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Turing patterns in multiplex networks

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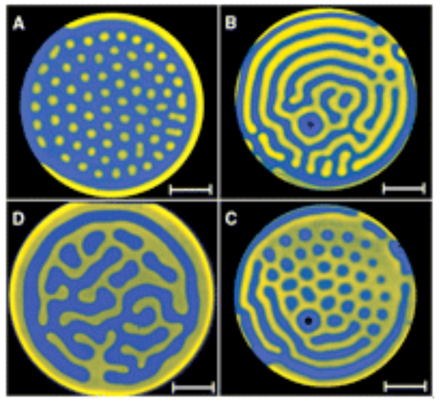
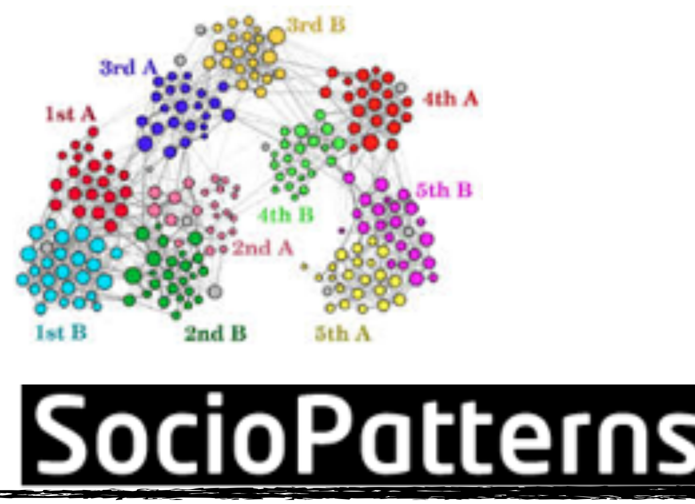
IAP VII/19 - DYSCO

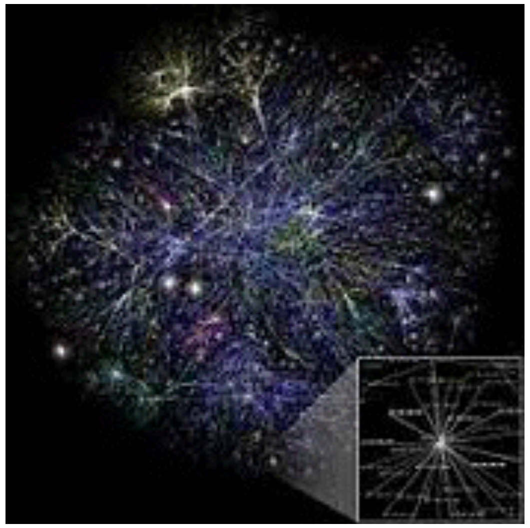
Patterns are ubiquitous

Animal kingdom



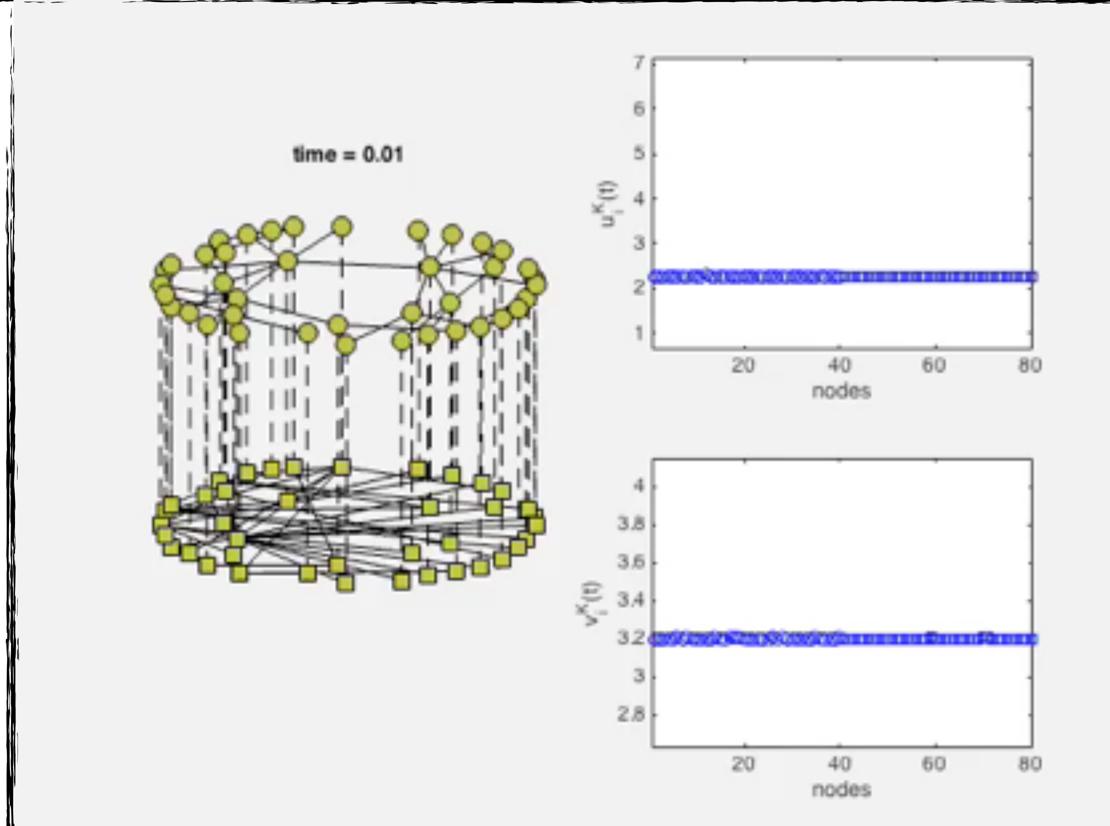
Chemistry





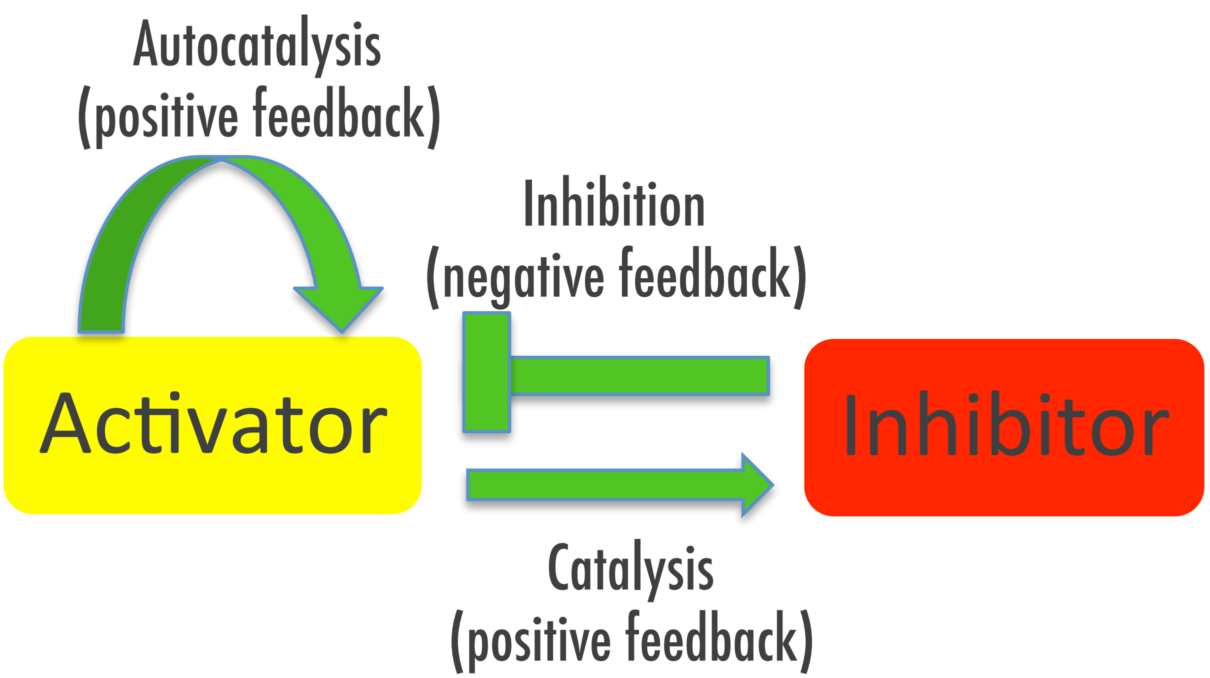
Internet

Twitter



One possible mechanism: Turing instability

Reaction



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Diffusion

$u(x,y,t)$: Amount of activator at time t and position (x,y)

$v(x,y,t)$: Amount of inhibitor at time t and position (x,y)

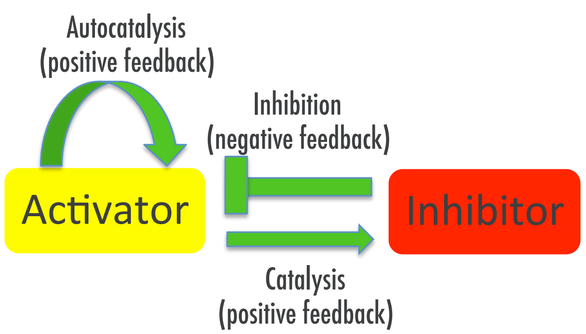
$$\begin{cases} \dfrac{\partial u}{\partial t} = f(u,v) + D_u \nabla^2 u \\ \dfrac{\partial v}{\partial t} = g(u,v) + D_v \nabla^2 v \end{cases}$$

$$(x,y) \in \Omega$$

+ boundary conditions
+ initial condition

One possible mechanism: Turing instability

Reaction



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$$(x,y) \in \Omega$$

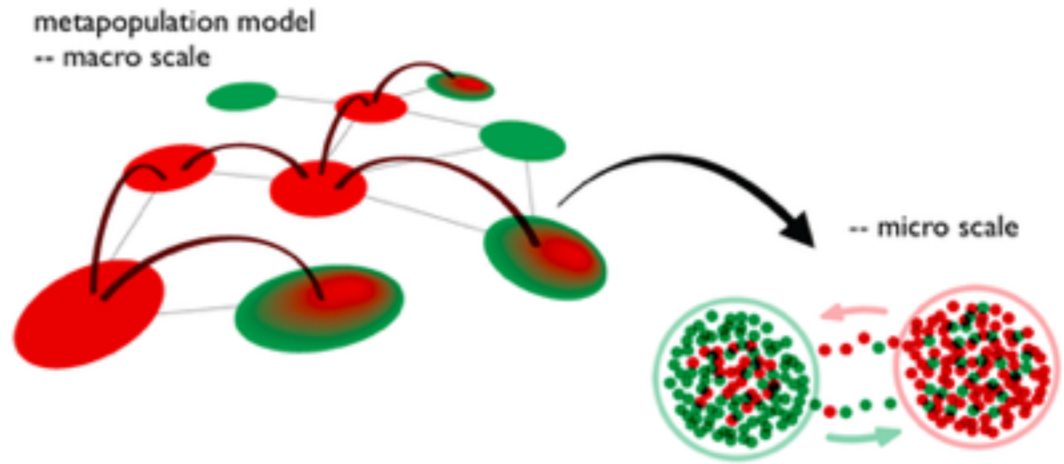
+ boundary conditions
+ initial condition

Diffusion can drive an instability by perturbing a homogeneous stable fixed point. Hence as the perturbation grows, non linearities enter into the game yielding an asymptotic, spatially inhomogeneous, steady state (stationary pattern).

A.M.Turing, *The chemical basis of morphogenesis*, Phil. Trans. R Soc London B, *237, (1952), pp.37*

Extension to networks

world flights map



Metapopulation models

e.g. in the framework of ecology:
May R., *Will a large complex system be stable?* Nature, 238, pp. 413, (1972)

Reactions occur at each node.
Diffusion occurs across edges.

Patterns : sets of nodes whose asymptotic state is far from the homogeneous equilibrium.

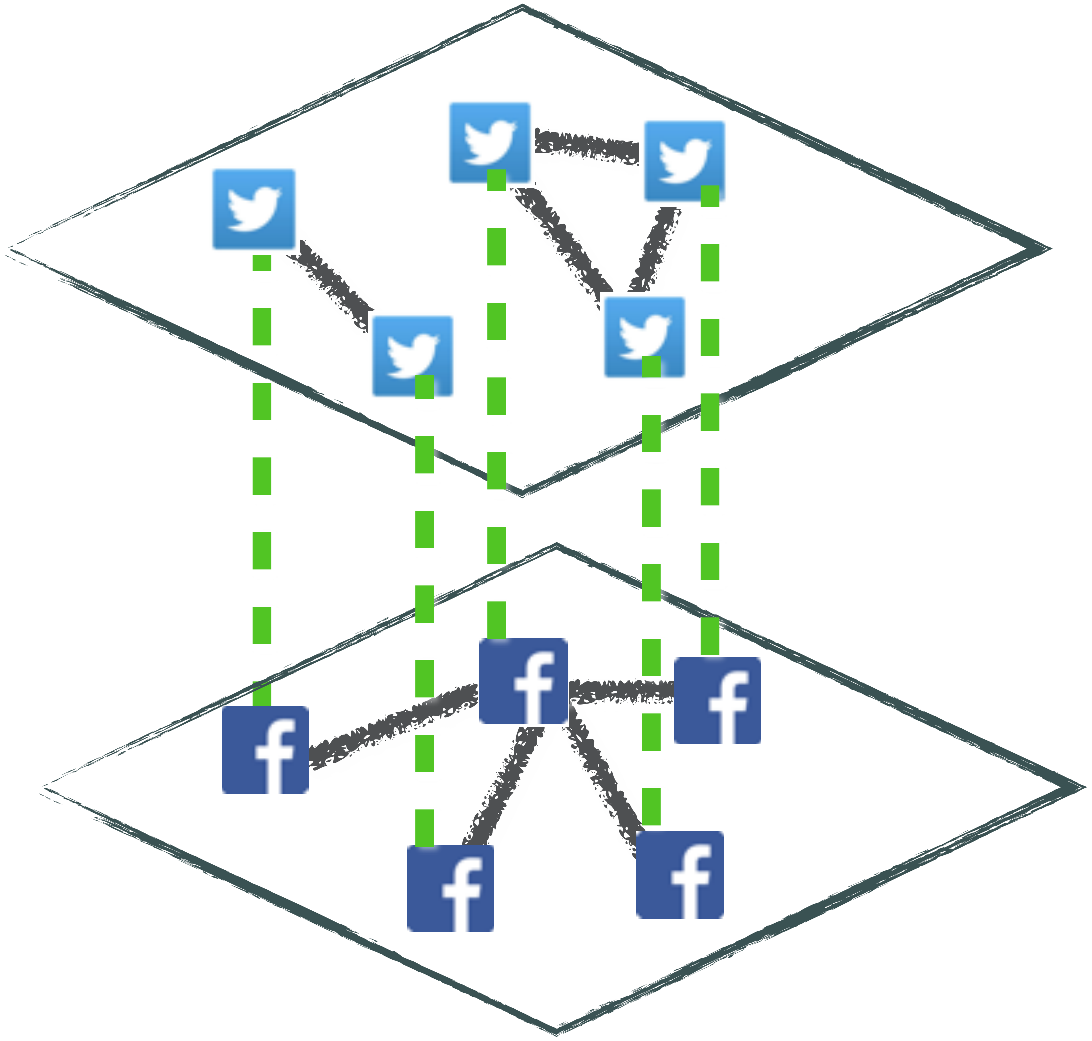
Nakao H. and Mikhailov A. S., Turing patterns in network-organized activator–inhibitor systems, Nat. Phys., 6, pp. 544 (2010)

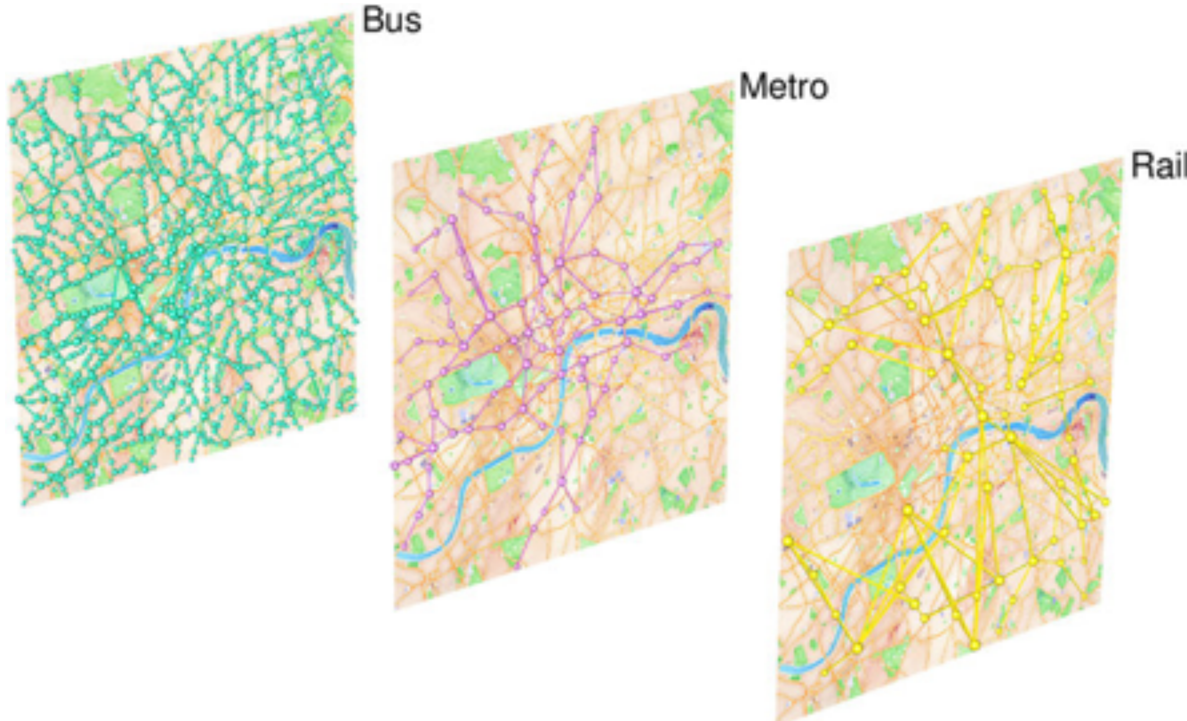
Systems composed by layers of networks: Multiplexes

<u>Social networks</u>

layers=different social networks

nodes=same agent in each SN





<u>Transportation networks</u>

layers=different modalities

nodes=same spatial location

Turing instabilities on multiplex networks

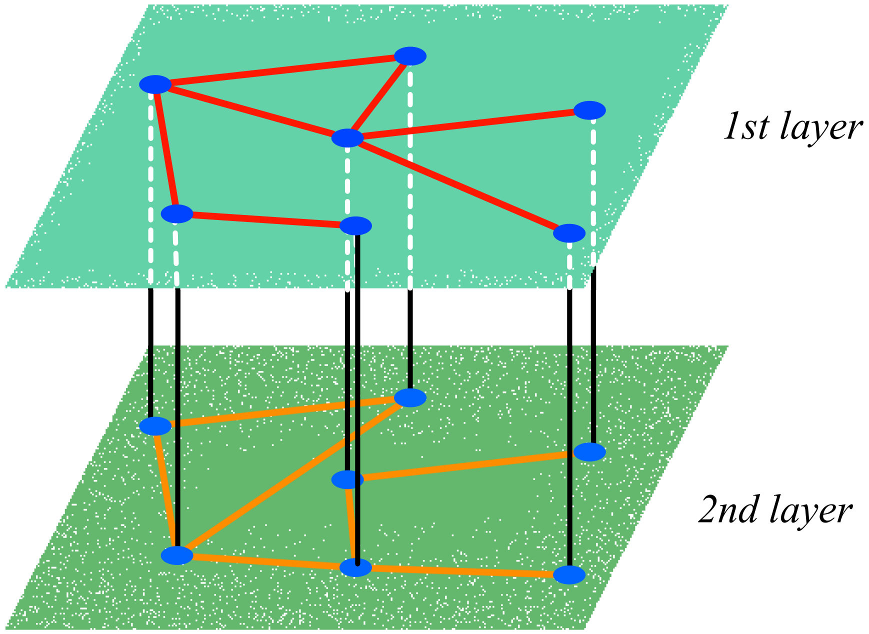
With K=1,2 (K=3 should be read K=1)

adjacency matrix of layer K

degree of ith note in layer K

$$L_{ij}^K = A_{ij}^K - \delta_{ij} k_i^K$$

Laplacian matrix of layer K



The same Ω nodes are present in each layer

$D_{u,v}^K$ **inter**-layer diffusion coefficient

$D_{u,v}^{12}$ **intra**-layer diffusion coefficient

$$\begin{cases} \dot{u}_i^K &= f(u_i^K, v_i^K) + D_u^K \sum_{j=1}^{\Omega} L_{ij}^K u_j^K + D_u^{12}\left(u_i^{K+1} - u_i^K\right) \\ \dot{v}_i^K &= g(u_i^K, v_i^K) + D_v^K \sum_{j=1}^{\Omega} L_{ij}^K v_j^K + D_v^{12}\left(v_i^{K+1} - v_i^K\right) \end{cases}$$

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General strategy

1) Determine an homogeneous solution $(u_j^K, v_j^K) = (\hat{u}, \hat{v})$ (assumed to be stable when $D_u^K = D_v^K = D_{u,v}^{12} = 0$)

General strategy

2) Linearize around this solution

$$u_j^K = \hat{u} + \delta u_j^K$$
$$v_j^K = \hat{v} + \delta v_j^K$$

$$\begin{pmatrix} \delta\dot{\boldsymbol{u}} \\ \delta\dot{\boldsymbol{v}} \end{pmatrix} = \tilde{\mathcal{J}} \begin{pmatrix} \delta\boldsymbol{u} \\ \delta\boldsymbol{v} \end{pmatrix}$$

$$\tilde{\mathcal{J}} = \begin{pmatrix} f_u \mathbf{I}_{2\Omega} + \boldsymbol{\mathcal{L}}_u + D_u^{12}\boldsymbol{\mathcal{I}} & f_v \mathbf{I}_{2\Omega} \\ g_u \mathbf{I}_{2\Omega} & g_v \mathbf{I}_{2\Omega} + \boldsymbol{\mathcal{L}}_v + D_v^{12}\boldsymbol{\mathcal{I}} \end{pmatrix}$$

Supra-Laplacian matrix $\boldsymbol{\mathcal{L}}_u + D_u^{12}\boldsymbol{\mathcal{I}}$

$$\boldsymbol{\mathcal{L}}_u = \begin{pmatrix} D_u^1 \mathbf{L}^1 & \mathbf{0} \\ \mathbf{0} & D_u^2 \mathbf{L}^2 \end{pmatrix}$$

$$\boldsymbol{\mathcal{I}} = \begin{pmatrix} -\mathbf{I}_\Omega & \mathbf{I}_\Omega \\ \mathbf{I}_\Omega & -\mathbf{I}_\Omega \end{pmatrix}$$

3) Study the spectrum of

$$\tilde{\mathcal{J}} = \begin{pmatrix} f_u \mathbf{I}_{2\Omega} + \boldsymbol{\mathcal{L}}_u + D_u^{12} \boldsymbol{\mathcal{I}} & f_v \mathbf{I}_{2\Omega} \\ g_u \mathbf{I}_{2\Omega} & g_v \mathbf{I}_{2\Omega} + \boldsymbol{\mathcal{L}}_v + D_v^{12} \boldsymbol{\mathcal{I}} \end{pmatrix}$$

to determine the existence of eigenvalues such that

$$\Re\lambda(D_{u,v}^{12}, D_{u,v}^{K}) > 0$$

Very hard for generic topologies

Small intra-layer diffusion case

Assume $D_v^{12} = \epsilon << 1 \qquad D_u^{12} = \mathcal{O}(\epsilon)$

$$\tilde{\mathcal{J}} = \begin{pmatrix} f_u \mathbf{I}_{2\Omega} + \boldsymbol{\mathcal{L}}_u & f_v \mathbf{I}_{2\Omega} \\ g_u \mathbf{I}_{2\Omega} & g_v \mathbf{I}_{2\Omega} + \boldsymbol{\mathcal{L}}_v \end{pmatrix} + \epsilon \begin{pmatrix} \frac{D_u^{12}}{D_v^{12}} \boldsymbol{L}^1 & \mathbf{0} \\ \mathbf{0} & \boldsymbol{L}^2 \end{pmatrix}$$

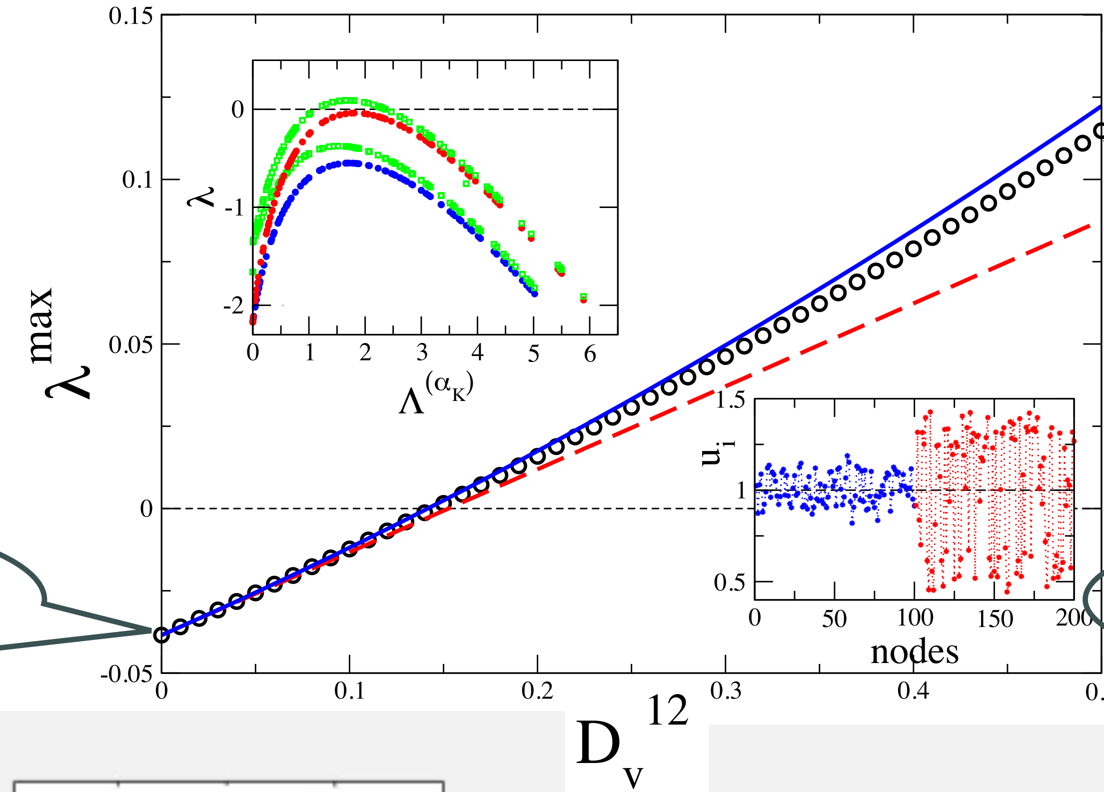
$$= \tilde{\mathcal{J}}_0 + \epsilon \boldsymbol{\mathcal{D}}_0$$

Perturbative approach to compute the spectrum

$$\lambda^{max}(\epsilon) = \lambda_0^{max} + \epsilon (U_0 \boldsymbol{\mathcal{D}}_0 V_0)_{k_{max} k_{max}} + \mathcal{O}(\epsilon^2)$$

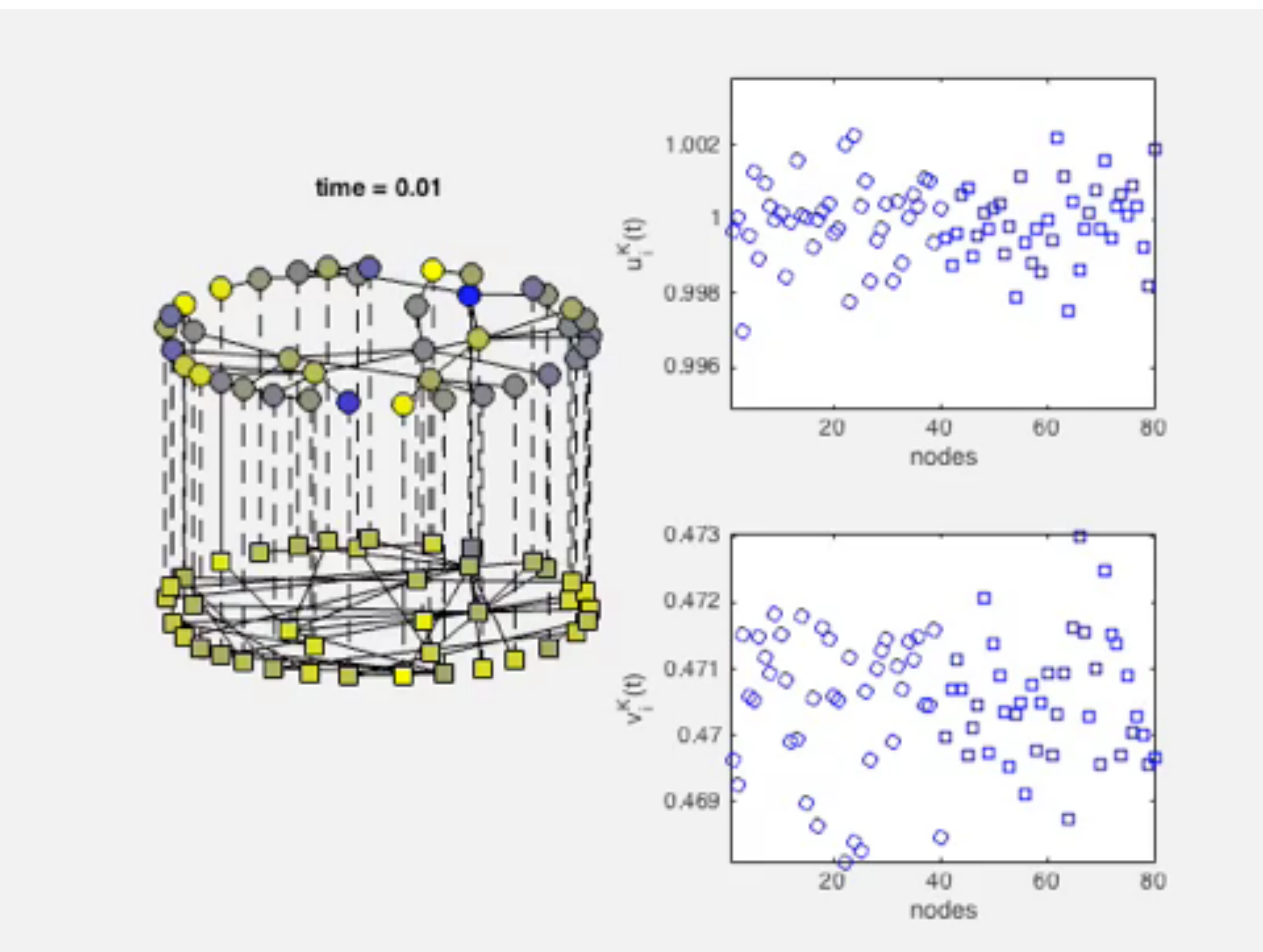
$$\lambda_0^{max} = \max \lambda_k(\epsilon = 0) \qquad k_{max} = \arg\max \lambda_k(\epsilon = 0)$$

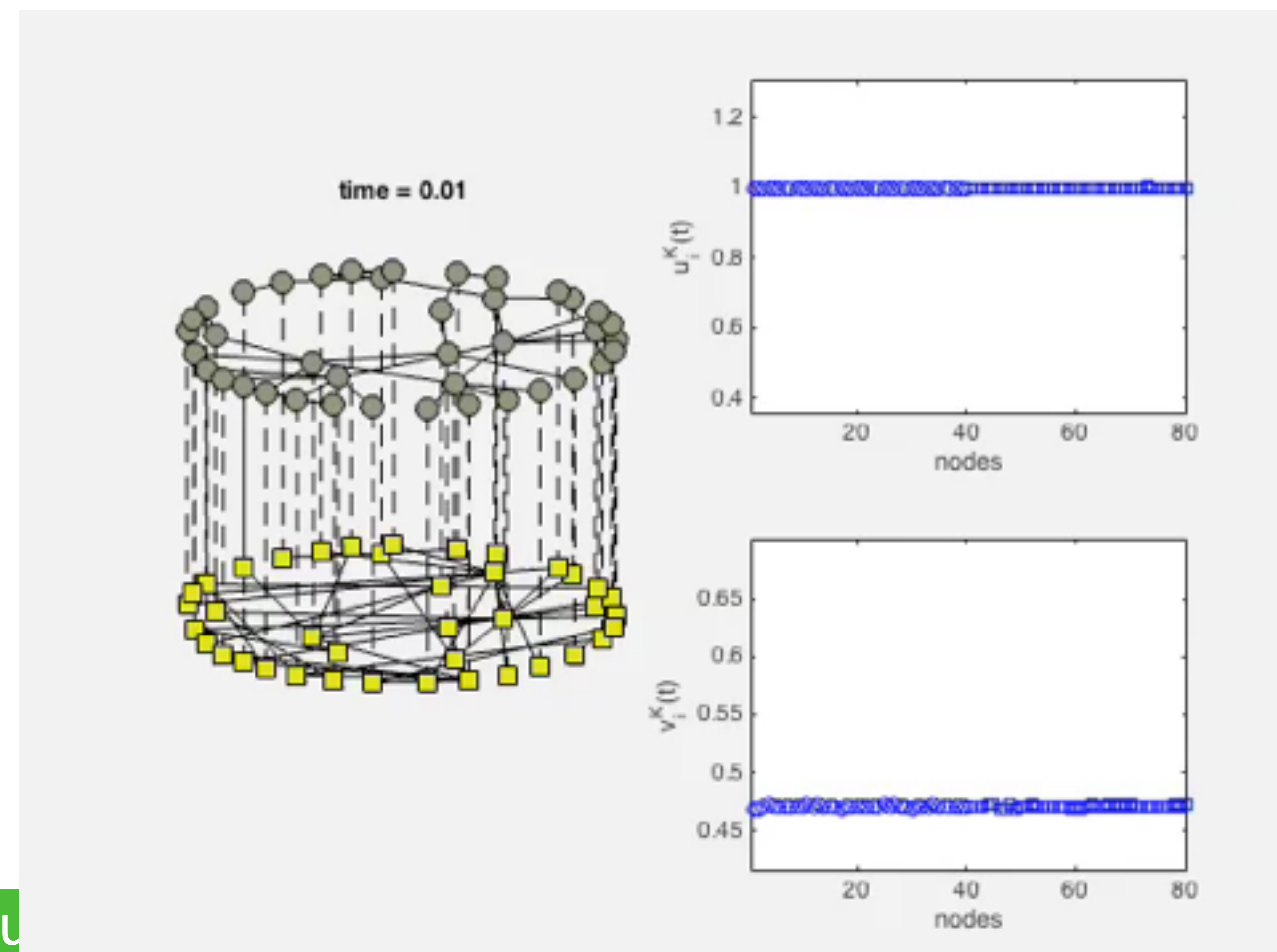
Small intra-layer diffusion case: onset of patterns



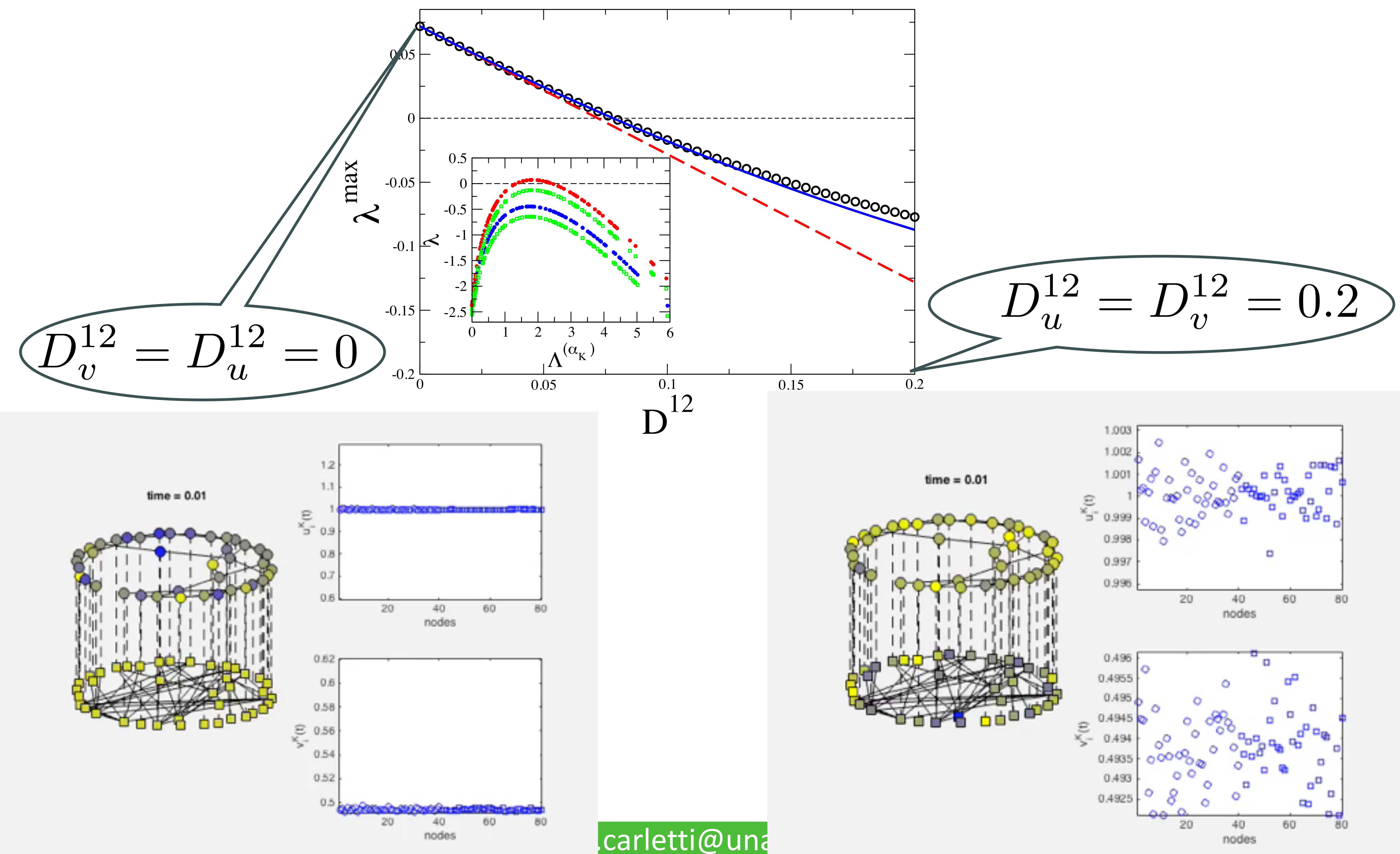
$D_v^{12} = D_u^{12} = 0$

$D_u^{12} = 0 \quad D_v^{12} = 0.5$





Small intra-layer diffusion case: destruction of patterns



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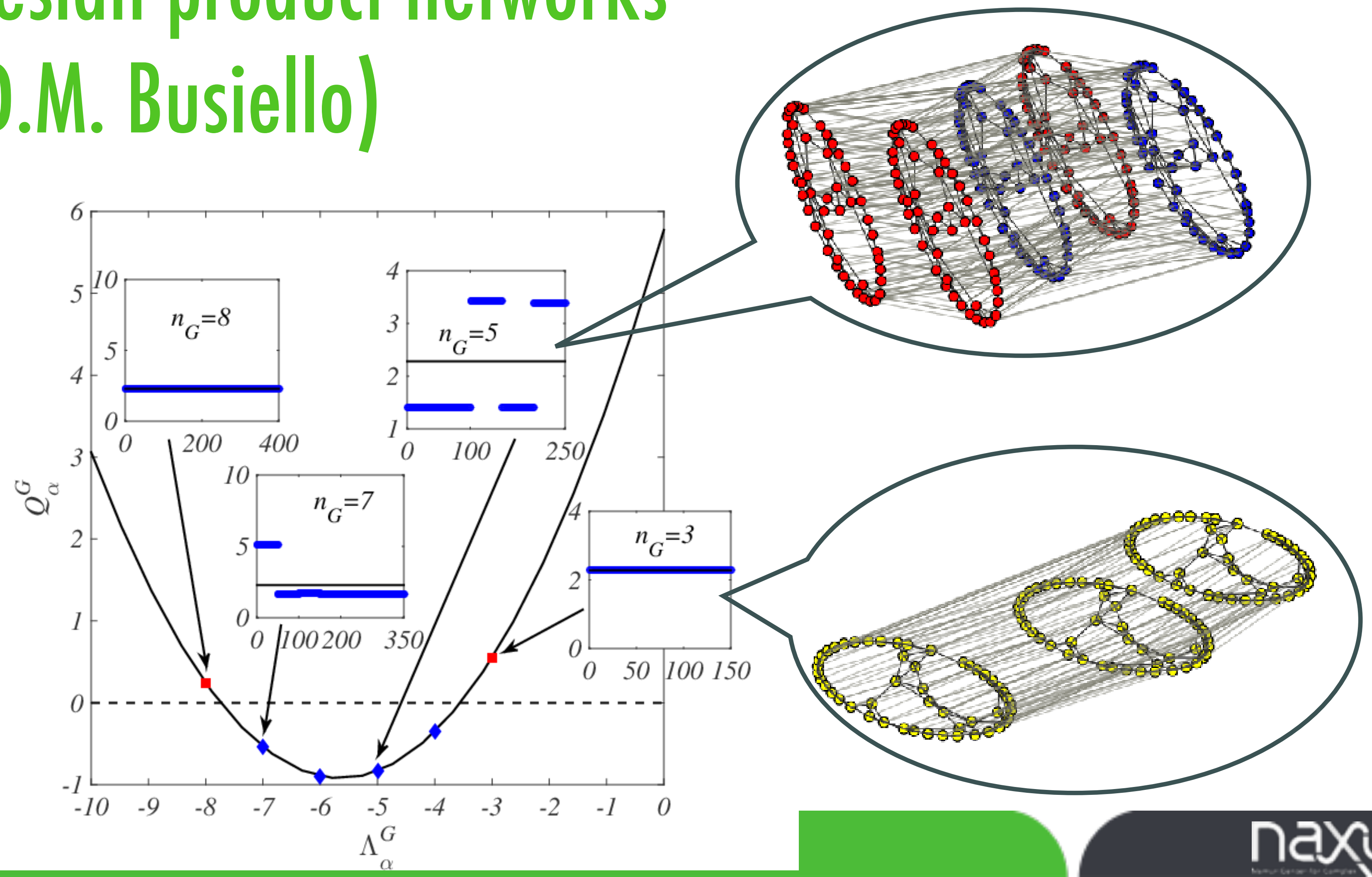
Remarks

1) The complementary case - large intra-layer diffusion - can be handled as well

2) The completely degenerate case - same network on all layers - can be handled using Cartesian product networks
(see talk Friday morning D.M. Busiello)



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