June the 6th, 2016, Logroño, Spain

Timoteo Carletti

A journey in the zoo of Turing patterns

Namur Center for Complex Systems

UNIVERSITÉ DE NAMUR

Acknowledgements

Belgian team: M. Asllani, J. Petit, G. Planchon

Italian team: D. Fanelli, D.M. Busiello, M. Galanti, F. Di Patti

IAP VII/19 - DYSCO

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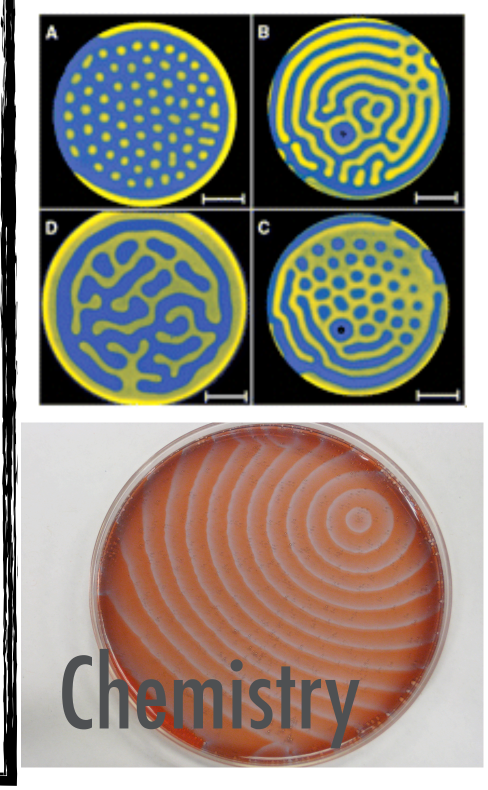
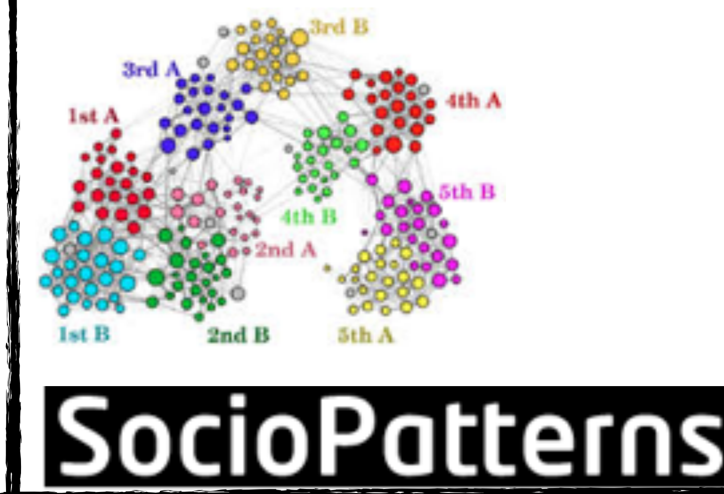
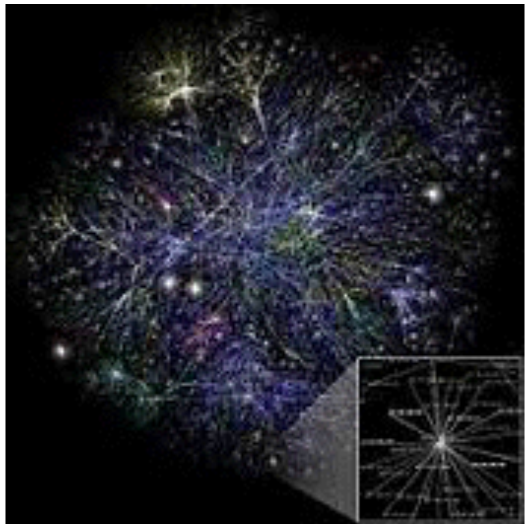
Patterns are ubiquitous

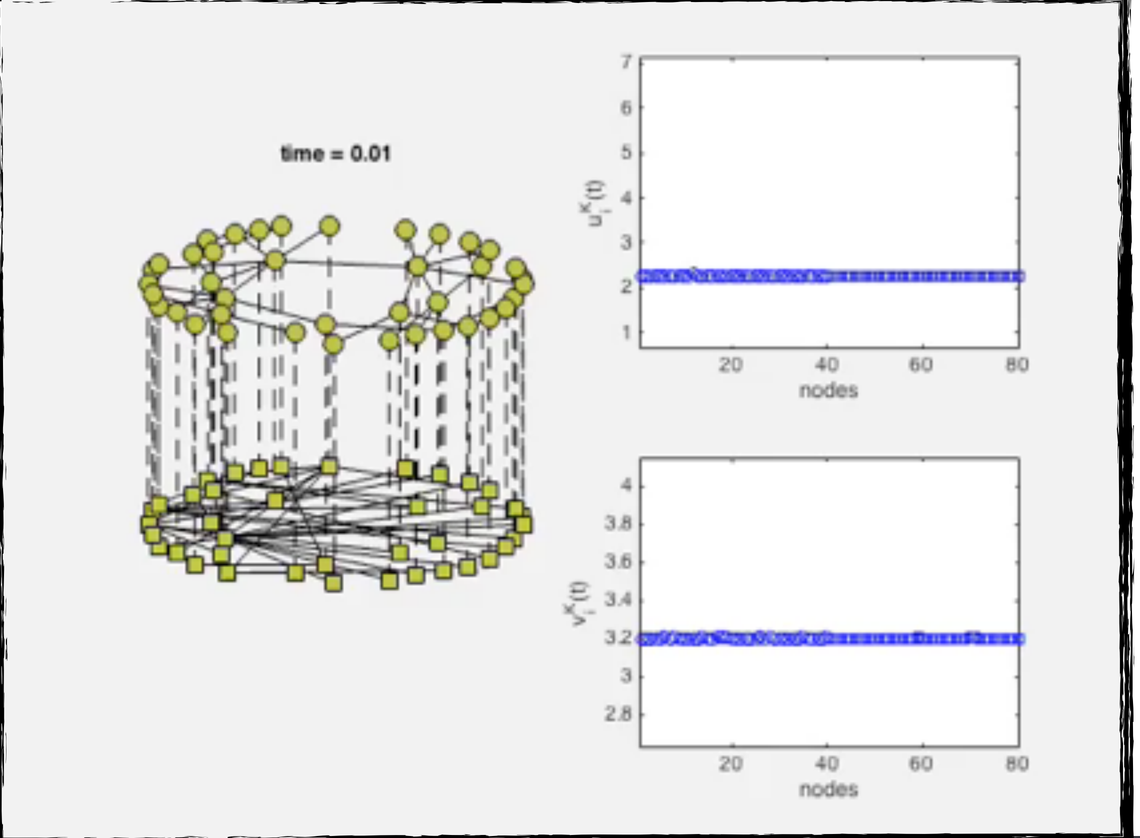






Internet

Twitter



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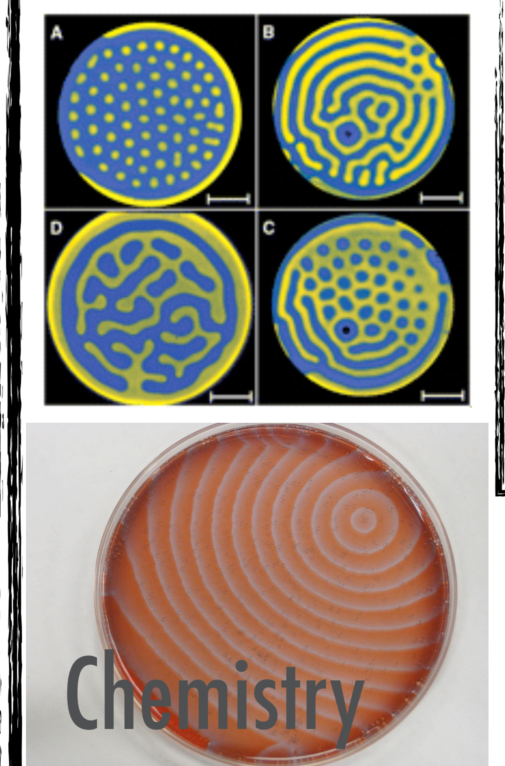

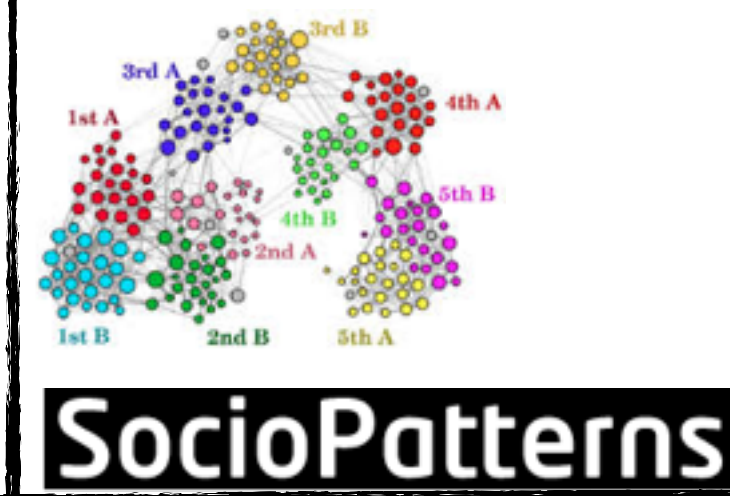

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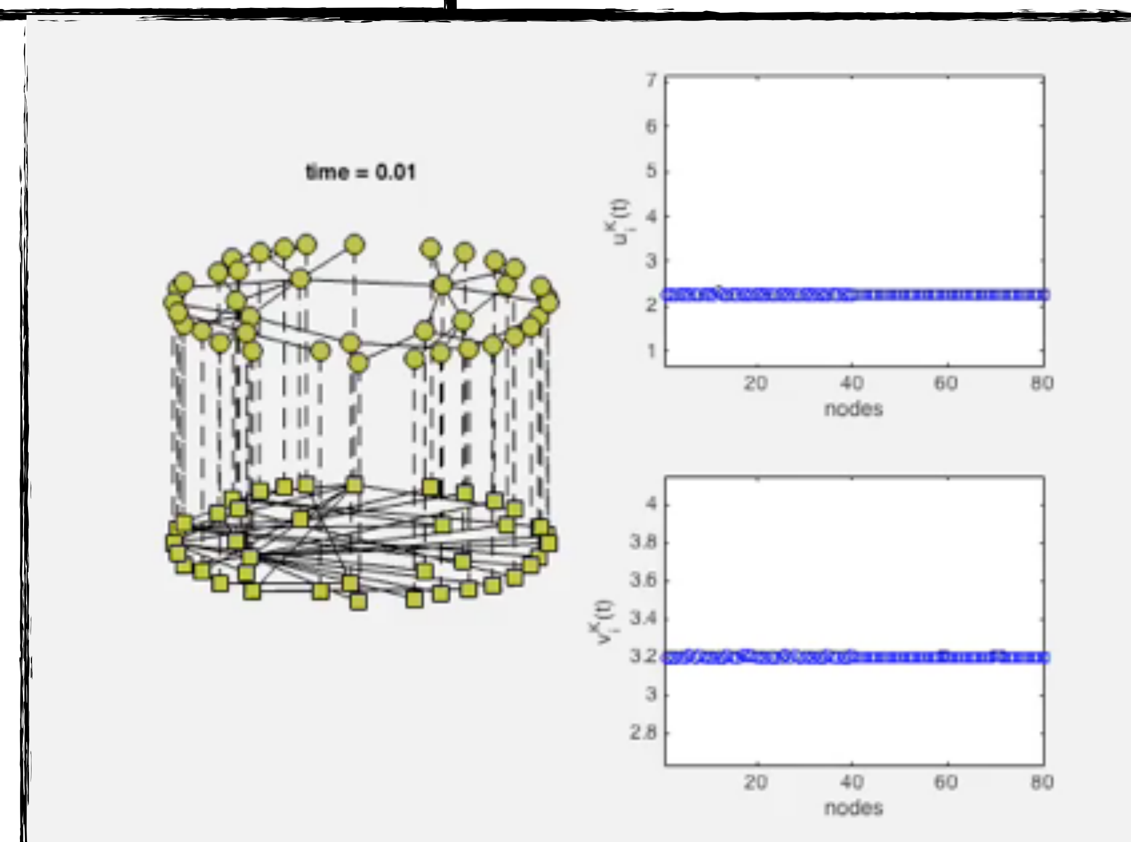
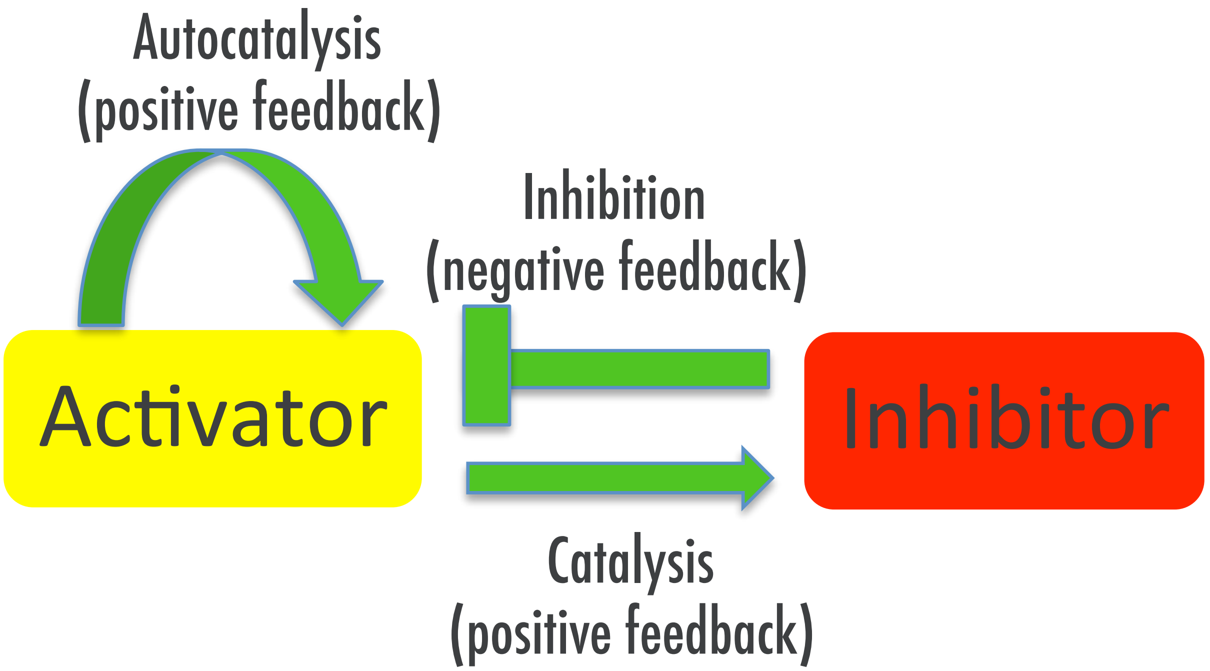


One possible mechanism: Turing instability

Reaction



+

Diffusion

$u(x,y,t)$: Amount of activator at time t and position (x,y)

$v(x,y,t)$: Amount of inhibitor at time t and position (x,y)

$$\begin{cases} \frac{\partial u}{\partial t} = f(u,v) + D_u \nabla^2 u \\ \frac{\partial v}{\partial t} = g(u,v) + D_v \nabla^2 v \end{cases}$$

$$(x,y) \in \Omega$$

+ boundary conditions
+ initial condition

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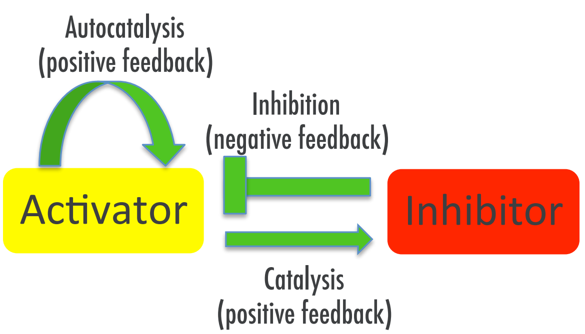

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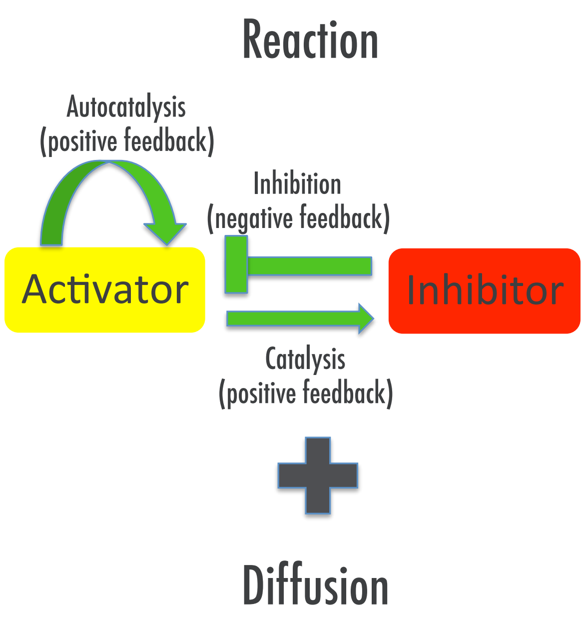
Diffusion can drive an instability by perturbing a homogeneous stable (in absence of diffusion) fixed point.

One possible mechanism: Turing instability



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Diffusion can drive an instability by perturbing a homogeneous stable (in absence of diffusion) fixed point.

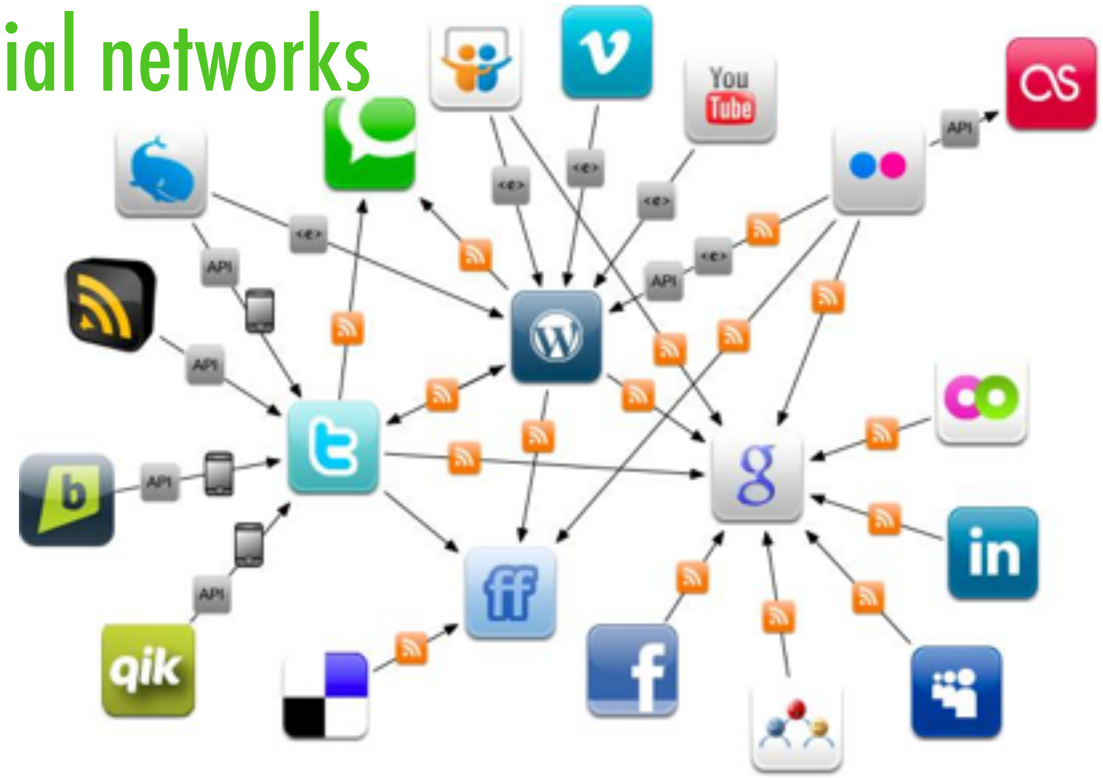
Hence as the perturbation grows, non-linearities enter into the game yielding an asymptotic, spatially inhomogeneous, steady state (stationary pattern) or time varying one (wave like pattern).

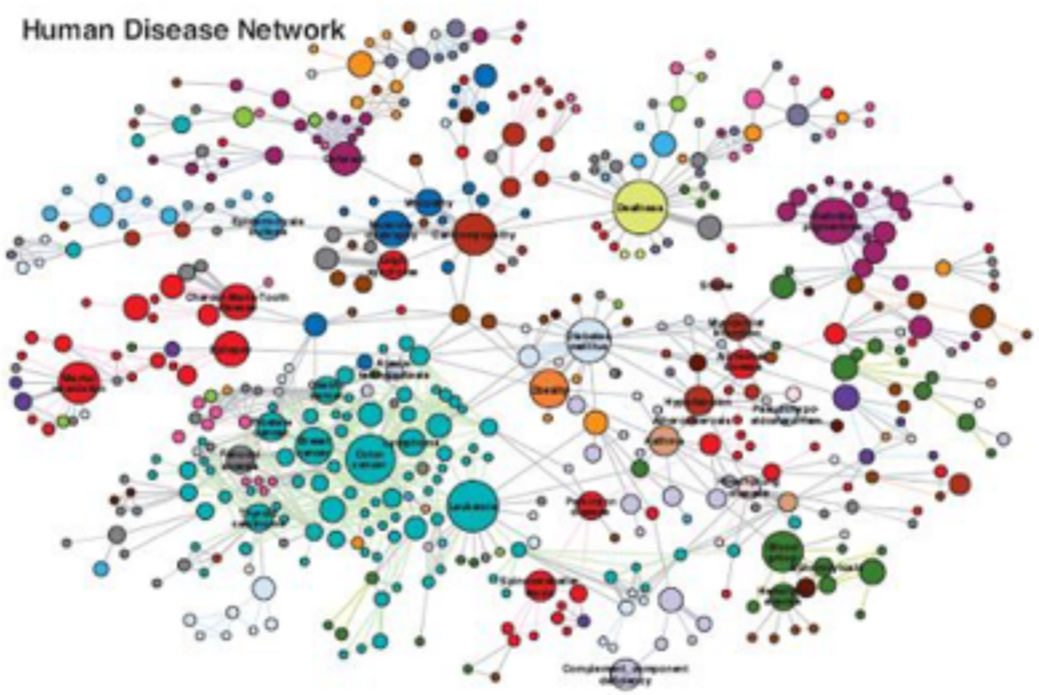
A.M.Turing, *The chemical basis of morphogenesis*, Phil. Trans. R Soc London B, *237, (1952), pp.37*

Networks are everywhere

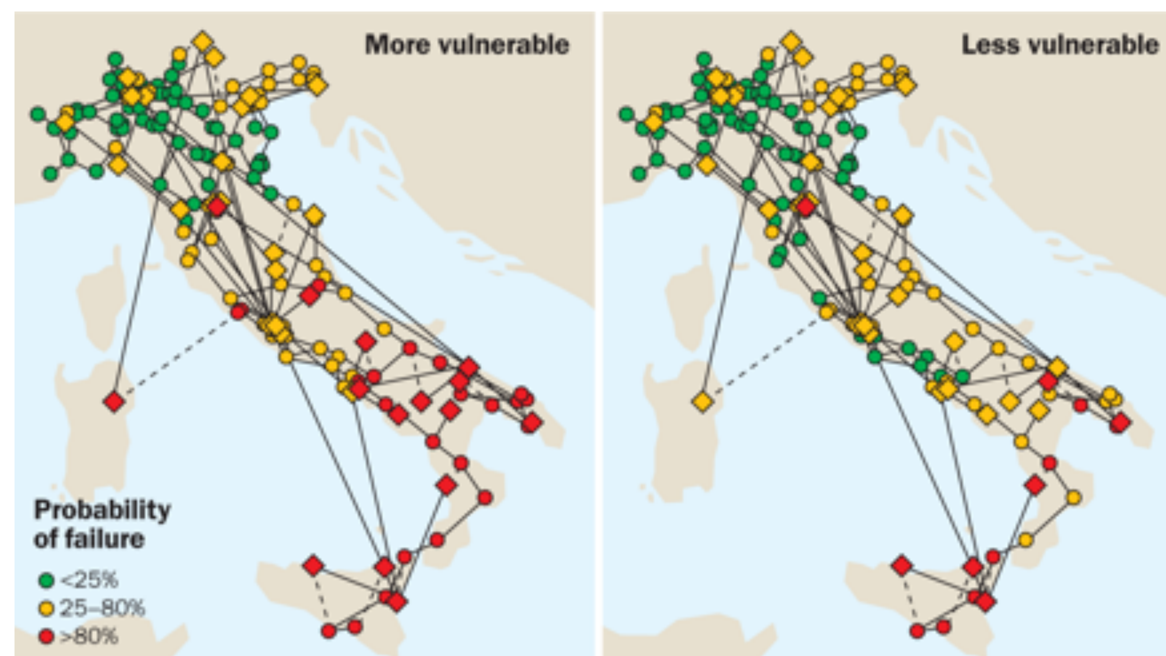
world flights map

social networks





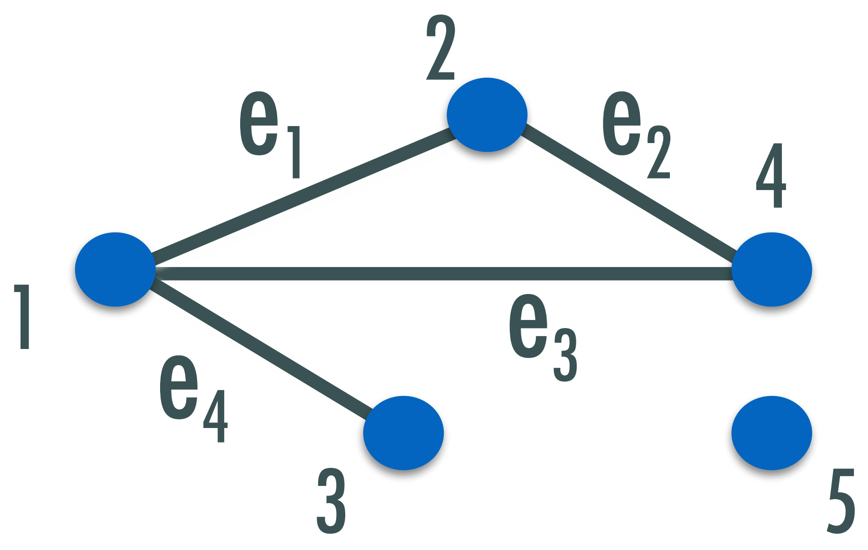
proteins networks



technological networks

(complex) Networks: some definitions

A network is a set of nodes connected by links (edges)



Ex.: 5 nodes and 4 edges (undirected)

A network can be described by its Adjacency matrix

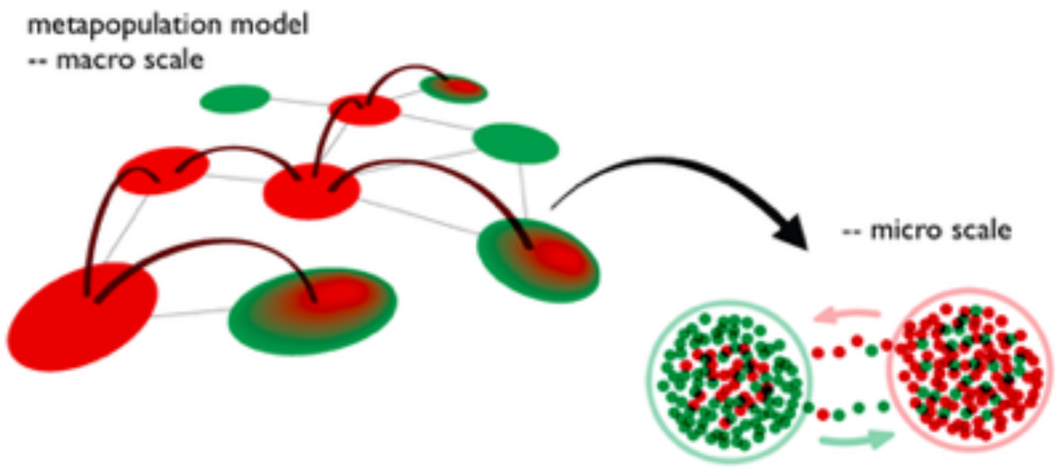
$$A_{ij} = \begin{cases} 1 & \text{if nodes } i \text{ and } j \text{ are linked} \\ 0 & \text{otherwise} \end{cases}$$

The number of links entering (going out) from each node is called in-degree (out-degree)

Ex.: degree node 1 = 3
degree nodes 2 & 4 = 2
degree node 3 = 1
degree node 5 = 0

A network is said to be complex if the degree distribution is not trivial, i.e. not constant (lattice) nor Poissonian (random, Erdős-Rényi)

Extension to networks



Metapopulation models

e.g. in the framework of ecology:
May R., *Will a large complex system be stable?* Nature, 238, pp. 413, (1972)

Reactions occur at each node. Diffusion occurs across edges.

Patterns : sets of nodes whose asymptotic state is far from the homogeneous equilibrium.

Nakao H. and Mikhailov A. S.,
Turing patterns in network-organized activator–inhibitor systems, Nature Physics, 6, pp. 544 (2010)

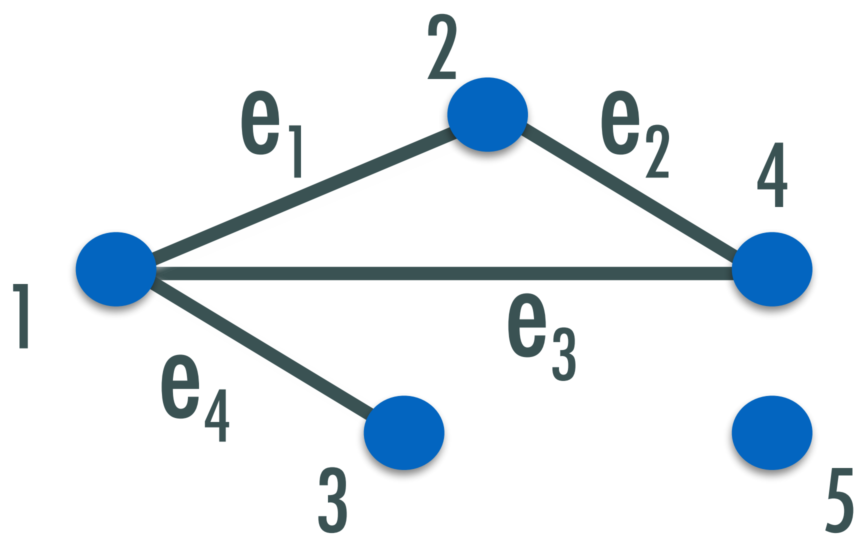
Reaction term:

$$\begin{cases} \dot{u}_i(t) & = f(u_i(t), v_i(t)) \\ \dot{v}_i(t) & = g(u_i(t), v_i(t)) \quad \forall i = 1, \dots, n \text{ and } t > 0. \end{cases}$$

At each node i=1,...,n, "species" u and v react through some non-linear functions f and g depending on the quantities available at node i-th (metapopulation assumption)

Diffusion term:

Diffusive transport of species into a certain node i is given by the sum of incoming fluxes to node i from other connected nodes j, fluxes are proportional to the concentration difference between the nodes (Fick's law).



Ex.: consider the amount of u in node 1,
u can enter from 2, 3 and 4
u can leave 1 to go to 2, 3 and 4

$$u_2 + u_3 + u_4 - 3u_1 = \sum_j A_{1j} u_j - k_1 u_1 = \sum_j \left(A_{1j} - \delta_{1j} k_j\right) u_j := \sum_j L_{1j} u_j$$

L is called Laplacian matrix of the network

Nakao H. and Mikhailov A. S.,
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The model:

$$\begin{cases} \dot{u}_i(t) & = f(u_i(t), v_i(t)) + D_u \sum_{j=1}^{n} L_{ij} u_j(t) \\ \dot{v}_i(t) & = g(u_i(t), v_i(t)) + D_v \sum_{j=1}^{n} L_{ij} v_j(t) \quad \forall i = 1, \dots, n \text{ and } t > 0. \end{cases}$$

Du and Dv are the diffusion coefficients of species u and v

Observe that because the network is undirected, the matrices A and L are symmetric

General strategy

1) Assume there exists a spatially homogeneous solution:

$$(u_i, v_i) = (\hat{u}, \hat{v}) \quad \forall i = 1, \dots, n$$

which moreover is stable when there is no diffusion:

$$D_u = D_v = 0$$

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which moreover is stable when there is no diffusion:

$$D_u = D_v = 0$$

2) Linearize around this solution

$$\begin{aligned} u_i &= \hat{u} + \delta u_i \\ v_i &= \hat{v} + \delta v_i \end{aligned} \qquad \begin{pmatrix} \dot{\delta \boldsymbol{u}} \\ \dot{\delta \boldsymbol{v}} \end{pmatrix} = \tilde{\mathcal{J}} \begin{pmatrix} \delta \boldsymbol{u} \\ \delta \boldsymbol{v} \end{pmatrix}$$

$$\tilde{\mathcal{J}} = \begin{pmatrix} f_u \mathbf{I}_n + D_u \boldsymbol{L} & f_v \mathbf{I}_n \\ g_u \mathbf{I}_n & g_v \mathbf{I}_n + D_v \boldsymbol{L} \end{pmatrix}$$

General strategy

3) Prove that (possibly) the spatially homogeneous solution:

$$(u_i, v_i) = (\hat{u}, \hat{v}) \quad \forall i = 1, \dots, n$$

turns out to be unstable once the diffusion is in action

$$D_u > 0 \text{ and } D_v > 0$$

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Sketch of the proof

i) Let $L\vec{\phi}^\alpha = \Lambda^\alpha \vec{\phi}^\alpha, \; \alpha = 1, \ldots, n \qquad \vec{\phi}^\alpha = (\phi_1^\alpha, \ldots, \phi_n^\alpha)$

$$\sum_i \phi_i^\alpha \phi_i^\beta = \delta_{\alpha\beta} \qquad \Lambda^\alpha \leq 0$$

ii) decompose the solution on the eigenbasis and use the ansatz

$$\delta u_i(t) = \sum_{\alpha=1}^{n} c_\alpha \phi_i^\alpha e^{\lambda_\alpha t}$$

General strategy

iii) λ_α (called relation dispersion) is solution of

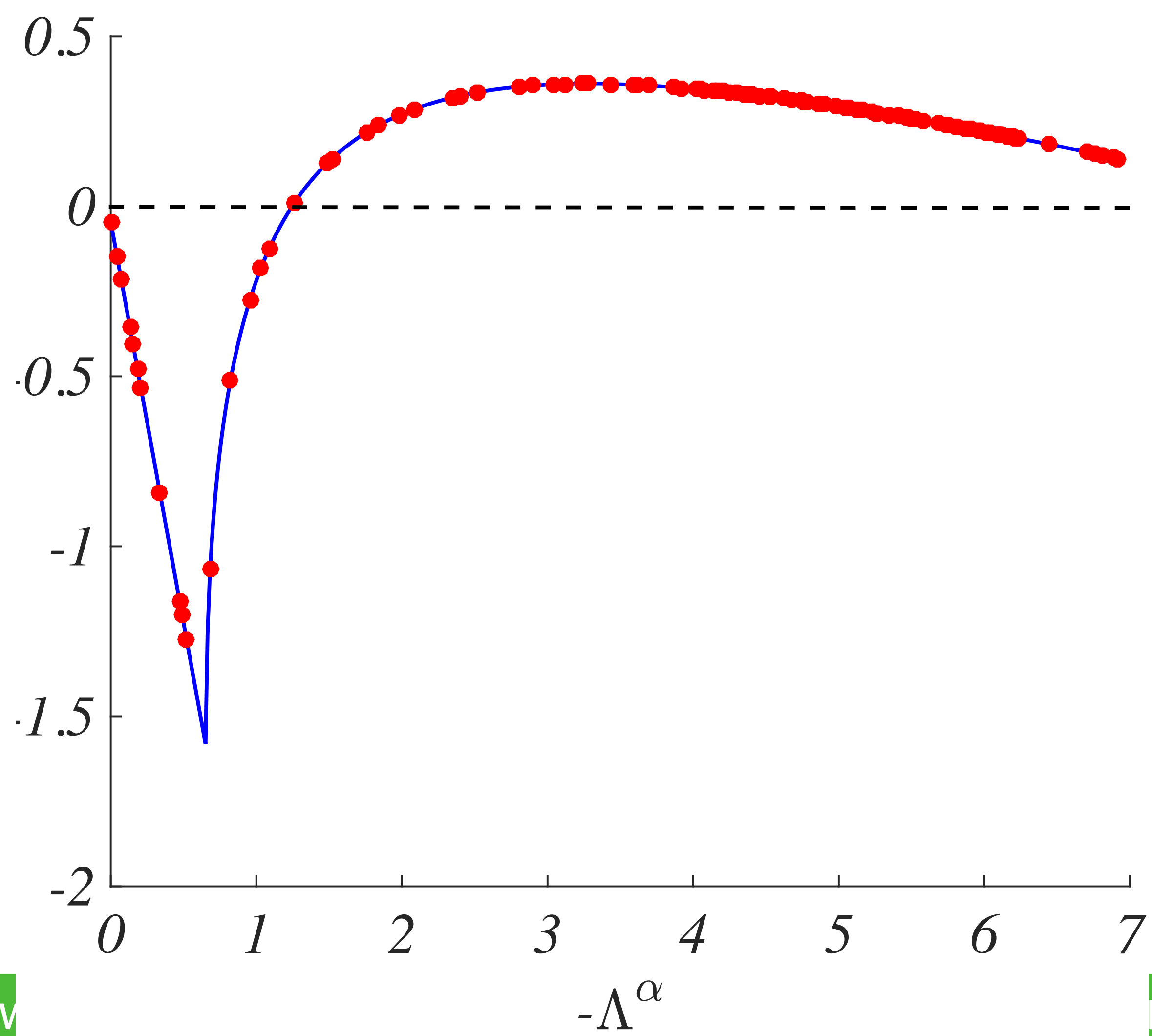
$$\det\left[\lambda_\alpha - \begin{pmatrix} f_u + D_u\Lambda^\alpha & f_v \\ g_u & g_v + D_v\Lambda^\alpha \end{pmatrix}\right] = 0$$

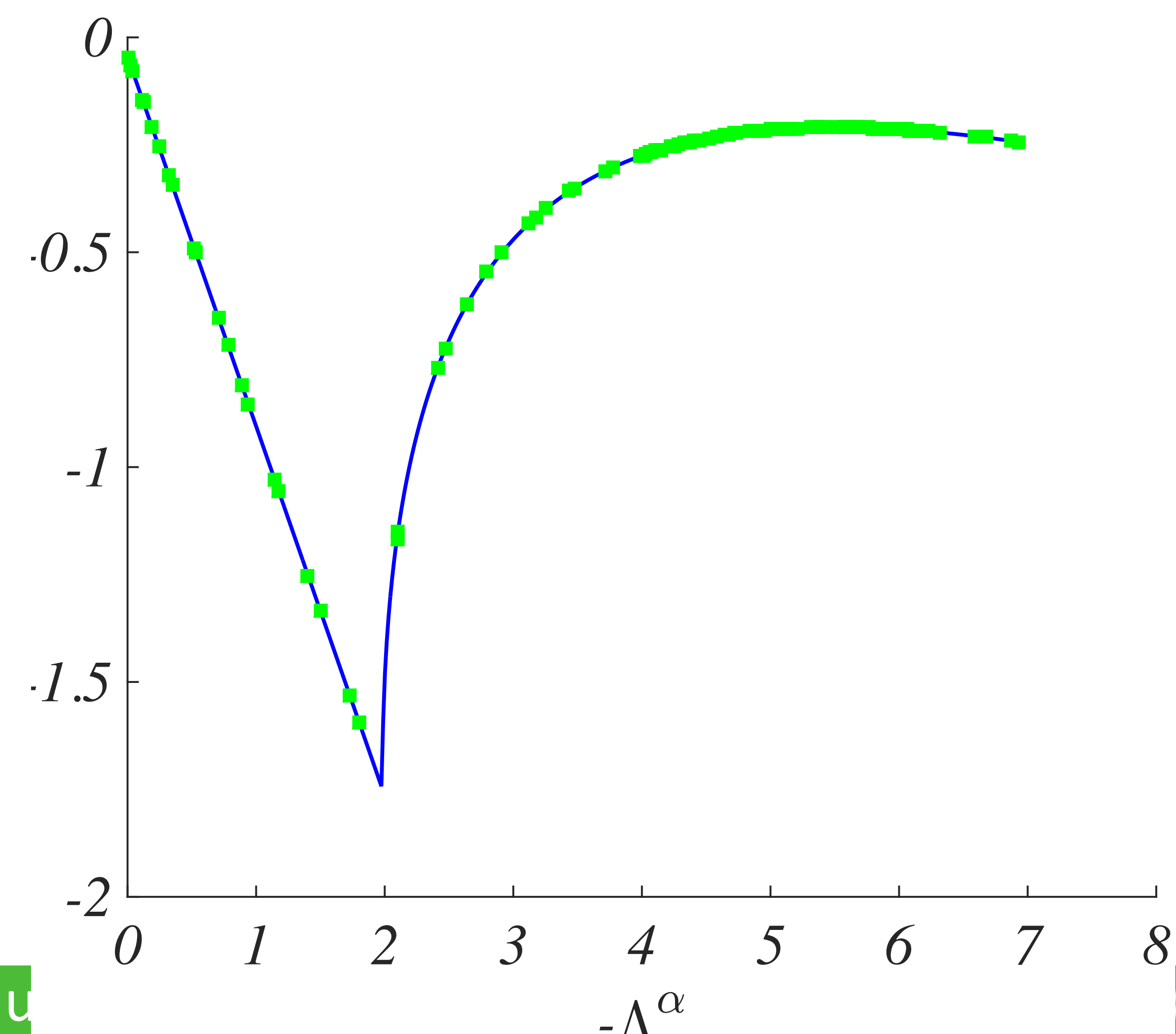
General strategy

iii) λ_α (called relation dispersion) is solution of

$$\det\left[\lambda_\alpha - \begin{pmatrix} f_u + D_u\Lambda^\alpha & f_v \\ g_u & g_v + D_v\Lambda^\alpha \end{pmatrix}\right] = 0$$

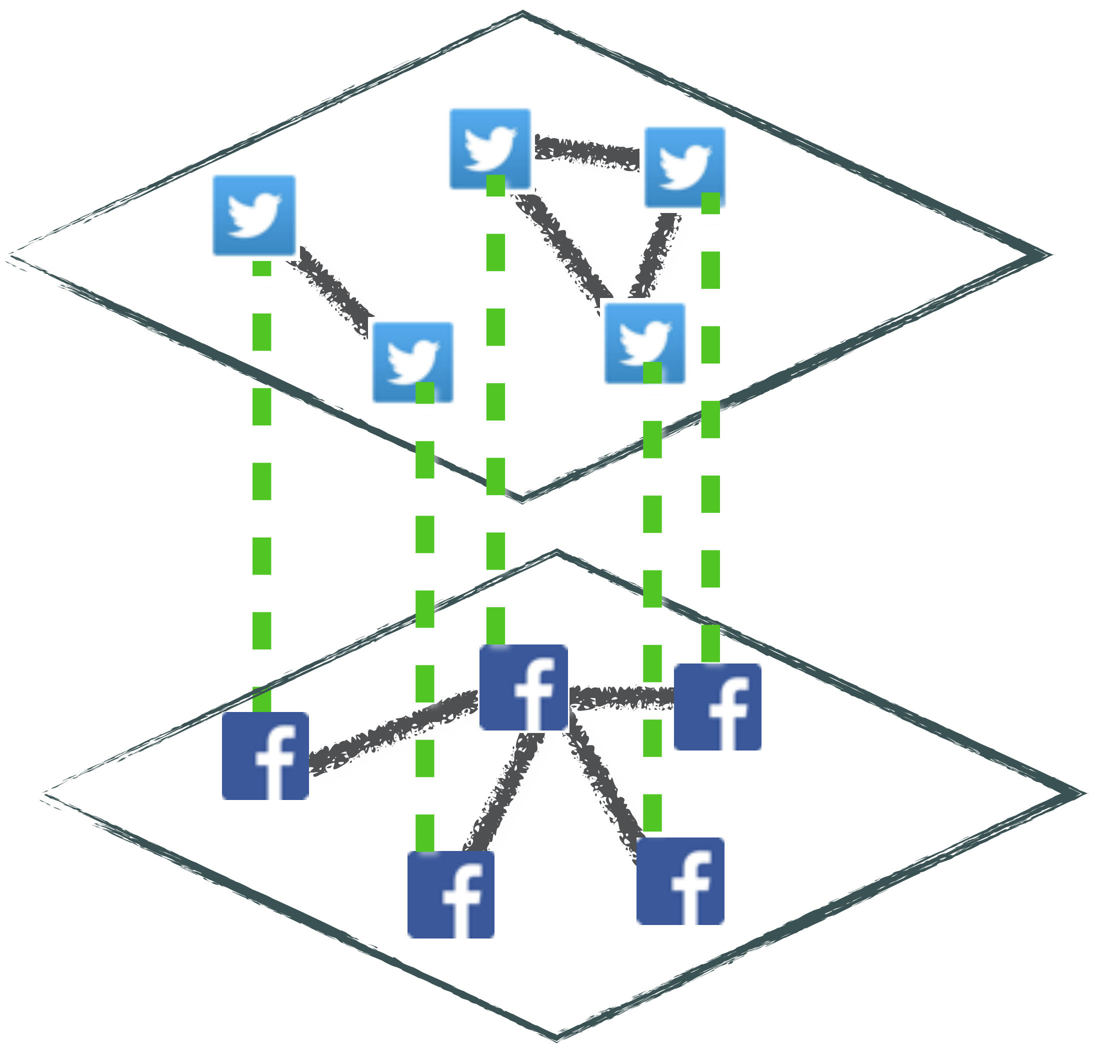
iv) if there exists Λ^{α_c} such that $\Re\lambda_{\alpha_c} > 0$ then Turing patterns do emerge.

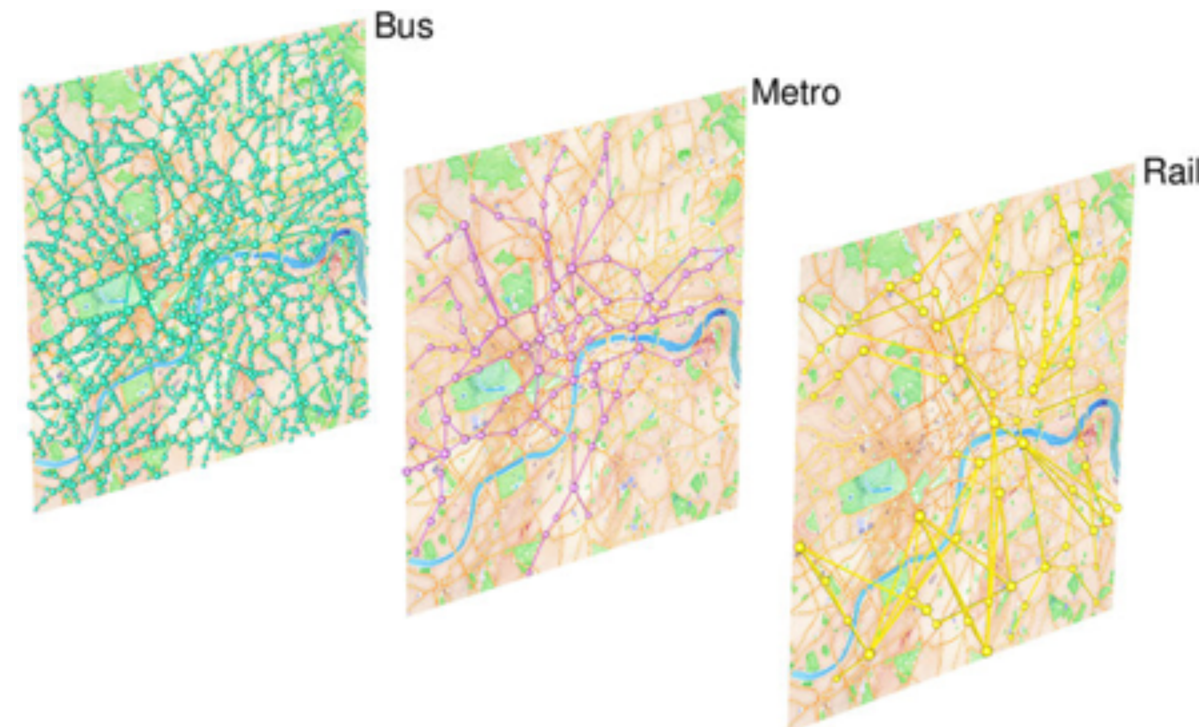




Systems composed by layers of networks: Multiplexes

Social networks
layers=different social networks
nodes=same agent in each SN





Transportation networks
layers=different modalities
nodes=same spatial location

Turing instabilities on multiplex networks

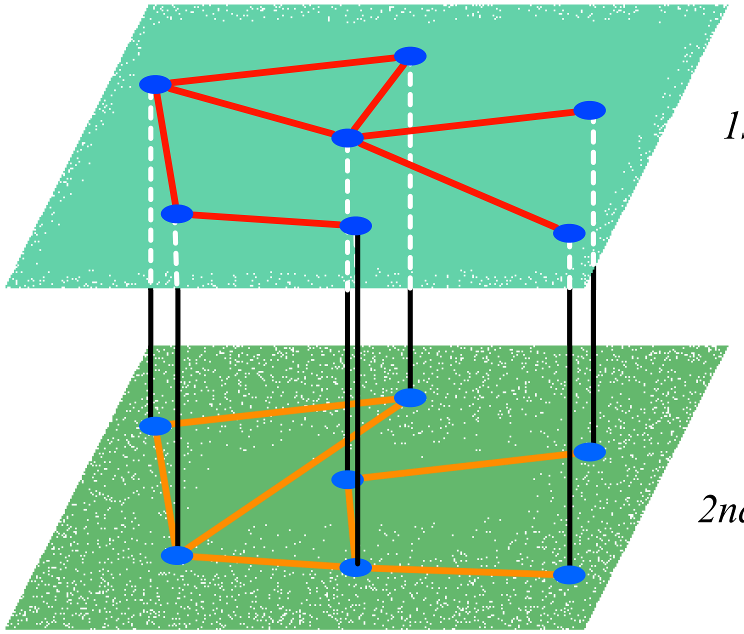
With K=1,2 (K=3 should be read K=1)

adjacency matrix of layer K

degree of i-th note in layer K

$$L_{ij}^K = A_{ij}^K - \delta_{ij} k_i^K$$

Laplacian matrix of layer K



The same Ω nodes are present in each layer

$D_{u,v}^K$ **inter**-layer diffusion coefficient

$D_{u,v}^{12}$ **intra**-layer diffusion coefficient

$$\begin{cases} \dot{u}_i^K &= f(u_i^K, v_i^K) + D_u^K \sum_{j=1}^{\Omega} L_{ij}^K u_j^K + D_u^{12} \left(u_i^{K+1} - u_i^K\right) \\ \dot{v}_i^K &= g(u_i^K, v_i^K) + D_v^K \sum_{j=1}^{\Omega} L_{ij}^K v_j^K + D_v^{12} \left(v_i^{K+1} - v_i^K\right) \end{cases}$$

1) Assume there exists a spatially homogeneous solution:

$$(u_i^K, v_i^K) = (\hat{u}, \hat{v}) \quad \forall i = 1, \ldots, n$$

General strategy

1) Assume there exists a spatially homogeneous solution:

$$(u_i^K, v_i^K) = (\hat{u}, \hat{v}) \quad \forall i = 1, \ldots, n$$

2) Linearize around this solution

$$u_j^K = \hat{u} + \delta u_j^K$$
$$v_j^K = \hat{v} + \delta v_j^K$$

$$\begin{pmatrix} \delta \dot{\boldsymbol{u}} \\ \delta \dot{\boldsymbol{v}} \end{pmatrix} = \tilde{\mathcal{J}} \begin{pmatrix} \delta \boldsymbol{u} \\ \delta \boldsymbol{v} \end{pmatrix}$$

$$\tilde{\mathcal{J}} = \begin{pmatrix} f_u \mathbf{I}_{2\Omega} + \boldsymbol{\mathcal{L}}_u + D_u^{12} \boldsymbol{\mathcal{I}} & f_v \mathbf{I}_{2\Omega} \\ g_u \mathbf{I}_{2\Omega} & g_v \mathbf{I}_{2\Omega} + \boldsymbol{\mathcal{L}}_v + D_v^{12} \boldsymbol{\mathcal{I}} \end{pmatrix}$$

Supra-Laplacian matrix $\boldsymbol{\mathcal{L}}_u + D_u^{12} \boldsymbol{\mathcal{I}}$

$$\boldsymbol{\mathcal{L}}_u = \begin{pmatrix} D_u^1 \mathbf{L}^1 & \mathbf{0} \\ \mathbf{0} & D_u^2 \mathbf{L}^2 \end{pmatrix} \qquad \boldsymbol{\mathcal{I}} = \begin{pmatrix} -\mathbf{I}_\Omega & \mathbf{I}_\Omega \\ \mathbf{I}_\Omega & -\mathbf{I}_\Omega \end{pmatrix}$$

General strategy

3) Study the spectrum of

$$\tilde{\mathcal{J}} = \begin{pmatrix} f_u \mathbf{I}_{2\Omega} + \boldsymbol{\mathcal{L}}_u + D_u^{12} \boldsymbol{\mathcal{I}} & f_v \mathbf{I}_{2\Omega} \\ g_u \mathbf{I}_{2\Omega} & g_v \mathbf{I}_{2\Omega} + \boldsymbol{\mathcal{L}}_v + D_v^{12} \boldsymbol{\mathcal{I}} \end{pmatrix}$$

to determine the existence of eigenvalues such that

$$\Re\lambda(D_{u,v}^{12}, D_{u,v}^{K}) > 0$$

General strategy

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$$\tilde{\mathcal{J}} = \begin{pmatrix} f_u \mathbf{I}_{2\Omega} + \boldsymbol{\mathcal{L}}_u + D_u^{12}\boldsymbol{\mathcal{I}} & f_v \mathbf{I}_{2\Omega} \\ g_u \mathbf{I}_{2\Omega} & g_v \mathbf{I}_{2\Omega} + \boldsymbol{\mathcal{L}}_v + D_v^{12}\boldsymbol{\mathcal{I}} \end{pmatrix}$$

to determine the existence of eigenvalues such that

$$\Re\lambda(D^{12}_{u,v}, D^{K}_{u,v}) > 0$$

Very hard for generic topologies, however ...

Small intra-layer diffusion case

Assume $D_v^{12} = \epsilon << 1 \qquad D_u^{12} = \mathcal{O}(\epsilon)$

$$\tilde{\mathcal{J}} = \begin{pmatrix} f_u \mathbf{I}_{2\Omega} + \boldsymbol{\mathcal{L}}_u & f_v \mathbf{I}_{2\Omega} \\ g_u \mathbf{I}_{2\Omega} & g_v \mathbf{I}_{2\Omega} + \boldsymbol{\mathcal{L}}_v \end{pmatrix} + \epsilon \begin{pmatrix} \frac{D_u^{12}}{D_v^{12}} \boldsymbol{L}^1 & \mathbf{0} \\ \mathbf{0} & \boldsymbol{L}^2 \end{pmatrix}$$

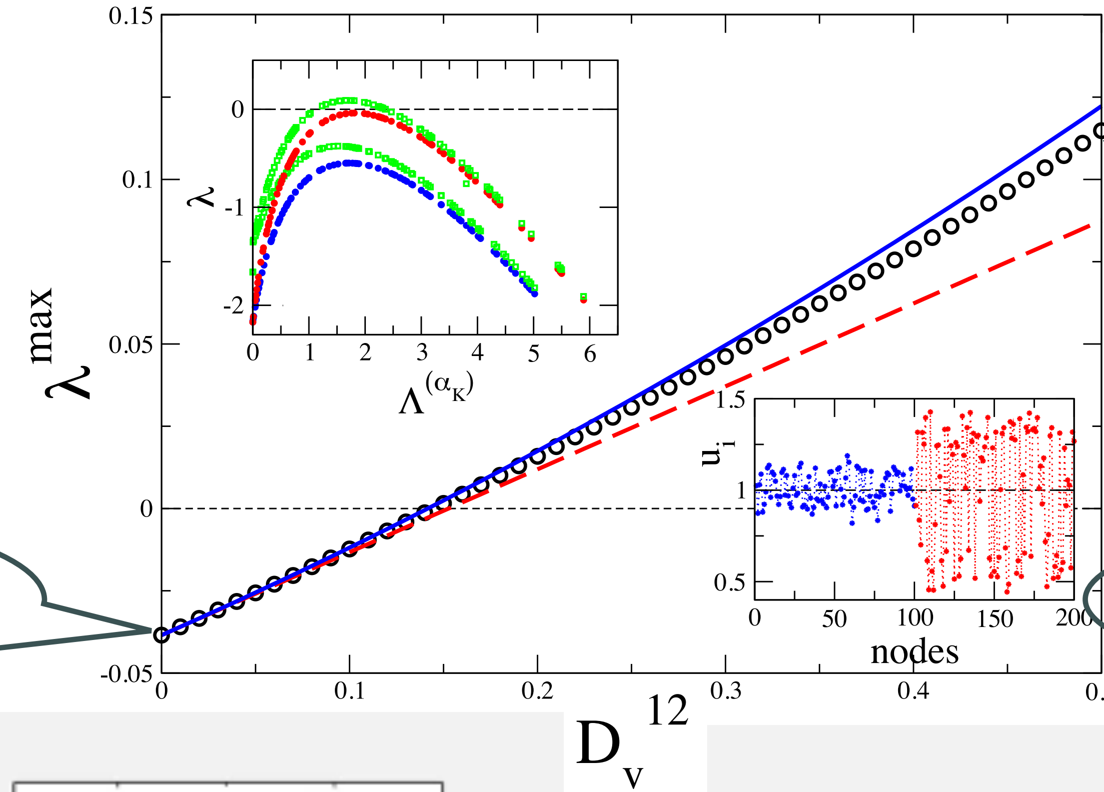
$$= \tilde{\mathcal{J}}_0 + \epsilon \boldsymbol{\mathcal{D}}_0$$

Perturbative approach to compute the spectrum

$$\lambda^{max}(\epsilon) = \lambda_0^{max} + \epsilon (U_0 \boldsymbol{\mathcal{D}}_0 V_0)_{k_{max} k_{max}} + \mathcal{O}(\epsilon^2)$$

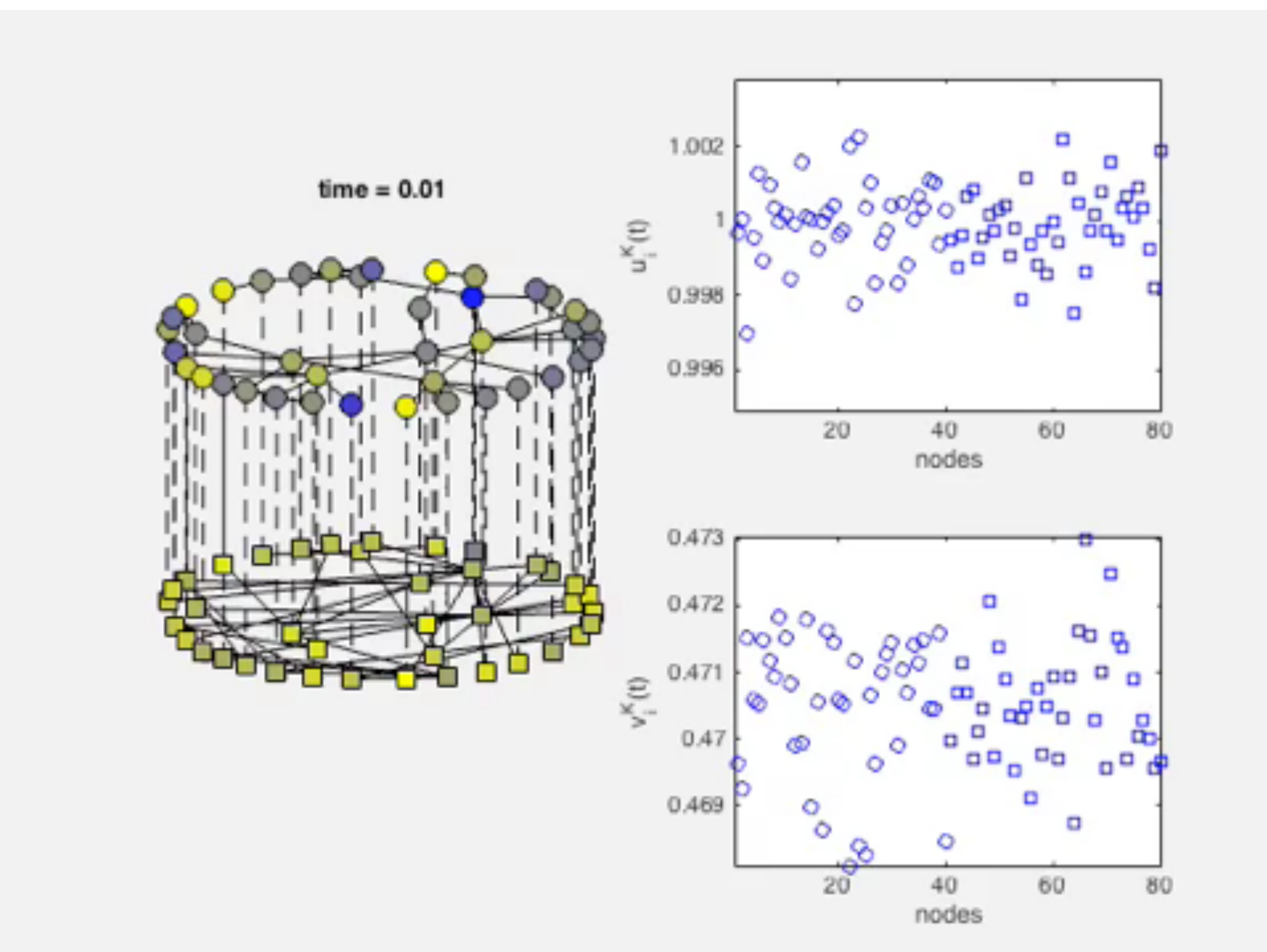
$$\lambda_0^{max} = \max \lambda_k(\epsilon = 0) \qquad k_{max} = \arg\max \lambda_k(\epsilon = 0)$$

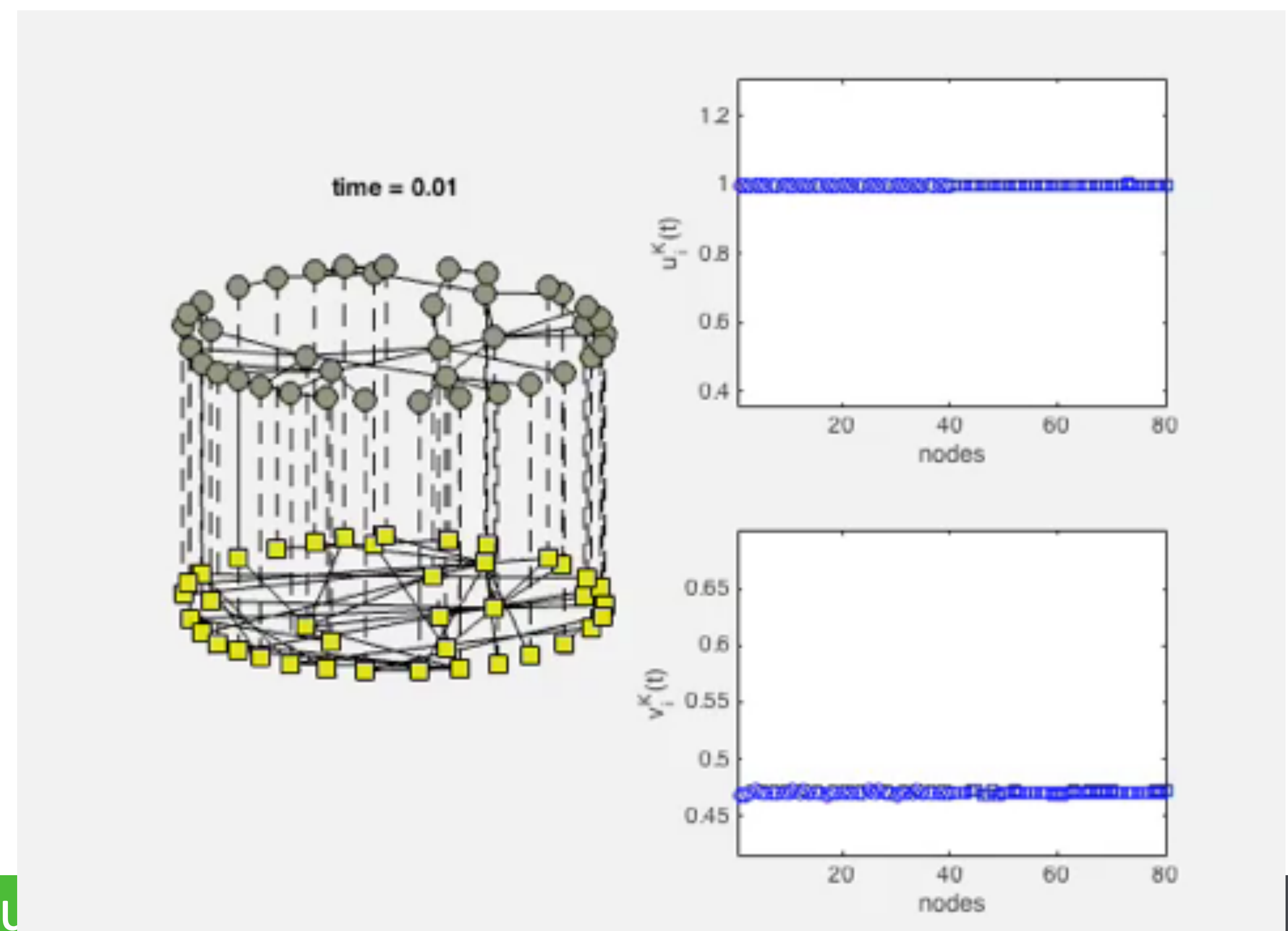
Small intra-layer diffusion case: onset of patterns



$$D_v^{12} = D_u^{12} = 0$$

$$D_u^{12} = 0 \quad D_v^{12} = 0.5$$





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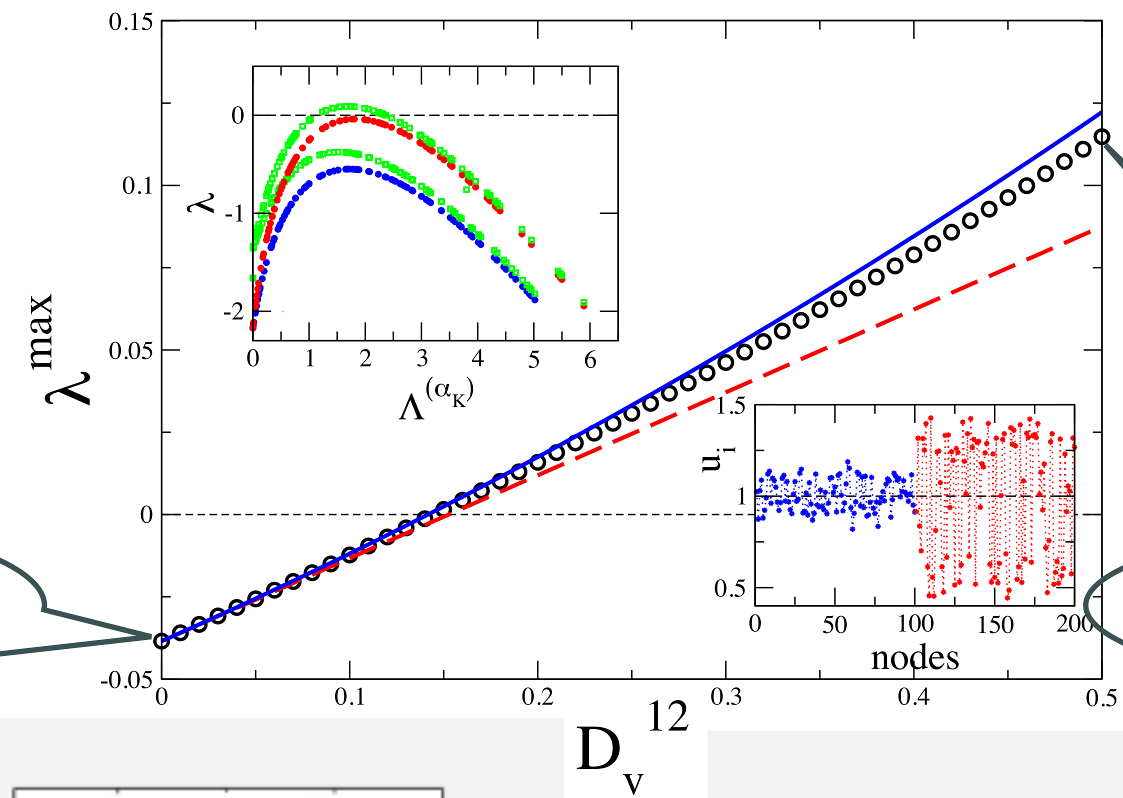

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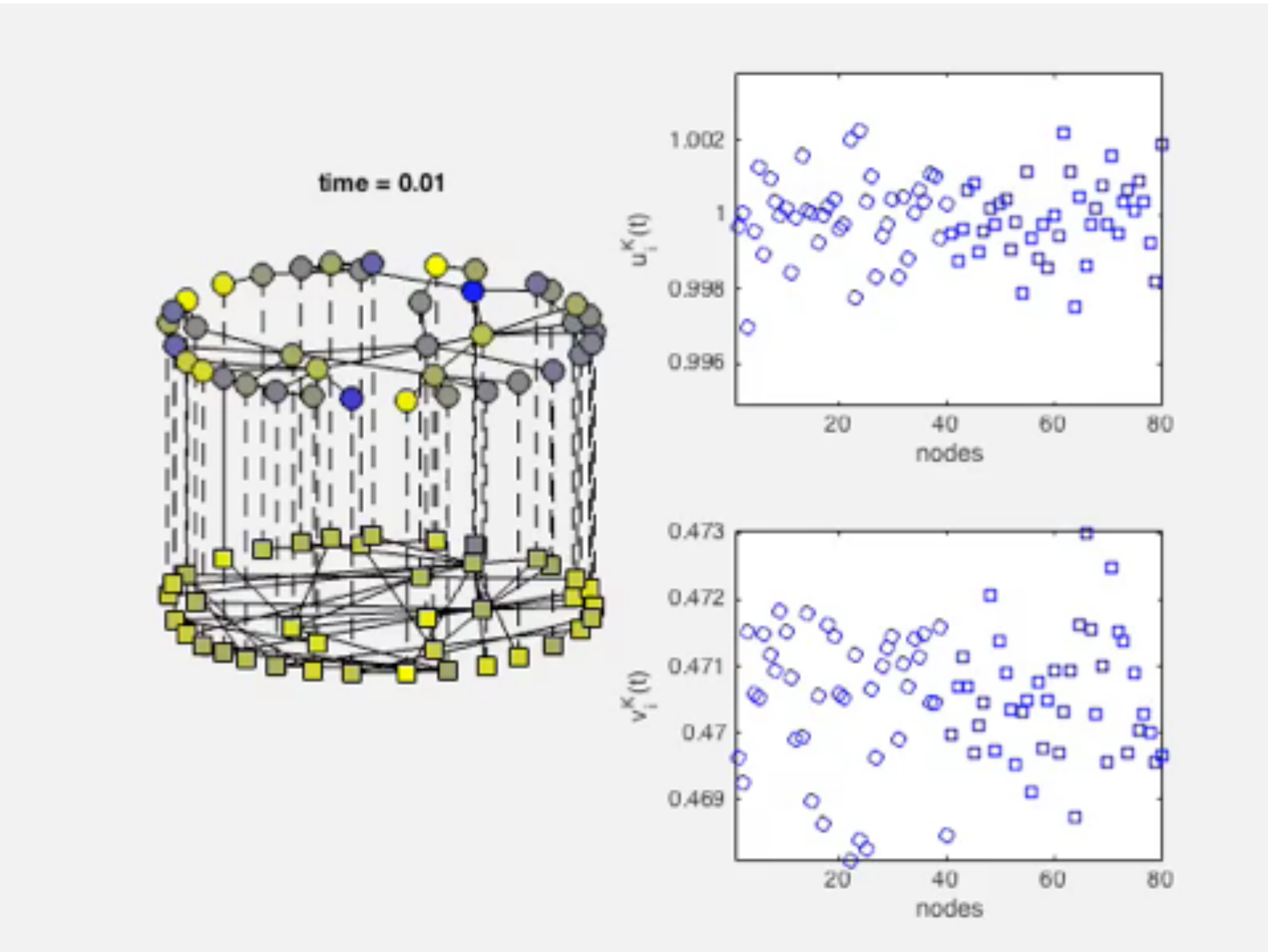

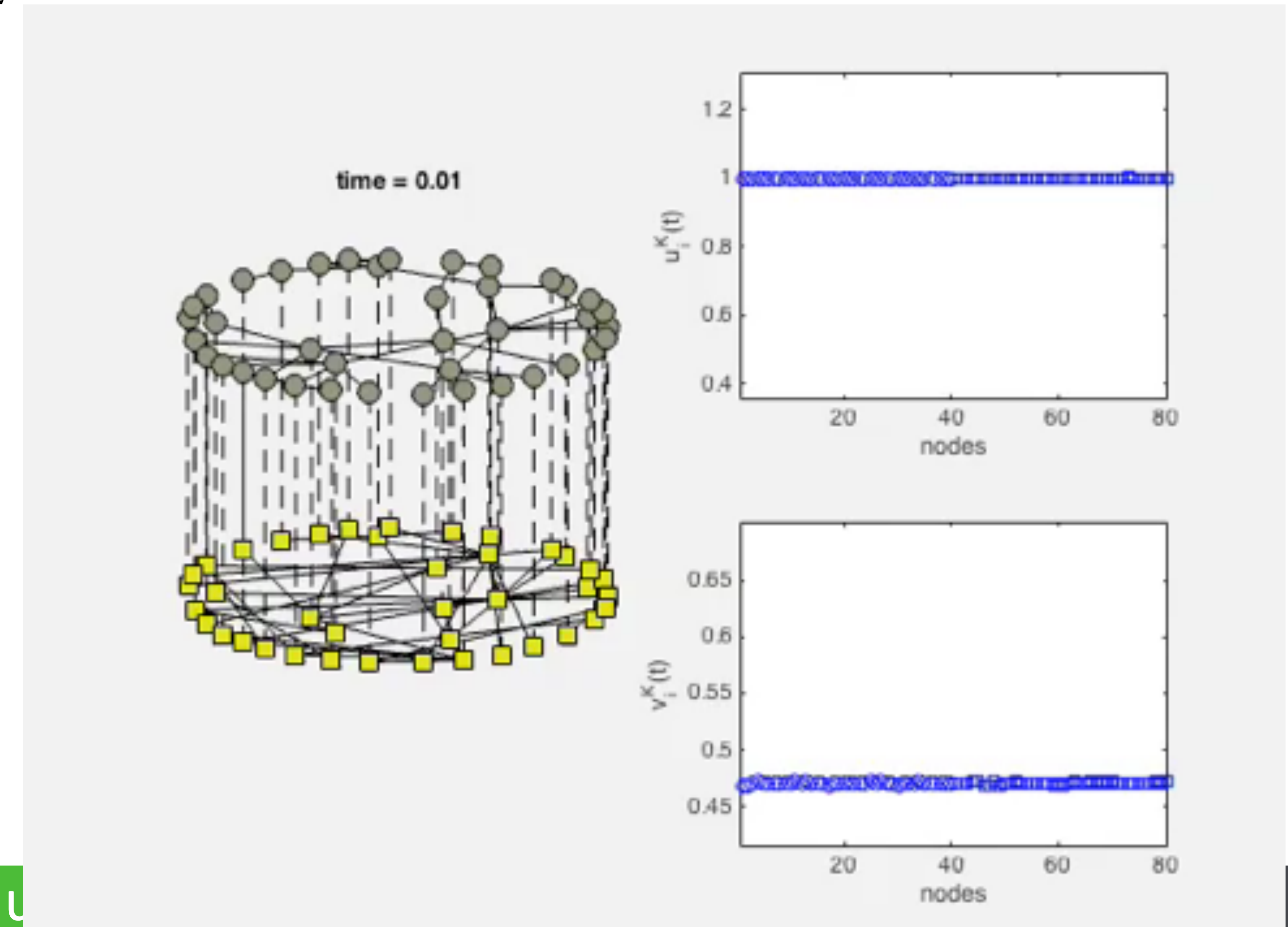

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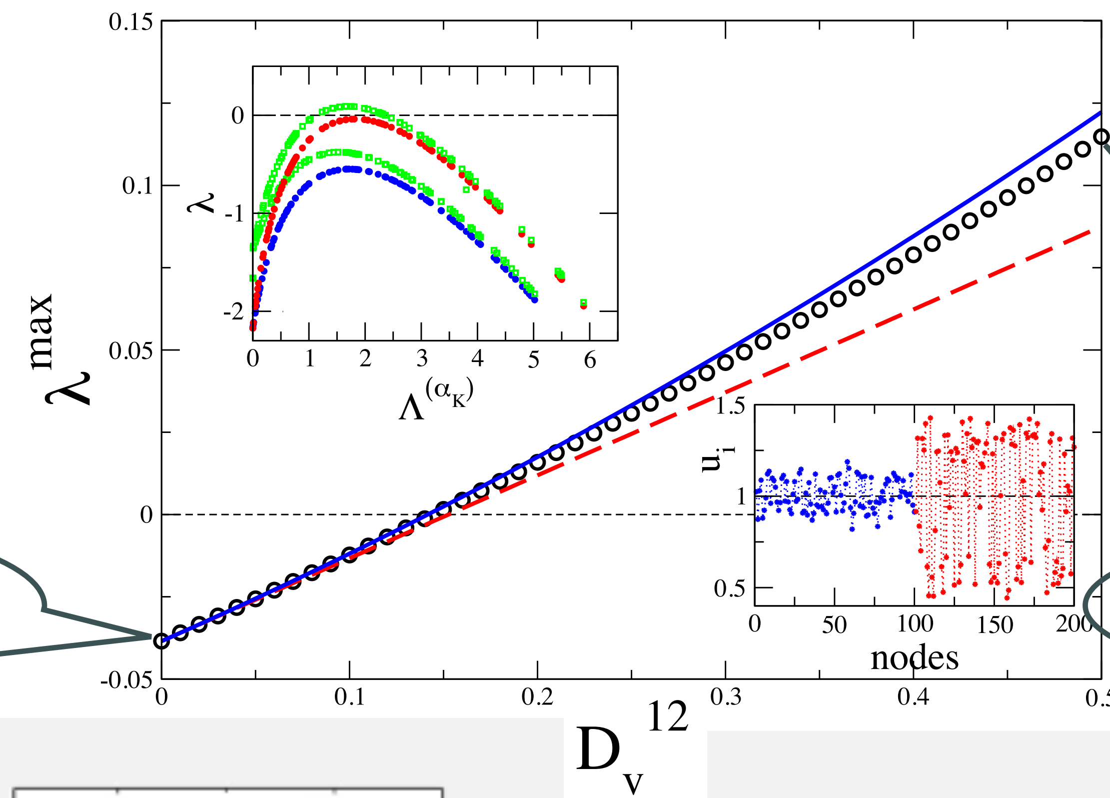
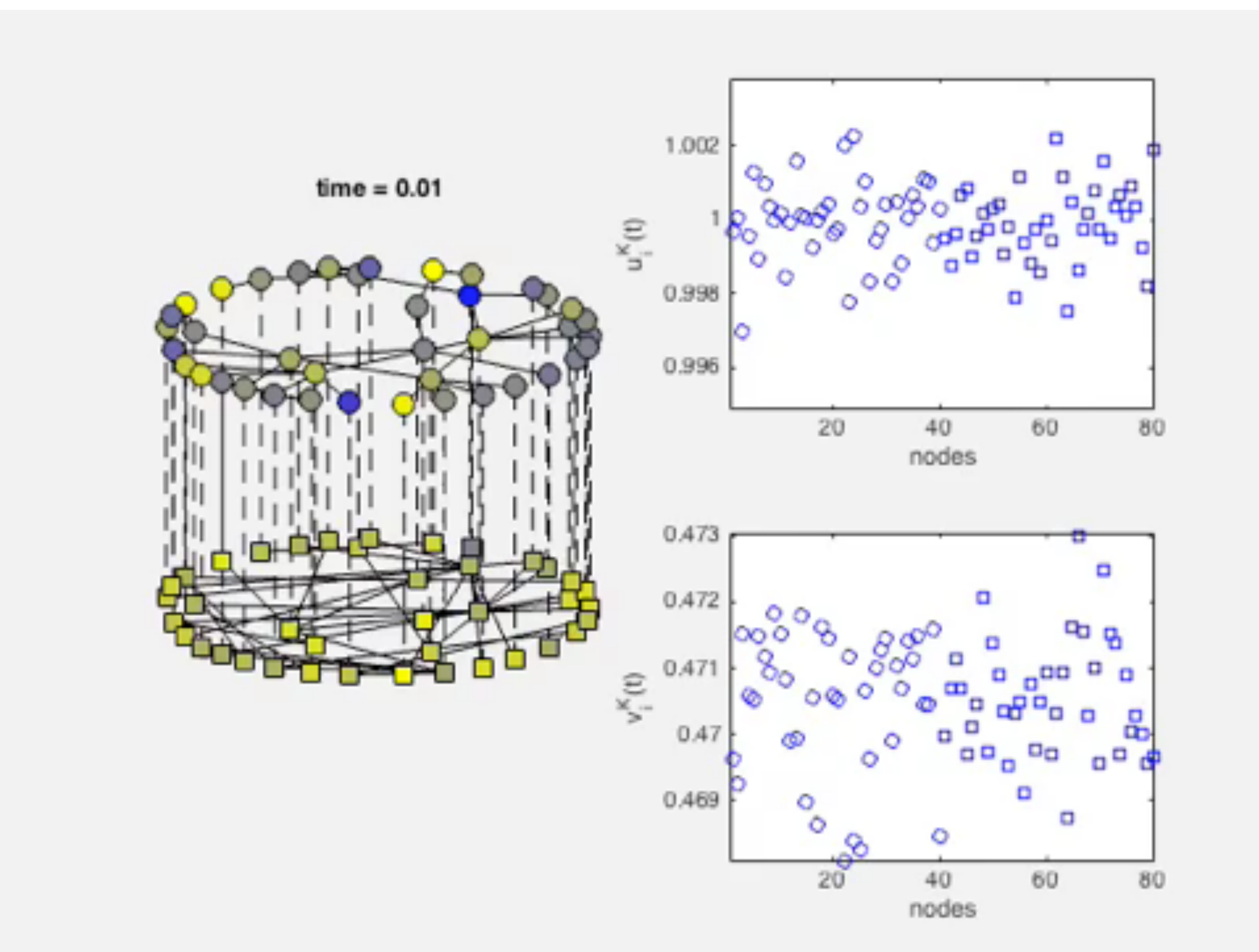
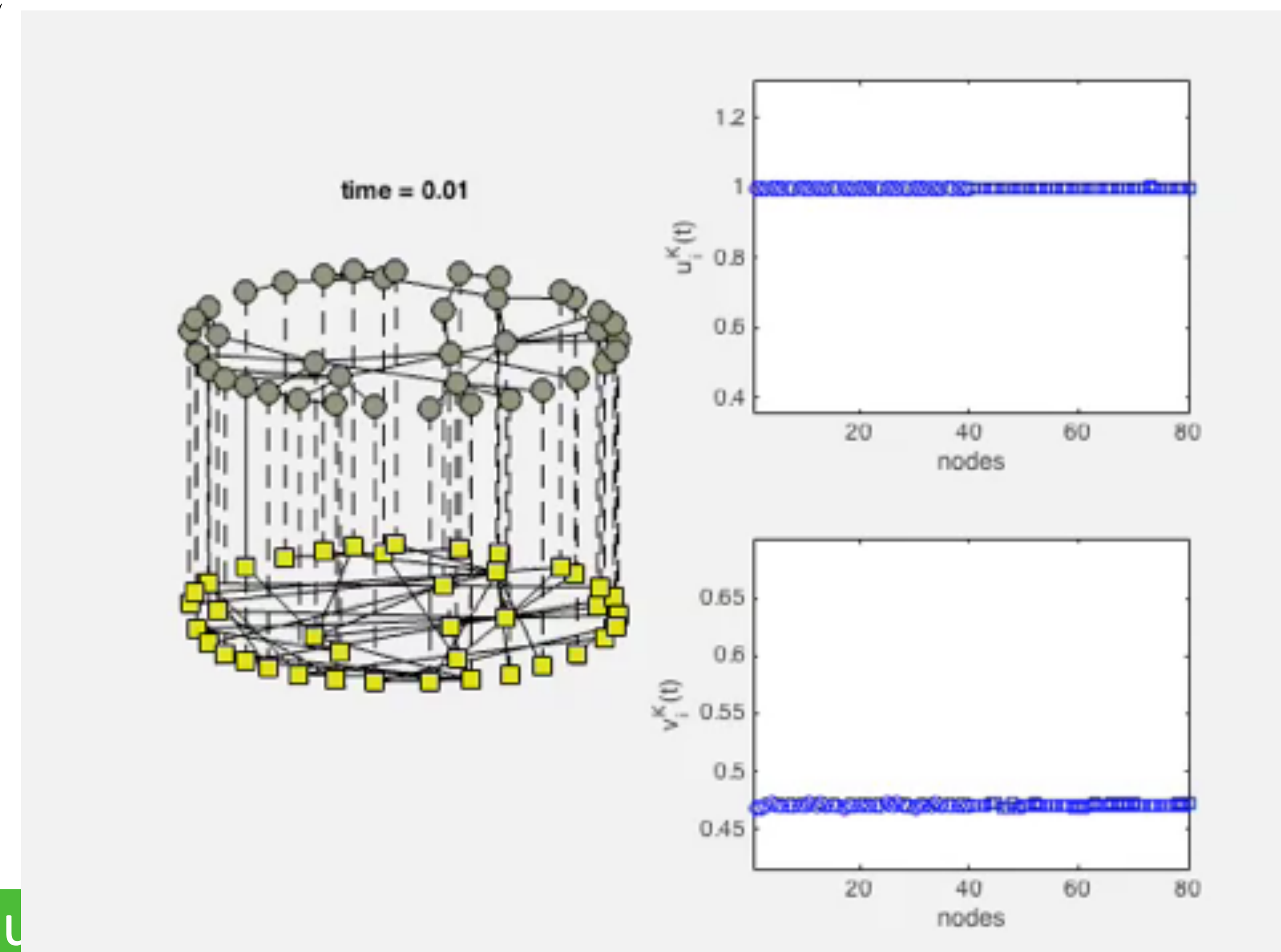


$D_v^{12} = D_u^{12} = 0$

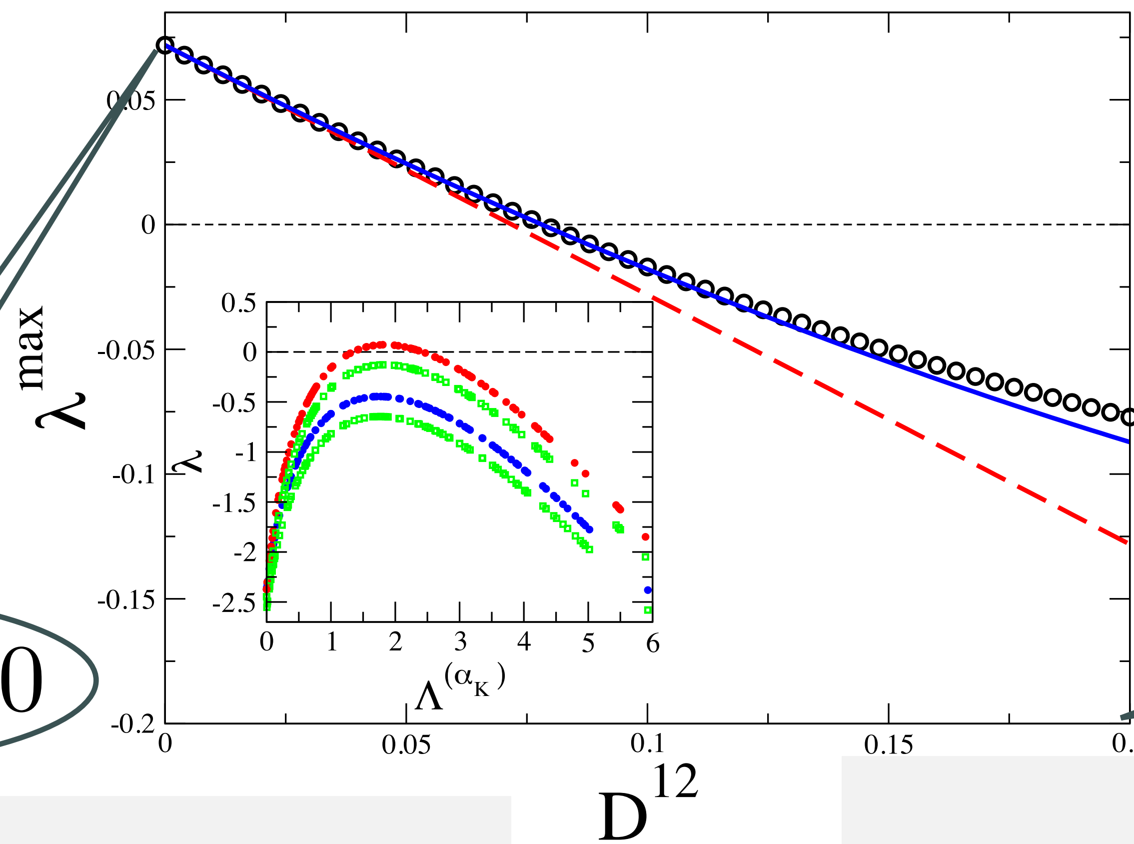
$D_u^{12} = 0 \quad D_v^{12} = 0.5$



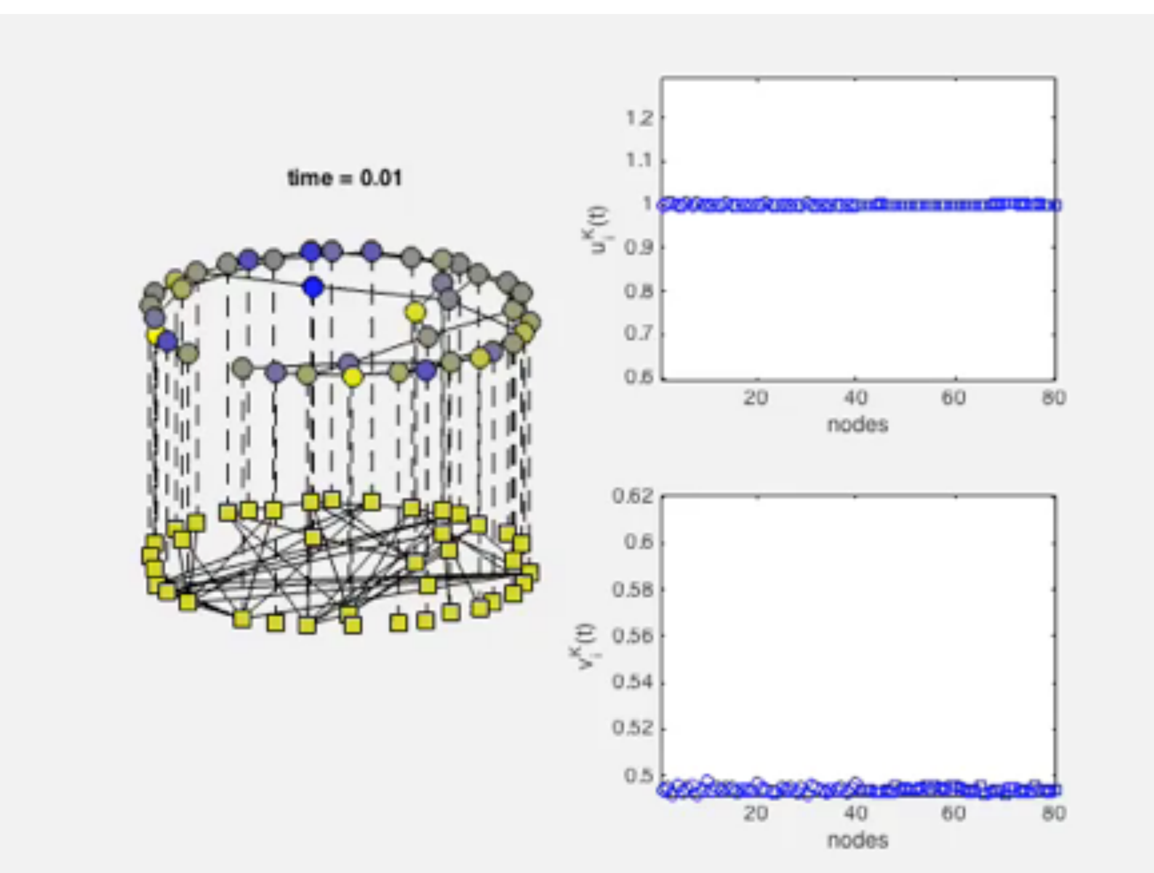


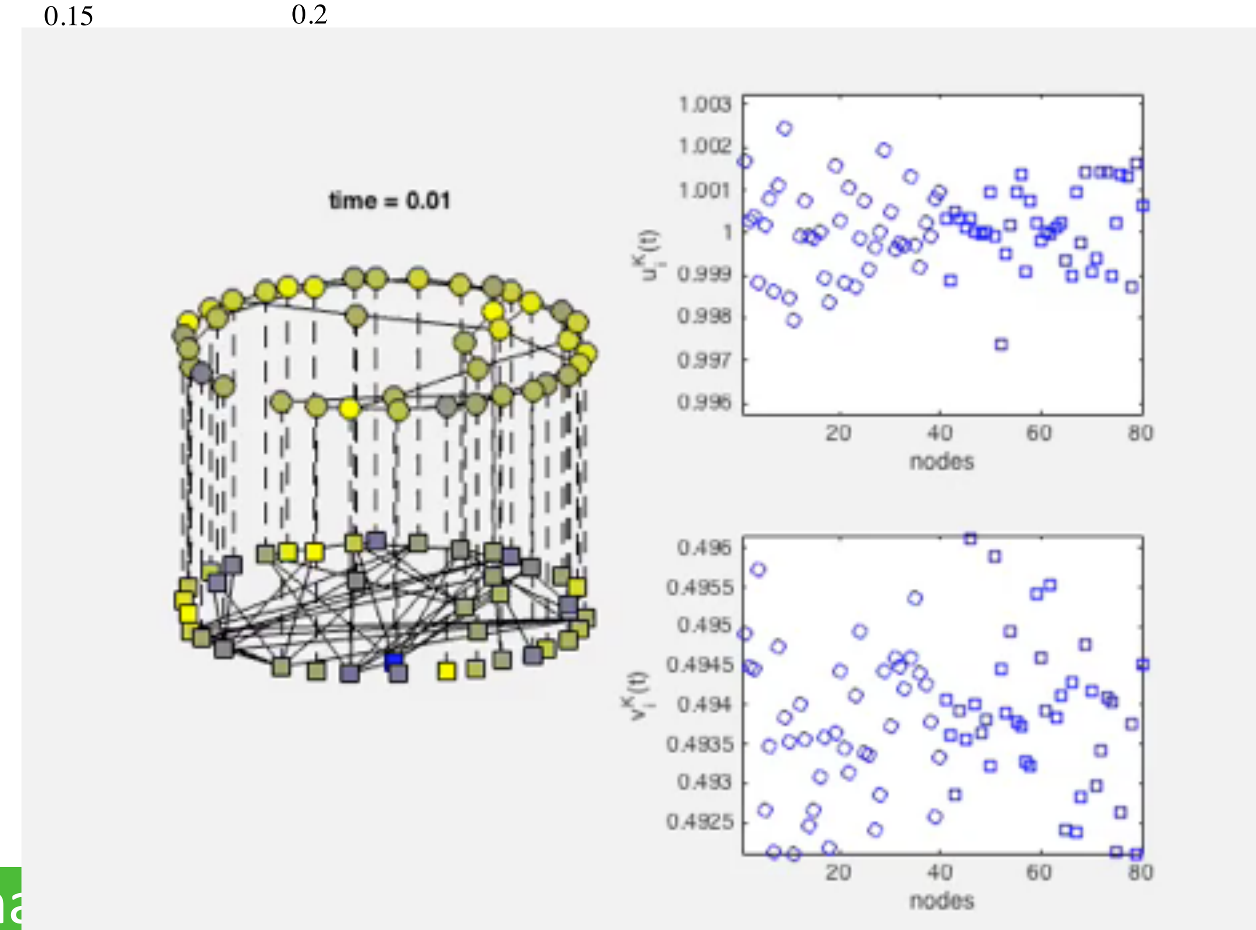
Small intra-layer diffusion case: destruction of patterns

$D_v^{12} = D_u^{12} = 0$



$D_u^{12} = D_v^{12} = 0.2$





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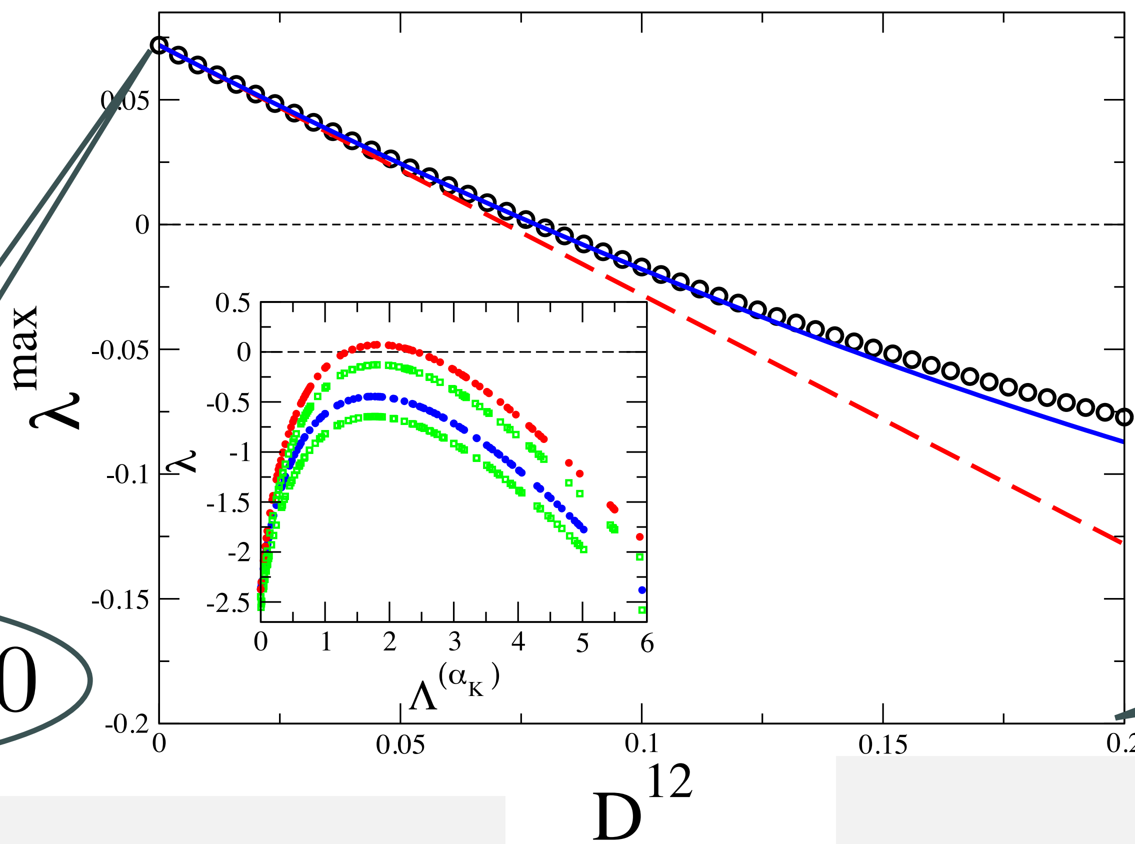
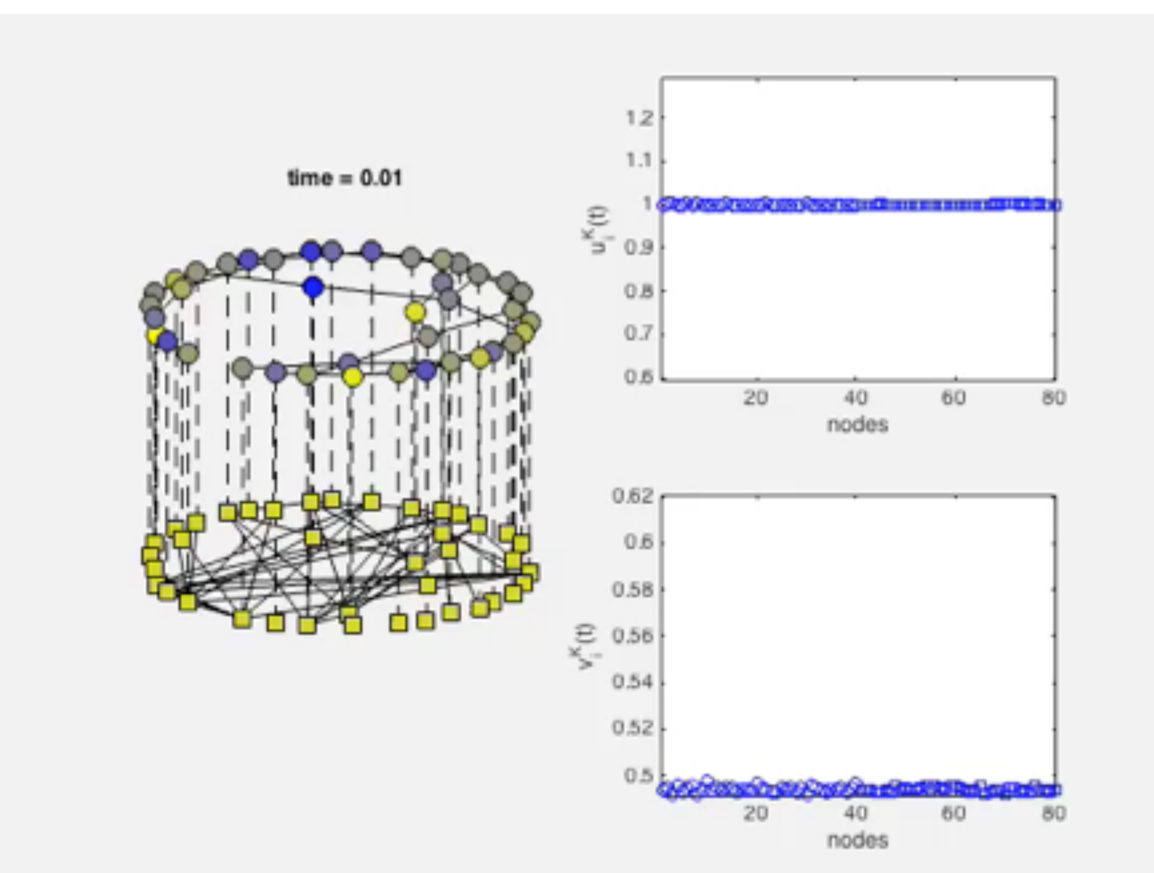
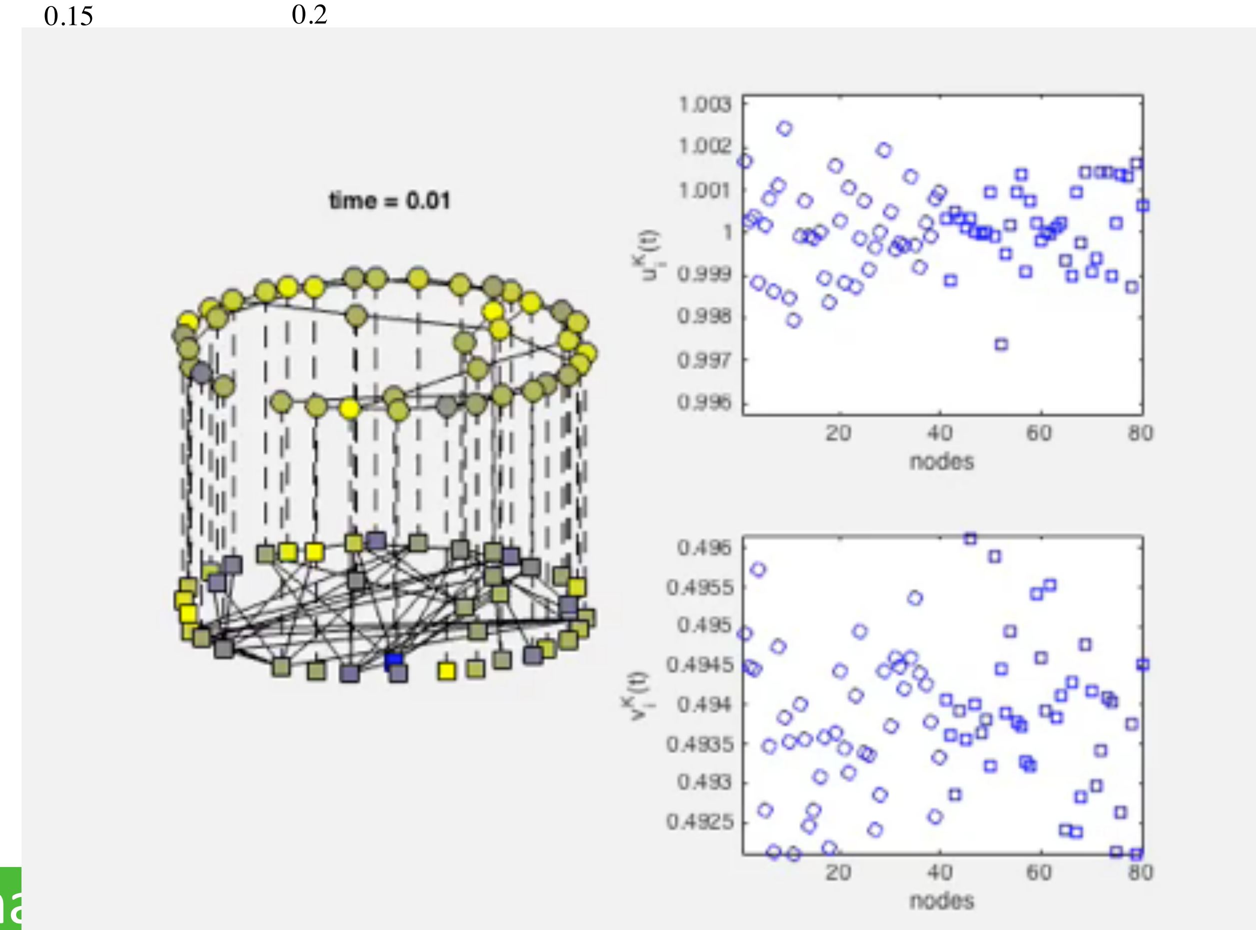


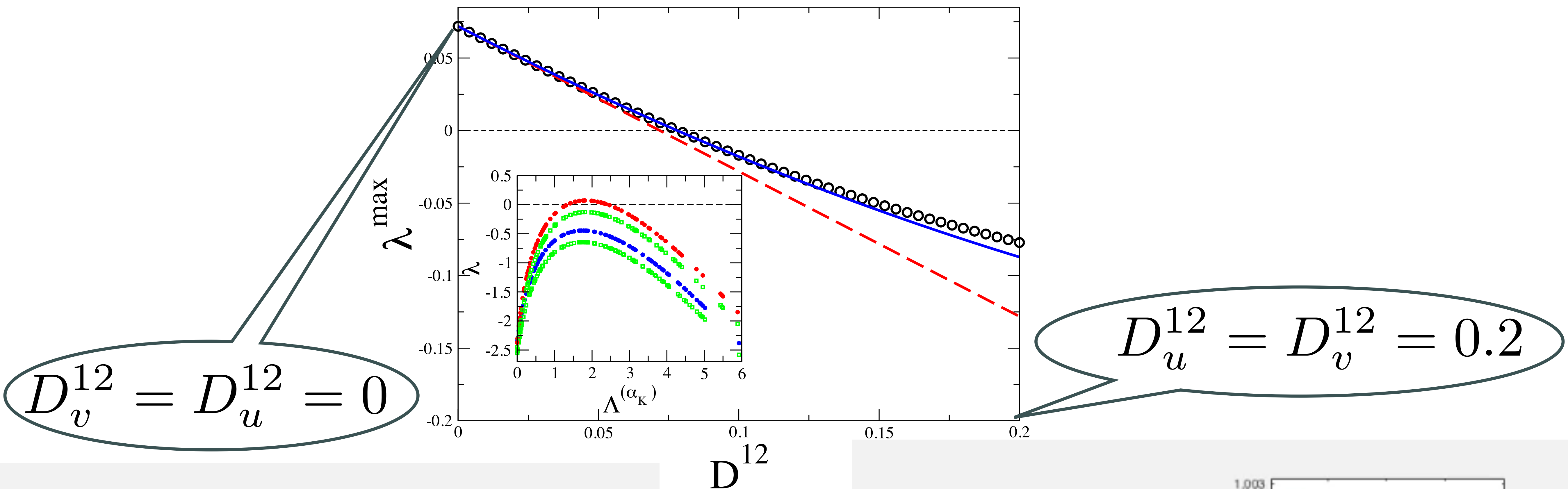
$D_v^{12} = D_u^{12} = 0$

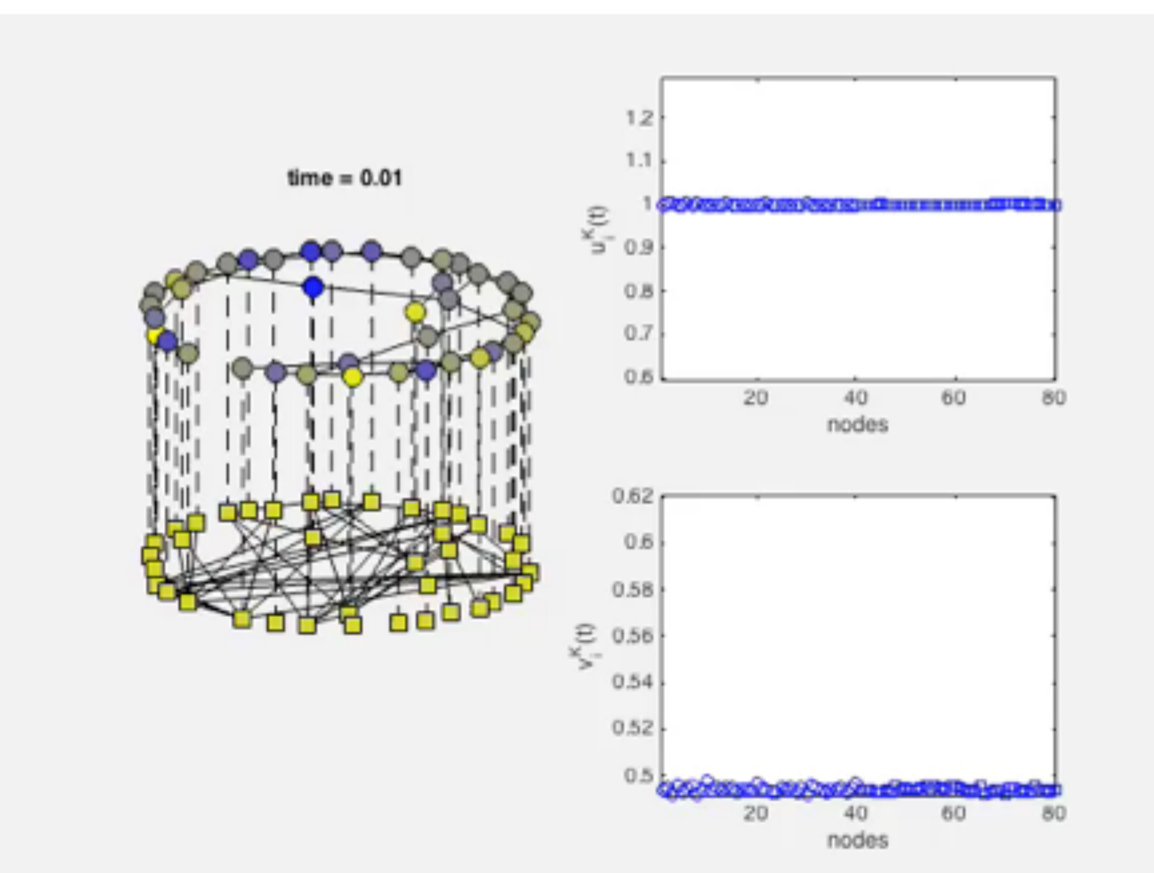
$D_u^{12} = D_v^{12} = 0.2$

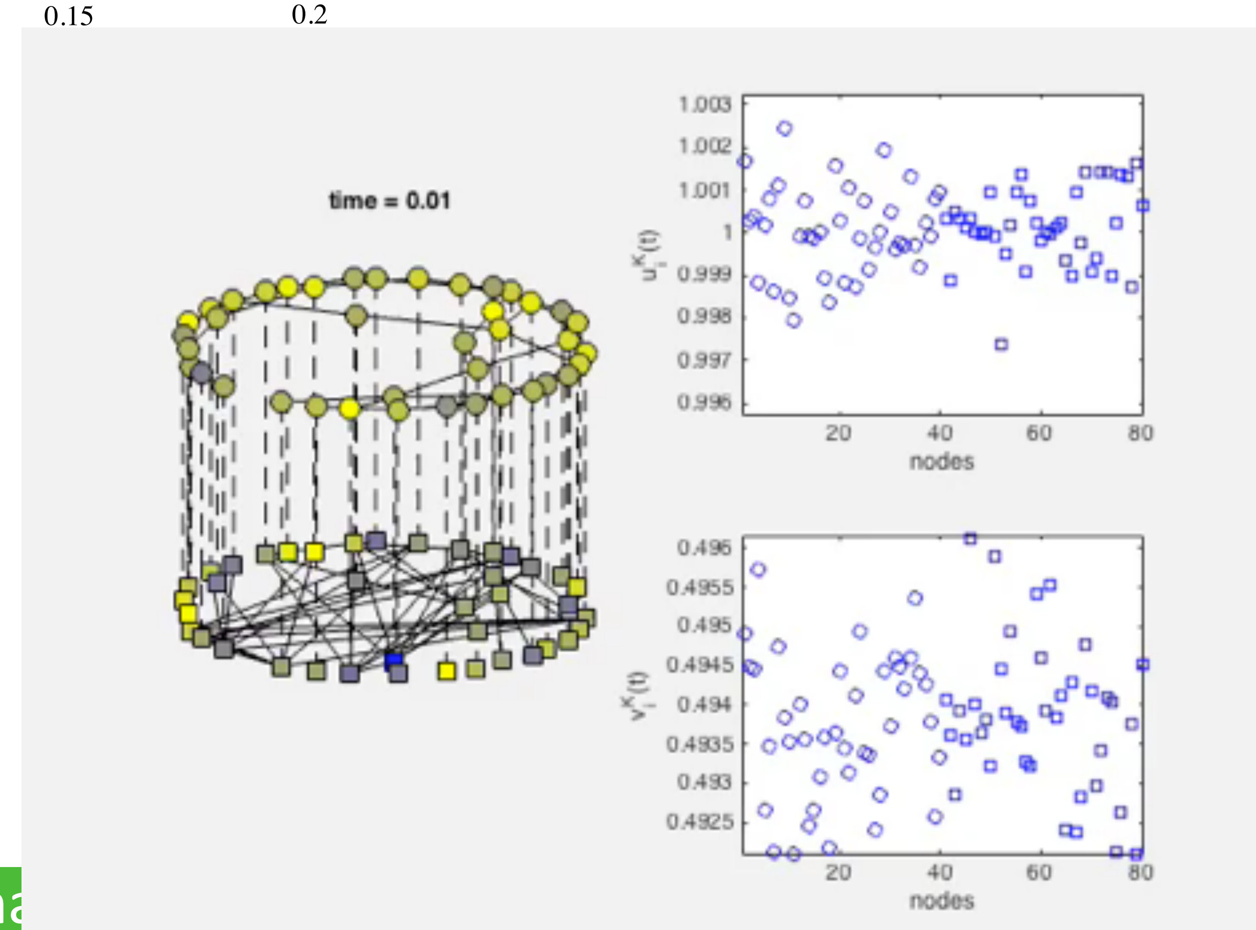




Small intra-layer diffusion case: destruction of patterns



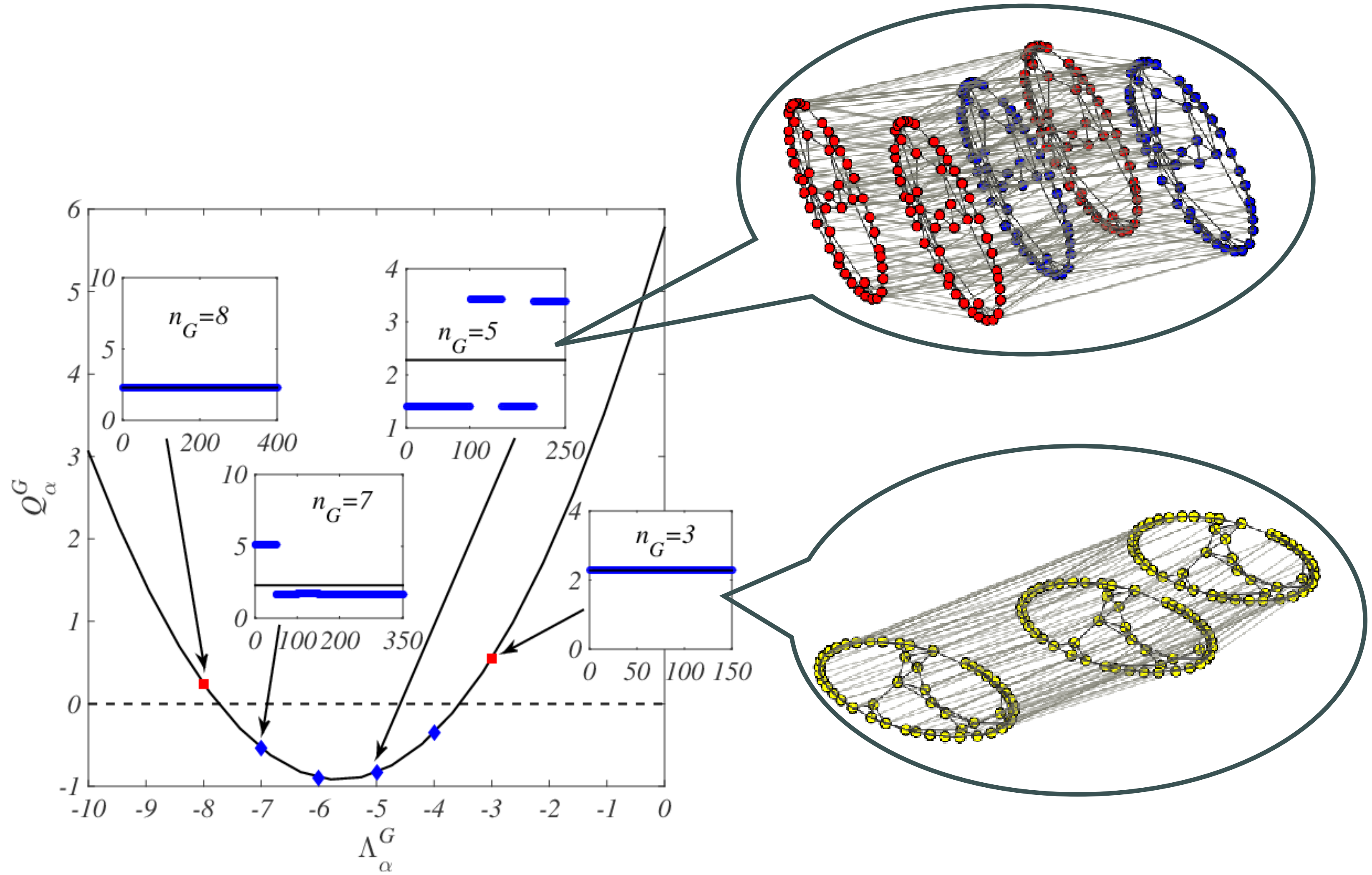




Remarks

1) The large intra-layer diffusion - can be handled as well

2) One can be interested in the effect of adding/removing layers



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Movement across links takes time, so the diffusion part should contain a delay term.

Also reactions can take time, so the reaction part should contain a delay term.

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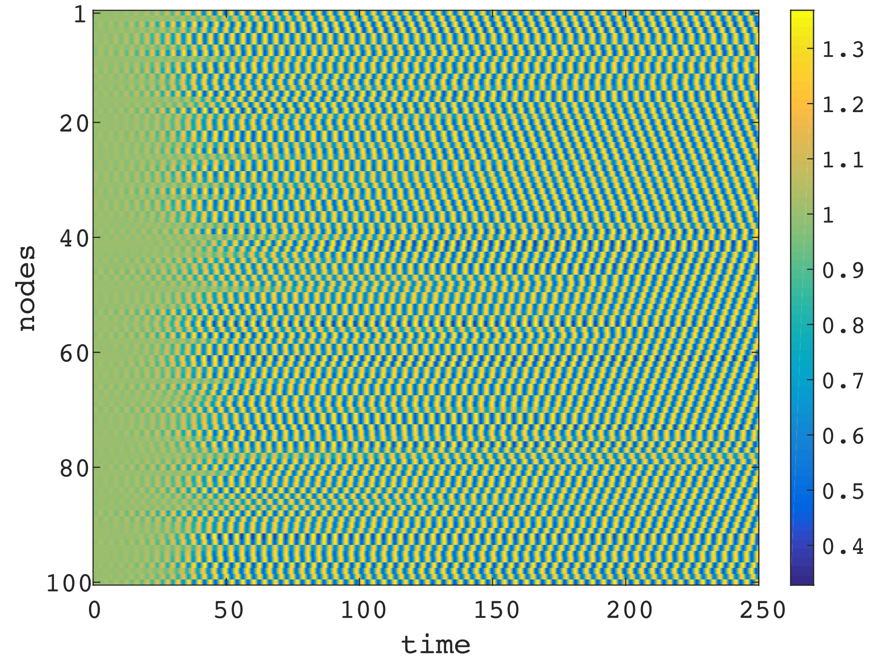

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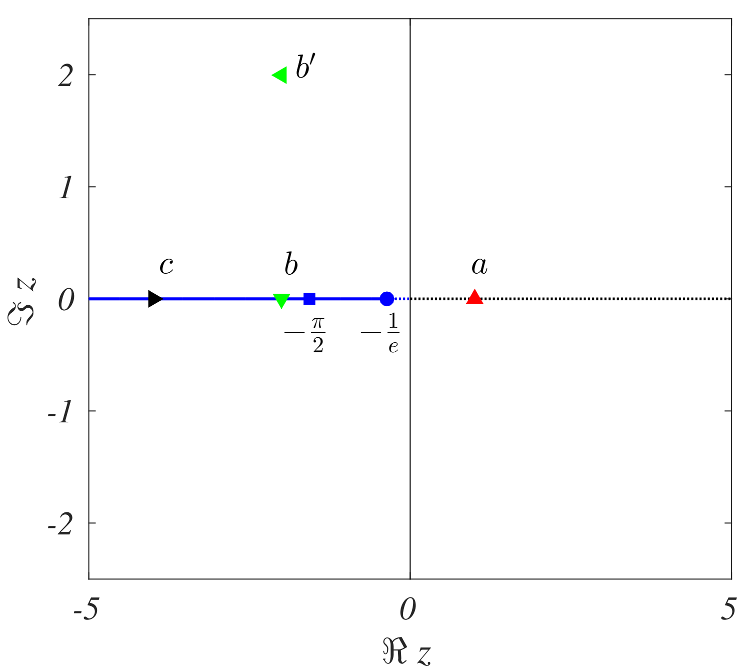
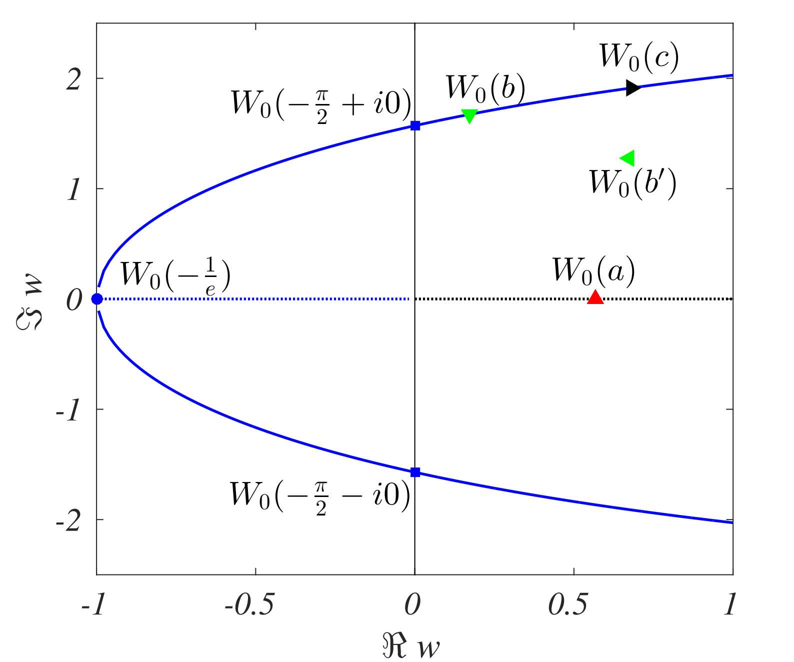
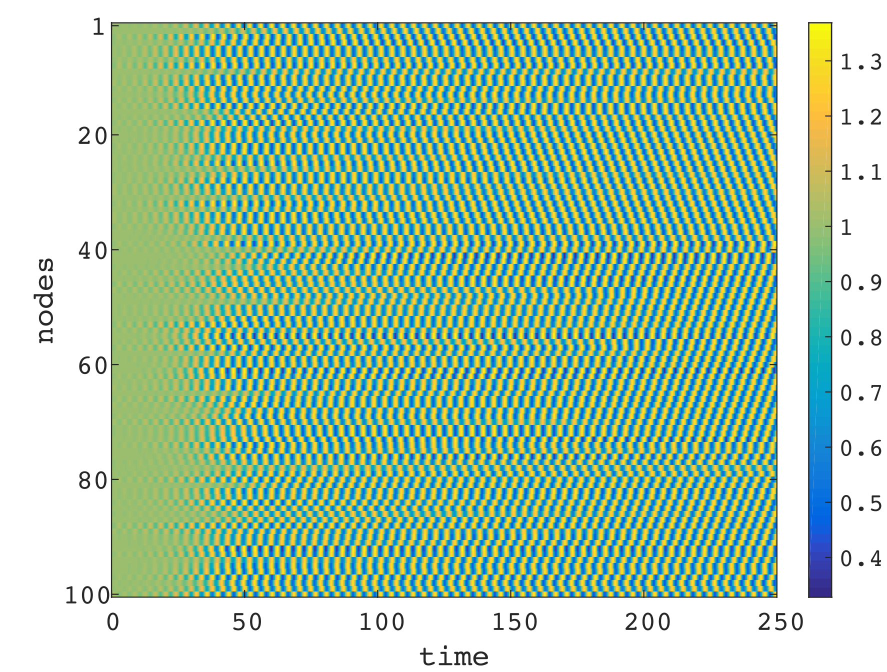
The relation dispersion can be analytically computed using the Lambert W-function







Some papers

Tune the topology to create or destroy patterns, M. Asllani, T. Carletti, D. Fanelli, preprint (2016)

Pattern formation in a two-component reaction-diffusion system with delayed processes on a network, J. Petit, M. Asllani, D. Fanelli, B. Lauwens, T. Carletti, in press Physica A, (2016)

Delay induced Turing-like waves for one species reaction–diffusion model on a network, J. Petit, T. Carletti, M. Asllani, D. Fanelli, Europhysics Letters. **111**, 5, pp. 58002, (2015)

Turing instabilities on Cartesian product networks, M. Asllani, D.M. Busiello, T. Carletti, D. Fanelli, G. Planchon, Scientific Reports. **5**, pp. 12927, (2015)

Turing patterns in multiplex networks, M. Asllani, D.M. Busiello, T. Carletti, D. Fanelli, G. Planchon, Physical Review E ,**90**, 4, pp. 042814, (2014)

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A journey in the zoo of Turing patterns

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