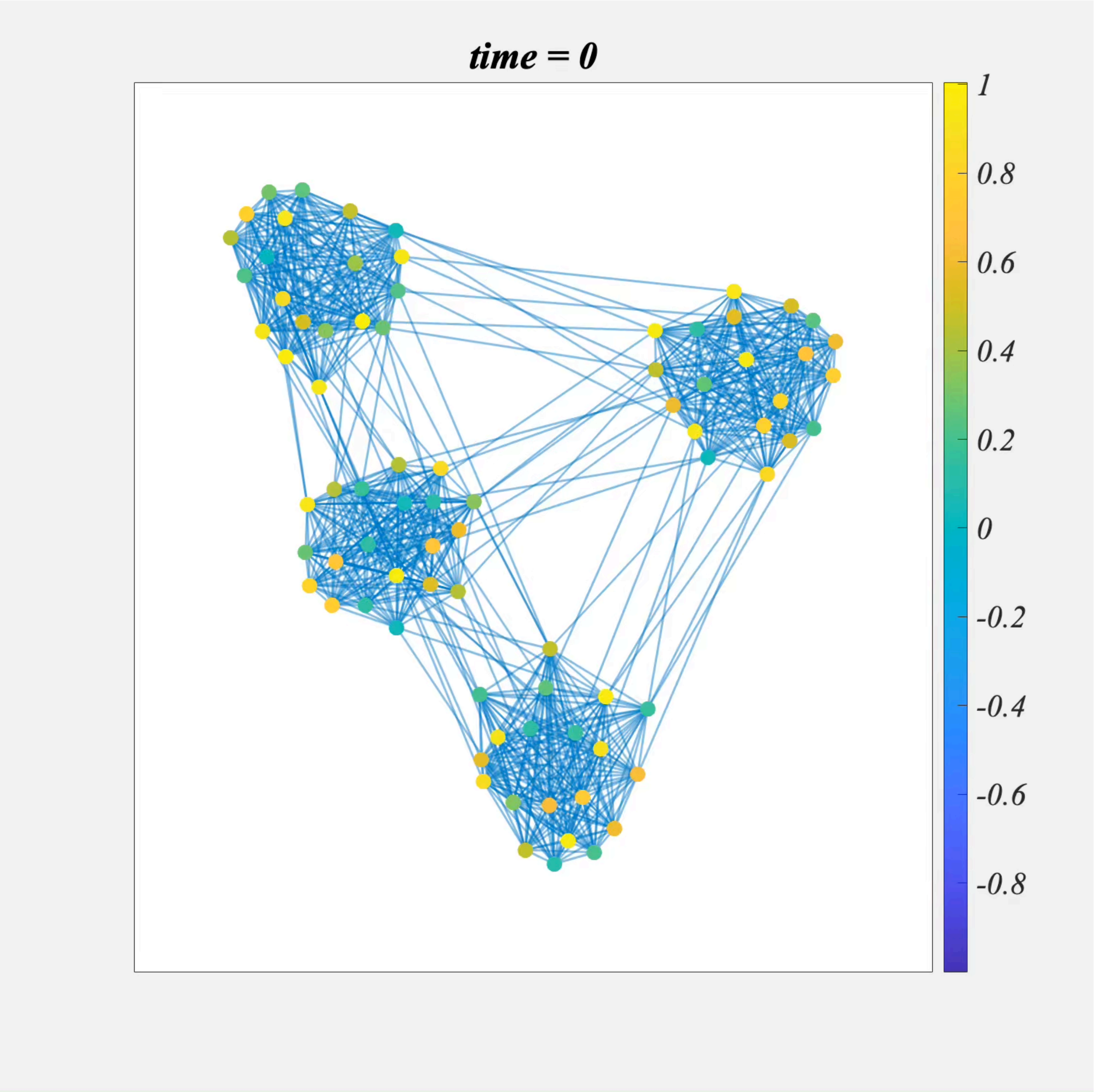
Stuart - Landau oscillator : no synch

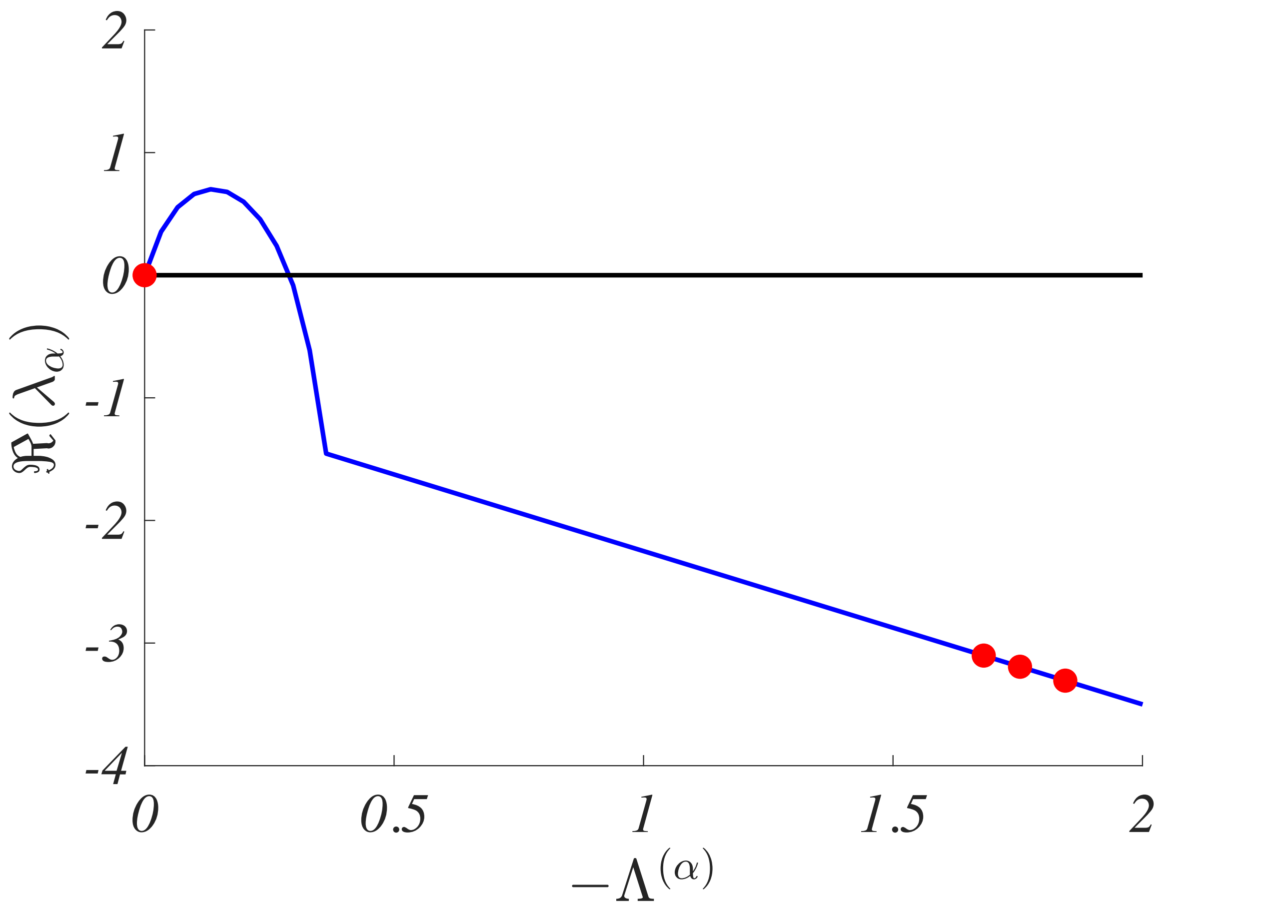


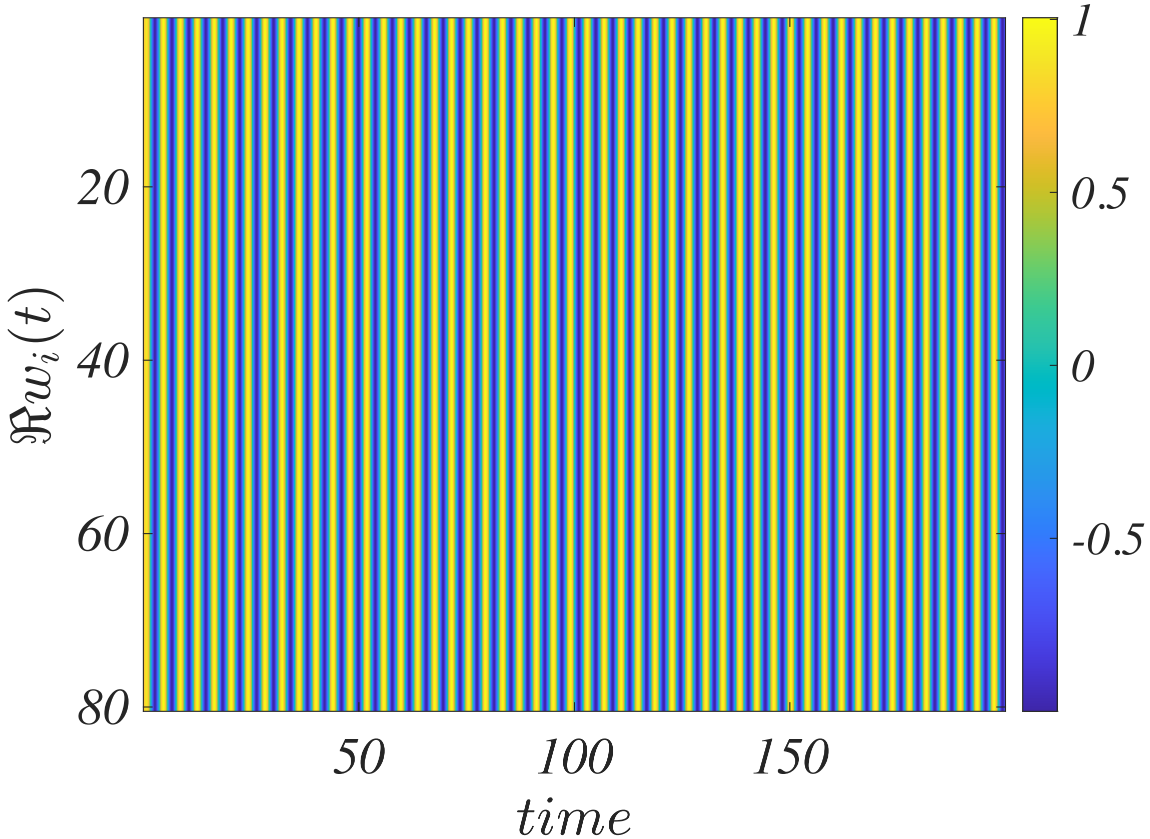




Stuart - Landau oscillator : synch

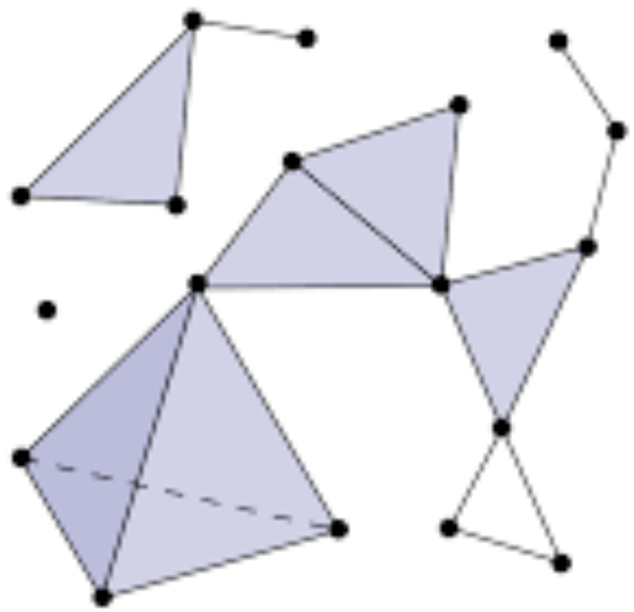






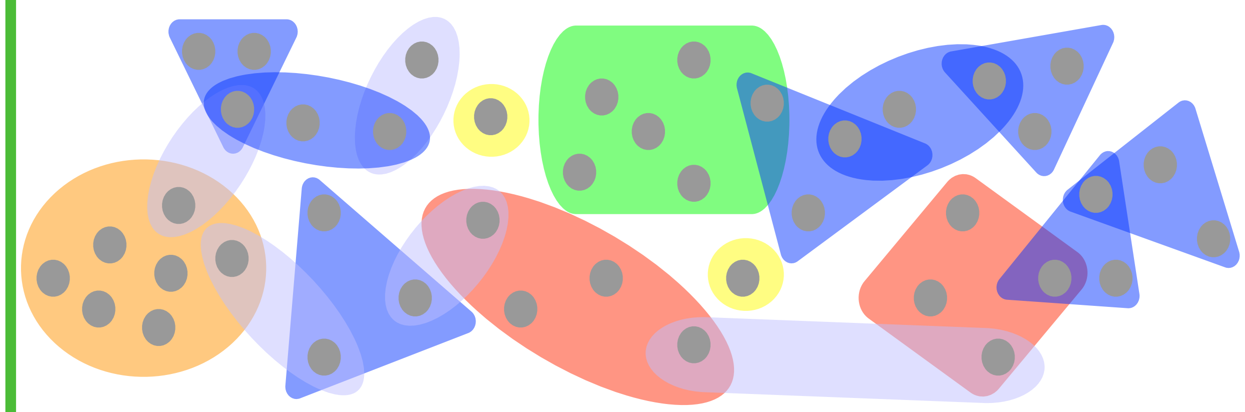
Simplicial complexes and Hypergraphs

Simplicial complexes



d-simplex = d+1 nodes
(all linked together)
0-simplex = node
1-simplex = link
2-simplex = triangle
3-simplex = tetrahedron

Hypergraphs



hyperedge = set of nodes

Global Synchronisation : beyond networks

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PAPER

Dynamical systems on hypergraphs

Timoteo Carletti[1,4], **Duccio Fanelli**[2] **and Sara Nicoletti**[2,3]

[1] naXys, Namur Institute for Complex Systems, University of Namur, rempart de la Vierge, 8 B5000 Namur, Belgium
[2] Università degli Studi di Firenze, Dipartimento di Fisica e Astronomia, CSDC and INFN, via G. Sansone 1, 50019 Sesto Fiorentino, Italy
[3] Dipartimento di Ingegneria dell'Informazione, Università di Firenze, Via S. Marta 3, 50139 Florence, Italy
[4] Author to whom any correspondence should be addressed.

E-mail: timoteo.carletti@unamur.be

Keywords: hypergraphs, master stability function, synchronisation, Turing patterns, dynamical systems

$$\dot{\mathbf{x}}_i = \mathbf{f}(\mathbf{x}_i) + \sigma_1 \sum_{j=1}^{N} a_{ij}^{(1)} \mathbf{g}^{(1)}(\mathbf{x}_i, \mathbf{x}_j) + \sigma_2 \sum_{j=1}^{N} \sum_{k=1}^{N} a_{ijk}^{(2)} \mathbf{g}^{(2)}(\mathbf{x}_i, \mathbf{x}_j, \mathbf{x}_k),$$

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ARTICLE

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https://doi.org/10.1038/s41467-021-21486-9 OPEN

Stability of synchronization in simplicial complexes

L. V. Gambuzza[1,12], F. Di Patti[2,12], L. Gallo[3,4,12], S. Lepri[2], M. Romance[5], R. Criado[5], M. Frasca[1,6,13]✉, V. Latora[3,4,7,8,13]✉ & S. Boccaletti[2,9,10,11,13]✉

Synchronisation non invasive

$$\mathbf{g}^{(d)}(\mathbf{x}, \mathbf{x}, \ldots, \mathbf{x}) \equiv 0 \ \forall d$$

$$\begin{cases} \dot{\eta}_1 = \mathrm{JF}\eta_1 \\ \dot{\eta}_i = (\mathrm{JF} - \sigma_1 \lambda_i \mathrm{JG}^{(1)})\eta_i - \sigma_2 \sum_{j=2}^{N} \tilde{\mathcal{L}}_{ij}^{(2)} \mathrm{JG}^{(2)} \eta_j, \end{cases}$$

i) Assumptions on $\mathbf{g}^{(2)}$

ii) Assumptions on $a_{ijk}^{(2)}$

Global Synchronisation time varying higher-order networks

J. Phys. Complex. **5** (2024) 015020 https://doi.org/10.1088/2632-072X/ad3262

Journal of Physics: Complexity

PAPER

Global synchronization on time-varying higher-order structures

Md Sayeed Anwar[1], Dibakar Ghosh[1] and Timoteo Carletti[2,*]

$$\dot{\vec{x}}_i = \vec{f}(\vec{x}_i) + \sum_{d=1}^{D} q_d \sum_{j_1,\dots,j_d=1}^{n} A^{(d)}_{ij_1\dots j_d}(t)\,\vec{g}^{(d)}(\vec{x}_i, \vec{x}_{j_1}, \dots, \vec{x}_{j_d})$$

Global Topological Synchronisation

PHYSICAL REVIEW LETTERS **130**, 187401 (2023)

Editors' Suggestion

Global Topological Synchronization on Simplicial and Cell Complexes

Timoteo Carletti,[1] Lorenzo Giambagli,[1,2] and Ginestra Bianconi[3,4]
[1]*Department of Mathematics and naXys, Namur Institute for Complex Systems, University of Namur, Rue Grafé 2, B5000 Namur, Belgium*
[2]*Department of Physics and Astronomy, University of Florence, INFN and CSDC, 50019 Sesto Fiorentino, Italy*
[3]*School of Mathematical Sciences, Queen Mary University of London, London, E1 4NS, United Kingdom*
[4]*The Alan Turing Institute, 96 Euston Road, London, NW1 2DB, United Kingdom*

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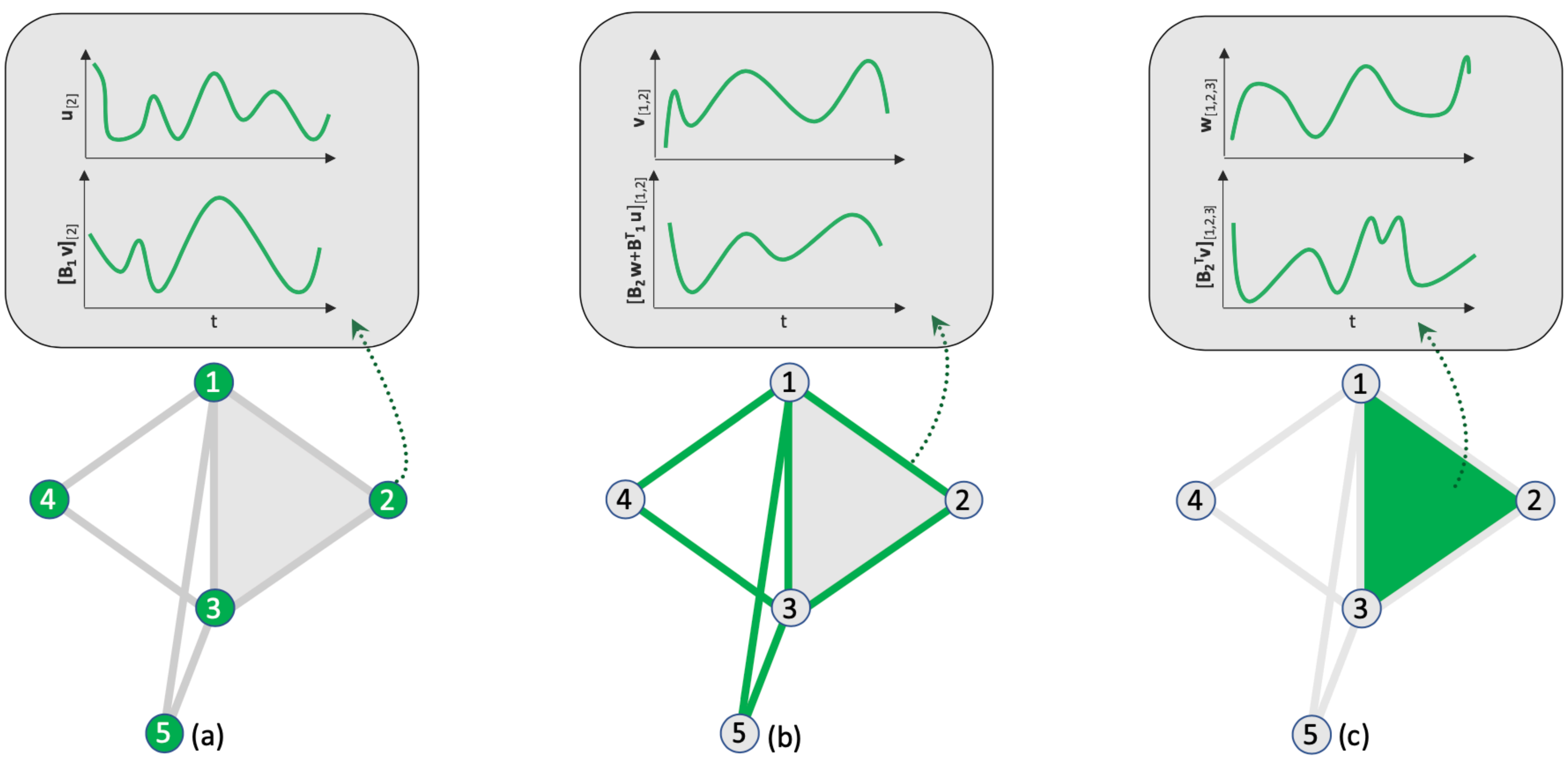
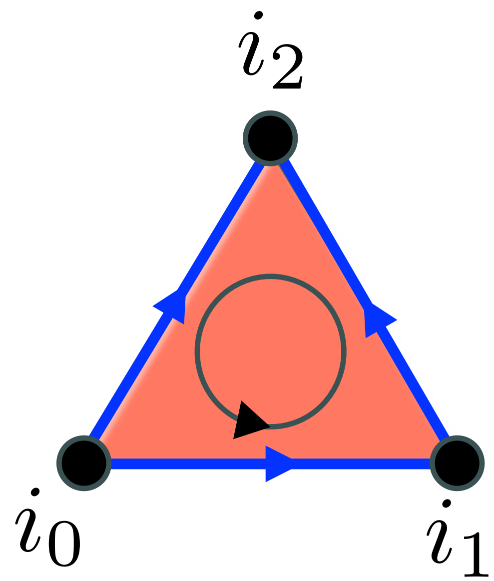


Simplicial complex: an example

$k = 2$ Three nodes, hence a triangle $\sigma^{(2)} = [i_0, i_1, i_2]$



$\sigma_1^{(1)} = [i_0, i_1]$ $\sigma_2^{(1)} = [i_1, i_2]$ $\sigma_3^{(1)} = [i_0, i_2]$

Incidence matrices

$$\mathbf{B}_1(\sigma_i^{(0)}, \sigma_j^{(1)}) = \begin{array}{c} \\ i_0 \\ i_1 \\ i_2 \end{array} \begin{array}{c} \begin{array}{ccc} [i_0, i_1] & [i_1, i_2] & [i_0, i_2] \end{array} \\ \begin{pmatrix} -1 & 0 & -1 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \end{array}$$

$$\mathbf{B}_1 \in M^{N_0 \times N_1}$$

$$\mathbf{B}_2 \in M^{N_1 \times N_2}$$

$$\mathbf{B}_2(\sigma_i^{(1)}, \sigma_j^{(2)}) = \begin{array}{c} \\ [i_0, i_1] \\ [i_1, i_2] \\ [i_0, i_2] \end{array} \begin{array}{c} [i_0, i_1, i_2] \\ \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \end{array}$$

timoteo.carletti@unamur.be

Simplicial complexes

$$\sigma_i^{(k)} = [i_0, \dots, i_k]$$

$$\mathbf{B}_k \in M^{N_{k-1} \times N_k}$$

Incidence matrix

$$B_k(\sigma_i^{(k-1)}, \sigma_j^{(k)}) = 1 \text{ if } \sigma_i^{(k-1)} \sim \sigma_j^{(k)}$$

$$B_k(\sigma_i^{(k-1)}, \sigma_j^{(k)}) = -1 \text{ if } \sigma_i^{(k-1)} \not\sim \sigma_j^{(k)}$$

$$B_k(\sigma_i^{(k-1)}, \sigma_j^{(k)}) = 0 \text{ otherwise}$$

$$\mathbf{L}_k = \mathbf{B}_k^\top \mathbf{B}_k + \mathbf{B}_{k+1} \mathbf{B}_{k+1}^\top$$

Hodge Laplace matrix

Dynamics on simplicial complexes

$$\sigma_i^{(k)} = [i_0, \dots, i_k]$$

$$\mathbf{x} : C_k \to \mathbb{R}^d$$ k-cochain

$$\mathbf{x}(-\sigma_i^{(k)}) = -\mathbf{x}(\sigma_i^{(k)})$$

$$\mathbf{x}_i = \mathbf{x}(\sigma_i^{(k)}) = (x_i^1, \dots, x_i^d)$$

Dynamical system on a simplex

$$\frac{d\mathbf{x}_i}{dt} = \mathbf{f}(\mathbf{x}_i)$$

$$\mathbf{f}(-\mathbf{x}_i) = -\mathbf{f}(\mathbf{x}_i)$$

Global Topological Synchronisation

Dynamical system on a simplicial complex

$$\frac{d\mathbf{x}_i}{dt} = \mathbf{f}(\mathbf{x}_i) - \sigma \sum_{j=1}^{N_k} L_k(i,j)\mathbf{h}(\mathbf{x}_j) \qquad \mathbf{h}(-\mathbf{x}_i) = -\mathbf{h}(\mathbf{x}_i)$$

Reference orbit $\mathbf{s}(t)$ solution of $\frac{d\mathbf{s}}{dt} = \mathbf{f}(\mathbf{s})$

Synchronisation : $\mathbf{x}_i(t) = \mathbf{s}(t) \quad \forall i = 1, \dots, N_k$

$$\left.\frac{d\mathbf{x}_i}{dt}\right|_{\mathbf{x}_i=\mathbf{s}} = \mathbf{f}(\mathbf{x}_i) + \sigma \sum_{j=1}^{n} L_{ij}\mathbf{h}(\mathbf{x}_j)\Big|_{\mathbf{x}_i=\mathbf{s}} \overset{?}{\neq} 0$$

Global Topological Synchronisation

Necessary condition $\mathbf{L}_k u = 0 \quad \Leftrightarrow \quad \mathbf{B}_k u = 0$ and $\mathbf{B}_{k+1}^\top u = 0$

$$\sum_j B_k(\sigma_i^{(k-1)}, \sigma_j^{(k)}) = 0 \Rightarrow \sigma_i^{(k-1)}$$ even numb incident $\sigma_j^{(k)}$

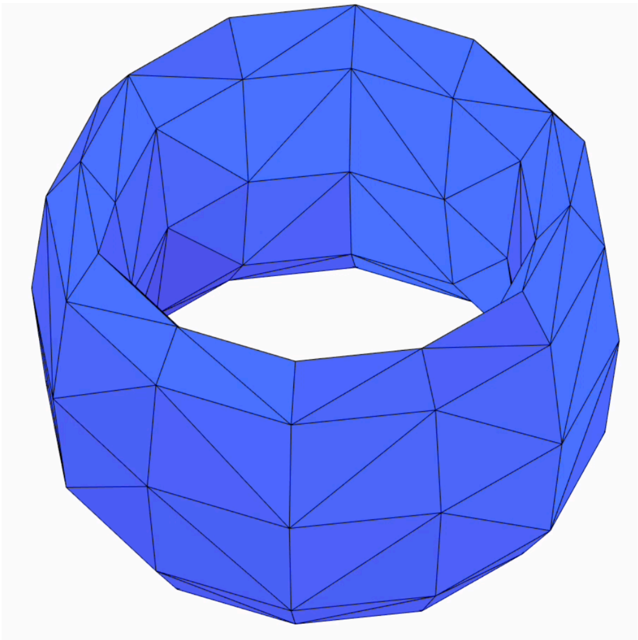
half coherent / half incoherent

Balanced simplicial complex

$$\sum_i B_{k+1}(\sigma_i^{(k)}, \sigma_j^{(k+1)}) = 0 \Rightarrow \sigma_j^{(k+1)}$$ contains even numb $\sigma_i^{(k)}$

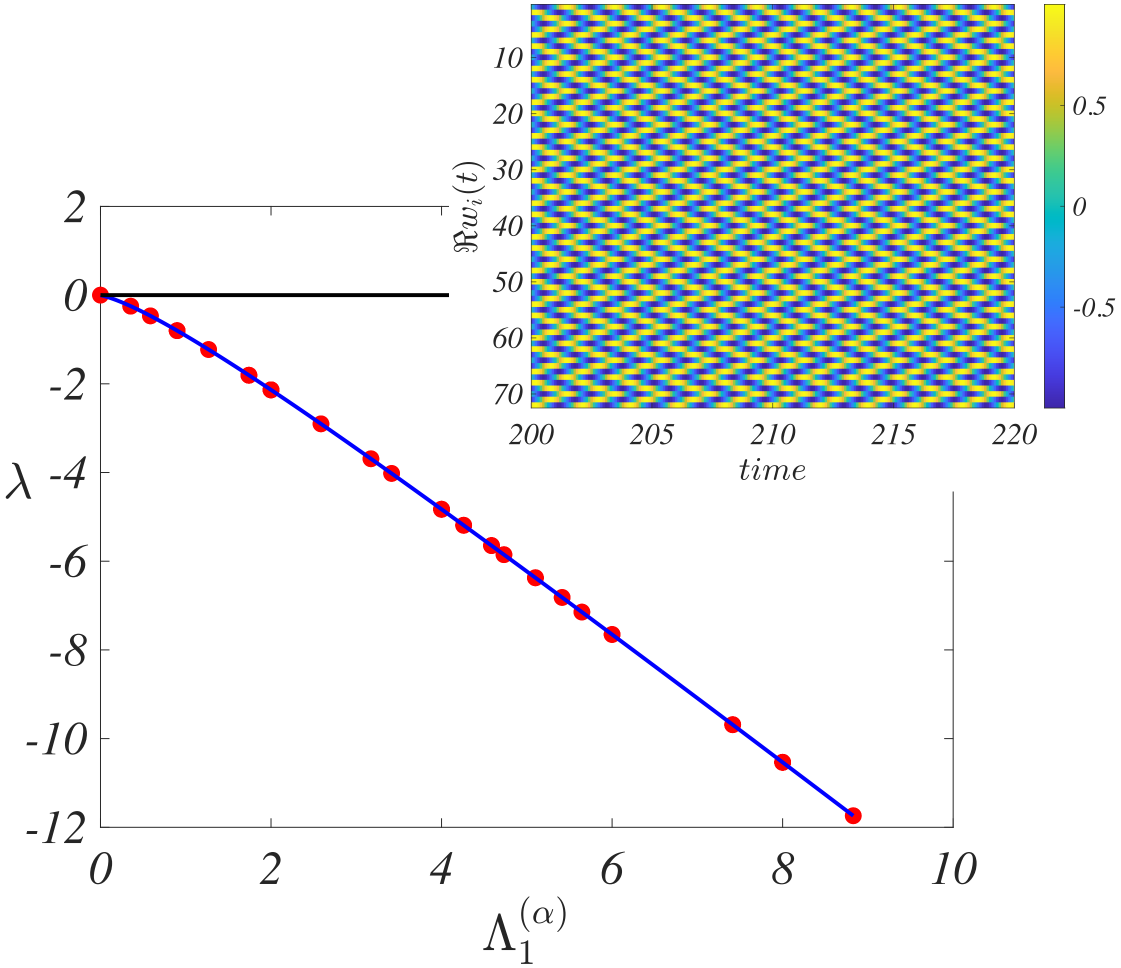
impossible if k is odd

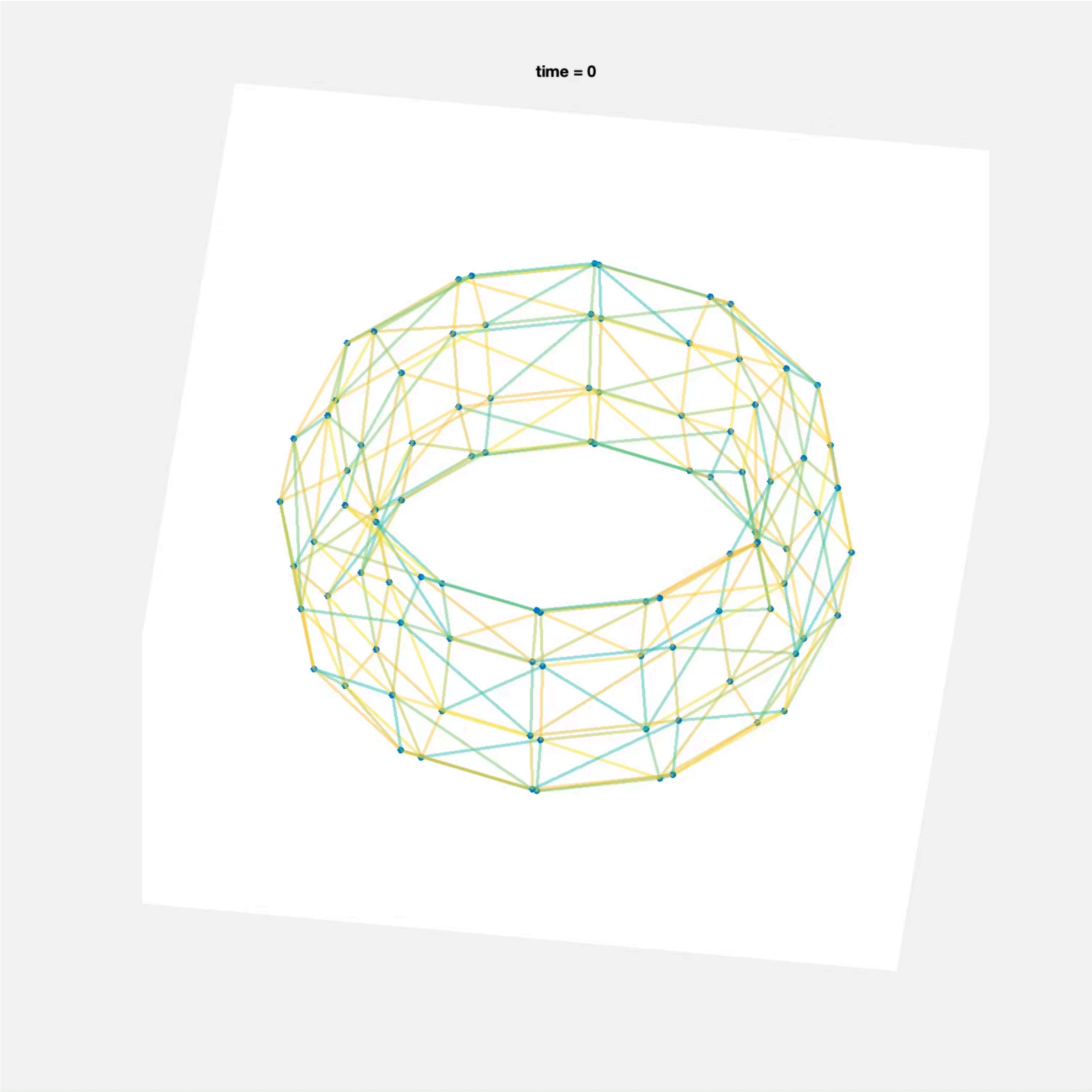
Global Topological Synchronisation : Stuart-Landau on 2D-triangulated torus



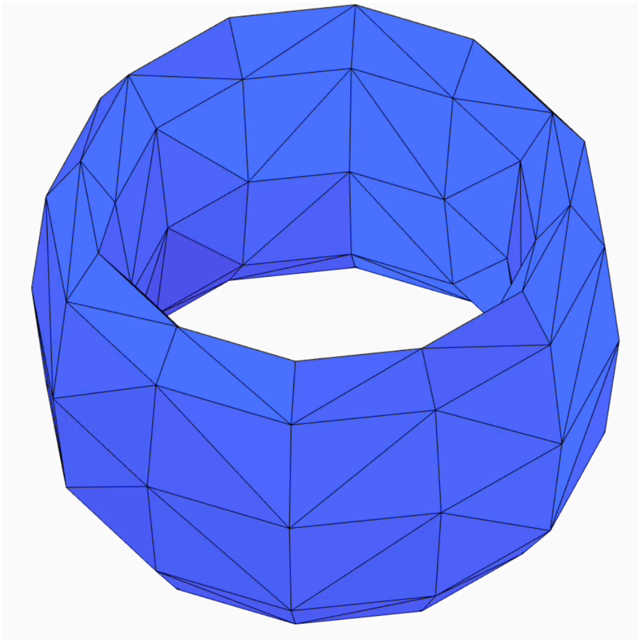
no global synch
for links (k=1)

$$\mathbf{B}_1(1,\dots,1)^\top = 0$$
$$\mathbf{B}_2^\top(1,\dots,1)^\top \neq 0$$





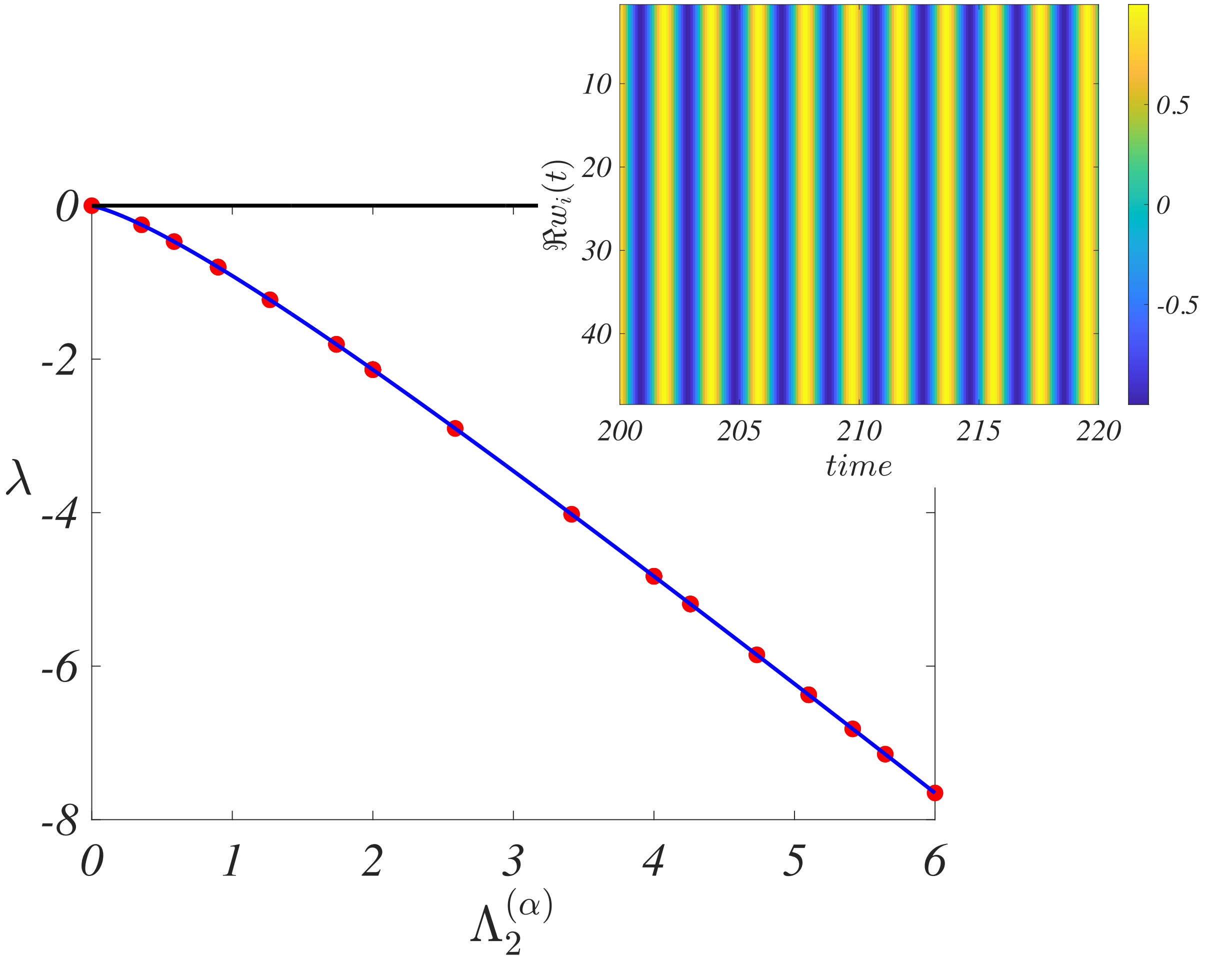
Global Topological Synchronisation : Stuart-Landau on 2D-triangulated torus

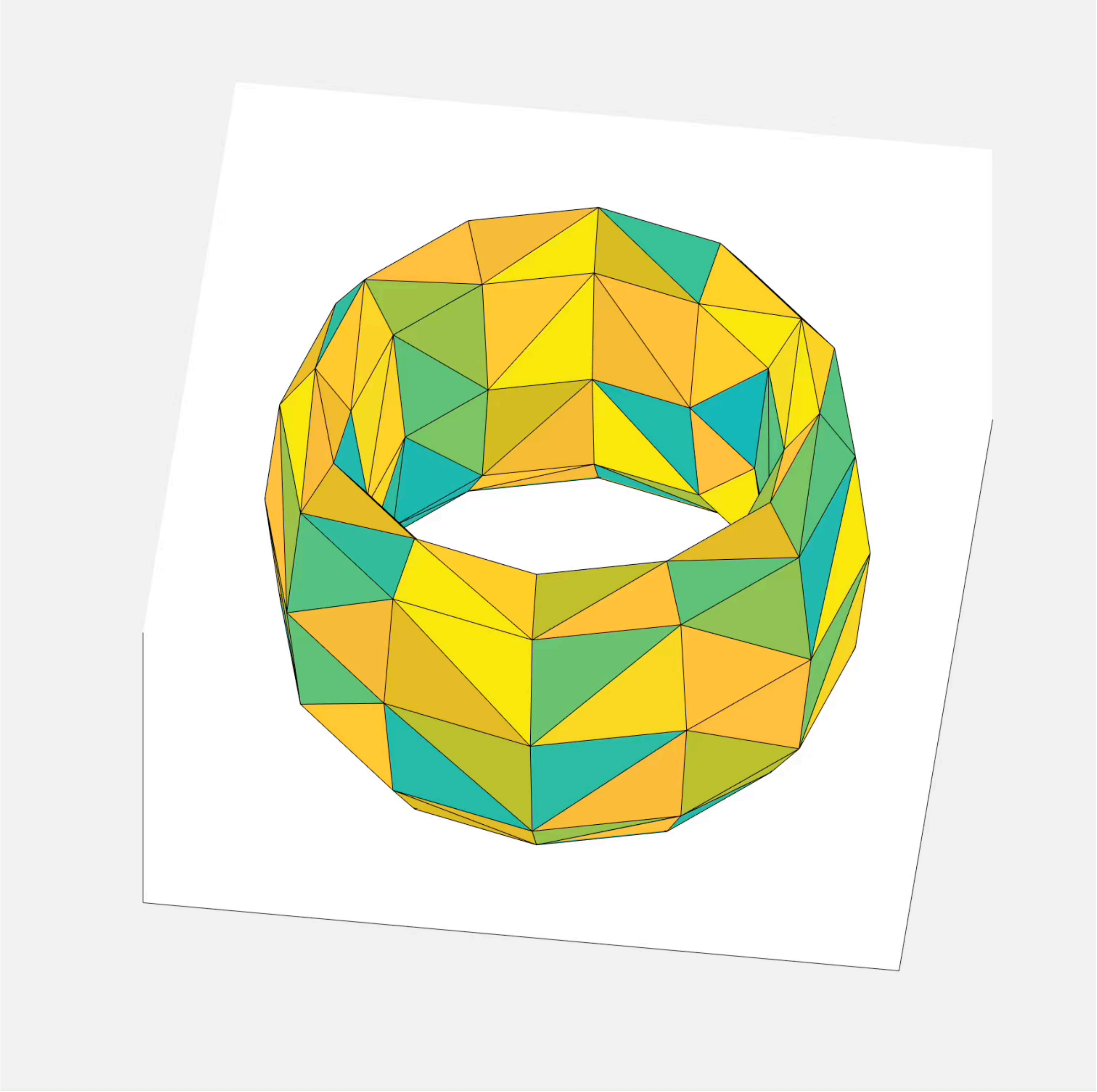


global synch
for faces (k=2)

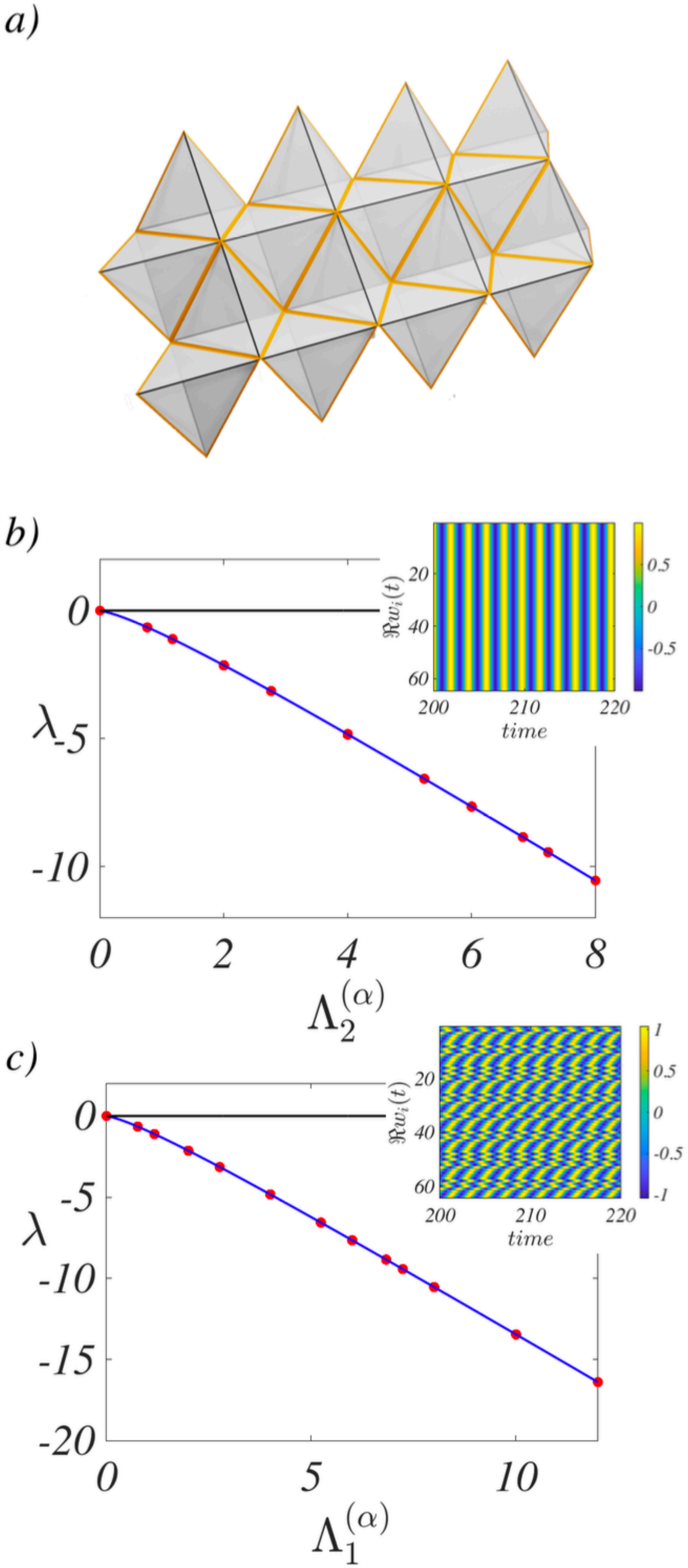
$$\mathbf{B}_2(1,\dots,1)^\top = 0$$

$$\mathbf{B}_3^\top(1,\dots,1)^\top = 0$$





Global Topological Synchronisation : Stuart-Landau on the waffle



global synch
for faces (k=2)

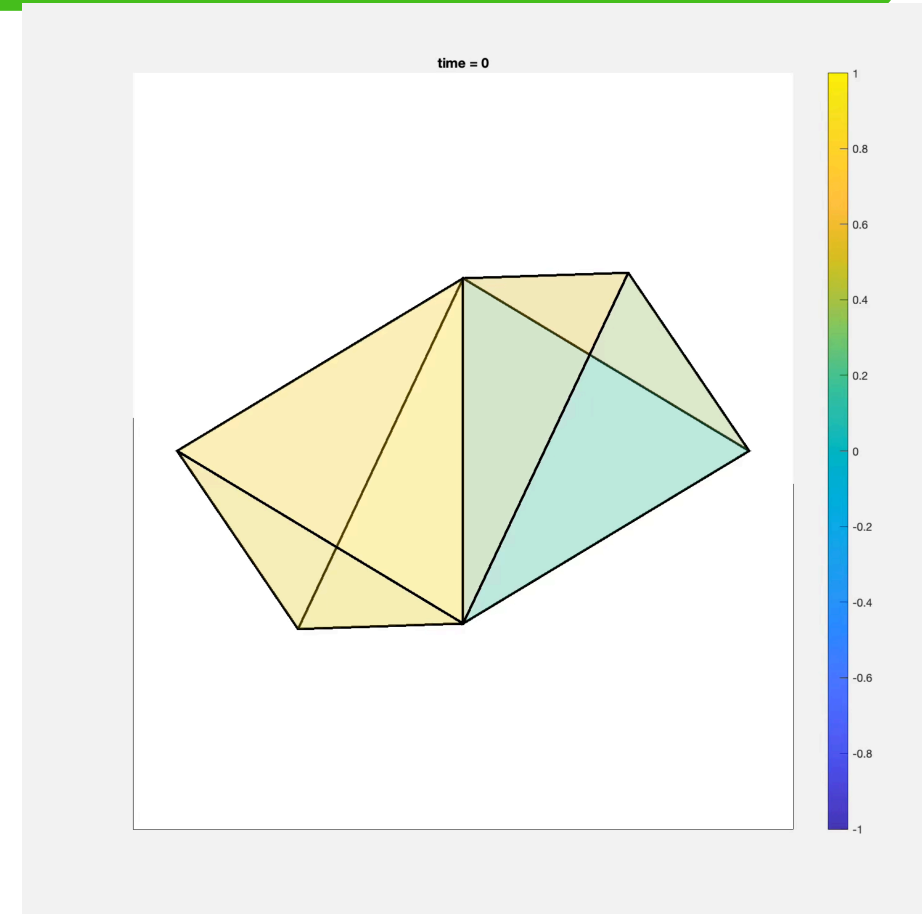
$$\mathbf{B}_2(1,\ldots,1)^\top = 0$$

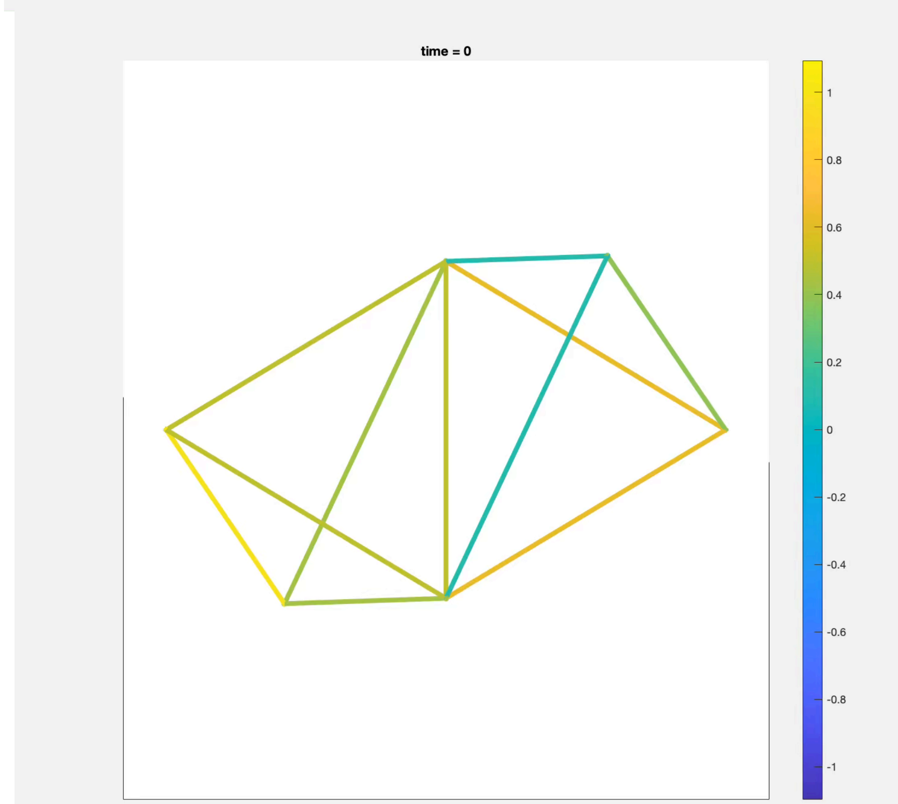
$$\mathbf{B}_3^\top(1,\ldots,1)^\top = 0$$

no global synch
for links (k=1)

$$\mathbf{B}_1(1,\ldots,1)^\top = 0$$

$$\mathbf{B}_2^\top(1,\ldots,1)^\top \neq 0$$

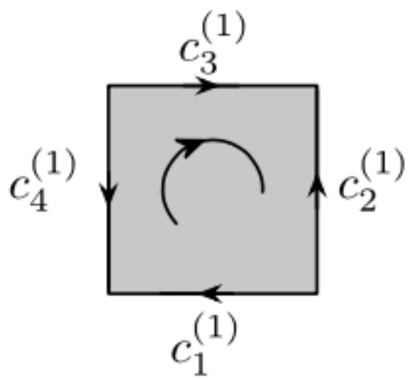




Global Topological Synchronisation on cell complexes

The topological obstruction
<u>does not exist</u> for cell complexes

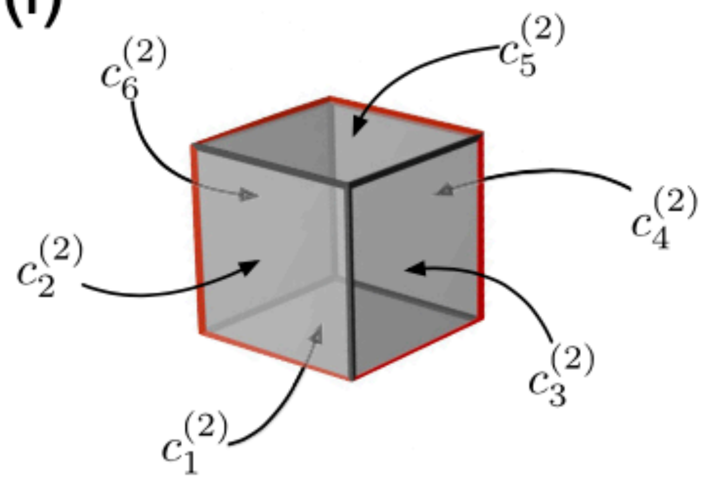
(e)



$$\mathbf{B}_2 = \begin{pmatrix} -1 \\ -1 \\ 1 \\ 1 \end{pmatrix}$$

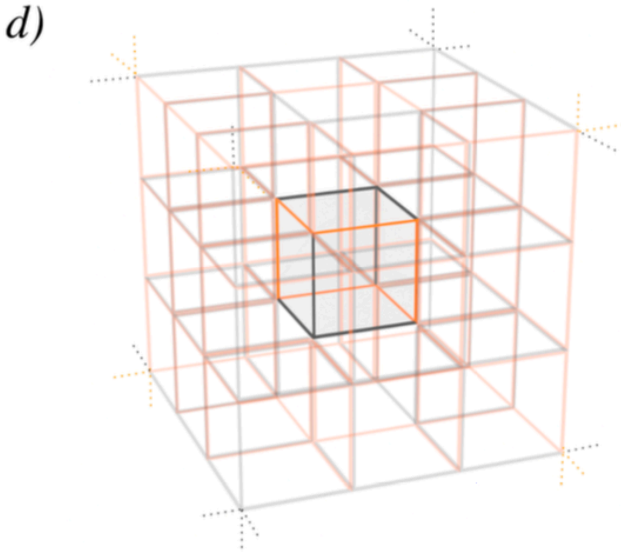
$$(1,1,1,1)\mathbf{B}_2 = 0$$

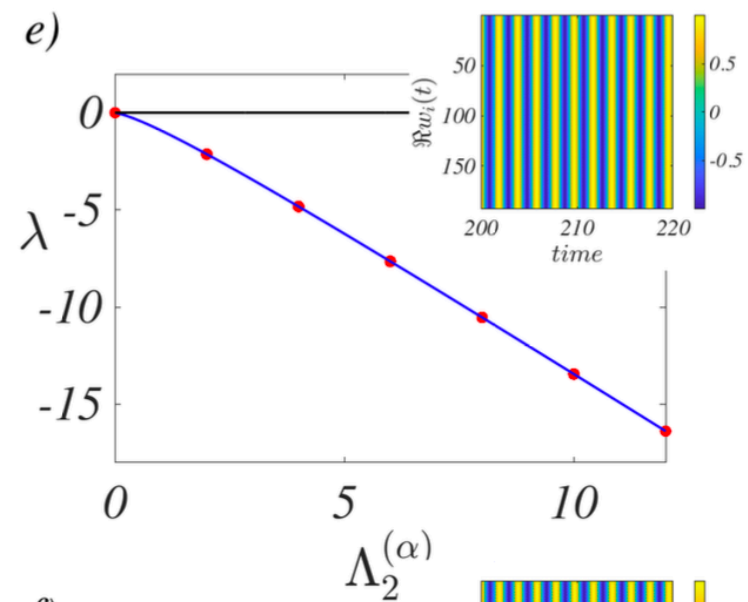
(f)



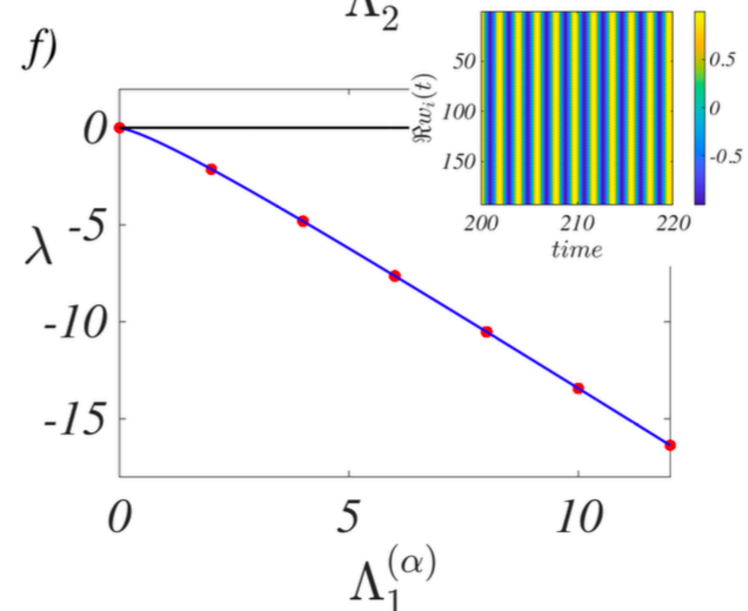
$$\mathbf{B}_3 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$$

$$(1,1,1,1,1,1)\mathbf{B}_3 = 0$$



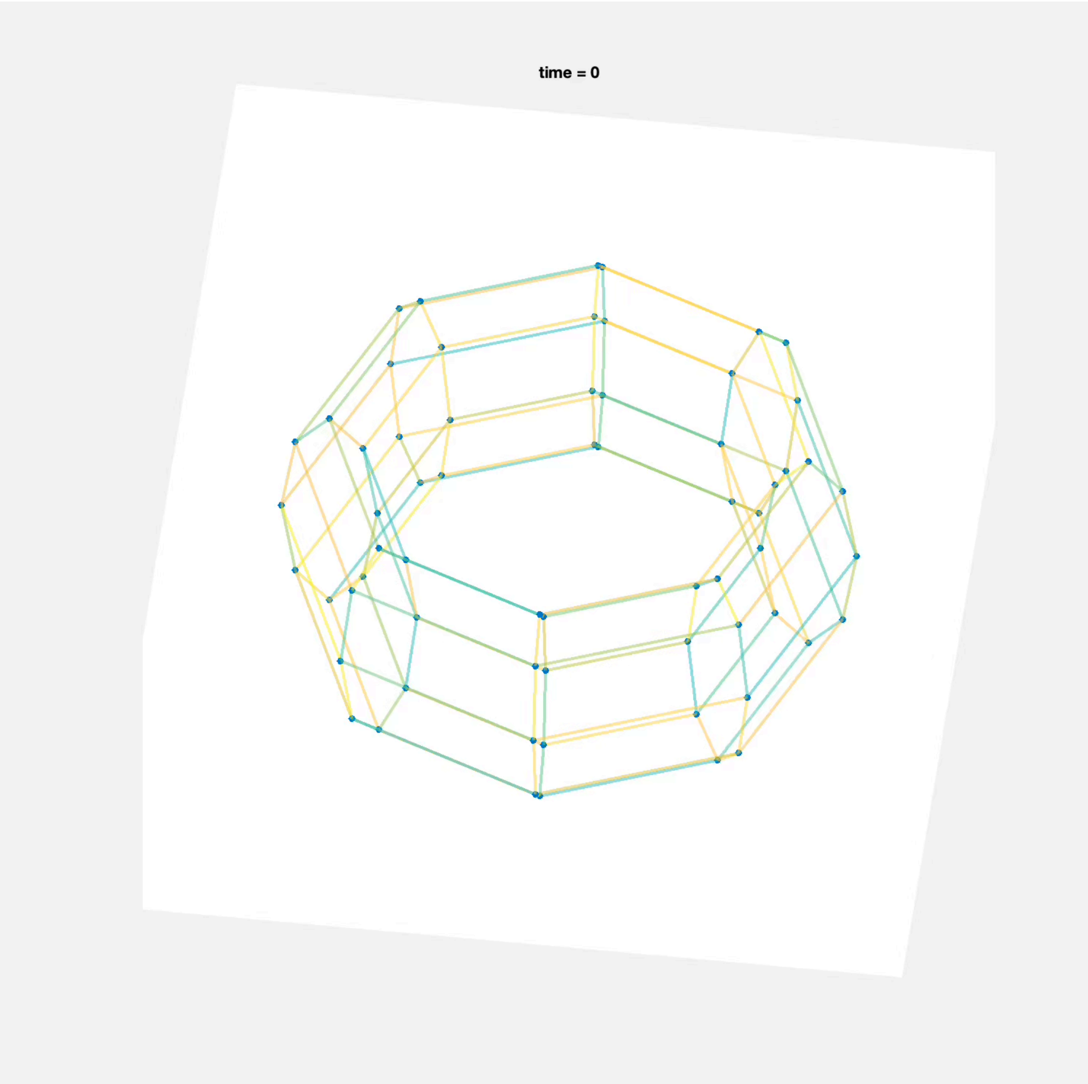


global synch for faces

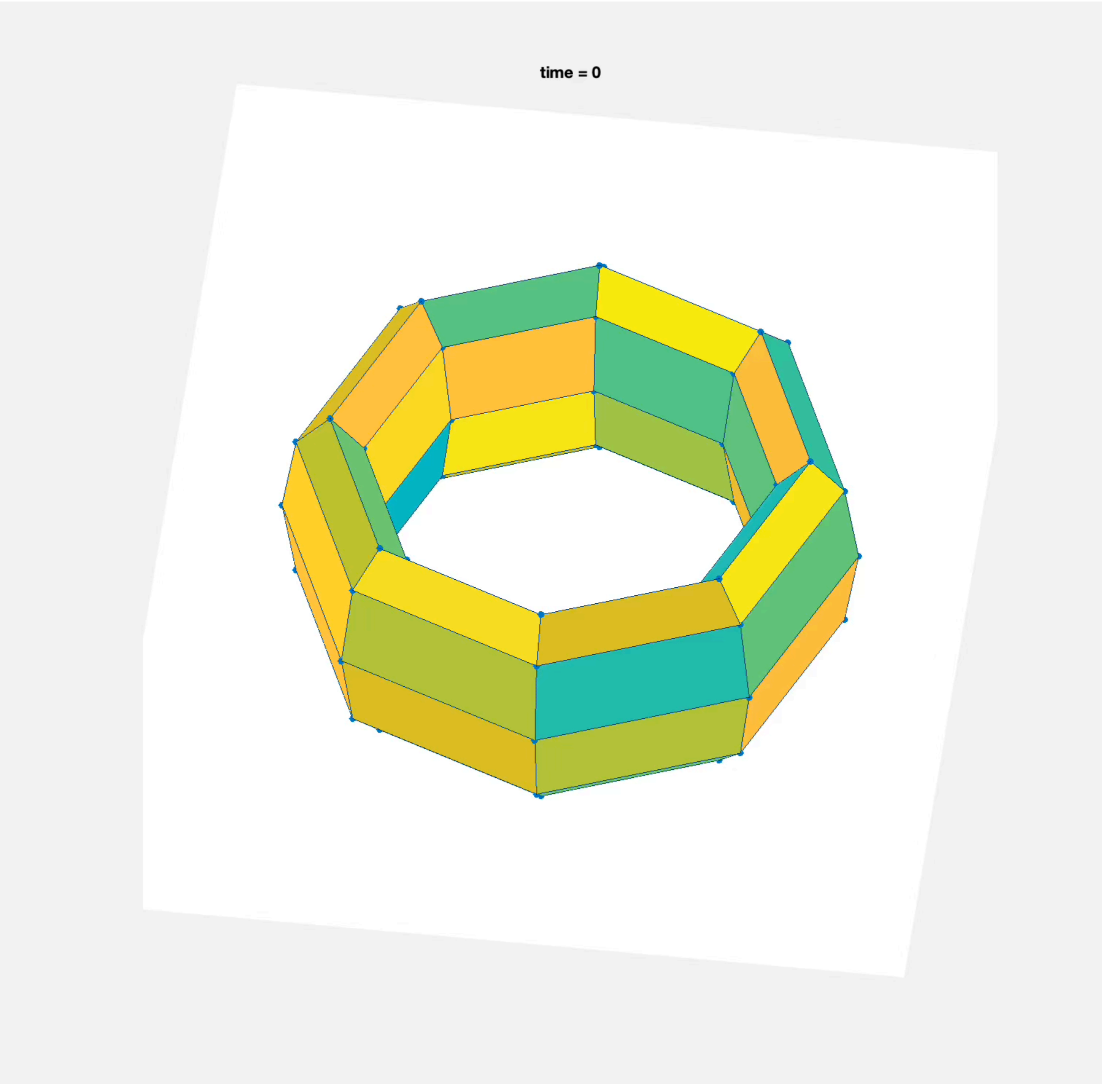


global synch for links

Global Topological Synchronisation on cell complexes



global synch
for links (k=1)



global synch
for faces (k=2)

Timoteo Carletti

Global synchronization on networks and beyond

Thank you

Any questions?

naxys
Namur Institute for Complex Systems

UNIVERSITÉ DE NAMUR