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Timoteo Carletti & Duccio Fanelli

New insight the Collatz conjecture. A contracting Markov walk on a directed graph.

Namur Center for Complex Systems

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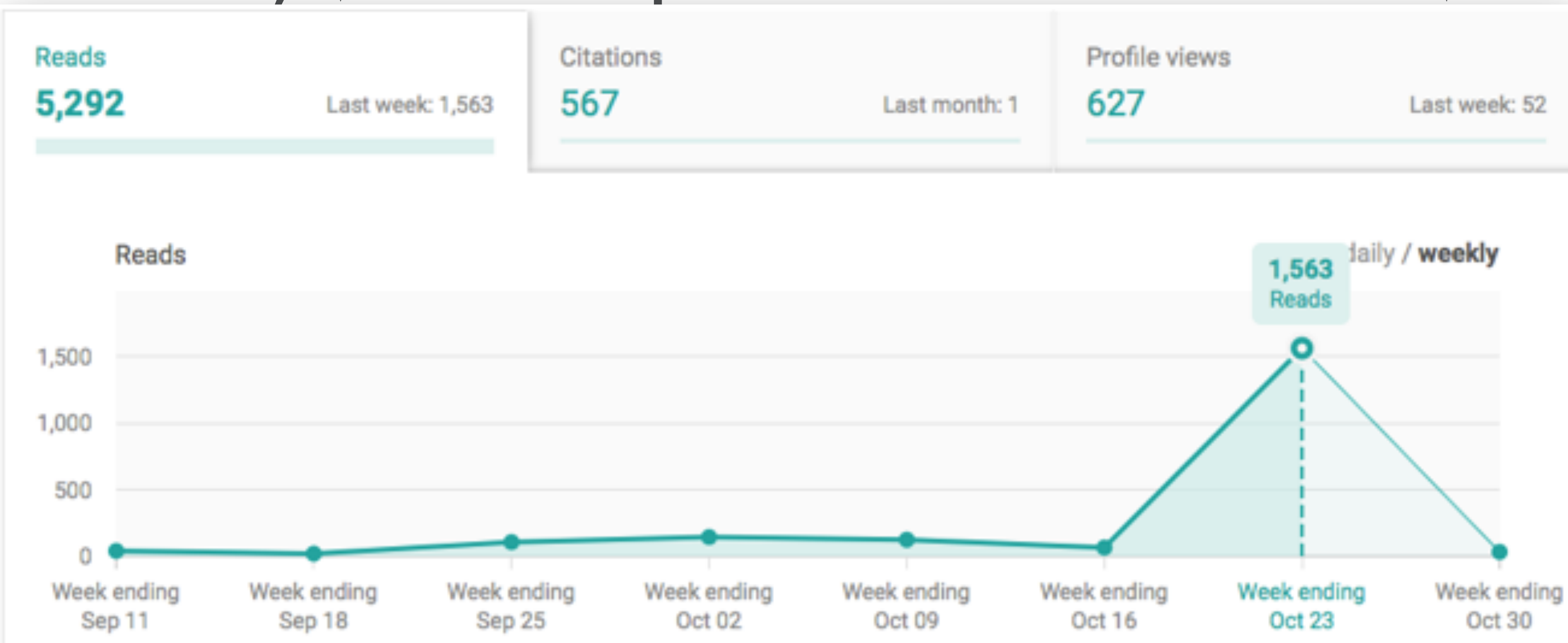

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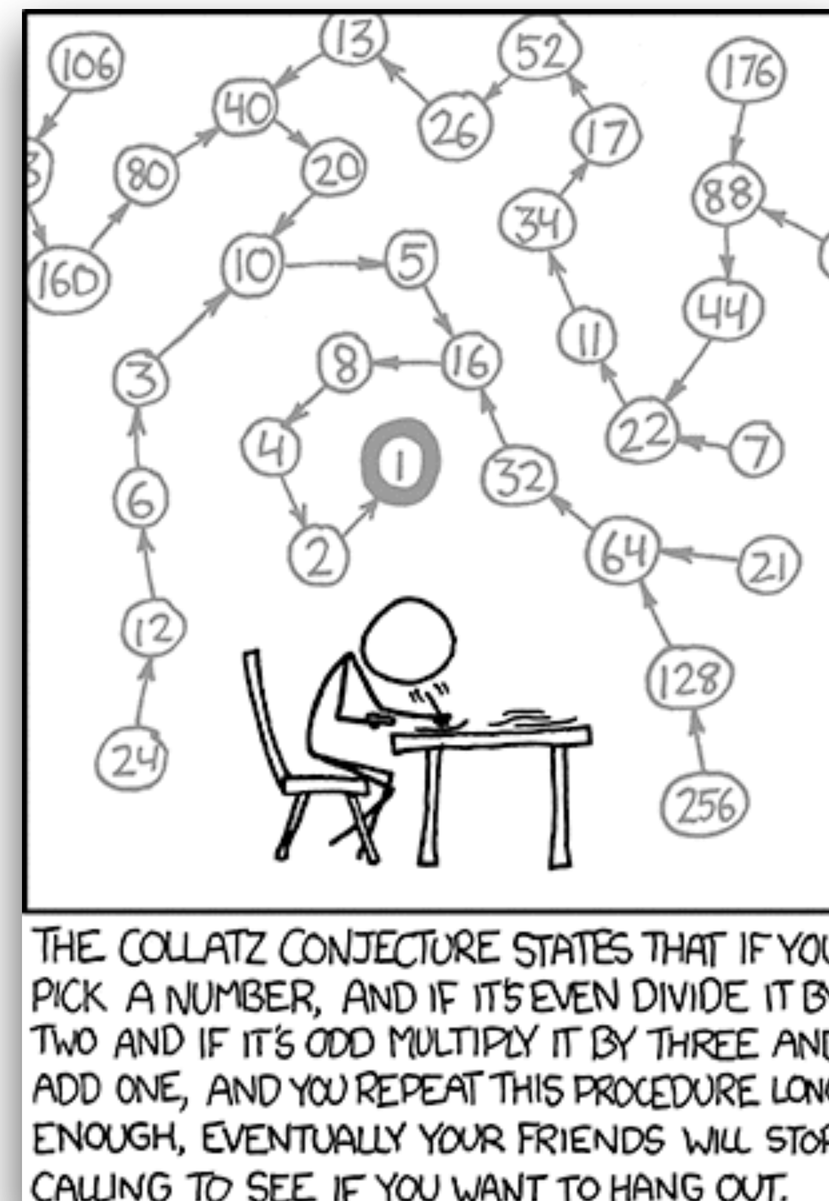

The Collatz map T (Lothar Collatz, 1937)

$$\forall n \in \mathbb{N} \quad T(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ 3n+1 & \text{if } n \text{ is odd.} \end{cases}$$

Conjecture

For any given positive integer $n_0 \in \mathbb{N}$ the iterates of the Collatz map $n_k = T^{\circ k}(n)$ eventually converge to the period-3 cycle {1,2,4} as $k \to \infty$.

J.C. Lagarias, "*The Ultimate Challenge: the 3x+1 problem*". American Mathematical Society, (2010).

Numerically verified up to 1.9×10^{17}

T. Oliveira E Silva, Math. Comp. 68 No. 1 (1999), 371

timoteo.carletti@unamur.be

Main ideas of the proof

1) Introduce the third iterate of the Collatz map and consider the equivalence classes of integer numbers modulo 8;
2) Define a finite state Markov chain with a suitable invariant probability measure, whose transition probabilities reflect the deterministic map;
3) Prove a weak version of the conjecture: diverging orbits cannot exist due to the bound on the average stationary distribution;
4) Prove the conjecture by excluding other periodic orbits.

timoteo.carletti@unamur.be

1) The third iterate of the Collatz map S=T∘3

$$\mathcal{B}(k,8) := \{n \in \mathbb{N} : \exists m \in \mathbb{N},\; n = 8m+k\} \quad k \in \{0,\dots,7\}$$

$$\forall n \in \mathbb{N} \quad S(n) = \begin{cases} \frac{n}{8} & \text{if } n \in \mathcal{B}(0,8) \\ \frac{6n+2}{8} & \text{if } n \in \mathcal{B}(1,8) \\ \frac{6n+4}{8} & \text{if } n \in \mathcal{B}(2,8) \\ \frac{36n+20}{8} & \text{if } n \in \mathcal{B}(3,8) \\ \frac{6n+8}{8} & \text{if } n \in \mathcal{B}(4,8) \\ \frac{6n+2}{8} & \text{if } n \in \mathcal{B}(5,8) \\ \frac{6n+4}{8} & \text{if } n \in \mathcal{B}(6,8) \\ \frac{36n+20}{8} & \text{if } n \in \mathcal{B}(7,8) \end{cases} \; .$$

The 3-cycle {1,2,4} becomes a fixed point

1) The third iterate of the Collatz map S=T∘3

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- $\mathcal{B}(0,8)$: contraction by 1/8
- $\mathcal{B}(1,8)$: contraction by 3/4
- $\mathcal{B}(2,8)$: contraction by 3/4
- $\mathcal{B}(3,8)$: expansion by 9/2
- $\mathcal{B}(4,8)$: contraction by 3/4
- $\mathcal{B}(5,8)$: contraction by 3/4
- $\mathcal{B}(6,8)$: contraction by 3/4
- $\mathcal{B}(7,8)$: expansion by 9/2

The 3-cycle {1,2,4} becomes a fixed point

timoteo.carletti@unamur.be

2) A finite states Markov process

- given a probability space $(\mathbb{N}, \mu)$
- 8 symbols alphabet: $\mathcal{B}(k, 8) \quad k = 0, \ldots, 7$
- Transition probabilities: $q^*_{ij} = P\left[S(x) \in \mathcal{B}(j, 8) | x \in \mathcal{B}(i, 8)\right]$

$$q^*_{ij} := \frac{\mu\left[\mathcal{B}(i, 8) \cap S^{-1}\mathcal{B}(j, 8)\right]}{\mu\left[\mathcal{B}(i, 8)\right]} \, (i, j = 0, \ldots, 7)$$

2) A finite states Markov process

$$q_{ij}^{(m)} := \frac{\mu\left[\mathcal{B}(i,8)\cap S^{-m}\mathcal{B}(j,8)\right]}{\mu\left[\mathcal{B}(i,8)\right]}\,(i,j=0,\dots,7)$$

$$q_{ij}^{(m)} \text{ versus } q_{ij}^{*} \ ?$$

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$$q_{ij}^{(m)} \text{ versus } q_{ij}^{*} \ ?$$

If μ is an invariant measure for S then

$$q_{ij}^{(m)} = \left(q_{ij}^{*}\right)^{m}$$

2) A finite states Markov process

- μ_{inv} invariant (probability) measure

$$q^*_{ij} := \frac{\mu_{inv}\left[\mathcal{B}(i,8)\cap S^{-1}\mathcal{B}(j,8)\right]}{\mu_{inv}\left[\mathcal{B}(i,8)\right]}$$

<u>Chapman-Komogorov equation</u>

Let $p_{m\,0}(j|i)$ be the probability to be in $\mathcal{B}(j,8)$ after m steps being initially in $\mathcal{B}(i,8)$. Then:

$$p_{m\,0}(j|i) = \sum_{k_1,\dots,k_{m-1}} p_{m\,m-1}(j|k_{m-1})p_{m-1\,m-2}(k_{m-1}|k_{m-2})\dots p_{1\,0}(k_1|i)$$

2) Markov process. Invariant measure

Let $n \in \mathbb{N}$, there exist $k \geq 1$ and $s_0, \dots, s_{k-1} \in \{0, \dots, 7\}$

$$n = s_{k-1} 8^{k-1} + \dots + s_1 8 + s_0$$

$$\mu_{inv}(s_{k-1} \dots s_1 s_0) = \frac{1}{2^k} \frac{1}{7} \frac{1}{8^{k-2}} \nu(s_0)$$

Note: $\mu_{inv} > 0$ on all integers.

$$\nu(s_0) = \frac{1}{6} \text{ if } s_0 = 0, 2, 4, 6 \text{ and } \nu(s_0) = \frac{1}{12} \text{ if } s_0 = 1, 3, 5, 7.$$

2) Markov process. Invariant measure

Theorem

For any integer $m \geq 1$ and for all $j = 0, \dots, 8^m - 1$ we have

$$\mu_{inv}\left[S^{-1}\mathcal{B}(j, 8^m)\right] = \mu_{inv}\left[\mathcal{B}(j, 8^m)\right]$$

$$\mu_{inv}(\mathbb{N}) = 1$$

Step 1: compute $\mu_{inv}\left[\mathcal{B}(j, 8^m)\right]$

Step 2: compute $\mu_{inv}\left[S^{-1}\mathcal{B}(j, 8^m)\right]$ by induction on *m*

2) Markov process. Invariant measure (step 2)

Proposition 4. *Let* $j = 0, \dots, 7$, *then* $S^{-1}\mathcal{B}(j,8)$ *is the union of disjoint congruence classes mod64,* $\mathcal{B}(l_j, 64)$, *where the indexes* l_j *depend on the mod8 congruence class* j.

In explicit form:

$$\begin{aligned}
S^{-1}\mathcal{B}(0,8) &= \mathcal{B}(0,64) \cup \mathcal{B}(10,64) \cup \mathcal{B}(42,64) \cup \mathcal{B}(3,64) \cup \mathcal{B}(19,64) \cup \mathcal{B}(35,64) \cup \mathcal{B}(51,64) \cup \mathcal{B}(20,64) \cup \mathcal{B}(52,64) \cup \\
&\quad \cup \mathcal{B}(21,64) \cup \mathcal{B}(53,64) \\
S^{-1}\mathcal{B}(1,8) &= \mathcal{B}(1,64) \cup \mathcal{B}(33,64) \cup \mathcal{B}(22,64) \cup \mathcal{B}(54,64) \cup \mathcal{B}(8,64) \\
S^{-1}\mathcal{B}(2,8) &= \mathcal{B}(2,64) \cup \mathcal{B}(34,64) \cup \mathcal{B}(12,64) \cup \mathcal{B}(44,64) \cup \mathcal{B}(13,64) \cup \mathcal{B}(45,64) \cup \mathcal{B}(7,64) \cup \mathcal{B}(23,64) \cup \mathcal{B}(39,64) \cup \\
&\quad \cup \mathcal{B}(55,64) \cup \mathcal{B}(16,64) \\
S^{-1}\mathcal{B}(3,8) &= \mathcal{B}(25,64) \cup \mathcal{B}(57,64) \cup \mathcal{B}(14,64) \cup \mathcal{B}(46,64) \cup \mathcal{B}(24,64) \\
S^{-1}\mathcal{B}(4,8) &= \mathcal{B}(26,64) \cup \mathcal{B}(58,64) \cup \mathcal{B}(11,64) \cup \mathcal{B}(27,64) \cup \mathcal{B}(43,64) \cup \mathcal{B}(59,64) \cup \mathcal{B}(4,64) \cup \mathcal{B}(36,64) \cup \mathcal{B}(5,64) \cup \\
&\quad \cup \mathcal{B}(37,64) \cup \mathcal{B}(32,64) \\
S^{-1}\mathcal{B}(5,8) &= \mathcal{B}(17,64) \cup \mathcal{B}(49,64) \cup \mathcal{B}(6,64) \cup \mathcal{B}(38,64) \cup \mathcal{B}(40,64) \\
S^{-1}\mathcal{B}(6,8) &= \mathcal{B}(18,64) \cup \mathcal{B}(50,64) \cup \mathcal{B}(28,64) \cup \mathcal{B}(60,64) \cup \mathcal{B}(29,64) \cup \mathcal{B}(61,64) \cup \mathcal{B}(15,64) \cup \mathcal{B}(31,64) \cup \mathcal{B}(47,64) \cup \\
&\quad \cup \mathcal{B}(63,64) \cup \mathcal{B}(48,64) \\
S^{-1}\mathcal{B}(7,8) &= \mathcal{B}(9,64) \cup \mathcal{B}(41,64) \cup \mathcal{B}(30,64) \cup \mathcal{B}(62,64) \cup \mathcal{B}(56,64)\,. \qquad (9)
\end{aligned}$$

<u>Remark</u> (On the solution of congruence linear equations)

Given positive integers a, b and n, then the equation <u>*ax=b mod n*</u> has solution iff <u>*d=gcd(a,n)*</u> (greatest common divisor) is a divisor of *b*, in this case the number of distinct solutions is given by *d*.

timoteo.carletti@unamur.be

2) Markov process. Invariant measure (step 2)

Proposition 6. *For any $m \geq 1$ and for all $j = 0, \dots, 8^m - 1$ we have*

$$S^{-1}\mathcal{B}(j, 8^m) = A_e^{(m)}(j) \cup A_o^{(m)}(j)\,, \tag{14}$$

where $A_e(j)$ is the union of disjoint classes $\mathcal{B}(l, 8^{m+1})$ with l even and $A_o^{(m)}(j)$ is the union of disjoint classes $\mathcal{B}(l, 8^{m+1})$ with l odd. Moreover if j is even then $A_e^{(m)}(j)$ contains five elements and $A_o^{(m)}(j)$ six elements, while if j is odd then $A_e^{(m)}(j)$ contains three elements and $A_o^{(m)}(j)$ two elements.

2) Markov process. Invariant measure (step 2)

Proposition 6. *For any $m \geq 1$ and for all $j = 0, \dots, 8^m - 1$ we have*

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Compute $\mu_{inv}\left[S^{-1}\mathcal{B}(j, 8^m)\right]$

$$\mu_{inv}\left[S^{-1}\mathcal{B}(j, 8^m)\right] = \mu_{inv}\left[A_e^{(m)}(j)\right] + \mu_{inv}\left[A_o^{(m)}(j)\right]$$

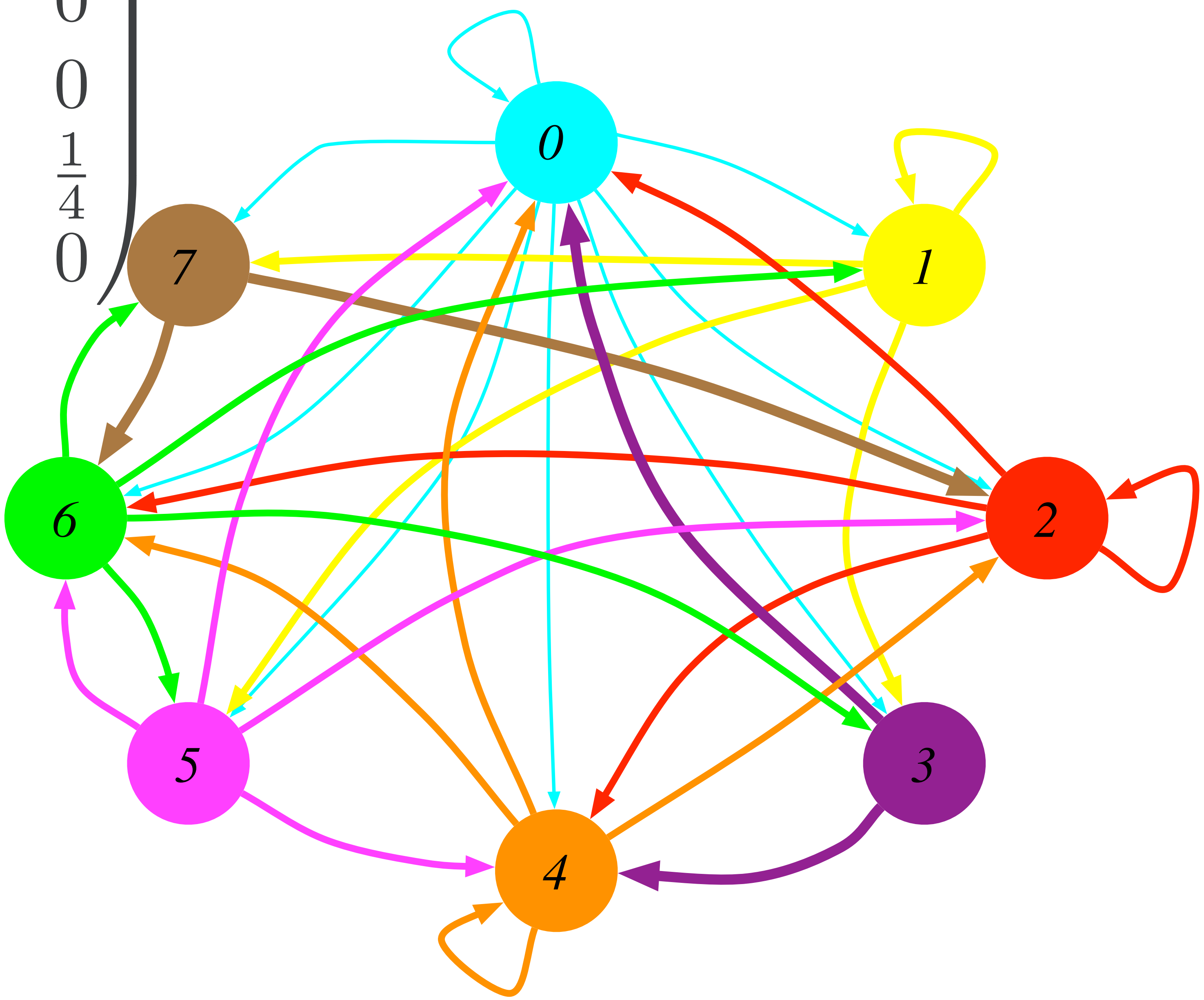
$$= \frac{1}{8^m}\frac{1}{6}\#A_e^{(m)}(j) + \frac{1}{8^m}\frac{1}{12}\#A_o^{(m)}(j)$$

(j even) $$= \frac{1}{8^m}\frac{1}{6}5 + \frac{1}{8^m}\frac{1}{12}6 = \frac{1}{8^{m-1}}\frac{1}{6} = \mu_{inv}\left[\mathcal{B}(j, 8^m)\right]$$

2) Markov process. Transition probabilities

$$Q^* = \begin{pmatrix} \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} \\ 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} \\ \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 \\ \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 \\ 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 \end{pmatrix}$$

$$q^*_{ij} := \frac{\mu_{inv}\left[\mathcal{B}(i,8) \cap S^{-1}\mathcal{B}(j,8)\right]}{\mu_{inv}\left[\mathcal{B}(i,8)\right]}$$



2) Markov process. Some properties

- Q^* is a stochastic matrix $\quad \forall i = 0, \dots, 7 \quad \sum_j q^*_{ij} = 1$

- The Markov process with stochastic matrix Q^* is ergodic.

- The left eigenvector associated to $\lambda = 1$ is $\vec{P}_S Q^* = \vec{P}_S$

$$\vec{P}_S = (1/6, 1/12, 1/6, 1/12, 1/6, 1/12, 1/6, 1/12)$$

$\mu_{inv}(\mathcal{B}(k,8)) = P_{S,k}$ invariant (stationary) distribution

3) Weak Collatz conjecture: all orbits are bounded

Ergodic Theorem

For μ_{inv} - almost every $n_0 \in \mathbb{N}$

$$\lim_{k\to\infty} \frac{\#\{0 \leq j \leq k-1 : S^{\circ j}(n_0) \in \mathcal{B}(i,8)\}}{k} = \mu_{inv}\left(\mathcal{B}(i,8)\right) \quad \forall i \in \{0,\dots,7\}$$

3) Weak Collatz conjecture: all orbits are bounded

Given any $n_0 \in \mathbb{N}$, then there exists a positive integer M(n_0) such that for $S^{\circ k}(n_0) \leq M(n_0)$ all k>0.

Proof

Assume there exists a diverging orbit for S.

By the ergodic theorem

$$\lim_{k\to\infty} \frac{\#\{0 \leq j \leq k-1 : S^{\circ j}(n_0) \in \mathcal{B}(i,8)\}}{k} = \mu_{inv}\left(\mathcal{B}(i,8)\right) \quad \forall i \in \{0,\dots,7\}$$

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$$\forall n \in \mathbb{N} \quad S(n) = \begin{cases} \frac{n}{8} & \text{if } n \in \mathcal{B}(0,8) \\ \frac{6n+2}{8} & \text{if } n \in \mathcal{B}(1,8) \\ \frac{6n+4}{8} & \text{if } n \in \mathcal{B}(2,8) \\ \frac{36n+20}{8} & \text{if } n \in \mathcal{B}(3,8) \\ \frac{6n+8}{8} & \text{if } n \in \mathcal{B}(4,8) \\ \frac{6n+2}{8} & \text{if } n \in \mathcal{B}(5,8) \\ \frac{6n+4}{8} & \text{if } n \in \mathcal{B}(6,8) \\ \frac{36n+20}{8} & \text{if } n \in \mathcal{B}(7,8) \end{cases} .$$

$$\mu_{inv}\left[\mathcal{B}(j,8)\right] = \frac{1}{6} \text{ if } j \in \{0,2,4,6\}$$

$$\mu_{inv}\left[\mathcal{B}(j,8)\right] = \frac{1}{12} \text{ if } j \in \{1,3,5,7\}$$

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average contracting/expanding factor:

$$f_{Q^*} = \left(\frac{1}{8}\right)^{1/6} \left(\frac{3}{4}\right)^{2/3} \left(\frac{9}{2}\right)^{1/6} = \frac{3}{4} < 1$$

Contradiction with the unboundedness of the orbit.

timoteo.carletti@unamur.be

3) Weak Collatz conjecture: all orbits are bounded

The result of the ergodic theorem holds for μ_{inv}- a.e. $n_0 \in \mathbb{N}$

3) Weak Collatz conjecture: all orbits are bounded

The result of the ergodic theorem holds for μ_{inv}- a.e. $n_0 \in \mathbb{N}$

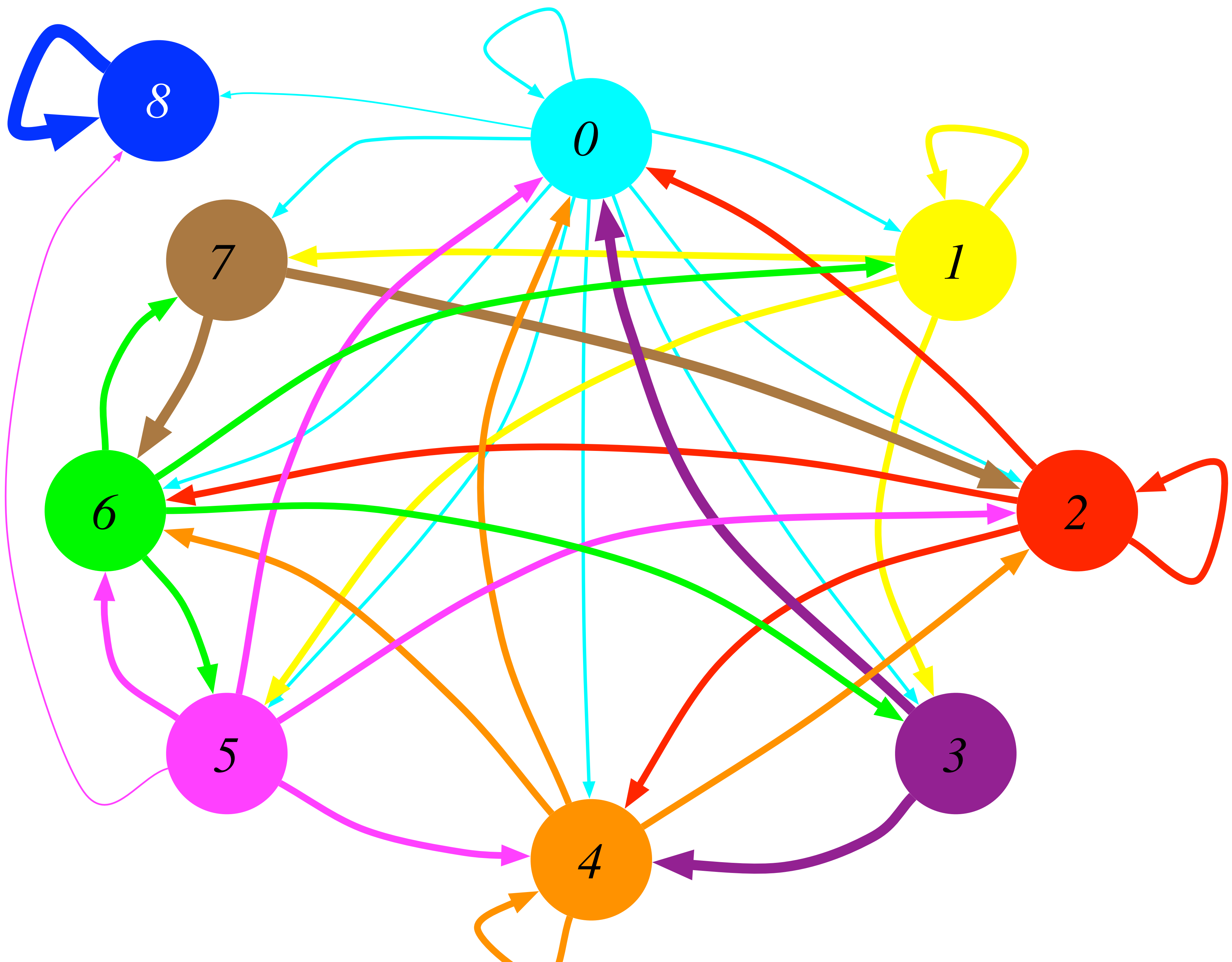
The existence of zero μ_{inv} measure sets can be ruled out because $\mu_{inv}(n_0) > 0$ for all $n_0 \in \mathbb{N}$

So ALL the orbits are bounded

4) Markov process with absorbing state

Let us observe that once we have proved the weak Collatz conjecture we can conclude using the following finite state Markov process

$$\begin{cases} \mathcal{C}_0 &= \mathcal{B}(0,8) \\ \mathcal{C}_1 &= \mathcal{B}(1,8) \setminus \{1\} \\ \mathcal{C}_2 &= \mathcal{B}(2,8) \setminus \{2\} \\ \mathcal{C}_3 &= \mathcal{B}(3,8) \\ \mathcal{C}_4 &= \mathcal{B}(4,8) \setminus \{4\} \\ \mathcal{C}_5 &= \mathcal{B}(5,8) \\ \mathcal{C}_6 &= \mathcal{B}(6,8) \\ \mathcal{C}_7 &= \mathcal{B}(7,8) \\ \mathcal{C}_8 &= \{1,2,4\} \end{cases}$$



4’) Collatz cycle: {1,2,4} is the unique attracting cycle

No periodic orbits of period different from 1 exist for S.

Proof

timoteo.carletti@unamur.be

4') Collatz cycle: {1,2,4} is the unique attracting cycle

No periodic orbits of period different from 1 exist for S.

Proof

Assume there exists a L-periodic orbit for S with L > 1. Observe it is a possible realisation of the Markov chain, hence it should infinitely visit the 8 classes.

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Assume there exists a L-periodic orbit for S with L > 1. Observe it is a possible realisation of the Markov chain, hence it should infinitely visit the 8 classes.

Invoking the ergodic theorem this orbit should have $f_{Q^*} < 1$

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Contradicting the periodicity (that would require $f_{Q^*} = 1$)

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Assume there exists a L-periodic orbit for S with L > 1. Observe it is a possible realisation of the Markov chain, hence it should infinitely visit the 8 classes.

Invoking the ergodic theorem this orbit should have $f_{Q^*} < 1$

Contradicting the periodicity (that would require $f_{Q^*} = 1$)

The existence of zero μ_{inv} measure sets can be ruled out as previously done

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New insight the Collatz conjecture. A contracting Markov walk on a directed graph.

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