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Perspective article: Topology shapes dynamics on higher-order networks

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ABSTRACT

Higher-order networks capture the many-body interactions present in complex systems, revealing novel phenomena that shed new light on the interplay between the systems' topology and dynamics. The emerging theory of higher-order topological dynamics, combining higher-order structures with discrete topology and non-linear dynamics, has the potential to play a fundamental role for the understanding of complex systems such as the brain and climate, as well as for the development of a new generation of Artificial Intelligence algorithms. A new theoretical framework to describe network dynamics that goes beyond the node-centered description adopted so far thus emerges. In this theoretical framework, the dynamics of a network is encoded by topological signals, i.e., variables assigned to the nodes as well as to the edges (like fluxes), to the triangles, or even to other higher-order cells of higher-order networks. One important challenge is to model and process these signals with algebraic operators such as the Hodge Laplacian and the Topological Dirac operator in order to formulate a deeper physical and mathematical theory of complex systems. Recently, it has been shown that topological signals lead to the emergence of novel types of dynamical states and collective phenomena such as topological synchronization, topological pattern formation and triadic percolation. They offer novel paradigms to understand how topology shapes dynamics, how dynamics learns the underlying network topology, and how topology varies dynamically. This Perspective aims at guiding physicists, mathematicians, computer scientists and network scientists from different disciplines in the growing field of topological signals, as well as at delineating challenges that must be addressed by future research.

Understanding, modeling and predicting the emergent behavior of complex systems are among the biggest challenges of current scientific research. Major examples include brain function, epidemic spreading, and climate change. Network science has deeply transformed the theory of complex systems providing powerful theoretical frameworks by representing them as graphs or networks. Networks encode relevant information about the complex systems they represent [1, 2] and their statistical and combinatorial properties strongly affect the unfolding of dynamical processes and critical phenomena defined on them [3–6]. The success of network science is undoubtedly rooted in the simplicity of its basic assumption: a complex system can be merely described in terms of the interactions between its elements. However, this assumption also highlights an important limitation of conventional network representations, as they encode pairwise interactions only.

As a matter of fact, representing a complex system by using just pairwise interactions is an approximation of the reality. Assuming the presence of many-body interactions is certainly more appropriate for many, if not all, systems. In high-energy physics, vertices of Feynman diagrams involve creation and annihilation of more than two particles, with quantum

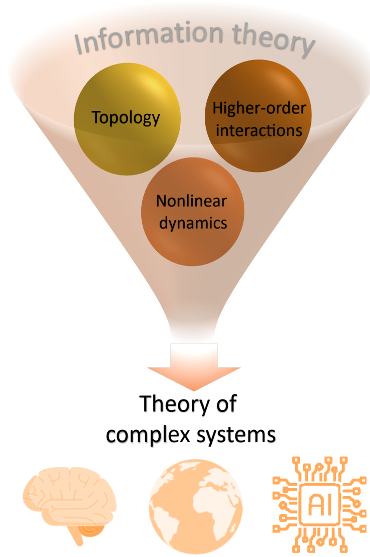

Figure 1. The emerging field of higher-order topological dynamics of complex systems. This field combines higher-order interactions, topology and non-linear dynamics giving rise to new emergent phenomena encoding information which can dramatically transform our understanding of complex systems such as the brain and the climate, and can allow for the formulation of new efficient AI algorithms inspired by physics.

chromodynamics notably admitting four-gluon vertices. Moreover, many-body wave functions display strong higher-order quantum correlations and, due to entanglement, cannot be fully described by two-point correlation functions. In inferential problems, a general multivariate distribution must involve higher-order interactions, as for instance in higher-order graphical models. Likewise, in network science pairwise networks cannot capture the many-body interactions in the brain [7–12], in social networks [13–16], ecosystems [17] and in inferential financial models [18, 19].

Addressing this limitation by allowing for higher-order interactions leads to the formulation of higher-order networks that include interactions between two or more nodes [20–27]. Being formed by building blocks such as triangles, tetrahedra, hypercubes, etc., higher-order networks and, in particular, simplicial complexes allow for a topological description of complex systems that changes dramatically our understanding of the interplay between structure and dynamics.

Topology involves the study of shapes and of their invariant properties such as the Betti numbers and the Euler characteristics. Topology is at the core of Topological Data Analysis (TDA) [8, 28–32] and persistent homology, and has wide applications including brain network analysis, [7–12], filtering algorithms [18, 19], as well as homological percolation [33, 34]. In particular, TDA has already shown to be key to detecting new higher-order aspects of brain networks [7–12] and to offer a very powerful set of tools to characterize different states of brain activity. Weighted homology and cohomology are also key to enhancing the representation power of simplicial complexes that can encode hypergraph data without loss of information [35]. Topological filtering algorithms [18, 19] have demonstrated to be a very powerful Data Science tool, in particular for financial data. Moreover, topology can be used to analyse higher-order network structure by the study of the so-called homological percolation, which characterises the emergence of large cycles and of higher-dimensional holes [11, 34].

Topology is not only crucial to characterize the structure of complex systems, but it is also essential to capture their higher-order dynamics. Specifically, higher-order topology unlocks fundamental mechanisms for higher-order topological diffusion [36–39], higher-order topological synchronization [40–45], topological pattern formation [46, 47], triadic percolation [48, 49], triadic neural networks [50, 51] and topological machine learning algorithms [52–56]. The new emergent field of topological dynamics of higher-order networks combines topology with nonlinear dynamics, giving rise to an entire new set of phenomena. These phenomena encode information and can be key to transforming our understanding of complex phenomena in neuroscience and climate, and to formulating a new generation of physics-inspired machine learning algorithms (See Figure 1).

This Perspective focuses on the paradigm shift that the adoption of higher-order networks implies for the understanding of the intricate interplay between topology and dynamics in complex systems. We focus on recent developments of the field by outlining the key results obtained so far, and the open challenges that future research must address. Accompany code, movies and material can be found in the repository [57].

Topological signals: beyond the node-centred view of network dynamics Dynamical systems defined on network topologies are widely studied [3, 4]. Yet most of the works so far make the implicit assumption that the dynamical state of a

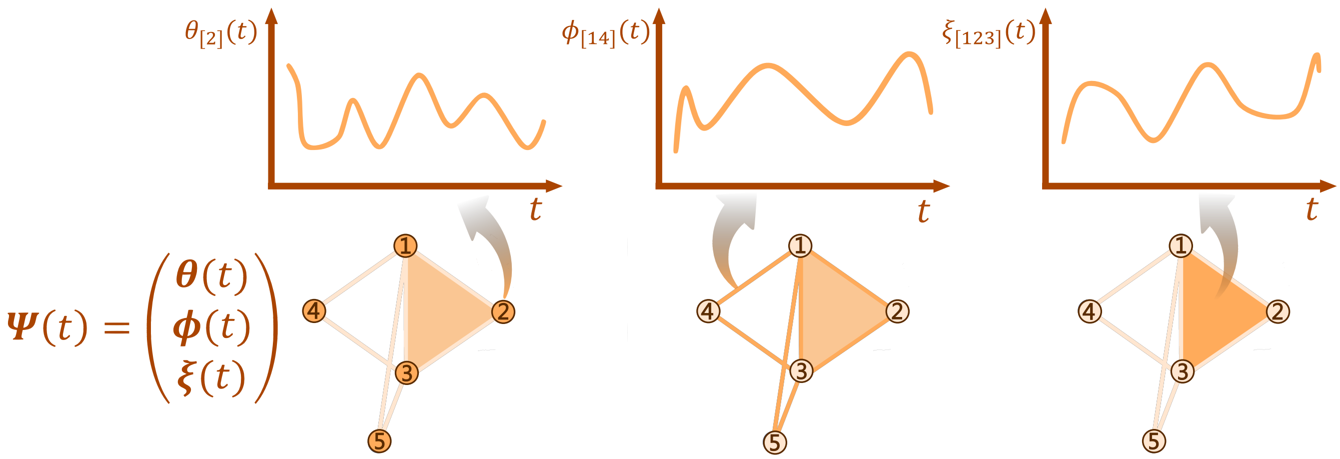

Figure 2. The dynamical state of a higher-order network. Going beyond the node-centered view of network dynamics, the dynamics of a higher-order network is captured by a topological spinor $\mathbf{\Psi}$, assigning a dynamical variable to each node, edge, triangle and higher-order simplices.

network is defined exclusively by variables associated with its nodes. While this is a valid assumption in some cases, e.g., in epidemic spreading, however, it generally represents a rather limiting one. For example, dynamical variables associated with the edges of a network, such as fluxes, are ubiquitous.

A growing interest of the scientific community concerns topological signals, i.e., dynamical variables associated not only with the nodes, but also the edges, triangles, or other higher-order structures. Important examples are synaptic signals between neurons or edge signals at the level of brain regions [10, 11], and in general biological transportation networks [58]. Other examples include currents at different locations in the ocean [39], the influence of volcanic activity on teleconnections in the climate [59], as well as the velocity of winds at a given altitude and geographical location. Citation counts obtained as a function of time by a collaboration of several authors are other examples of topological signals of higher dimensions.

Abstracting from these examples, we can describe the dynamical state of a simplicial complex formed by nodes, edges and triangles as comprising topological signals defined on each dimension. These are encoded in the *topological spinor* (see Box 1 and Figure 2).

While in other fields of physics, such as gauge theory [60] and quantum information [61], having dynamical variables associated with edges of plaquettes is widely accepted, in network science and machine learning the description of the dynamical state of complex systems with a topological spinor represents a radically new perspective that leads to unprecedented research questions. On one side, research is exploring new algorithms [52, 55, 62] for treating and processing topological signal data, and new topological neural network architectures to make predictions on topological signals. On the other side, research is building on the powerful tools provided by discrete topology and discrete exterior calculus [63] to characterize collective behavior of topological signals [40, 41, 46, 47, 64].

Topology shapes dynamics: topological synchronization Synchronization [65–68] refers to the emergence of a collective and ordered dynamical motion of an extensive number of oscillators. Synchronization is universal in characterizing the dynamics of complex natural and man-made systems, ranging from brain dynamics to power grids and Josephson junctions. The two most fundamental models that capture the synchronization transition on complex networks are the Kuramoto model [65] and global synchronization [69, 70] of coupled identical oscillators. Both approaches are traditionally defined only for node topological signals, i.e., oscillators located on the nodes of the network and coupled via edges. While the Kuramoto model describes global as well as different types of partial synchronization of heterogeneous oscillators, i.e., oscillators that in absence of interactions have different phases, global synchronization involves identical oscillators which, however, may follow an arbitrary, even chaotic, dynamics.

Going beyond conventional dyadic networks, higher-order topological synchronization allows us to treat not only synchronization of node topological signals, but also of topological signals of higher-order structures including, most relevantly for the applications, edge topological signals. The higher-order topological synchronization includes the Topological Kuramoto model [40, 64], and topological global synchronization [41], which display a phenomenology that changes drastically our understanding of the interplay between topology and dynamics.

The Topological Kuramoto model (see Box 2) reveals a very surprising connection with topology. On one side we can say that the *topology shapes the dynamics* and on the other side that *the dynamics learns the underlying topology* of the higher-order network. In fact, the Topological Kuramoto model displays striking differences compared to the traditional node-based Kuramoto model which reveals its deep connections with topology. Firstly, the synchronization dynamics of the n-dimensional topological signal is only possible if the higher-order network has at least one n-dimensional hole. Secondly, the synchronized state is localized on the holes of the higher-order structure. If the Topological Kuramoto model is defined on a simplicial or cell complex that contains more than a single n-dimensional hole, the synchronization dynamics might be driven by an harmonic eigenvector localized on a single hole or by a linear combination of the harmonic eigenvectors localized on different holes (see Figure 3). Changing the homology of the simplicial complex by filling some holes, or modulating their geometry by changing the metric matrices (weights associated with nodes, edges, and higher-order simplices) as well as changing the intrinsic random frequencies of oscillators, can change the nature of the synchronized state. Relevantly, brain research has uncovered that the higher-dimensional holes capture important dynamical information [7–9, 12]. Therefore, an interesting working hypothesis is whether the localized dynamics on the holes of a simplicial complex can be used to store information [71], and whether control theory [72] can provide key mechanisms for driving the dynamics towards a specific hole, or from one hole to another.

The higher-order Topological Kuramoto dynamics, defined in Eq. (1), entails one linear transformation of the signal induced by a boundary operator, a non-linear transformation due to the application of the sine function, concatenated by another linear transformation induced by another boundary operator. These dynamical transformations are also at the basis of simplicial neural architectures [54, 73], especially when weighted boundary matrices are adopted [63].

This important interplay between topology and dynamics can be even enriched by considering weighted and directed versions of this model [43, 44]. An interesting outcome is revealed by the investigation of global topological synchronization [41], i.e., the conditions that allow topological oscillators sitting on higher-order simplices to globally synchronize. For node-based oscillators, the global synchronized state always exists. Its existence is ensured by the fact that the constant eigenvector is always a harmonic eigenvector of the graph Laplacian. As such, a basic question is whether the global synchronized state is dynamically stable. Moving to global topological synchronization of oscillators placed on higher-order topological signals, we observe that the globally synchronized state is not in general guaranteed to exist. Also, the synchronized dynamics will be localized on the harmonic eigenvectors of dimension n. As a result, if the simplicial complex does not contain a harmonic eigenvector constant on all the n-dimensional simplices, then it is not possible to observe global topological synchronization of order $n > 0$. We thus infer that only some simplicial and cell complexes can sustain global synchronization. However, some cell complexes such as the square lattice tessellation of the torus can sustain global synchronization for topological signals of any order n (see Figure 3) and an appropriate choice of the weights of the simplicial complex can further facilitate the global synchronized state [74]. Hodge Laplacians also offer a way to revisit higher-order diffusion on simplicial complexes, by providing an extension for the notion of the spectral dimension [37, 75], and to define higher-order random walks [39] which allows for a separation of diffusion of the irrotational and solenoidal component of the dynamics [76] and control [36]. Moreover, Hodge Laplacians provide a spectral principle for community detection [77] related to clique communities [78] and k-connectedness [20].

The Topological Dirac operator and higher-order dynamics While the Hodge Laplacians are suitable to treat topological signals of a given dimension, e.g., edge signals or triangle signals, they fall short in describing how topological signals can cross-talk. The Topological Dirac operator (see Box 3) allows the treatment of topological signals of different dimensions simultaneously and coherently. Originally defined in lattice gauge theory to define staggered [60] and Dirac-Kälher fermions on lattices [80], the Dirac operator has then been adopted in the context of non-commutative geometry [81], in quantum graphs [82] and in quantum computation [83, 84] over the years. Only very recently it has been understood that the Topological Dirac operator has not only relevance for quantum physics but has also important applications for the study of complex systems and the collective phenomena involving topological signals of different dimensions [85].

The Topological Dirac operator is rooted in theoretical physics [85–88] and can be used to formulate a Topological Dirac equation [85], where the eigenstates have a topological interpretation and are defined on nodes as well as links and higher-order simplices. For this equation, the matter-antimatter symmetry is broken for states of particles and antiparticles of energy $E = m$ (see Figure 4). Moreover, the Topological Dirac operator can be used to define the mass [86] of simple and higher-order networks and is at the basis of an information theory coupling matter with geometry [88]. The Topological Dirac operator can be coupled to metric matrices leading to symmetric and asymmetric weighted Dirac operators [35]. Moreover, as for the continuous Dirac operator, the Topological Dirac operator can be coupled to gamma matrices to extend the Dirac operator to topological signals comprising for instance two-dimensional node signal and two-dimensional edge signals or even a two-dimensional node and a one-dimensional edge signals [47, 85].

Recently it is emerging that the Dirac operator is a very versatile algebraic operator that has wide applications in the study of complex systems, and that can become foundational to study dynamics on networks and simplicial complexes. Here we provide few illustrative examples of its use in the field of higher-order topological dynamics.

The Topological Dirac operator naturally defines Dirac synchronization [42, 45, 89] locally coupling node and edge

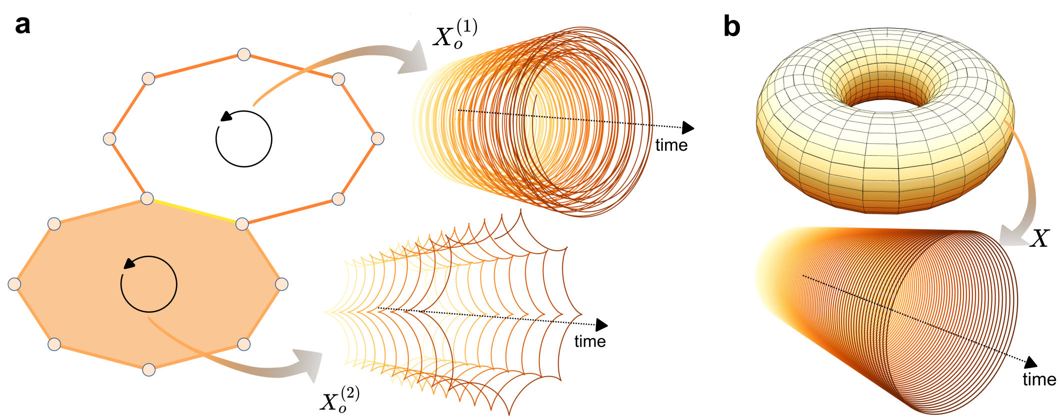

Figure 3. Topological Kuramoto and global synchronization. The synchronization of the topological signals is driven by the presence of n dimensional holes in the higher-order network. In the topological Kuramoto model [40] empty and full cells (panel a shows the case of edge synchronization for a cell complex with one empty cell or hole) display a different dynamical state as evident from the dynamics of the local complex order parameters defined in Box 2. Panel (a) represents $X_o^{(1)}$ and $X_o^{(2)}$ localized on the two cycles of the higher-order structure. In the limit in which there are no n-dimensional holes the dynamics of the n-order topological Kuramoto model freezes. While the higher-order topological Kuramoto can achieve synchronization also if the holes of the cell complex are localized, global synchronization requires the existence of a harmonic eigenvector constant on each cell of the cell complex, which is achieved if there is a single delocalized hole, such as for the square lattice tessellation of the torus (panel b) displaying a global complex order parameter $X = \sum_\alpha e^{\mathrm{i}\phi_\alpha}/N_n$ oscillating in phase while keeping its absolute value $|X|$=1 (panel b).

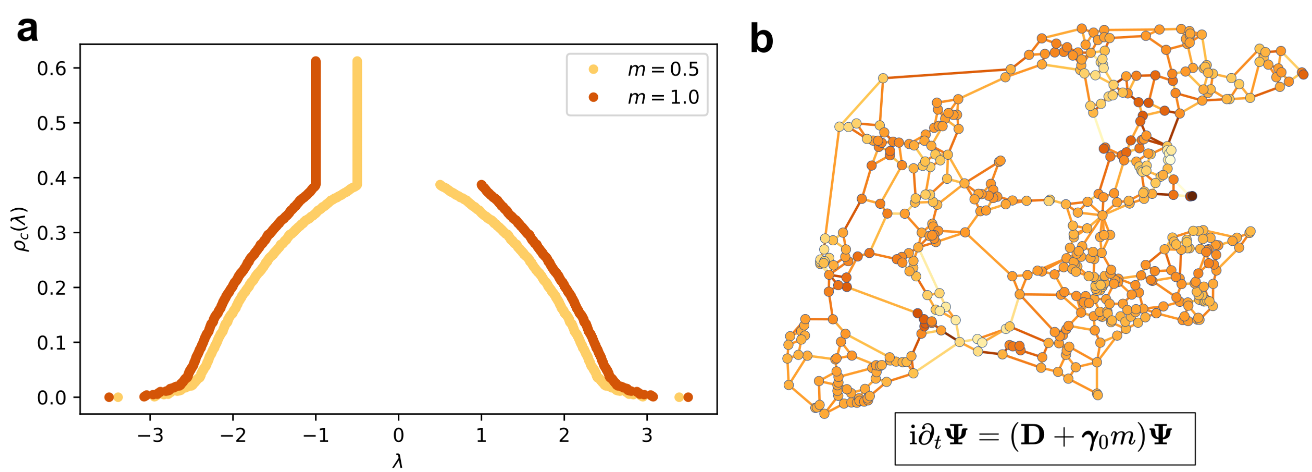

Figure 4. The properties of the Topological Dirac operator and the Topological Dirac equation. Panel a represents the spectrum of the Topological Dirac equation with mass $m = 1/2$ and $m = 1$ of the fungal network shown in panel b. Panel b represents the eigenstate of the Topological Dirac operator with energy $E = -1.1929$ and $m = 1$, associated to the different colors of the nodes and edges of the network. In panel a the spectrum is described by the cumulative distribution of eigenvalues $\rho_c(\lambda)$ indicating for positive values of λ the number of eigenvalues μ of the Topological Dirac operator with $\mu \geq \lambda$ and for negative values of λ the number of eigenvalues μ of the Topological Dirac operator with $\mu \leq \lambda$. It is apparent that the spectrum is symmetric with the only exception of the eigenvalues of energy E with $|E| = m$ that are not chiral. The fungi network is 'Pp_M_Tokyo_U_N_26h_6' from Ref. [79].

topological signals inducing a discontinuous synchronization transition on fully connected as well as on random networks. A peculiar property of Dirac synchronization is that on a network, the order parameter of the dynamics includes linear combinations of node and edge signals. Moreover, Dirac synchronization leads to the emergence of spontaneous rhythms, i.e., fluctuations of the order parameter, which is promising for the study of biological rhythms in the brain and rhythmic behaviour in the climate system, as for example the see-saw relationship of the East Asian - Australian summer monsoon [90].

The Topological Dirac operator is also key to defining topological patterns that extend the notion of Turing patterns in the continuum [91] or on networks [92, 93] to topological signals defined on nodes, edges and even squares of cell complexes [46, 47]. On a network, these topological patterns involve node and edge signals, and constitute an important generalization of the Turing instability existing in the continuum, so far discretized only by considering node signals. Interestingly, patterns formed by one node and one edge signals can only be static, however, by using the 3-way Dirac operator acting on two-dimensional node and one-dimensional edge topological signals (or, as a matter of fact, one-dimensional node and two-dimensional edge topological signals), dynamical patterns emerge as well [47]. When the dynamics of node and edge signals is coupled exclusively by the Dirac operator, Dirac patterns emerge with a very distinct dynamical signature [47].

The spectral properties of the Dirac operator encode topological features of the simplicial complexes, and its weighted version reveals the geometric degree of freedom of the simplicial complex. Moreover the Dirac operator can be defined for single as well as for multiplex networks [94]. The strong interplay between the structure of the simplicial complex and the spectral properties of the Dirac operator is at the core of its growing popularity in persistent homology [83, 84, 95, 96]. Moreover due to its ability to treat coupled topological signals of different dimensions, the Dirac operator is used also in machine learning research, for signal processing on simplicial complexes and hypergraphs [53, 97] and for formulating Gaussian kernels [98], and simplicial neural networks [99, 100].

Therefore, the Topological Dirac operator, that in its continuum version has played such an important role in physics, is now also emerging as a fundamental algebraic topology tool for the study of complex systems and for the formulation of machine learning algorithms. The research in the field is only at its infancy and we believe that it has great potential for new discoveries in complex systems as well as in machine learning rooted in and inspired by theoretical physics.

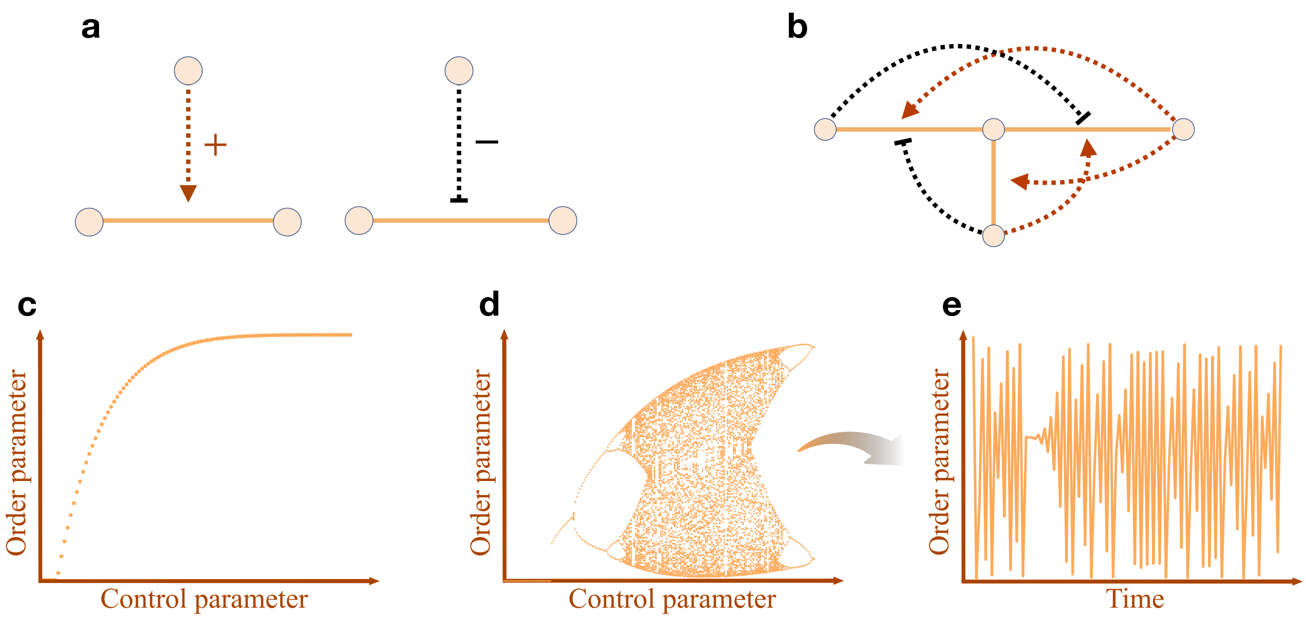

Figure 5. Signed triadic interactions and the phase diagram of triadic percolation. Triadic interactions occur when one node regulates the interactions between two other nodes either positively or negatively (panel a) leading to higher-order networks with triadic interactions (panel b). The inclusion of triadic interactions turns percolation into a fully-fledged dynamical process allowing the temporal modulation of the giant component. In particular, the phase diagram of percolation, which in absence of triadic interactions leads to a second-order phase transition (panel c), under the presence of triadic interactions can be predicted to be an orbit diagram (panel d), demonstrating that the percolation order parameter undergoes a route to chaos. Thus for some control parameter values the order parameter changes in time following a chaotic time series (panel e).

Topology is dynamical: triadic interactions Until now, we have only discussed models in which the topology of the simplicial complexes is time independent. However, there are many situations in complex systems in which the topology changes in time, as it is typically observed in brain resting activity and climate. Basic motifs of such time-evolving networks are triadic interactions occurring when nodes regulate the interactions among other pairs of nodes. Triadic interactions can be signed, i.e., the regulator node can either enhance or inhibit the interactions between other two nodes. A network with triadic interactions is at the same time a higher-order network and a network of networks. Indeed, triadic interactions include more than two nodes and can be combined such that a single interaction is regulated by more than one node. Typical examples are neuron-glia interactions in the brain, transcription networks, and ecological interactions [17]. While in each scientific domain (neuroscience, molecular biology, ecology, finance) triadic interactions are well established, only recently it is becoming clear that their role in modulating interactions leads to novel dynamical processes including triadic percolation [48], triadic Hopfield model [50], more general neuron-astrocyte models of associative memory [51], and information propagation in non-equilibrium signaling networks [101].

Triadic interactions can be used to regulate on or off structural edges, leading to triadic percolation [48], a fully-fledged dynamical process in which the giant component changes in time. Specifically, triadic percolation implements a two-step algorithm: (i) nodes are active when they belong to the giant component of the structural network formed by nodes and active structural edges, and (ii) active nodes regulate the structural edges through their associated triadic interactions. The resulting percolation process has an order parameter, given by the fraction of nodes in the giant component, that is time varying and can be proven to undergo a route to chaos in the universality class of the logistic map. Therefore, the phase diagram instead of describing the standard second-order percolation transition is represented by an orbit diagram. This reveals that the percolating phase is characterized by non-stationary dynamics of the order parameter (see Figure 5). Thus the network can display a blinking behavior (i.e., periodic activation of different sets of nodes) or, in the infinite network limit, a chaotic dynamics of the order parameter. In spatial networks defined on a $2D$ torus, triadic percolation [49] gives rise to complex spatio-temporal patterns and to a giant component whose topology changes in time displaying, for some parameter values intermittency between patterns of different topology. These results open new scenarios for understanding the spatio-temporal modulation of the giant component, e.g., in neuroscience and climate.

Higher-order networks are starting to reveal how network topology, which is key in traditional physics fields such as high-energy physics and condensed matter, is also key to capturing higher-order network dynamics. This emerging theoretical framework discloses innovative findings in various disciplines ranging from physics, computer science, Earth science, neuroscience to finance. In this Perspective we have covered fundamental aspects of this nascent field of research, and the first interdisciplinary applications, by outlining the key challenges that emerge from the main results obtained so far.

The growing field of higher-order topological dynamics opens interesting perspectives for the development of a more comprehensive theory of complexity by combining higher-order networks with topology and non-linear processes, and by interpreting these interactions under the lens of information theory.

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Box 1

Mathematically speaking each topological signal defined on simplices of dimension n is a n-cochain C^n. If the simplicial complex is composed of N_0 nodes, N_1 edges and N_2 triangles, its dynamics is encoded by a *topological spinor* $\mathbf{\Psi} \in C^0 \oplus C^1 \oplus C^2$ [85] (see Figure 2) which can be represented as the vector $\mathbf{\Psi} = (\boldsymbol{\theta}^\top, \boldsymbol{\phi}^\top, \boldsymbol{\xi}^\top)^\top$, where $\boldsymbol{\theta} \in \mathbb{R}^{N_0}, \boldsymbol{\phi} \in \mathbb{R}^{N_1}, \boldsymbol{\xi} \in \mathbb{R}^{N_2}$ are respectively topological signals defined on nodes, edges and triangles. The natural operators acting on topological signals are the boundary operators $\mathbf{B}_{[n]}$ [102]. On an unweighted network, the boundary operator $\mathbf{B}_{[1]}$ represents the divergence, its transpose $\mathbf{B}_{[1]}^\top$ represents the gradient and $\mathbf{B}_{[2]}^\top$ represents the curl. The Hodge Laplacians $\mathbf{L}_{[n]}$ [102, 103] are defined through the boundary operators and describe diffusion from n-simplices to n-simplices through $n-1$ or $n+1$ simplices. Hence the 1-Hodge Laplacian describes diffusion from edges to edges passing through nodes or passing through triangles. The boundary operators and the Hodge Laplacians are pivotal to uncover the interplay between structure and dynamics of higher-order networks. The key topological and spectral properties of the Hodge Laplacian $\mathbf{L}_{[n]}$ are that the dimension of its kernel is given by the n-th Betti number β_n and that its harmonic eigenvectors can be chosen on a basis in which they are mostly localized along n-dimensional holes. Moreover, the Hodge Laplacians obey Hodge decomposition. For an edge signal, this fact implies that there is a unique way to decompose these topological signals into a sum of an irrotational (curl free), a solenoidal (divergence free) and an harmonic component. For the discussion of higher-order diffusion and random walks using the Hodge-Laplacians and the boundary operators see Refs. [36, 37, 39, 76]. Note that the properties of this type of diffusion processes are distinct from the ones defined on hypergraphs [104, 105] as diffusion on simplicial complexes is allowed to go at most one dimension up or one dimension down.

Box 2

The higher-order Topological Kuramoto model [40] captures the topological synchronization of the n-dimensional topological signal $\boldsymbol{\phi}$ of elements ϕ_α describing non-identical oscillators placed on n-dimensional simplices. In this model the topological signal $\boldsymbol{\phi}$ follows the dynamical equation

$$\frac{d\boldsymbol{\phi}}{dt} = \boldsymbol{\omega} - \sigma \mathbf{B}_{[n]}^\top \sin(\mathbf{B}_{[n]}\boldsymbol{\phi}) - \sigma \mathbf{B}_{[n+1]} \sin(\mathbf{B}_{[n+1]}^\top \boldsymbol{\phi}) \,, \tag{1}$$

where the sine function is taken element-wise, $\mathbf{B}_{[n]}$ are the boundary matrices, σ is the coupling constant and $\boldsymbol{\omega}$ is the vector of intrinsic frequencies drawn from a random unimodal distribution, typically a Gaussian or Lorentzian distribution. The Topological Kuramoto model reduces for $n=0$ to the standard node-based Kuramoto model [65]. It leads to a continuous synchronization transition for any $n \geq 0$, while the adaptive modulation of its coupling constants with the global order parameters gives rise to explosive discontinuous transitions [40, 64]. The Topological Kuramoto model is a gradient flow of the Hamiltonian $\mathscr{H}$

$$\mathscr{H} = \boldsymbol{\omega}^\top \boldsymbol{\phi} - \sigma \mathbf{1}_{N_{n-1}}^\top \cos(\mathbf{B}_{[n]}\boldsymbol{\phi}) - \sigma \mathbf{1}_{N_{n+1}}^\top \cos(\mathbf{B}_{[n+1]}^\top \boldsymbol{\phi}), \tag{2}$$

where $\mathbf{1}_{N_n}$ is the N_n column vector whose elements are all ones. This Hamiltonian has the fundamental state that is highly degenerate and includes all the eigenvectors in the kernel of the Hodge-Laplacian $\mathbf{L}_{[n]}$. The Topological Kuramoto dynamics can be associated to global topological complex order parameters [40] given by $X_\pm = \sum_\alpha e^{\mathrm{i}\phi_\alpha^\pm}/N_{n\pm 1}$ where $\boldsymbol{\phi}^+ = \mathbf{B}_{n+1}^\top \boldsymbol{\psi}$ and $\boldsymbol{\phi}^- = \mathbf{B}_n \boldsymbol{\psi}$. These order parameters indicate when the non-harmonic modes freeze. In Figure 3 we indicate instead some local complex order parameters associated with the two octagons of the figure and given by $X_o = \sum_{\alpha \in \mathscr{O}} e^{\mathrm{i}(-1)^{f(\alpha)}\phi_\alpha}/N_o$ where the sum is extended to all the edges α incident to the octagon $\mathscr{O}$, formed by $N_o = 8$ edges, and where $f(\alpha) = 0$ if the edge α is oriented clockwise and $f(\alpha) = 1$ if the edge α is oriented anti-clockwise. As apparent from Figure 3 this order parameter can help us distinguish between filled and unfilled cycles just by considering the topological dynamics of the edge signals.

Box 3

On an unweighted simplicial complex of dimension 2, the Topological Dirac operator [85] $\mathbf{D}: C^0 \oplus C^1 \oplus C^2 \to C^0 \oplus C^1 \oplus C^2$ is given by

$$\mathbf{D} = \begin{pmatrix} 0 & \mathbf{B}_{[1]} & 0 \\ \mathbf{B}_{[1]}^\top & 0 & \mathbf{B}_{[2]} \\ 0 & \mathbf{B}_{[2]}^\top & 0 \end{pmatrix} \quad \text{with} \quad \mathbf{D}^2 = \mathscr{L} = \begin{pmatrix} \mathbf{L}_{[0]} & 0 & 0 \\ 0 & \mathbf{L}_{[1]} & 0 \\ 0 & 0 & \mathbf{L}_{[2]} \end{pmatrix}. \tag{3}$$

Hence, the Topological Dirac operator can be interpreted as the "square root" of the Laplacian and admits both positive and negative eigenvectors related by chirality. Note, however, that the harmonic eigenvectors break the chiral symmetry. On a 2-dimensional simplicial complex, the Topological Dirac operator $\mathbf{D}$ maps the topological spinor $\boldsymbol{\Psi} = (\boldsymbol{\theta}^\top, \boldsymbol{\phi}^\top, \boldsymbol{\xi}^\top)^\top$ into another topological spinor $\boldsymbol{\Phi} = ((\boldsymbol{\theta}')^\top, (\boldsymbol{\phi}')^\top, (\boldsymbol{\xi}')^\top)^\top = ((\mathbf{B}_{[1]}\boldsymbol{\phi})^\top, (\mathbf{B}_{[1]}^\top \boldsymbol{\theta} + \mathbf{B}_{[2]}\boldsymbol{\xi})^\top, (\mathbf{B}_{[2]}^\top \boldsymbol{\phi})^\top)^\top$, by projecting topological signals one dimension up or down thus allowing them to cross-talk. The Topological Dirac operator can be adopted to define a Topological Dirac equation [85] given by $\mathrm{i}\partial_t \boldsymbol{\Psi} = (\mathbf{D} + \boldsymbol{\gamma}_0 m)\boldsymbol{\Psi}$ where, for a 2-dimensional simplicial or cell complex, $\boldsymbol{\gamma}_0$ is the $\mathscr{N} \times \mathscr{N}$ diagonal matrix with diagonal blocks $(\mathbf{I}_{N_0}, -\mathbf{I}_{N_1}, \mathbf{I}_{N_2})$ (see Fig. 4). The Topological Dirac equation can be mapped to the original Dirac equation by adopting the dictionary listed in Table 1. As discussed in the main text the Topological Dirac operator has great potential to treat topological signals and is increasingly adopted to study the topology and the dynamics of higher-order networks.

Continuum Dirac operator	Topological Dirac operator
Spinor	Topological spinor (with topological interpretation)
Chirality	Chirality valid only for non-harmonic eigenvectors
Dirac operator as square-root of the Laplacian	Topological Dirac operator as square-root of the Gauss-Bonnet Laplacian $\mathscr{L}$

Table 1. Dictionary that can be used to map the Topological Dirac operator and the corresponding Topological Dirac equation [85] to the original Continuum Dirac operator and corresponding Dirac equation. The results for the Topological Dirac operator refer to the scenario in which the operator is acting on a topological spinor having a single topological signal in each dimension.

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