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Asymptotic stability of infinite-dimensional semilinear systems: Application to a nonisothermal reactor

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Abstract

The concept of asymptotic stability is studied for a class of infinite-dimensional semilinear Banach state space (distributed parameter) systems. Asymptotic stability criteria are established, which are based on the concept of strictly m-dissipative operator. These theoretical results are applied to a nonisothermal plug flow tubular reactor model, which is described by semilinear partial differential equations, derived from mass and energy balances. In particular it is shown that, under suitable conditions on the model parameters, some equilibrium profiles are asymptotically stable equilibriums of such model.

Keywords: Nonlinear contraction semigroup; Asymptotic stability; Strict m-dissipativity; Nonlinear infinite dimensional systems; Partial differential equation; Nonisothermal plug flow reactor; Tubular chemical reactor

1. Introduction

Stability is one of the most important aspects of system theory. The fundamental theory of stability, established by Lyapunov, is extensively developed for finite-dimensional systems. Many results on the asymptotic behavior of nonlinear infinite-dimensional systems are known, for which the dissipativity property plays an important role, see e.g. [10,12,6,18,17]. Here we are interested in the asymptotic stability of a class of infinite-dimensional semi-linear systems which contains tubular reactor models.

Tubular reactors cover a large class of processes in chemical and biochemical engineering (see e.g. [13]). They are typically reactors in which the medium is not homogeneous (like fixed-bed reactors, packed-bed reactors, fluidized-bed reactors,…) and possibly involve different phases (liquid/solid/gas). Such systems are sometimes called “Diffusion–convection–reaction” systems since their dynamical model typically includes these three terms. In particular the dynamics of nonisothermal axial dispersion or plug flow tubular reactors are described by semilinear partial differential equations (PDEs) derived from mass and energy balances.

The objective of this paper is basically twofold. The first goal is to develop a theory concerning the asymptotic stability of semilinear distributed parameter systems. The second goal is to apply this theory to a nonisothermal plug flow reactor model.

More specifically, this paper is dedicated to the asymptotic stability of a class of semilinear infinite-dimensional Banach state space (distributed parameter) systems. This theory is applied to a nonisothermal plug flow tubular reactor model, which is described by semilinear partial differential equations containing nonlinear terms which stem from the Arrhenius law. In particular it is shown that, under suitable conditions on the system parameters, some equilibrium profiles are asymptotically stable equilibriums of such model. Roughly speaking the main stability condition is that a Lipschitz constant of the nonlinear part of the model is sufficiently small with respect to the decay rate of the (exponentially stable) semigroup generated by the linear part of the model. This stability condition is illustrated by means of some numerical experiments.

In order to make this paper as readable as possible, in particular for non-experts in infinite-dimensional systems, we have

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written it with sufficient details especially concerning the basic concepts and results of infinite-dimensional system theory.

The paper is organized as follows. Section 2 contains some properties concerning nonlinear contraction semigroup theory. Notably an asymptotic stability criterion in Banach space is stated, which is based on the concept of strictly m-dissipative operator. In Section 3, asymptotic stability criteria are established for a class of semilinear infinite-dimensional systems by applying the result stated in the previous section. Section 4 deals with an infinite-dimensional state space description of a nonisothermal plug flow reactor model. Section 5 is devoted to the asymptotic stability analysis of this model.

2. Asymptotic stability in Banach space

In order to study the asymptotic stability of semilinear infinite-dimensional systems, some preliminary concepts and results are needed, that are related to the theory of nonlinear abstract differential equations on Banach spaces. More specifically this section is devoted to some properties of nonlinear contraction semigroup theory [6,16,19,5,17] and to a paramount known result related to LaSalle's invariance principle: see Theorem 5. In addition the concept of *strict* m-dissipativity is introduced, which leads to the stability Theorem 6.

Let us consider a real reflexive Banach space X equipped with the norm $\|\cdot\|$. Let D be a nonempty closed subset of X.

Definition 1. A (*strongly continuous*) *nonlinear contraction semigroup on* D is a family of operators $\Gamma(t): D \to D$, $t \geqslant 0$, satisfying:

(i) $\Gamma(t+s) = \Gamma(t)\Gamma(s)$ for every $s, t \geqslant 0$; $\Gamma(0) = I$.
(ii) $\|\Gamma(t)x - \Gamma(t)x'\| \leqslant \|x - x'\|$ for every $t \geqslant 0$ and every $x,\ x' \in D$.
(iii) For every $x \in D$, $\Gamma(t)x \to x$ as $t \to 0^+$.

The *infinitesimal generator* $\mathscr{A}_\Gamma$ of the nonlinear contraction semigroup $\Gamma(t)$ (on D) is defined on its domain

$$D(\mathscr{A}_\Gamma) = \left\{ x \in D : \lim_{t\to 0+} t^{-1}[\Gamma(t)x - x] \text{ exists} \right\}$$

by

$$\mathscr{A}_\Gamma x = \lim_{t\to 0+} t^{-1}[\Gamma(t)x - x], \quad \text{for every } x \in D(\mathscr{A}_\Gamma).$$

Definition 2. Let $\mathscr{A}$ be an operator with domain $D(\mathscr{A})$.

(i) $\mathscr{A}$ is said to be *dissipative* if, for all $x,\ x' \in D(\mathscr{A})$ and for all $\lambda > 0$,

$$\|x - x'\| \leqslant \|(x - x') - \lambda(\mathscr{A}x - \mathscr{A}x')\|,$$

or equivalently, for all $x,\ x' \in D(\mathscr{A})$, there exists a bounded linear functional f on X such that $f(x - x') = \|x - x'\|^2 = \|f\|^2$ and $f(\mathscr{A}x - \mathscr{A}x') \leqslant 0$.

In addition $\mathscr{A}$ is said to be *strictly dissipative* if the conditions above hold with strict inequalities, for all $x,\ x' \in D(\mathscr{A})$ such that $x \neq x'$.

(ii) $\mathscr{A}$ is said to be (*strictly*) *m-dissipative* if it is (strictly) dissipative and $\mathscr{R}(I - \mathscr{A}) = X$, where $\mathscr{R}(T)$ denotes the range of an operator T.

Comment 3. (a) By [17, p. 99], if $\mathscr{A}$ is a dissipative operator, then for all $\lambda > 0$, $(I - \lambda\mathscr{A})^{-1}$ is well-defined and non-expansive on $\mathscr{R}(I - \lambda\mathscr{A})$, i.e. for all $y,\ y' \in \mathscr{R}(I - \lambda\mathscr{A})$,

$$\|(I - \lambda\mathscr{A})^{-1}y - (I - \lambda\mathscr{A})^{-1}y'\| \leqslant \|y - y'\|.$$

(b) By [17, Proposition 2.109, p. 100; Corollary 2.120, pp. 106–107], if $\mathscr{A}$ is a m-dissipative operator, then $\mathscr{A}$ is the generator of a unique nonlinear contraction semigroup $\Gamma(t)$ on $D := \overline{D(\mathscr{A})}$. Hence for any $x_0 \in D$, the corresponding state trajectory $t \mapsto x(t, x_0) := \Gamma(t)x_0 \in D$ exists on $(0, \infty)$ and is differentiable almost everywhere whenever $x_0 \in D(\mathscr{A})$ such that $(\mathrm{d}/\mathrm{d}t)\Gamma(t)x_0 = \mathscr{A}\Gamma(t)x_0$ for almost all t in $(0, \infty)$ (see e.g. [17, Proposition. 2.98(iii), p. 93]). In other words, for any $x_0 \in D(\mathscr{A})$, $x(t, x_0) := \Gamma(t)x_0$ is the unique solution of the following nonlinear abstract Cauchy problem:

$$\begin{cases} \dot{x}(t) = \mathscr{A}x(t), & t > 0, \\ x(0) = x_0. \end{cases} \tag{1}$$

(c) Assuming that $\mathscr{A}$ is dissipative, the conclusions of (b) follow also from a weaker condition than the m-dissipativity one (see [17, Corollary 2.120]). This condition is given as follows:

$$\overline{conv(D(\mathscr{A}))} \subset \bigcap_{\lambda > 0} \mathscr{R}(I - \lambda\mathscr{A}), \tag{2}$$

where $\overline{conv(S)}$ denotes the closure of the convex hull of the set S.

Definition 4. Consider the system (1) and assume that $\mathscr{A}$ generates a nonlinear contraction semigroup $\Gamma(t)$. Consider an equilibrium point $\overline{x}$ of (1), i.e. $\overline{x} \in D(\mathscr{A})$ and $\mathscr{A}\overline{x} = 0$. $\overline{x}$ is an *asymptotically stable equilibrium point* of (1) on D if

$$\forall x_0 \in D, \quad \lim_{t\to\infty} x(t, x_0) := \lim_{t\to\infty} \Gamma(t)x_0 = \overline{x}.$$

In what follows we also need the following important result (see Theorem 5), which is strongly related to the well-known Lasalle's invariance principle, see e.g. [17,12]. In order to state this result, the following concepts and notations are needed. If $\Gamma(t)$ is a nonlinear semigroup of contractions on D, for any $x_0 \in D$, the *orbit* $\gamma(x_0)$ *through* x_0 is defined by

$$\gamma(x_0) := \{\Gamma(t)x_0 : t \geqslant 0\},$$

and the ω-*limit set* $\omega(x_0)$ of x_0 is the set of all states in D which are the limits at infinity of converging sequences in the orbit through x_0, i.e. more specifically $x \in \omega(x_0)$ if and only if $x \in D$ and there exists a sequence $t_n \to \infty$ such that

$$x = \lim_{n\to\infty} \Gamma(t_n)x_0.$$

Observe that $\omega(x_0)$ is $\Gamma(t)$-invariant, i.e. for all $t \geqslant 0$, $\Gamma(t)\omega(x_0) \subset \omega(x_0)$. Recall that the distance between a point

$x \in X$ and a subset $\Omega \subset X$ is given by

$$d(x, \Omega) := \inf\{\|x - y\| : y \in \Omega\}.$$

Theorem 5 below is a slightly modified version of [12, Theorem 3]. The present version can be deduced from [17, Proposition 3.60, Theorems 3.61 and 3.65].

Theorem 5. *Consider the system*

$$\begin{cases} \dot{x}(t) = \mathscr{A}_\Gamma x(t), & t > 0, \\ x(0) = x_0, \end{cases} \tag{3}$$

where $\mathscr{A}_\Gamma$ is a dissipative operator such that (2) *holds, and let $\Gamma(t)$ be the nonlinear contraction semigroup on $D = \overline{D(\mathscr{A}_\Gamma)}$, generated by $\mathscr{A}_\Gamma$. Assume that $\overline{x}$ is an equilibrium point of* (3) *and $(I - \lambda\mathscr{A}_\Gamma)^{-1}$ is compact for some $\lambda > 0$. Then for any $x_0 \in D$, $x(t, x_0) := \Gamma(t)x_0$ converges, as $t \to \infty$, to $\omega(x_0) \subset \{z : \|z - \overline{x}\| = r\}$, where $r \leqslant \|x_0 - \overline{x}\|$, i.e.*

$$\lim_{t\to\infty} d(x(t, x_0), \omega(x_0)) = 0.$$

It turns out that, under an additional condition, viz. the strict dissipativity of the state operator $\mathscr{A}_\Gamma$, the ω-limit set reduces to the point $\overline{x}$, obviously leading to the asymptotic stability of the latter.

Theorem 6. *Let us consider the system* (3) *under the assumptions of Theorem* 5. *If in addition $\mathscr{A}_\Gamma$ is strictly dissipative, then $x(t, x_0) \to \overline{x}$ as $t \to \infty$ i.e. $\overline{x}$ is an asymptotically stable equilibrium point of* (3) *on D.*

Proof. Without loss of generality assume that $\overline{x} = 0$. Consider any $x_0 \in D(\mathscr{A}_\Gamma)$. By the first paragraph of the proof of [12, Theorem 5, pp. 100, 104], $\omega(x_0) \subset D(\mathscr{A}_\Gamma)$. Let us prove that $\omega(x_0) = \{0\}$. Consider any $x \in \omega(x_0)$ and recall that $\omega(x_0)$ is a $\Gamma(t)$-invariant subset of the sphere $\{z : \|z\| = r\}$. Hence, for all $t \geqslant 0$, $\Gamma(t)x \in \omega(x_0) \subset D(\mathscr{A}_\Gamma)$ and $\|x\| = \|\Gamma(t)x\| = r$. Observe that, in view of the inclusion

$$\omega(x_0) \subset \{z : \|z\| = r\},$$

it is enough to prove that $x = 0$, i.e. that the sphere $\{z : \|z\| = r\}$ reduces to the origin. In order to get a contradiction, assume that $x \neq 0$. Then,

$$\text{for almost all } t > 0, \quad \frac{\mathrm{d}}{\mathrm{d}t}\Gamma(t)x = \mathscr{A}_\Gamma\Gamma(t)x. \tag{4}$$

Moreover, for every $h < 0$ and every bounded linear functional f such that

$$f(\Gamma(t)x) = \|\Gamma(t)x\|^2 = \|f\|^2 = r^2,$$

there holds

$$\begin{aligned} f(\Gamma(t+h)x - \Gamma(t)x) &\leqslant f(\Gamma(t+h)x) - f(\Gamma(t)x) \\ &\leqslant \|f\|\|\Gamma(t+h)x\| - f(\Gamma(t)x) \\ &\leqslant r\|\Gamma(t+h)x\| - r\|\Gamma(t)x\| \\ &\leqslant r\ (\|\Gamma(t+h)x\| - \|\Gamma(t)x\|). \end{aligned}$$

Therefore, one has

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\|\Gamma(t)x\|^2 &= \lim_{h\to 0^-} \frac{\|\Gamma(t+h)x\|^2 - \|\Gamma(t)x\|^2}{h} \\ &= \lim_{h\to 0^-} \frac{\|\Gamma(t+h)x\| - \|\Gamma(t)x\|}{h} \\ &\quad \cdot (\|\Gamma(t+h)x\| + \|\Gamma(t)x\|) \\ &\leqslant \lim_{h\to 0^-} \frac{f(\Gamma(t+h)x - \Gamma(t)x)}{rh} \cdot 2r \\ &\leqslant 2f(\mathscr{A}\Gamma(t)x). \end{aligned}$$

Using the strict dissipativity of the operator $\mathscr{A}_\Gamma$, it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\Gamma(t)x\|^2 \leqslant 2f(\mathscr{A}_\Gamma\Gamma(t)x) < 0 \quad \text{a.e. in } (0, \infty). \tag{5}$$

Therefore, for all $t \geqslant 0$, $\|\Gamma(t)x\| < \|x\|$. This contradicts the fact that $\|\Gamma(t)x\| = \|x\| = r$. Hence

$$\text{for all } x_0 \in D(\mathscr{A}_\Gamma), \quad \omega(x_0) = \{0\}. \tag{6}$$

Now let us consider any $x_0 \in D$ (not necessarily in $D(\mathscr{A}_\Gamma)$). Let $\varepsilon > 0$ be arbitrarily fixed. By the density of $D(\mathscr{A}_\Gamma)$ in D, there exists $y_0 \in D(\mathscr{A}_\Gamma)$ such that $\|x_0 - y_0\| < \varepsilon/2$. It follows, by the fact that $\Gamma(t)$ is a contraction semigroup, that,

$$\text{for all } t \geqslant 0, \quad \|x(t, x_0) - x(t, y_0)\| < \frac{\varepsilon}{2}. \tag{7}$$

Since $y_0 \in D(\mathscr{A}_\Gamma)$, it follows from (6) that $\omega(y_0) = \{0\}$, whence there exists $T > 0$ such that,

$$\text{for all } t > T, \quad \|x(t, y_0)\| < \frac{\varepsilon}{2}. \tag{8}$$

It follows from (7)–(8) that, for all $t > T$,

$$\|x(t, x_0)\| \leqslant \|x(t, x_0) - x(t, y_0)\| + \|x(t, y_0)\| < \varepsilon.$$

Consequently, 0 is an asymptotically stable equilibrium point of (3) on D. □

Comment 7. In view of its proof, especially inequality (5), Theorem 6 can also be proved by using LaSalle's invariance principle (see e.g. [17, Theorem 3.64, p. 161]) instead of Theorem 5. Indeed, the function $\mathscr{V}(x) := \|x\|^2$ can be shown to be a Lyapunov function such that the set $\{x : \dot{\mathscr{V}}(x) = 0\}$ reduces to 0.

3. Asymptotic stability of semilinear systems

Numerous research works are concerned with semilinear evolution equations in Banach spaces and application to many classes of partial differential equations: see e.g. [18,20,16,21,19,5,1] and many other references. The idea in this section is to apply Theorems 5 and 6 to a class of semilinear systems by writing the state operator $\mathscr{A}_\Gamma$ in Eq. (3) as a sum of two operators, one being the generator of an exponentially stable C_0-semigroup of linear operators and the other being a continuous Lipschitz nonlinear operator. Some criteria are given which guarantee the (strict) m-dissipativity of this sum and the compactness of its resolvent operator, and therefore the asymptotic stability of the origin: see Theorem 12 and Corollary 13.

For this purpose the following preliminary lemma is needed:

Lemma 8. (i) *If N is a Lipschitz operator on $D \subset X$, with Lipschitz constant l_N, then $N - l_N I$ is a dissipative operator; whence for all $l > l_N$, $N - lI$ is a strictly dissipative operator.*

(ii) *If $\mathscr{A}_1$ is a m-dissipative operator and $\mathscr{A}_2$ is a Lipschitz (strictly) dissipative operator on X, then $\mathscr{A}_1 + \mathscr{A}_2$ is a (strictly) m-dissipative operator.*

Comment 9. Properties (i) and (ii) are well-known. For (i) see e.g. [18, Lemma 6.1, p. 245] and for (ii) see e.g. [16, Corollary 3.8.2].

Let us consider the following class of semilinear systems:

$$\begin{cases} \dot{x}(t) = A_0 x(t) + N_0(x(t)), \\ x(0) = x_0 \in D(A_0) \cap F, \end{cases} \tag{9}$$

where the following assumptions hold:

(A1) the linear operator A_0, defined on its domain $D(A_0)$, is the infinitesimal generator of a C_0-semigroup of bounded linear operators $S_0(t)$ on a Banach space X such that

$$\|S_0(t)\| \leqslant M e^{\omega t},$$

for all $t \geqslant 0$, for some $\omega < 0$ and $M \geqslant 1$, i.e. the C_0-semigroup $S_0(t)$ is exponentially stable (the growth constant of $S_0(t)$, denoted by ω_0, is negative).

(A2) there exists a positive constant λ such that

$$(I - \lambda A_0)^{-1} \quad \text{is compact.} \tag{10}$$

(A3) N_0 is a Lipschitz continuous nonlinear operator defined on a closed subset F of X, with Lipschitz constant l_0.

Let us introduce the following notations:

$$\mathscr{A} := A_0 + N_0 \quad \text{and} \quad D := \overline{D(\mathscr{A})} = \overline{D(A_0) \cap F}. \tag{11}$$

First let us assume that the nonlinear operator N_0 is defined everywhere, i.e. $F = X$ (in this case $D = X$). Before stating a stability theorem for this class of systems, the following lemmas are needed.

Lemma 10. *Consider a semilinear system described by* (9) *and satisfying conditions* (A1) *and* (A3) *with $M = 1$ and $F = X$. Assume that $-\omega \geqslant l_0$ (respectively, $-\omega > l_0$). Then the operator $\mathscr{A}$ is m-dissipative (respectively, strictly m-dissipative).*

Proof. The idea is to write the operator $\mathscr{A}$ as follows:

$$\mathscr{A} = A_0 + l_0 I + N_0 - l_0 I.$$

By Lemma 8(i), the fact that N_0 is a Lipschitz operator implies that $N_0 - l_0 I$ is a Lipschitz dissipative operator. On the other hand, since A_0 is the generator of a C_0-semigroup $S_0(t)$ with $\|S_0(t)\| \leqslant e^{\omega t}$, the operator $A_0 + l_0 I$ is the generator of a C_0-semigroup $S_l(t)$ such that $\|S_l(t)\| \leqslant e^{(\omega + l_0)t}$ (see e.g. [11, Theorem 3.2.1, p. 110]). It follows from the fact that $-\omega \geqslant l_0$ that $S_l(t)$ is a contraction semigroup, whence $A_0 + l_0 I$ is m-dissipative. Consequently by Lemma 8(ii), the operator $\mathscr{A}$ is m-dissipative.

In order to show that the operator $\mathscr{A}$ is strictly m-dissipative when $-\omega > l_0$, it suffices to write it as $\mathscr{A} = A_0 + lI + N_0 - lI$, where $-\omega \geqslant l > l_0$, and to observe that the operator $N_0 - lI$ is strictly dissipative (see Lemma 8(i)) and $A_0 + lI$ is a m-dissipative operator. The conclusion follows by Lemma 8(ii). □

Lemma 11. *Consider a semilinear system given by* (9) *and satisfying conditions* (A1)–(A3) *with $M = 1$ and $F = X$, such that $-\omega \geqslant l_0$. Then the operator $(I - \lambda\mathscr{A})^{-1}$ is compact, where $\lambda > 0$ is a constant such that* (10) *holds.*

Proof. Let $\lambda > 0$ be such that (10) holds. In order to prove the compactness of the operator $(I - \lambda\mathscr{A})^{-1}$, let us consider any bounded sequence (v_n) in X and prove that the sequence $(u_n) := (I - \lambda\mathscr{A})^{-1} v_n$, defined in D, has a converging subsequence.

Observe that $v_n = u_n - \lambda\mathscr{A} u_n$ and consider a bounded linear functional f such that $f(u_n) = \|u_n\|^2 = \|f\|^2$. Then

$$f(v_n) = \|u_n\|^2 - \lambda f(\mathscr{A} u_n).$$

Using the fact that the operator $\mathscr{A}$ is dissipative, we have

$$\|u_n\|^2 \leqslant \|u_n\|^2 - \lambda f(\mathscr{A} u_n) = f(v_n) \leqslant \|u_n\| \|v_n\|.$$

It follows that the sequence (u_n) is bounded. Now consider the sequence (w_n) defined by

$$\begin{aligned} w_n &:= (I - \lambda A_0) u_n = (I - \lambda\mathscr{A} + \lambda N_0) u_n \\ &= v_n + \lambda N_0(u_n). \end{aligned}$$

Since N_0 is a Lipschitz operator, it follows from the fact that the sequence (u_n) is bounded, that so is the sequence $(N_0(u_n)))$. Thus, using the boundedness of (v_n), one can conclude that (w_n) is bounded; whence, by using the compactness of the operator $(I - \lambda A_0)^{-1}$, the sequence $(u_n) = ((I - \lambda A_0)^{-1} w_n)$ has a converging subsequence. □

The following theorem follows directly from Lemmas 10 and 11, Theorems 5 and 6, and Comment 3(c).

Theorem 12. *Consider a semilinear system given by* (9) *as in Lemma* 11 *such that $-\omega \geqslant l_0$. Let $\Gamma(t)$ be the nonlinear contraction semigroup on D, generated by $\mathscr{A}$. Assume that $\overline{x}$ is an equilibrium profile of* (9). *Then for any $x_0 \in D$, $x(t, x_0) := \Gamma(t) x_0$ converges, as $t \to \infty$, to $\omega(x_0) \subset \{z : \|z - \overline{x}\| = r\}$, where $r \leqslant \|x_0 - \overline{x}\|$, i.e.*

$$\lim_{t \to \infty} d(x(t, x_0), \omega(x_0)) = 0.$$

If in addition $-\omega > l_0$, then $x(t, x_0) \to \overline{x}$ as $t \to \infty$ i.e. $\overline{x}$ is an asymptotically stable equilibrium point of (9) *on D.*

By using a standard argument, viz. the use of an equivalent vector norm on X, Theorem 12 can be extended to the general case, i.e. $M \geqslant 1$. This extension is needed in several applications, especially for axial dispersion tubular reactor models: see [14].

Corollary 13. *Consider a semilinear system described by* (9) *and satisfying conditions* (A1)–(A3), *where* $F=X$. *Assume that* $-\omega > Ml_0$. *Let* $\Gamma(t)$ *be the nonlinear contraction semigroup on* D, *generated by the operator* $\mathscr{A}$ (*recall that the operator* $\mathscr{A}$ *and the subset* D *are given by* (11)). *Assume that* $\overline{x}$ *is an equilibrium profile of* (9). *Then for any* $x_0 \in D$,

$$x(t, x_0) := \Gamma(t)x_0 \to \overline{x} \quad \text{as } t \to \infty$$

i.e. $\overline{x}$ *is an asymptotically stable equilibrium point of* (9) *on* D.

Proof. Consider the norm given by

$$|x| := \sup\{\exp(-\omega t)\|S_0(t)x\| : \ t \geqslant 0\}$$

which is equivalent to the norm $\|\cdot\|$ (see e.g. [18, p. 277,21, p. 19]). It follows that the corresponding operator norm of the C_0-semigroup generated by A_0 satisfies the condition

$$|S_0(t)| \leqslant e^{\omega t} \quad \text{for all } t \geqslant 0.$$

In addition N_0 is a Lipschitz operator relatively to the norm $\|\cdot\|$ with Lipschitz constant l_0. Hence N_0 is also a Lipschitz operator with respect to $|\cdot|$, with Lipschitz constant $l_0' = l_0 M$. Moreover, $-\omega > l_0'$. The conclusion follows by Theorem 12. □

Comment 14. It is well-known that, for all $\omega \in (\omega_0, 0)$ (where $\omega_0 < 0$ denotes the growth constant of the semi-group $S_0(t)$), there exists a constant M_ω such that

$$\forall t \geqslant 0, \quad \|S_0(t)\| \leqslant M_\omega e^{\omega t}.$$

By the proof of [11, Theorem 2.1.6(e)], there exist a time t_ω and a constant $M_{0,\omega} \geqslant 1$ such that

$$\forall t \geqslant t_\omega, \quad \|S_0(t)\| \leqslant e^{\omega t}$$

and

$$\forall t \in [0, t_\omega], \quad \|S_0(t)\| \leqslant M_{0,\omega}.$$

The constants t_ω, $M_{0,\omega}$ and M_ω are given, respectively, by

$$t_\omega := \inf\{\tau \geqslant 0, \|S_0(t)\| \leqslant e^{\omega t}, \forall t \geqslant \tau\},$$

$$M_{0,\omega} = \sup_{t \in [0, t_\omega]} \|S_0(t)\|, \quad M_\omega = e^{-\omega t_\omega} M_{0,\omega}.$$

Therefore, the condition $-\omega > Ml_0$ in Corollary 13 can be replaced by the weaker condition

$$l_0 < \sup_{\omega_0 < \omega < 0} (-\omega M_\omega^{-1}) := \sup_{\omega_0 < \omega < 0} \left(\frac{-\omega e^{\omega t_\omega}}{M_{0,\omega}} \right).$$

In the previous stability results, viz. Theorem 12 and Corollary 13, it is assumed that N_0 is defined everywhere. This condition is needed in order to guarantee the m-dissipativity of $\mathscr{A}$ (see Lemma 8(ii)). However, observe that, in Theorems 5 and 6, m-dissipativity can be replaced by the weaker condition (2). This observation will be crucial in what follows for establishing the asymptotic stability of the system (9) when the nonlinear operator N_0 is defined on a convex closed subset $F \subset X$. This extension is needed for the plug flow reactor model treated in Sections 4 and 5. This point was overlooked in an earlier analysis, leading to erroneous stability conditions for that application in [3]. First the following technical concept is introduced:

Definition 15. Let $\mathscr{A}$ be a dissipative operator. Let X_0 be a subset of X. $\mathscr{A}$ is said to be in $Q(X_0)$ if $\overline{D(\mathscr{A})} \subset X_0$ and $X_0 \subset \mathscr{R}(I - \lambda\mathscr{A})$ for all $\lambda > 0$.

Now we are in a position to state the following important result, that is useful when the nonlinearity is not defined everywhere:

Theorem 16. *Let* F *be a closed convex subset of* X. *Consider a semilinear system given by* (9), *satisfying conditions* (A2) *and* (A3), *and such that* A_0 *is a closed dissipative operator whose domain* $D(A_0)$ *is a linear subspace of* X. *Assume that* $\mathscr{A} = A_0 + N_0$ *is dissipative, the restriction of* A_0 *on* $D(A_0) \cap F$ *is in* $Q(F)$ *and the condition*

$$\liminf_{\lambda \to 0^+} \lambda^{-1} d(F, x + \lambda N_0(x)) = 0 \quad \text{for } x \in D := \overline{D(\mathscr{A})},$$

holds. Then $\mathscr{A}$ *is the generator of a contraction semigroup* $\Gamma(t)$. *Assume that* $\overline{x}$ *is an equilibrium point of* (9). *Then for any* $x_0 \in D$,

$$\lim_{t \to \infty} d(x(t, x_0), \omega(x_0)) = 0,$$

where $x(t, x_0) = \Gamma(t)x_0$. *If in addition* $\mathscr{A}$ *is strictly dissipative, then* $\Gamma(t)x_0 \to \overline{x}$ *as* $t \to \infty$, *i.e.* $\overline{x}$ *is an asymptotically stable equilibrium point of* (9) *on* D.

Proof. First observe that the compactness of $(I - \lambda\mathscr{A})^{-1}$ follows from conditions (A2), (A3) and the dissipativity of $\mathscr{A}$ as in the proof of Lemma 11. In order to apply Theorem 5, it suffices to prove condition (2). Let us consider the restriction of A_0 on $D(A_0) \cap F$. For the sake of simplicity, we keep the same notation for this restriction. Using the assumption that the operator A_0 is closed dissipative on $D(A_0)$ and the closedness of F, it is easy to show that A_0 is closed dissipative on $D(A_0) \cap F$. Also since $A_0 \in Q(F)$ and $\overline{D(A_0) \cap F} \subset F$, then by [16, Theorem 3.8.1] (where $X_0 = F$, $A = A_0$ and $B = N_0$), $\mathscr{A} = A_0 + N_0 \in Q(F)$ with $D(\mathscr{A}) = D(A_0) \cap F$. Hence condition (2) holds since $D(A_0) \cap F$ is convex. Finally, asymptotic stability follows from Theorem 6 when $\mathscr{A}$ is strictly dissipative. □

In the sequel, the asymptotic stability theory developed above is applied to a nonisothermal plug flow reactor model: see Section 5. First the PDEs model is described together with its dimensionless infinite-dimensional state-space description.

4. Nonlinear plug-flow reactor model

4.1. Nonlinear PDE model

The dynamics of tubular reactors are typically described by nonlinear PDEs derived from mass and energy balances. As a case study, for the purpose of illustrating the main theoretical results derived in Sections 2 and 3, especially Theorem 16, we consider a nonisothermal plug flow reactor with the following chemical reaction:

$$A \longrightarrow bB,$$

where $b>0$ denotes the stoichiometric coefficient of the reaction. If the kinetics of the above reaction are characterized by first order kinetics with respect to the reactant concentration C_A (mol/l) and by an Arrhenius-type dependence with respect to the temperature T (K), the dynamics of the process in such a reactor are described by the following energy and mass balance PDEs, where C_B (mol/l) denotes the product concentration:

$$\frac{\partial T}{\partial \tau} = -v\frac{\partial T}{\partial \zeta} - \frac{\Delta H}{\rho C_p} k_0 C_A \exp\left(-\frac{E}{RT}\right) - \beta_0 (T - T_c), \tag{12}$$

$$\frac{\partial C_A}{\partial \tau} = -\frac{\partial C_A}{\partial \zeta} - k_0 C_A \exp\left(-\frac{E}{RT}\right), \tag{13}$$

$$\frac{\partial C_B}{\partial \tau} = -v\frac{\partial C_B}{\partial \zeta} + bk_0 C_A \exp\left(-\frac{E}{RT}\right), \tag{14}$$

where $\beta_0 := 4h(\rho C_p d)^{-1}$ and with the boundary conditions given, for $\tau \geqslant 0$, by

$$T(0,\tau) = T_{in}, \quad C_A(0,\tau) = C_{A,in}, \quad C_B(0,\tau) = 0. \tag{15}$$

The initial conditions are assumed to be given, for $0 \leqslant \zeta \leqslant L$, by

$$T(\zeta,0) = T_0(\zeta), \quad C_A(\zeta,0) = C_{A,0}(\zeta), \quad C_B(\zeta,0) = 0. \tag{16}$$

In the equations above, $v, \Delta H, \rho, C_p, k_0, E, R, h, d, T_c, T_{in}$ and $C_{A,in}$ hold for the superficial fluid velocity, the heat of reaction, the density, the specific heat, the kinetic constant, the activation energy, the ideal gas constant, the wall heat transfer coefficient, the reactor diameter, the coolant temperature, the inlet temperature, and the inlet reactant concentration, respectively. In addition τ, ζ and L denote the time and space independent variables, and the length of the reactor, respectively. Finally, T_0 and $C_{A,0}$ denote the initial temperature and reactant concentration profiles, respectively, such that $T_0(0) = T_{in}$ and $C_{A,0}(0) = C_{A,in}$.

Comment 17. In the particular case $T_0(\zeta) = T_{in}$ and $C_{A,0}(\zeta) = C_{A,in}$ for all $0 \leqslant \zeta \leqslant L$, the existence and the invariance properties of the state trajectories of such model on $[0,\infty)$ have been studied in [14] (see also the references therein), by using a dimensionless variable equivalent model. We shall also use such a model here: see notably Section 4.3.

Observe that in Eqs. (12)–(16), the product concentration C_B is known if the reactant concentration C_A and the temperature T are known. Therefore, we will only consider the two first state components, viz. the reactor temperature and the reactant concentration. Their dynamics will be described by means of an infinite dimensional system description derived from an equivalent nonlinear PDE dimensionless model. Such an approach is standard in tubular reactor analysis (see e.g. [14] and references therein) and is briefly developed in the following subsection.

4.2. Infinite dimensional system description

Let us consider two new state variables $x_1(t)$ and $x_2(t)$, $t \geqslant 0$, which are defined via the following state transformation given by

$$x_1 = \frac{T - T_{in}}{T_{in}}, \quad x_2 = \frac{C_{A,in} - C_A}{C_{A,in}}. \tag{17}$$

Let us consider also dimensionless time t and space z variables defined as follows:

$$t = \frac{\tau v}{L}, \quad z = \frac{\zeta}{L}.$$

Then we obtain the following equivalent representation of the model (12)–(13) (without considering the product concentration dynamics (14)):

$$\frac{\partial x_1}{\partial t} = -\frac{\partial x_1}{\partial z} - \beta(x_1 - x_c) + \alpha\delta(1 - x_2)\exp\left(\frac{\mu x_1}{1 + x_1}\right), \tag{18}$$

$$\frac{\partial x_2}{\partial t} = -\frac{\partial x_2}{\partial z} + \alpha(1 - x_2)\exp\left(\frac{\mu x_1}{1 + x_1}\right), \tag{19}$$

where x_c is the dimensionless coolant temperature

$$x_c = \frac{T_c - T_{in}}{T_{in}}$$

and the dimensionless parameters α, β, δ, and μ are related to the original parameters as follows:

$$\mu = \frac{E}{RT_{in}}, \quad \alpha = \frac{k_0 L}{v}\exp(-\mu), \tag{20}$$

$$\beta = \frac{4hL}{\rho C_p \, dv}, \quad \delta = -\frac{\Delta H}{\rho C_p}\frac{C_{A,in}}{T_{in}}. \tag{21}$$

Comment 18. From a physical point of view, one can observe that for all $z \in [0,1]$, and for all $t \geqslant 0$, $0 \leqslant T(z,t) \leqslant T_{max}$ and $0 \leqslant C(z,t) \leqslant C_{A,in}$ or equivalently

$$-1 \leqslant x_1(z,t) \leqslant x_{1,max} := \frac{T_{max} - T_{in}}{T_{in}}$$

and

$$0 \leqslant x_2(z,t) \leqslant 1,$$

where the upper bound T_{max} could possibly be equal to ∞. It turns out that the case $T_{max} < +\infty$ is the most interesting one in the stability analysis (see below). Moreover, this case is physically feasible: see [13, Theorem 4.1], which shows that under some physical assumptions, the temperature and the reactant concentration are bounded.

The equivalent state space description of the model (18)–(19) is given by the following semilinear abstract differential equation on the Banach space $H := L^2(0,1)\times L^2(0,1)$ (equipped with the norm $\|(x_1,x_2)\| := \max(\|x_1\|_2, \|x_2\|_2)$, where $\|\cdot\|_2$ denotes the usual norm on the Hilbert space $L^2(0,1)$):

$$\begin{cases} \dot{x}(t) = Ax(t) + N_0(x(t)) + B_0 x_c, \\ x(0) = x_0 \in D(A)\cap F, \end{cases} \tag{22}$$

where A is the linear (unbounded) operator defined on its domain

$$D(A) := \left\{ x \in H : x \text{ is a.c, } \frac{dx}{dz} \in H \text{ and } x(0) = 0 \right\} \tag{23}$$

(where a.c means that the function x is absolutely continuous) by

$$Ax := \begin{bmatrix} -\dfrac{d}{dz} - \beta I & 0 \\ 0 & -\dfrac{d.}{dz} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{24}$$

and the nonlinear operator N is defined on the closed convex subset

$$F := \{x \in H : x_1 \geqslant -1 \text{ and } 0 \leqslant x_2 \leqslant 1\}$$

(where the inequalities hold almost everywhere on $[0,\ 1]$) by

$$N_0(x) := \begin{bmatrix} \alpha\delta(1-x_2)\exp\left(\dfrac{\mu x_1}{x_1+1}\right) \\ \alpha(1-x_2)\exp\left(\dfrac{\mu x_1}{x_1+1}\right) \end{bmatrix}. \tag{25}$$

The operator $B_0 \in L(H)$ is the linear bounded operator defined by

$$B_0 = \beta \begin{bmatrix} I \\ 0 \end{bmatrix}. \tag{26}$$

5. Plug flow reactor model analysis

5.1. Stability analysis

In this part we are interested in the study of the asymptotic stability of any equilibrium profile, which is a solution of the equilibrium ordinary differential equations corresponding to the plug flow reactor PDEs model (22)–(26).

The local stability of any equilibrium profile with constant temperature has been studied in [4], where it is shown that the operator of the linearized system generates an exponentially stable semigroup. This stability analysis was done in the same spirit than the one developed e.g. in [7,8], [9, Section 2.2], for similar hyperbolic PDE models. Here we are interested in obtaining conditions on the model parameters for the global stability of such semilinear systems.

In terms of dimensionless variables, let us denote by $x_e := (x_{1,e}, x_{2,e})^{\mathrm{T}} \in H$ and $x_{c,e} \in L^2(0,1)$ any equilibrium profile, i.e. any solution of the following equations:

$$Ax_e + N_0(x_e) + B_0 x_{c,e} = 0.$$

By setting the coolant temperature x_c equal to its equilibrium value along the reactor, i.e. by taking $x_c := x_{c,e}$ in Eq. (22), the plug flow reactor model described by (22)–(26) can be rewritten as the following abstract differential equation on the Hilbert space H:

$$\begin{cases} \dot{x}(t) = A_0 x(t) + N_0(x(t)), \\ x(0) = x_0 \in D(A_0)\cap F, \end{cases} \tag{27}$$

where A_0 is the affine operator defined on its domain $D(A_0) = D(A)$ by $A_0 := A + \kappa$ with $\kappa := (\beta x_{c,e}, 0)^{\mathrm{T}}$.

In order to apply Theorem 16, the following lemmas will be useful. The first one is straightforward: see e.g. [23, Theorem 4.1 and Remark 4.1(ii)].

Lemma 19. *Consider the operator A given by* (23)–(24). *The operator A generates an exponentially stable C_0-semigroup $S(t)$. Moreover,*

for all $t \geqslant 0, \quad \|S(t)\| \leqslant 1.$

Hence A is m-dissipative and A_0 is closed dissipative.

Lemma 20. *There exists a positive constant λ such that the operator $(I-\lambda A_0)^{-1}$ is compact.*

Proof. Observe that it suffices to show the compactness of $(I-\lambda A)^{-1}$. Consider the operator $Q_\beta := -(d/dz) - \beta I$ defined on the domain $D(Q_\beta) = \{f \in L^2(0,\ 1) : f$ is absolutely continuous, $df/dz \in L^2(0,1)$ and $f(0) = 0\}$. A straightforward computation reveals that, for any $\lambda > 0$ and for any $g \in L^2(0,1)$,

$$((I-\lambda Q_\beta)^{-1} g)(z) = \frac{1}{\lambda}\exp\left(-\left(\frac{1}{\lambda}+\beta\right)z\right) \times \int_0^z \exp\left(\left(\frac{1}{\lambda}+\beta\right)\eta\right) g(\eta)\, d\eta.$$

Using this identity, it can be easily shown that $(I-\lambda Q_\beta)^{-1}$ is a Hilbert–Schmidt operator, whence it is compact. The compactness of the operator $(I-\lambda A)^{-1}$ follows by observing that

$$(I-\lambda A)^{-1} = \operatorname{diag}[(I-\lambda Q_\beta)^{-1}, (I-\lambda Q_0)^{-1}]. \qquad \square$$

Comment 21. Another way to show the compactness of $(I-\lambda Q_\beta)^{-1}$ is to prove that the operator Q_β^{-1} (which can be easily computed) is a Volterra operator and hence it is compact. Consequently the operator

$$(I-\lambda Q_\beta)^{-1} = Q_\beta^{-1}(Q_\beta^{-1} - \lambda I)^{-1}$$

is compact, since the operator $(Q_\beta^{-1} - \lambda I)^{-1}$ is bounded.

Lemma 22. *The restriction of A on $D(A)\cap F$ is in $Q(F)$. Moreover, assume that, for all $x \in F$,*

$$\lim_{h\to 0^+} h^{-1} d(F, x + h\kappa) = 0. \tag{28}$$

Then the restriction of A_0 on $D(A_0)\cap F$ is in $Q(F)$.

Proof. First observe that $D(A) \cap F$ is dense in F. Indeed, the latter property follows by the proof of [15, Lemma 4.1] (for axial dispersion tubular reactors), which can be easily adapted for the plug flow reactor case. Indeed, that proof is based on (1) the positivity of e^{At}, and (2) the e^{At}-invariance of F (see [14, proof of Theorem 4.1]).

Now let us prove that $F \subset \mathscr{R}(I - \lambda A)$ for all $\lambda > 0$. Let $\lambda > 0$, let $y \in F$, then there exists $x = (I - \lambda A)^{-1} y \in D(A)$. Recall that $\forall t > 0 \;\; e^{At} F \subset F$ (see [14, p. 10]). One has

$$(I - \lambda A)^{-1} y = \frac{1}{\lambda}\left(\frac{1}{\lambda} - A\right)^{-1} y = \frac{1}{\lambda}\int_0^{\infty} e^{-t/\lambda} e^{At} y \, dt.$$

By straightforward calculation, we can prove that for all $\tilde{y} \in F$,

$$\frac{1}{\lambda}\int_0^{\infty} e^{-t/\lambda} \tilde{y} \, dt \in F,$$

and in particular $\tilde{y} = e^{At} y \in F$. Then $F \subset \mathscr{R}(I - \lambda A)$ and consequently the restriction of A on $F \cap D(A)$ is in $Q(F)$. The second assertion follows directly from the first one and Theorem 3.8.1 in [16]. $\square$

Comment 23. It is easy to see that if $x_{c,e}$ is nonnegative then condition (28) holds.

Concerning the nonlinear operator, many of its properties were established in the framework of the trajectory analysis performed in an earlier work: see [14, Lemmas 3.1 and 3.2].

Lemma 24. *The following properties hold*:

(i) *The operator N_0 is a Lipschitz continuous operator on the subset F.*
(ii) *For any $x \in F$, the following identity holds*:

$$\lim_{h \to 0^+} \frac{1}{h} d(F, x + hN_0(x)) = 0.$$

In order to guarantee the strict dissipativity of the operator $A_0 + N_0$, we need the following condition:

$$l_0 := \max(l_1, l_2) < \beta, \tag{29}$$

where l_1 and l_2 are given by

$$\begin{aligned} l_1 &= |\delta| \frac{k_0 L}{v}\left[\exp\left(\frac{-E}{RT_{\max}}\right) + \frac{4RT_{\text{in}}}{Ee^2}\right] \quad \text{if } \frac{E}{2R} \leqslant T_{\max} \\ &= |\delta| \frac{k_0 L}{v} \exp\left(\frac{-E}{RT_{\max}}\right)\left[1 + \frac{ET_{\text{in}}}{RT_{\max}^2}\right] \quad \text{if } \frac{E}{2R} > T_{\max} \end{aligned} \tag{30}$$

and

$$\begin{aligned} l_2 &= \frac{k_0 L}{v}\left[G_m + \frac{4RT_{\text{in}}}{Ee^2}\right] \quad \text{if } \frac{E}{2R} \leqslant T_{\max} \\ &= \frac{k_0 L}{v}\left[G_m + \frac{ET_{\text{in}}}{RT_{\max}^2}\exp\left(\frac{-E}{RT_{\max}}\right)\right] \quad \text{if } \frac{E}{2R} > T_{\max}, \end{aligned} \tag{31}$$

where

$$G_m := \max_{0 \leqslant T \leqslant T_{\max}} \left|\exp\left(\frac{-E}{RT}\right) - \frac{4h}{k_0 \rho C_p d}\right|. \tag{32}$$

Lemma 25. *If condition (29) holds, then the operator $A_0 + N_0$ is strictly dissipative.*

Proof. First observe that the operator $A_0 + N_0$ can be also written as $\mathscr{A}_v + \mathscr{N}_v$ where the operator $\mathscr{A}_v$ is defined on its domain $D(\mathscr{A}_v) = D(A_0)$ by

$$\mathscr{A}_v x := \begin{bmatrix} -\frac{d.}{dz} - \beta I & 0 \\ 0 & -\frac{d.}{dz} - \beta I \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{33}$$

and the nonlinear operator $\mathscr{N}_v$ is given on the closed subset F by

$$\mathscr{N}_v(x) = \begin{bmatrix} \mathscr{N}_1(x) \\ \mathscr{N}_2(x) \end{bmatrix} := N_0(x) + \begin{bmatrix} \beta x_{c,e} \\ \beta x_2 \end{bmatrix}. \tag{34}$$

It is easy to see that $\mathscr{A}_v$ generates a C_0-semigroup $(e^{\mathscr{A}_v t})_{t \geqslant 0}$ such that $\|e^{\mathscr{A}_v t}\| \leqslant e^{-\beta t}$. Now let us prove that $\mathscr{N}_v$ is a Lipschitz operator. For all $x := (x_1, x_2)^T$ and $x' := (x_1', x_2')^T$ in F,

$$\begin{aligned} &\|\mathscr{N}_v(x) - \mathscr{N}_v(x')\| \\ &\quad = \max(\|\mathscr{N}_1(x) - \mathscr{N}_1(x')\|, \|\mathscr{N}_2(x) - \mathscr{N}_2(x')\|). \end{aligned}$$

Hence a Lipschitz constant l_0 of the operator $\mathscr{N}_v$ is given by $l_0 = \max(l_1, l_2)$ where l_1 and l_2 are Lipschitz constants of the operators $\mathscr{N}_1$ and $\mathscr{N}_2$, respectively.

Observe that, by using the state transformation (17), the operator $\mathscr{N}_2$ can be rewritten as

$$\mathscr{N}_2(x_1, x_2) = \frac{k_0 L}{v C_{A,\text{in}}} C_A G(T) + \beta,$$

where the operator G is defined by

$$G(T) := \exp\left(\frac{-E}{RT}\right) - \frac{\beta v}{k_0 L}.$$

It follows that

$$\begin{aligned} \|\mathscr{N}_2(x) - \mathscr{N}_2(x')\| &= \frac{k_0 L}{v C_{A,\text{in}}}\|C_A G(T) - C_A' G(T')\| \\ &\leqslant \frac{k_0 L}{v C_{A,\text{in}}}[G_m\|C_A - C_A'\| + C_{A,\text{in}} l_s \|T - T'\|], \end{aligned}$$

where l_s is a Lipschitz constant of the function $\exp(-E/RT)$ on $[0, T_{\max})$ (see below). Consequently,

$$\|\mathcal{N}_2(x) - \mathcal{N}_2(x')\| \leqslant \frac{k_0 L}{v}[G_m + l_s T_{\text{in}}]\|x - x'\|;$$

whence $\mathcal{N}_2$ is a Lipschitz operator on D.

Now let us compute l_s. Observe that the first derivative of the function $g(s) := \exp(-k/s)$, where k is a positive constant, is bounded on the interval $[0, T_{\max}]$. Therefore, the function g is a Lipschitz continuous function on $[0, T_{\max}]$. A Lipschitz constant of g can be estimated by

$$l_s := \sup\{g'(s), 0 \leqslant s \leqslant T_{\max}\},$$

where $g'(s) = k/s^2 \exp(-k/s)$, for $0 < s \leqslant T_{\max}$ and $g'(0) = 0$. The study of the function g' leads to

$$l_s = \begin{cases} g'\left(\frac{k}{2}\right) = \dfrac{4}{k\mathrm{e}^2} & \text{if } k \leqslant 2T_{\max}, \\ g'(T_{\max}) = \dfrac{k}{T_{\max}^2} \exp\left(-\dfrac{k}{T_{\max}}\right) & \text{if } k > 2T_{\max}. \end{cases}$$

When applying this result to the case $k = E/R$, the constant l_s is given by

$$l_s = \begin{cases} \dfrac{4R}{E\mathrm{e}^2} & \text{if } E \leqslant 2RT_{\max}, \\ \dfrac{E}{RT_{\max}^2} \exp\left(-\dfrac{E}{RT_{\max}}\right) & \text{if } E > 2RT_{\max}. \end{cases}$$

It follows that the constant l_2 given by (31) is a Lipschitz constant of $\mathcal{N}_2$. By using similar computations, it can be shown that the constant l_1 given by (30) is a Lipschitz constant of $\mathcal{N}_1$. Finally, by using Lemma 10, condition (29) implies the strict dissipativity of $A_0 + N_0$. □

An important consequence of Lemmas 20, 22, 24 and 25, and Theorem 16, is the following theorem giving a criterion of asymptotic stability for the plug flow reactor model.

Theorem 26. *Consider the system* (22)–(25) *such that conditions* (28) *and* (29) *hold. Then the operator* $\mathscr{A} := \mathscr{A}_0 + N_0$ *is the generator of a unique nonlinear contraction semigroup* $\Gamma(t)$ *on* F. *Moreover, for any* $x_0 \in F$,

$$x(t, x_0) := \Gamma(t)x_0 \to x_e \quad \text{as } t \to \infty$$

i.e. x_e *is an asymptotically stable equilibrium point of the system* (22) *on* F.

Comment 27. (a) Typically, in most cases, the condition $E > 2RT_{\max}$ is satisfied, since the activation energy E is very large. In this case, the asymptotic stability criterion of Theorem 26 reads as follows:

$$k_0 \exp\left(-\frac{E}{RT_{\max}}\right) K < \frac{4h}{\rho C_p d},$$

Table 1
Numerical values of process parameters

Process parameters	Symbols	Numerical values
Superficial fluid velocity	v	0.025 m/s
Length of the reactor	L	1 m
Activation energy	E	11250 cal/mol
Kinetic constant	k_0	$10^6\ \text{s}^{-1}$
Heat transfer coefficient	$\frac{4h}{\rho C_p d}$	$0.2\ \text{s}^{-1}$
Inlet reactant concentration	$C_{A,\text{in}}$	0.02 mol/L
Ideal gas constant	R	1.986 cal/(mol K)
Inlet temperature	T_{in}	340 K
	$\frac{\Delta H}{\rho C_p}$	−4250 K L/mol

where the constant K is defined as the maximum of the two following quantities:

$$|\delta|\left[1 + \frac{ET_{\text{in}}}{RT_{\max}^2}\right] \quad \text{and}$$

$$\left[G_m \exp\left(\frac{E}{RT_{\max}}\right) + \frac{ET_{\text{in}}}{RT_{\max}^2}\right].$$

From a physical point of view, the condition above means that the heat transfer coefficient must be large enough in order to dominate a weighted value of the kinetic constant $k_0 \exp(-E/RT_{\max})$.

(b) On the basis of the (local) stability results obtained e.g. in [4] or [7, p. 18], the stability condition (28), although natural from a theoretical point of view, could possibly appear to be conservative for the specific case studied here. However, this is how far one can get by using such theoretical tools. This specific point could be an interesting topic for further research, by using possibly classical PDE techniques like the methods of characteristics.

Yet, the theoretical results obtained in Sections 2 and 3 are interesting on their own and they could also be applied e.g. to axial dispersion tubular reactor models as well.

5.2. Numerical simulations

In this section, we are interested in numerical simulations of the open-loop controlled plug flow reactor model (12)–(16). The latter model can be approximated by several methods. Here we use the finite forward difference method. The parameter values used here are given in Table 1.

It is theoretically shown in the previous section that, under suitable conditions on the system parameters and on the coolant temperature at the equilibrium, an equilibrium profile is asymptotically stable for the plug flow reactor model. The equilibrium profile considered here corresponds to a constant coolant temperature profile. The choice of such equilibrium profile is motivated by the fact that it is optimal in view to minimize the reactant concentration at the reactor outlet and the difference between the coolant temperature and the ambient temperature [22]. Moreover, this equilibrium profile satisfies condition (28) for a coolant temperature larger than the inlet

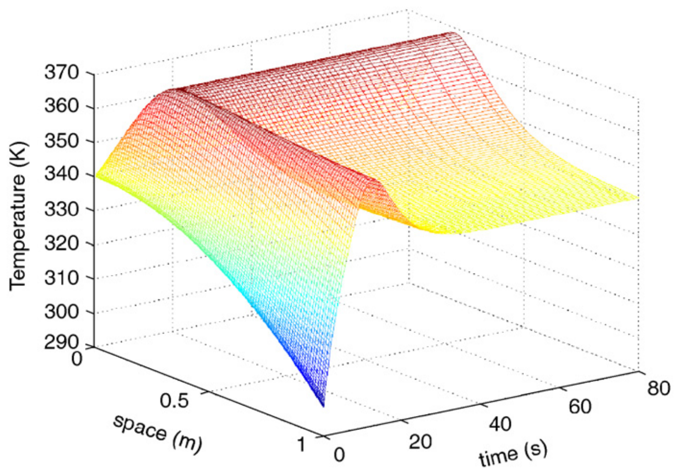

Fig. 1. Temperature using the initial conditions $T_0(z) = T_{\text{in}} \cos(0.2 * z)$ and $C_{A,0}(z) = C_{A,\text{in}} \cos(0.1 * z)$ and the coolant temperature $T_{c,e} = 340\,\text{K}$ as open-loop control.

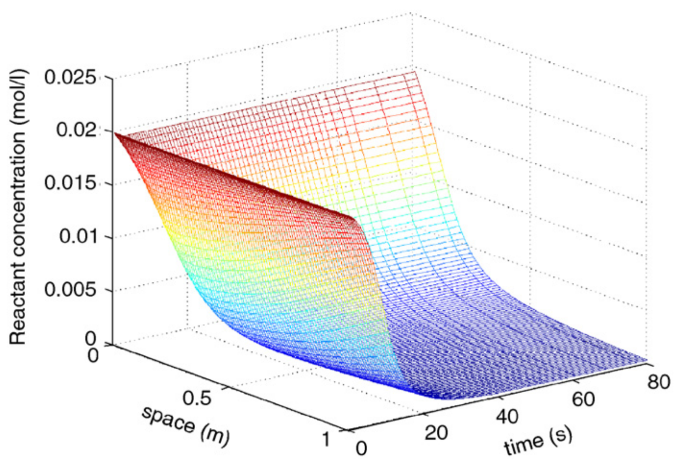

Fig. 2. Reactant concentration using the initial conditions $T_0(z) = T_{\text{in}} \cos(0.2 * z)$ and $C_{A,0}(z) = C_{A,\text{in}} \cos(0.1 * z)$ and the coolant temperature $T_{c,e} = 340\,\text{K}$ as open-loop control.

temperature (see Comment 23). Several initial state functions have been tested in the numerical simulations (see [2]). Here we have selected $T_0(z)=T_{\text{in}} \cos(0.2*z)$ and $C_{A,0}=C_{\text{in}} \cos(0.1*z)$ as initial state. The results of the simulations are illustrated in Figs. 1 and 2: they show that the system state converges to the chosen steady state. These observations agree with the result predicted by Theorem 26, since in this case $l_0=14$ and $\beta=20$.

6. Conclusion

In this paper we have studied the asymptotic stability property of a specific class of semilinear infinite-dimensional systems. First this property was studied for a general class of nonlinear infinite-dimensional state space systems by using the concept of strict m-dissipativity: see Theorem 6. This result was applied to a plug flow reactor model. The results can be summarized as follows. A semilinear model is asymptotically stable under the conditions that (a) its linear part generates an exponentially stable semigroup and has a compact resolvent, and (b) its nonlinear part is a Lipschitz operator with a Lipschitz constant less than some parameter, which depends on the linear part parameters: see Theorem 12 and Corollary 13. This result has been extended to nonlinearity defined on a closed convex proper subset of the state space: see Theorem 16. Finally, any equilibrium profile was shown to be asymptotically stable for a nonisothermal plug flow reactor model under some conditions on the system parameters and on the corresponding (constant) coolant temperature: see Theorem 26. This result was illustrated by some numerical simulations.

An interesting open question is the analysis of the asymptotic stability property when the condition $l_0 < \beta$ does not hold. Some preliminary numerical experiments seem to indicate that stability could be lost in that case.

Another interesting topic for further research is the application of the stability results proved in Sections 2 and 3, to axial dispersion tubular reactor models like those studied in [14,15,7, Chapter 4].

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References

[1] B. Abouzaïd, Régulation des systèmes á paramètres répartis avec contraintes sur la commande. Mémoire du Diplôme d'Etudes Supérieures Appprofondies, Université Chouaib Doukkali, El Jadida, Morocco, 2003.

[2] I. Aksikas, Analysis and LQ-optimal control of infinite-dimensional semilinear systems: application to a plug flow reactor, Ph.D. Thesis, Université Catholique de Louvain, 2005. http://edoc.bib.ucl.ac.be:81/ETD-db/collection/available/BelnUcetd-11302005-154241/

[3] I. Aksikas, J. Winkin, D. Dochain, Asymptotic stability of a nonisothermal plug flow reactor infinite-dimensional model, Proceedings of the Sixth IFAC Symposium on Nonlinear Control Systems, NOLCOS 2004, vol. 3, Stuttgart, Germany, pp. 1049–1054.

[4] I. Aksikas, J. Winkin, D. Dochain, Stability analysis of an infinite-dimensional linearized plug flow reactor model, Proceedings of the 43rd IEEE Conference on Decision and Control, CDC 2004, Atlantis, Paradise Island, Bahamas, pp. 2417–2422.

[5] A. Belleni-Morante, A. Mc Bride, Applied Nonlinear Semigroups: An Introduction, Wiley, New York, 1998.

[6] H. Brezis, Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert, Mathematics Studies, North-Holland, Amsterdam, 1973.

[7] P.D. Christofides, Nonlinear and Robust Control of Partial Differential Equation Systems: Methods and Application to Transport-reaction Processes, Birkhauser, Boston, 2001.
[8] P.D. Christofides, P. Daoutidis, Feedback control of hyperbolic PDE systems, AIChE J. 42 (1996) 6063–6086.
[9] P.D. Christofides, P. Daoutidis, Robust control of hyperbolic PDE systems, Chem. Eng. Sci. 53 (1998) 85–105.
[10] M.G. Crandall, A. Pazy, Semi-groups of nonlinear contractions and dissipative sets, J. Funct. Anal. 3 (1969) 376–418.
[11] R.F. Curtain, H.J. Zwart, An Introduction to Infinite-dimensional Linear Systems Theory, Springer, New York, 1995.
[12] C.M. Dafermos, M. Slemrod, Asymptotic behavior of nonlinear contraction semigroups, J. Funct. Anal. 13 (1973) 97–106.
[13] D. Dochain, Contribution to the analysis and control of distributed parameter systems with application to (bio)chemical processes and robotics, Thèse d'Agrégation de l'Enseignement Supérieur, Université Catholique de Louvain, Louvain-la-Neuve, Belgium, 1994.
[14] M. Laabissi, M.E. Achhab, J. Winkin, D. Dochain, Trajectory analysis of nonisothermal tubular reactor nonlinear models, Systems Control Lett. 42 (2001) 169–184.
[15] M. Laabissi, M.E. Achhab, J. Winkin, D. Dochain, Multiple equilibrium profiles for nonisothermal tubular reactor nonlinear models, Dynamics of Continuous, Discrete and Impulsive Systems 11 (2004) 339–352.
[16] V. Lasshmikantham, S. Leela, Nonlinear Differential Equations in Abstract Spaces, Pergamon, Oxford, 1981.
[17] Z. Luo, B. Guo, O. Morgül, Stability and Stabilization of Infinite Dimensional Systems with Applications, Springer, London, 1999.
[18] R.H. Martin, Nonlinear Operators and Differential Equations in Banach Spaces, Wiley, New York, 1976.
[19] I. Miyadera, Nonlinear semigroups, Amer. Math. Soc., 1992.
[20] N.H. Pavel, Nonlinear Evolution Operators and Semigroups: Applications to Partial Differential Equations, Springer, New York, 1981.
[21] A. Pazy, Semigroups of Linear Operators and Application to Partial Differential Equations, Applied Mathematical Sciences, vol. 44, Springer, New York, 1983.
[22] I.Y. Smets, D. Dochain, J.F. Van Impe, Optimal temperature control of a steady-state exothermic plug flow reactor, AIChE J. 48 (2) (2002) 279–286.
[23] J. Winkin, D. Dochain, P. Ligarius, Dynamical analysis of distributed parameter tubular reactors, Automatica 36 (2000) 349–361.