

November the 16th, 2016, ULB Brussels

# Timoteo Carletti

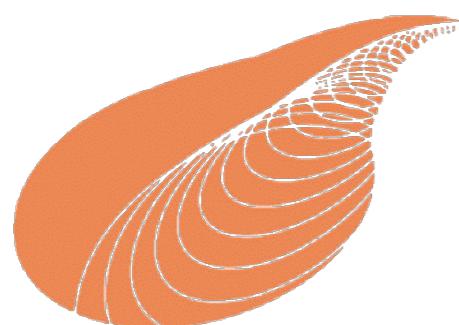
## A journey in the zoo of Turing patterns



## Acknowledgements

“Belgian” team: M. Asllani, J. Petit, G. Planchon

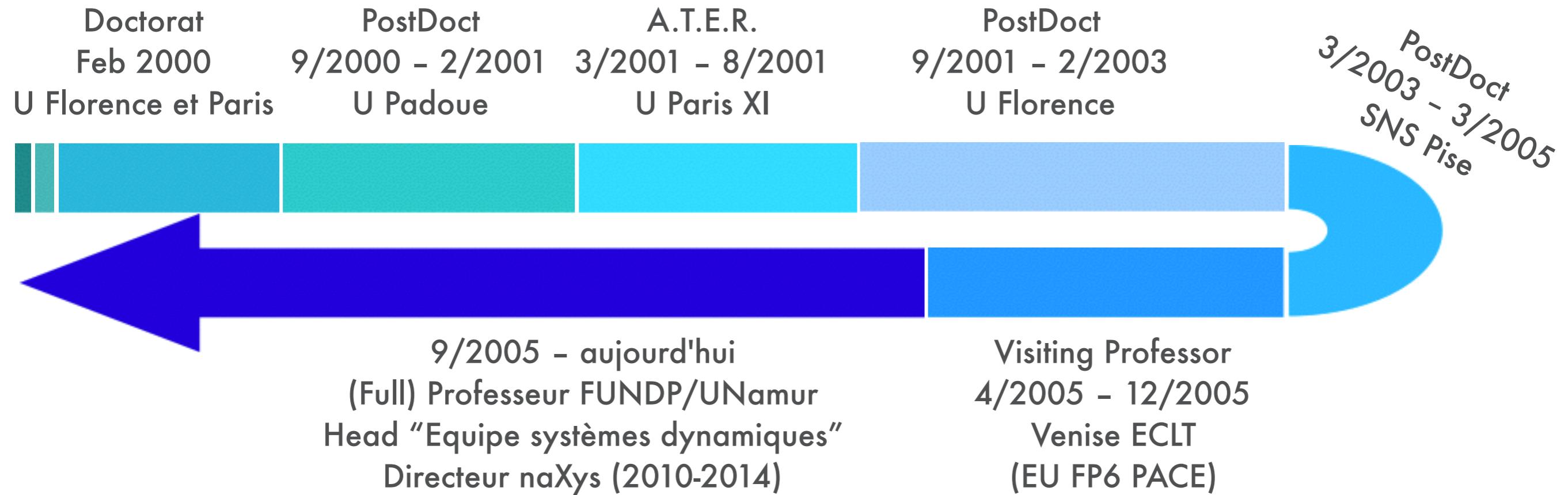
“Italian” team: D. Fanelli, D.M. Busiello, C. Cianci, M. Galanti, F. Di Patti



IAP VII/19 - DYSKO



# Who am I?



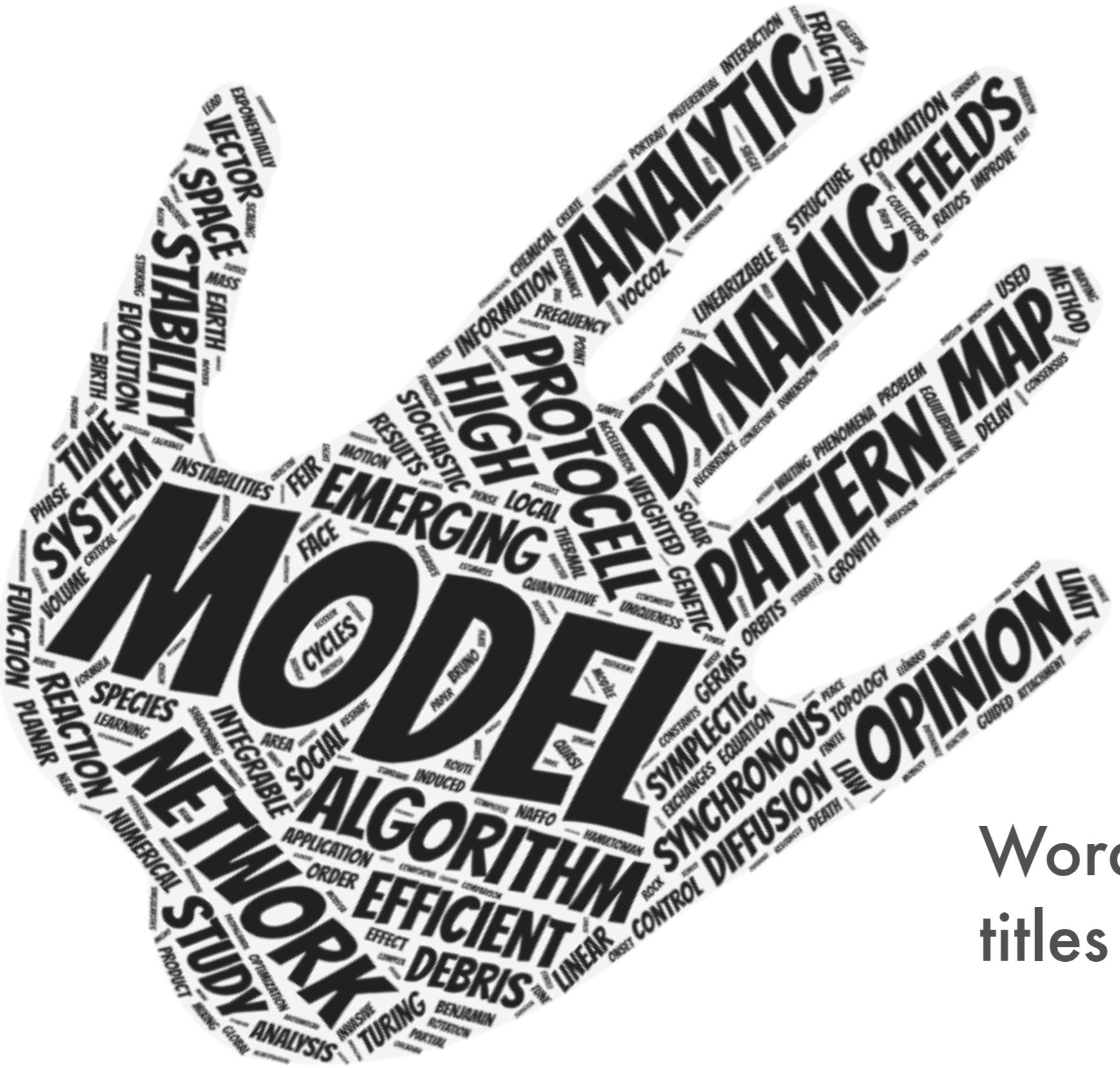
## PhD Thesis title:

*"Stability of orbits and Arithmetics for some discrete dynamical systems"*

## Most cited paper:

T.C., S. Marmi, *Linearization of Analytic and Non-Analytic Germs of Diffeomorphisms of  $(C, 0)$* , Bulletin SMF, **128**, (2000), pp. 69-85

# Who am I?



# Word cloud of the titles of my papers

# Pattern ? [Oxford dictionary]

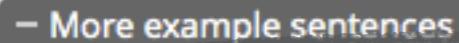
## pattern

★ Top 1000 frequently used words

Pronunciation: /'pat(ə)n/ 

### NOUN

1 A repeated decorative design:  
*'a neat blue herringbone pattern'*

 – More example sentences

*'Included are geometrics, florals and foliates, animals and nature motifs and other decorative repeat patterns.'*

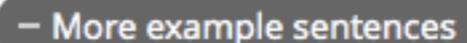
*'These aspects then become ornamented with Islamic-inspired decorative patterns and Islamic cultural artifacts.'*

*'It featured exuberant decorative patterns, designs in the brickwork and wooden attachments.'*

1 A repeated decorative design:  
*'a neat blue herringbone pattern'*

 + More example sentences  + Synonyms

1.1 An arrangement or design regularly found in comparable objects:  
*'the house had been built on the usual pattern'*

 – More example sentences

*'Structurally, the tumor cells were arranged in a medullary pattern composed of polygonal tumor cells.'*

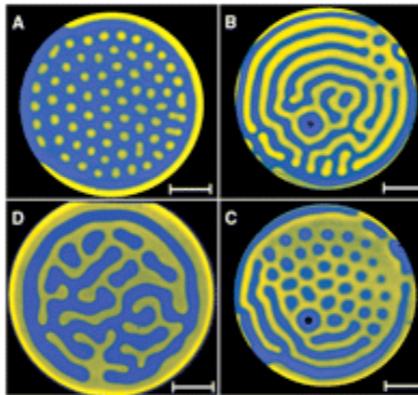
*'The hair-cells within the spiralling cochlear duct are arranged in a pattern like the bristles of a brush.'*

*'The fossils indicate the wings had feathers, arranged in a similar pattern to that of modern birds.'*

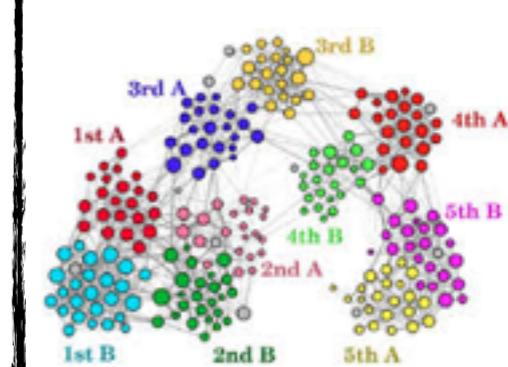
# Patterns are ubiquitous



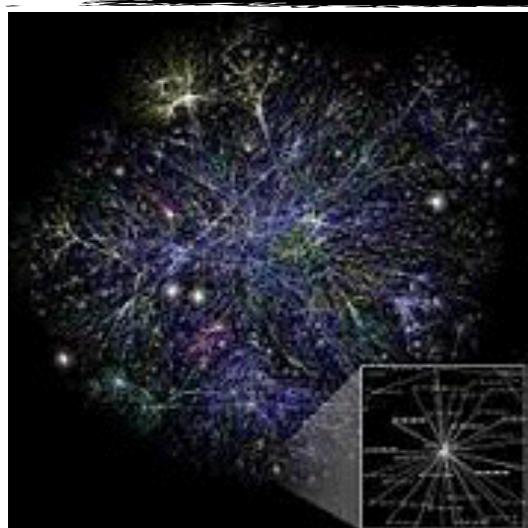
Animal kingdom



Chemistry



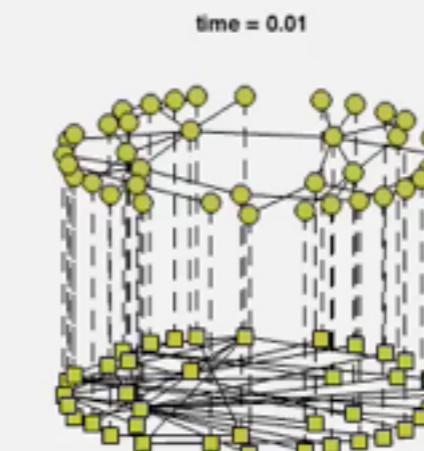
SocioPatterns



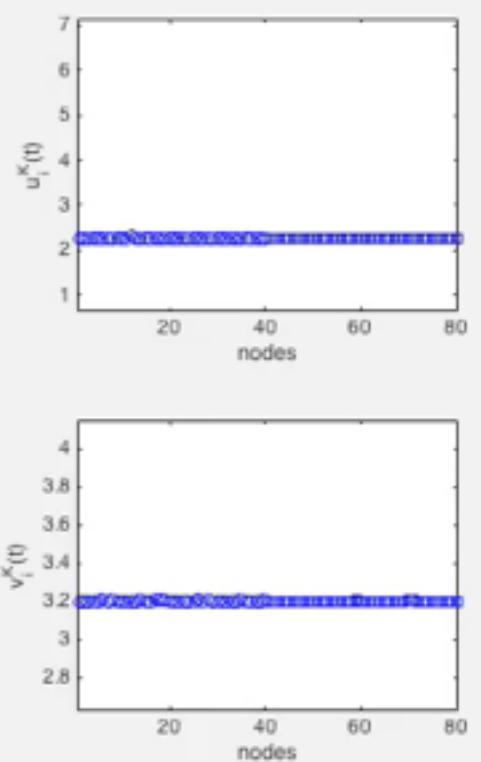
Internet



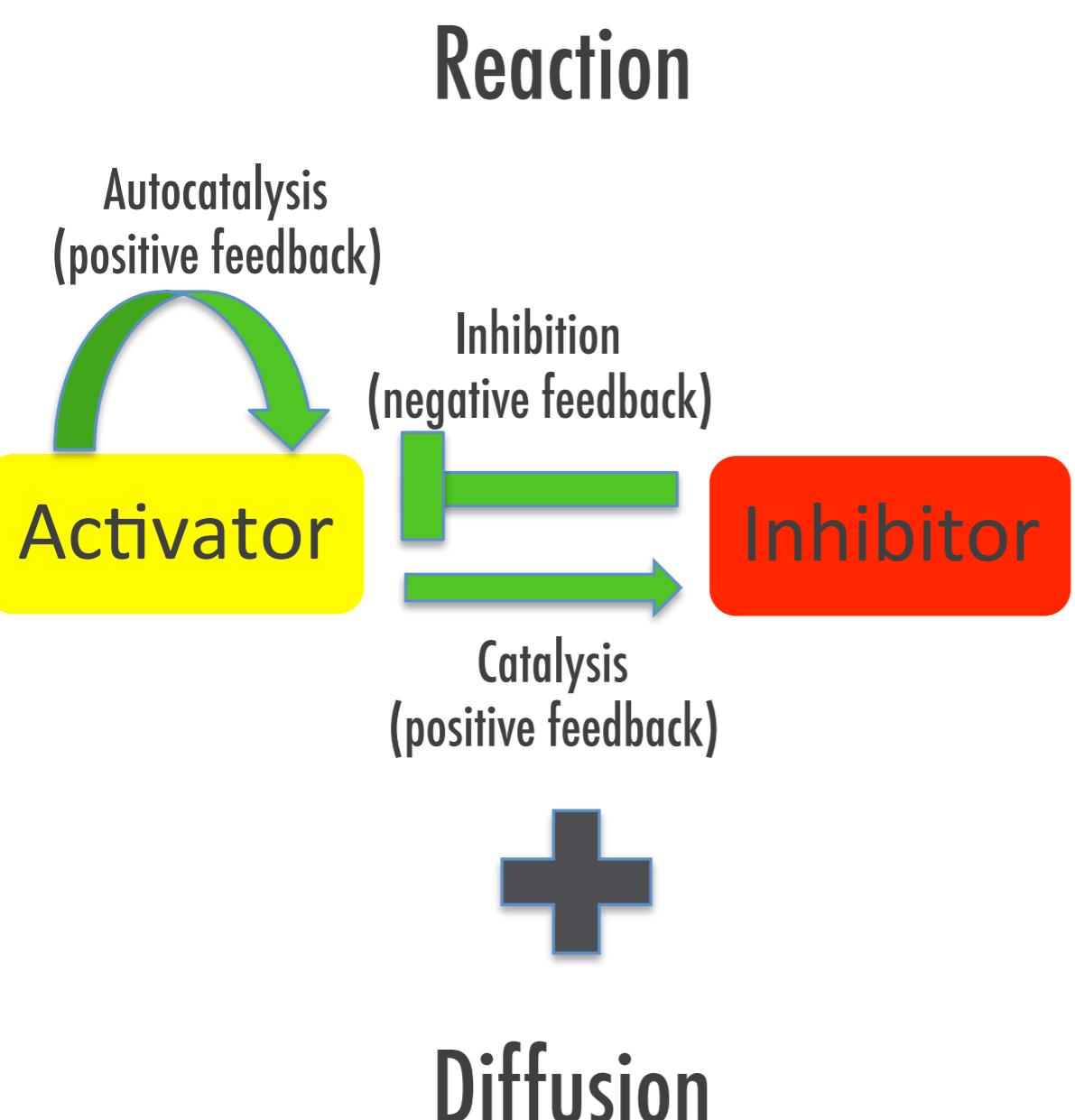
Twitter



Math models



# One possible mechanism: Turing instability



$u(x, y, t)$ : Amount of activator at time  $t$  and position  $(x, y)$

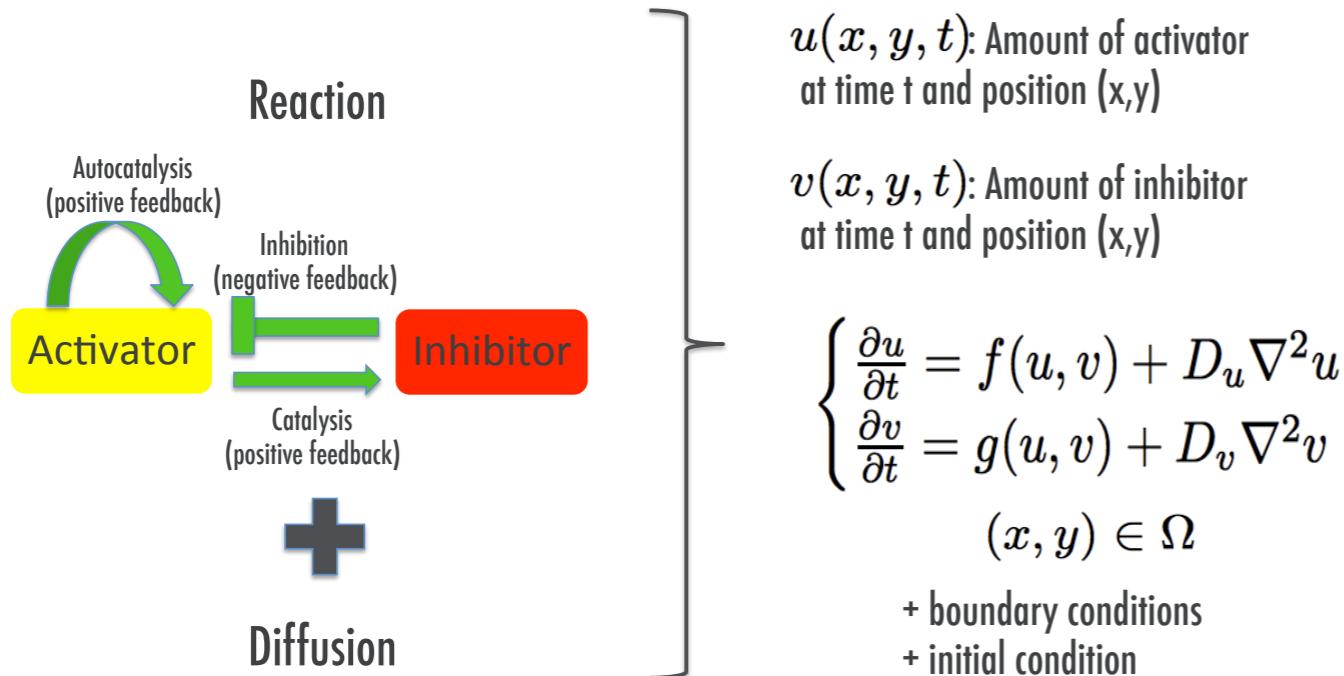
$v(x, y, t)$ : Amount of inhibitor at time  $t$  and position  $(x, y)$

$$\begin{cases} \frac{\partial u}{\partial t} = f(u, v) + D_u \nabla^2 u \\ \frac{\partial v}{\partial t} = g(u, v) + D_v \nabla^2 v \end{cases}$$

$$(x, y) \in \Omega$$

+ boundary conditions  
+ initial condition

# One possible mechanism: Turing instability



Diffusion can drive an instability by perturbing a homogeneous stable fixed point (in absence of diffusion)

Hence as the perturbation grows, non-linearities enter into the game yielding an asymptotic, spatially inhomogeneous, steady state (stationary pattern) or time varying one (wave like pattern).

A.M.Turing, *The chemical basis of morphogenesis*, Phil. Trans. R Soc London B, 237, (1952), pp.37

## Some mathematics for the Turing instability

$$\begin{cases} \frac{\partial u}{\partial t} = f(u, v) + D_u \nabla^2 u \\ \frac{\partial v}{\partial t} = g(u, v) + D_v \nabla^2 v \\ (x, y) \in \Omega \end{cases}$$

Assume  $\Omega$  to be a square/rectangle and to use periodic boundary conditions

1) Assume there exists a spatially homogeneous solution:

$$u(x, y, t) = \hat{u} \text{ and } v(x, y, t) = \hat{v} \quad \forall (x, y) \in \Omega \text{ and } \forall t \geq 0$$

which moreover is stable when there is no diffusion:

$$D_u = D_v = 0$$

## Some mathematics for the Turing instability

### 2) Linearise around this solution

$$\begin{cases} u(x, y, t) = \hat{u} + \delta u(x, y, t) \\ v(x, y, t) = \hat{v} + \delta v(x, y, t) \end{cases} \quad \begin{pmatrix} \dot{\delta u} \\ \dot{\delta v} \end{pmatrix} = \tilde{\mathcal{J}} \begin{pmatrix} \delta u \\ \delta v \end{pmatrix}$$

$$\tilde{\mathcal{J}} = \begin{pmatrix} f_u + D_u \nabla^2 & f_v \\ g_u & g_v + D_v \nabla^2 \end{pmatrix}$$

### 3) Prove that the spatially homogeneous solution:

$$u(x, y, t) = \hat{u} \text{ and } v(x, y, t) = \hat{v}$$

turns out to be unstable once the diffusion is in action

$$D_u > 0 \text{ and } D_v > 0$$

# Some mathematics for the Turing instability

## Sketch of the proof

i) decompose the solution on the Fourier modes (Laplacian eigenbasis) and use the ansatz

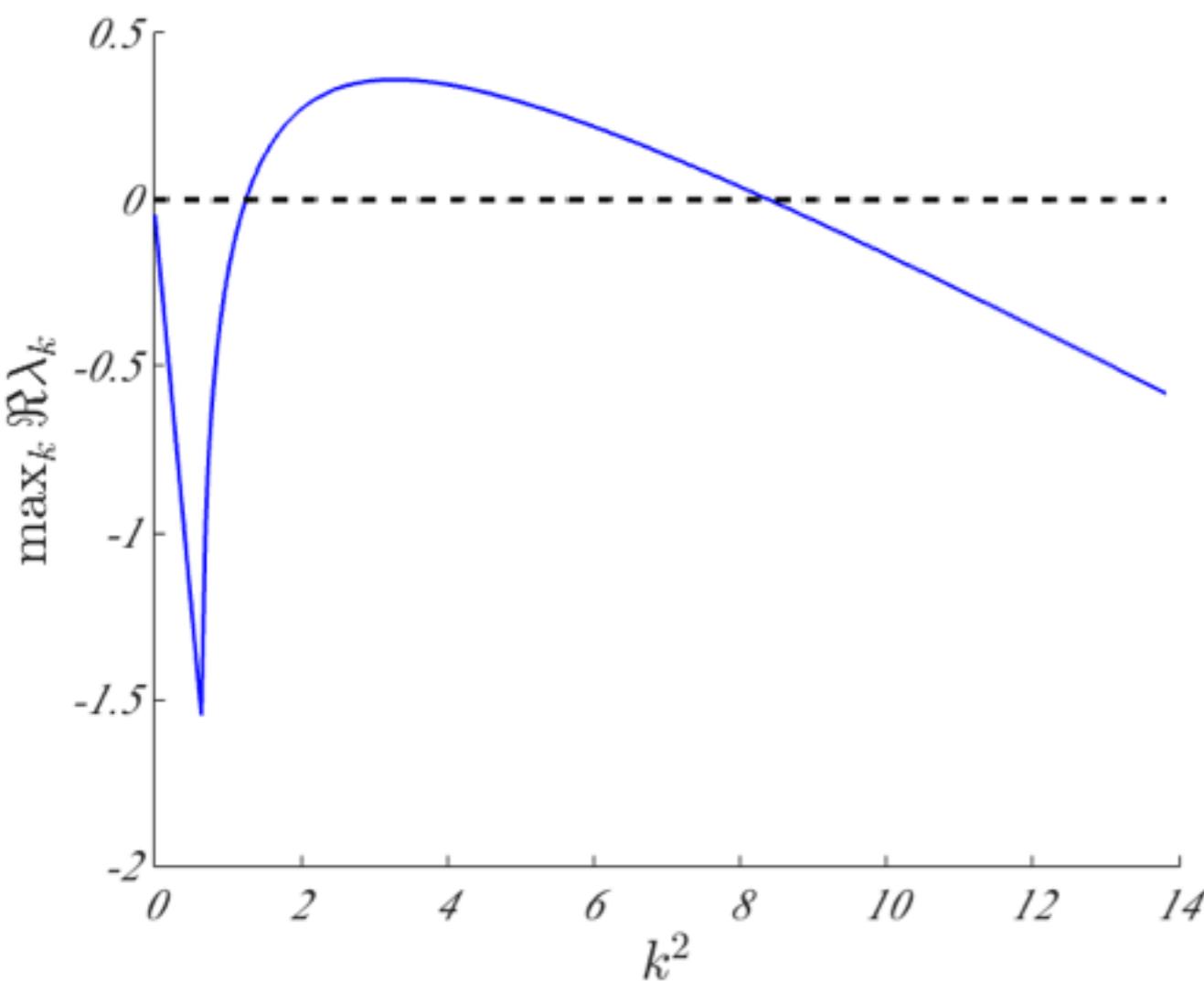
$$\delta u(x, y, t) = \sum_{k=(k_1, k_2)} c_k e^{2\pi i(k_1 x + k_2 y)} e^{\lambda_k t}$$

ii)  $\lambda_k$  is solution of

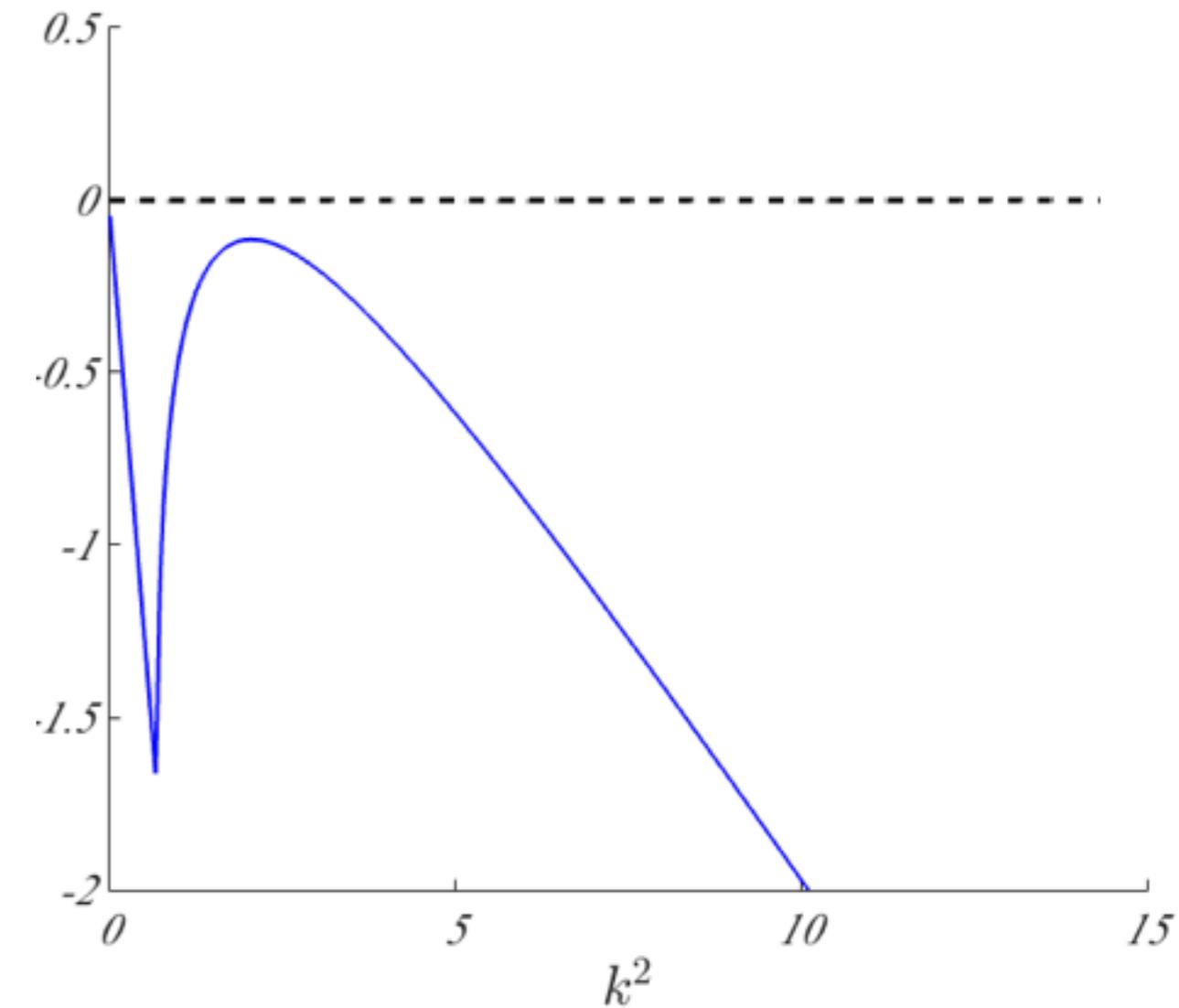
$$\det \left[ \lambda_k - \begin{pmatrix} f_u - D_u k^2 & f_v \\ g_u & g_v - D_v k^2 \end{pmatrix} \right] = 0$$

# Some mathematics for the Turing instability

iii) if there exists  $\hat{k}^2 \in (k_-^2, k_+^2)$  such that  $\Re \lambda_{\hat{k}} > 0$  then Turing patterns do emerge.



patterns do emerge

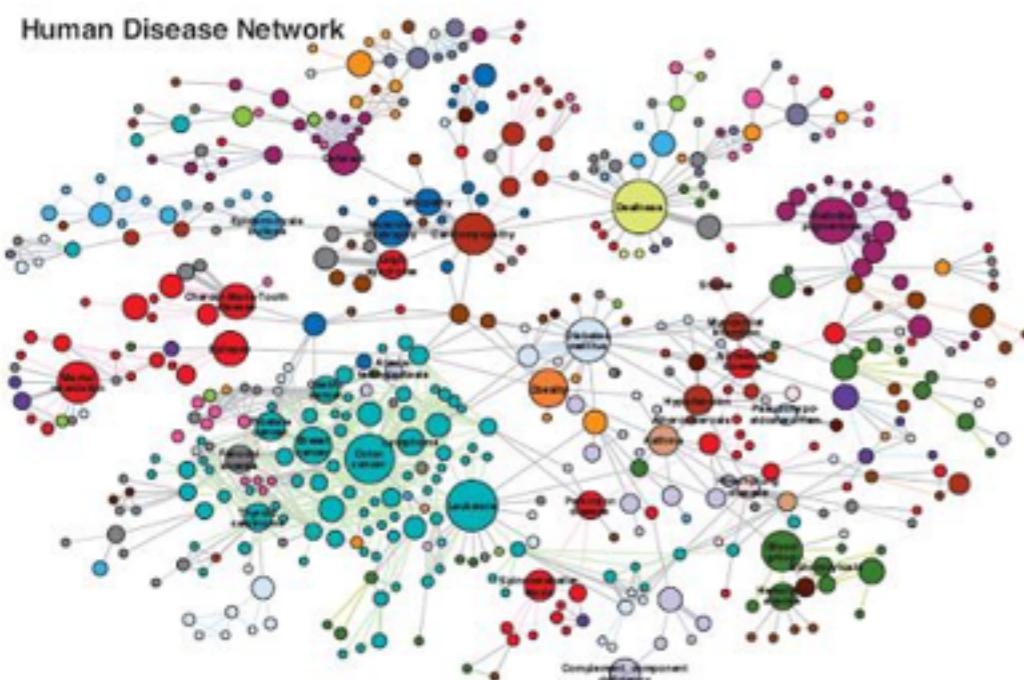


patterns do not emerge

# Networks are everywhere

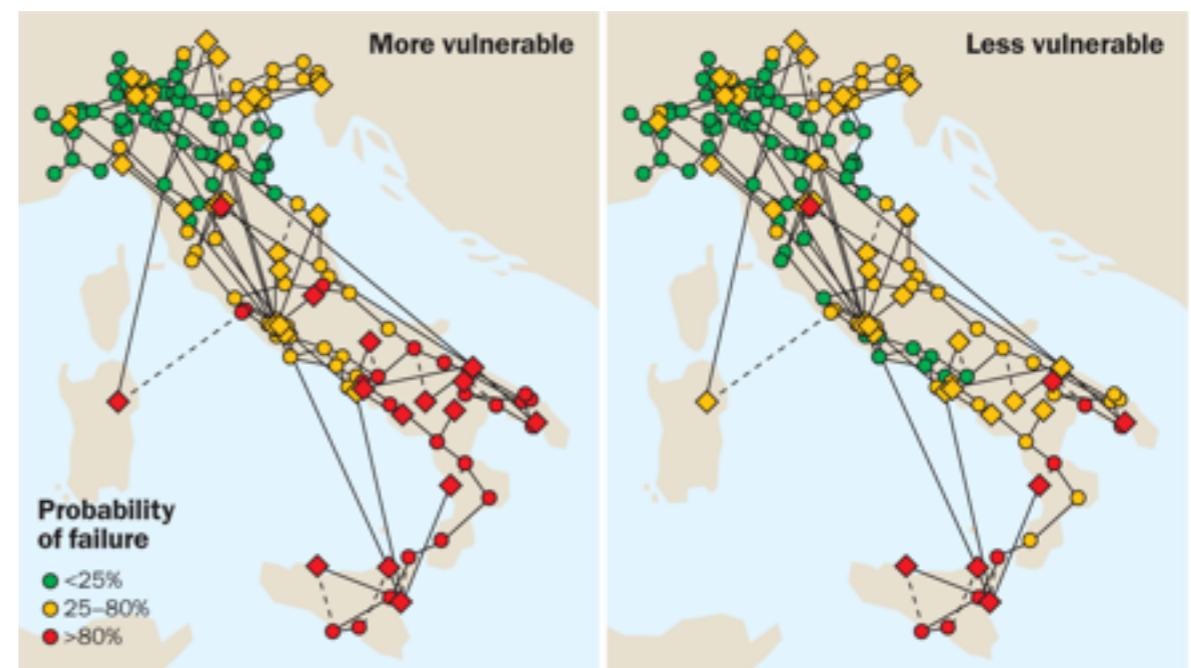
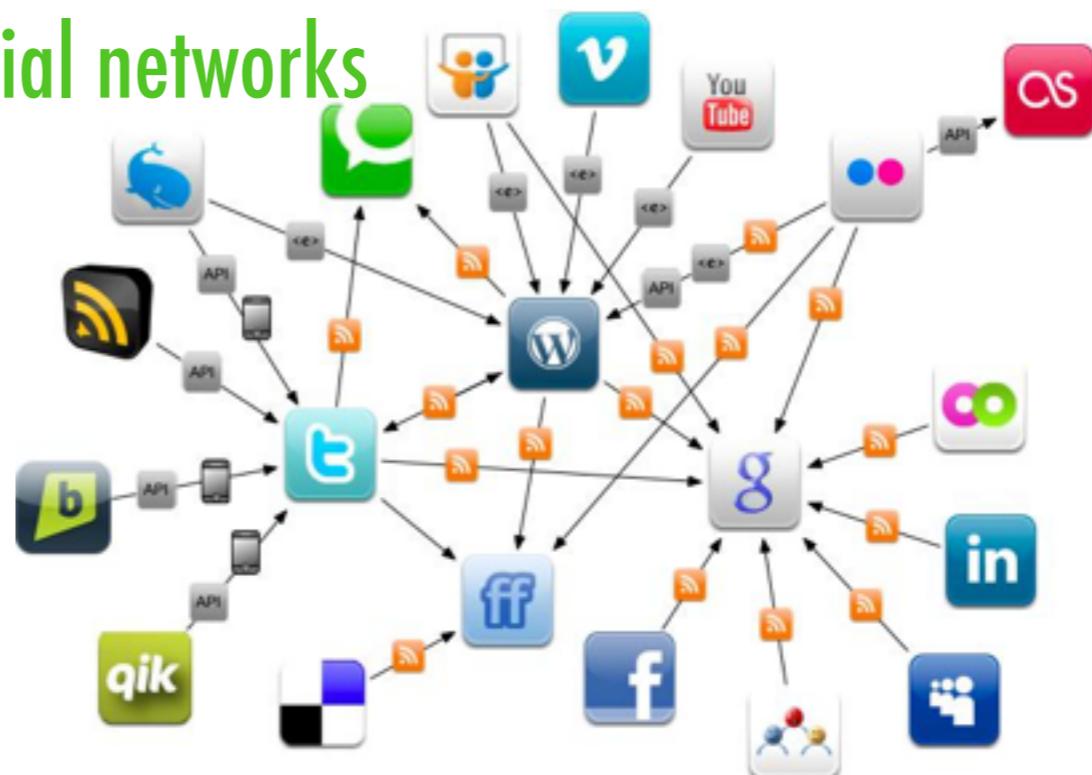


world flights map



proteins networks

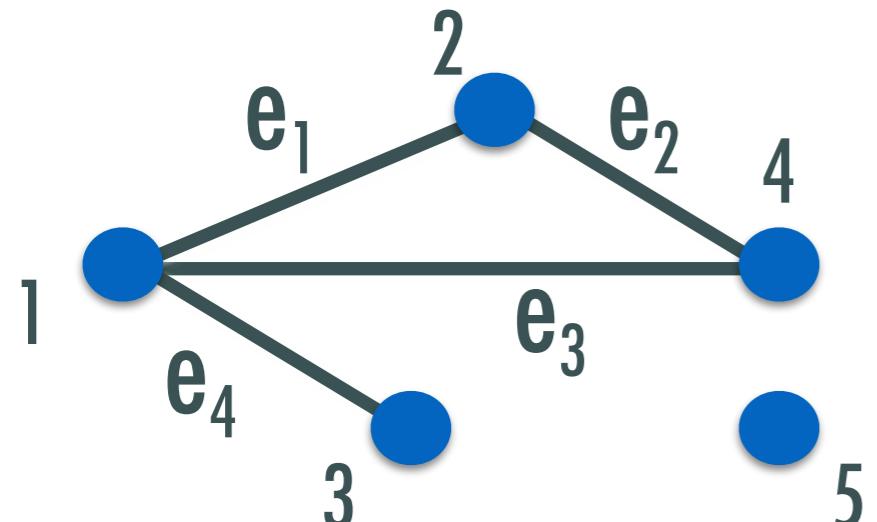
## social networks



technological networks

## (complex) Networks: some definitions

A network is a set of nodes connected by links (edges)



Ex.: 5 nodes and 4 edges (undirected)

Adjacency matrix

$$A_{ij} = \begin{cases} 1 & \text{if nodes } i \text{ and } j \text{ are linked} \\ 0 & \text{otherwise} \end{cases}$$

The number of links entering (going out) from each node is called in-degree (out-degree)

Ex.: "degree node 1" = 3

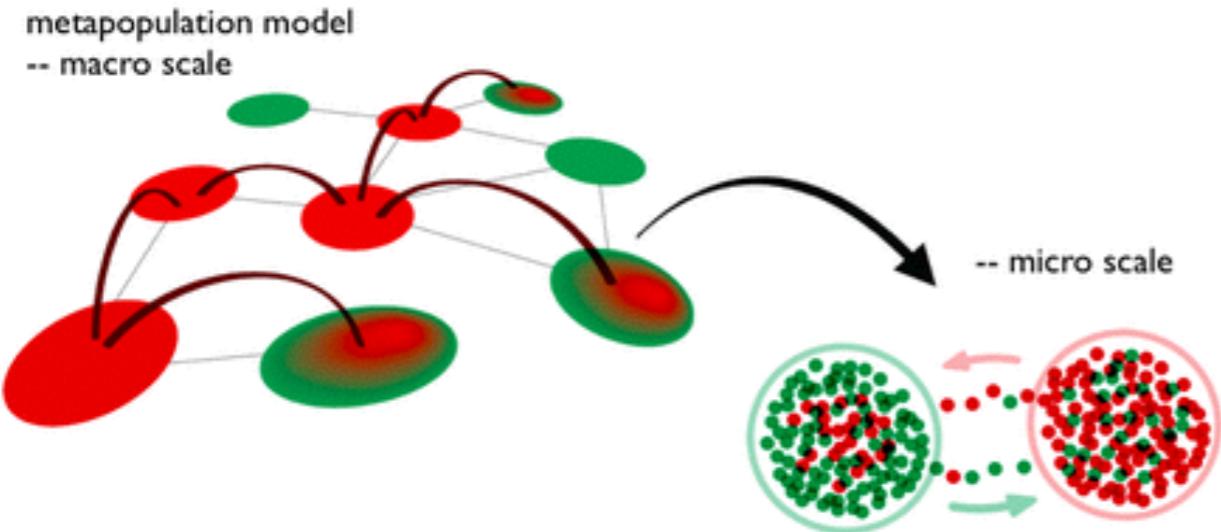
"degree nodes 2 & 4" = 2

"degree node 3" = 1

"degree node 5" = 0

A network is said to be complex if the degree distribution is not trivial, i.e. not constant (lattice) nor Poissonian (random, Erdős-Rényi)

## Extension to networks



**Metapopulation models**  
e.g. in the framework of ecology:  
May R., *Will a large complex system be stable?* Nature, 238, pp. 413, (1972)

Reactions occur at each node. Diffusion occurs across edges.

Patterns :

Sets of nodes whose asymptotic state is far from the homogeneous equilibrium.

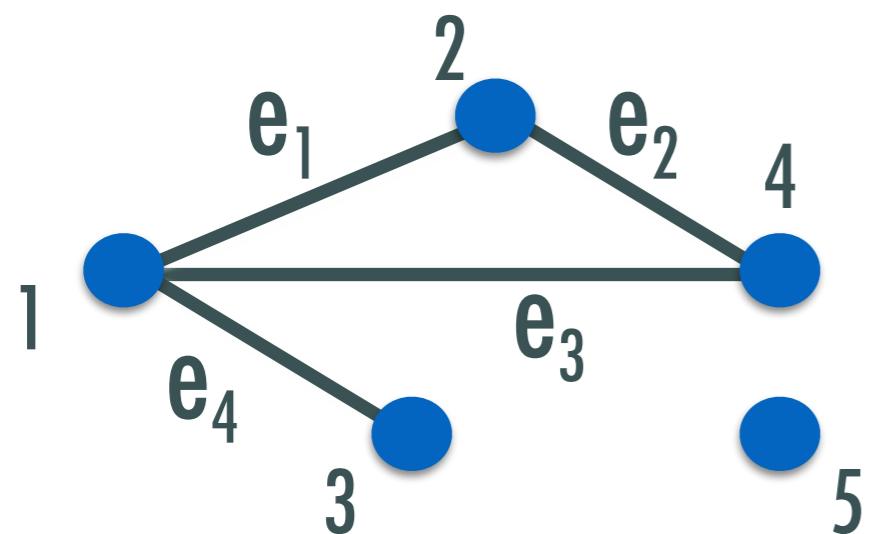
## Reaction term:

$$\begin{cases} \dot{u}_i(t) = f(u_i(t), v_i(t)) \\ \dot{v}_i(t) = g(u_i(t), v_i(t)) \end{cases} \quad \forall i = 1, \dots, n \text{ and } t > 0.$$

At each node  $i=1,\dots,n$ , “species”  $u$  and  $v$  react through some non-linear functions  $f$  and  $g$  depending on the quantities available at the  $i$ -th node (metapopulation assumption)

## Diffusion term:

Diffusive transport of species into a certain node  $i$  is given by the sum of incoming fluxes to node  $i$  from other connected nodes  $j$ , fluxes are proportional to the concentration difference between the nodes (Fick's law).



Ex.: consider the amount of  $u$  in node 1,  
 $u$  can enter from 2, 3 and 4  
 $u$  can leave 1 to go to 2, 3 and 4

$$u_2 + u_3 + u_4 - 3u_1 = \sum_j A_{1j}u_j - k_1u_1 = \sum_j (A_{1j} - \delta_{1j}k_j) u_j := \sum_j L_{1j}u_j$$

$L$  is called Laplacian matrix of the network

## The model:

$$\begin{cases} \dot{u}_i(t) &= f(u_i(t), v_i(t)) + D_u \sum_{j=1}^n L_{ij} u_j(t) \\ \dot{v}_i(t) &= g(u_i(t), v_i(t)) + D_v \sum_{j=1}^n L_{ij} v_j(t) \end{cases} \quad \forall i = 1, \dots, n \text{ and } t > 0.$$

D<sub>u</sub> and D<sub>v</sub> are the diffusion coefficients of species u and v

Observe that because the network is undirected, the matrices A and L are symmetric

## General strategy for the network case

1) Assume there exists a spatially homogeneous solution:

$$(u_i, v_i) = (\hat{u}, \hat{v}) \quad \forall i = 1, \dots, n$$

which moreover is stable when there is no diffusion:

$$D_u = D_v = 0$$

2) Linearize around this solution

$$\begin{aligned} u_i &= \hat{u} + \delta u_i \\ v_i &= \hat{v} + \delta v_i \end{aligned} \quad \left( \begin{array}{c} \dot{\delta u} \\ \dot{\delta v} \end{array} \right) = \tilde{\mathcal{J}} \left( \begin{array}{c} \delta u \\ \delta v \end{array} \right)$$

$$\tilde{\mathcal{J}} = \begin{pmatrix} f_u \mathbf{I}_n + D_u L & f_v \mathbf{I}_n \\ g_u \mathbf{I}_n & g_v \mathbf{I}_n + D_v L \end{pmatrix}$$

## General strategy for the network case

3) Prove that (possibly) the spatially homogeneous solution:

$$(u_i, v_i) = (\hat{u}, \hat{v}) \quad \forall i = 1, \dots, n$$

turns out to be unstable once the diffusion is in action

$$D_u > 0 \text{ and } D_v > 0$$

Sketch of the proof

i) Let  $L\vec{\phi}^\alpha = \Lambda^\alpha \vec{\phi}^\alpha$ ,  $\alpha = 1, \dots, n$        $\vec{\phi}^\alpha = (\phi_1^\alpha, \dots, \phi_n^\alpha)$

$$\sum_i \phi_i^\alpha \phi_i^\beta = \delta_{\alpha\beta} \quad \Lambda^\alpha \leq 0$$

ii) decompose the solution on the eigenbasis and use the ansatz

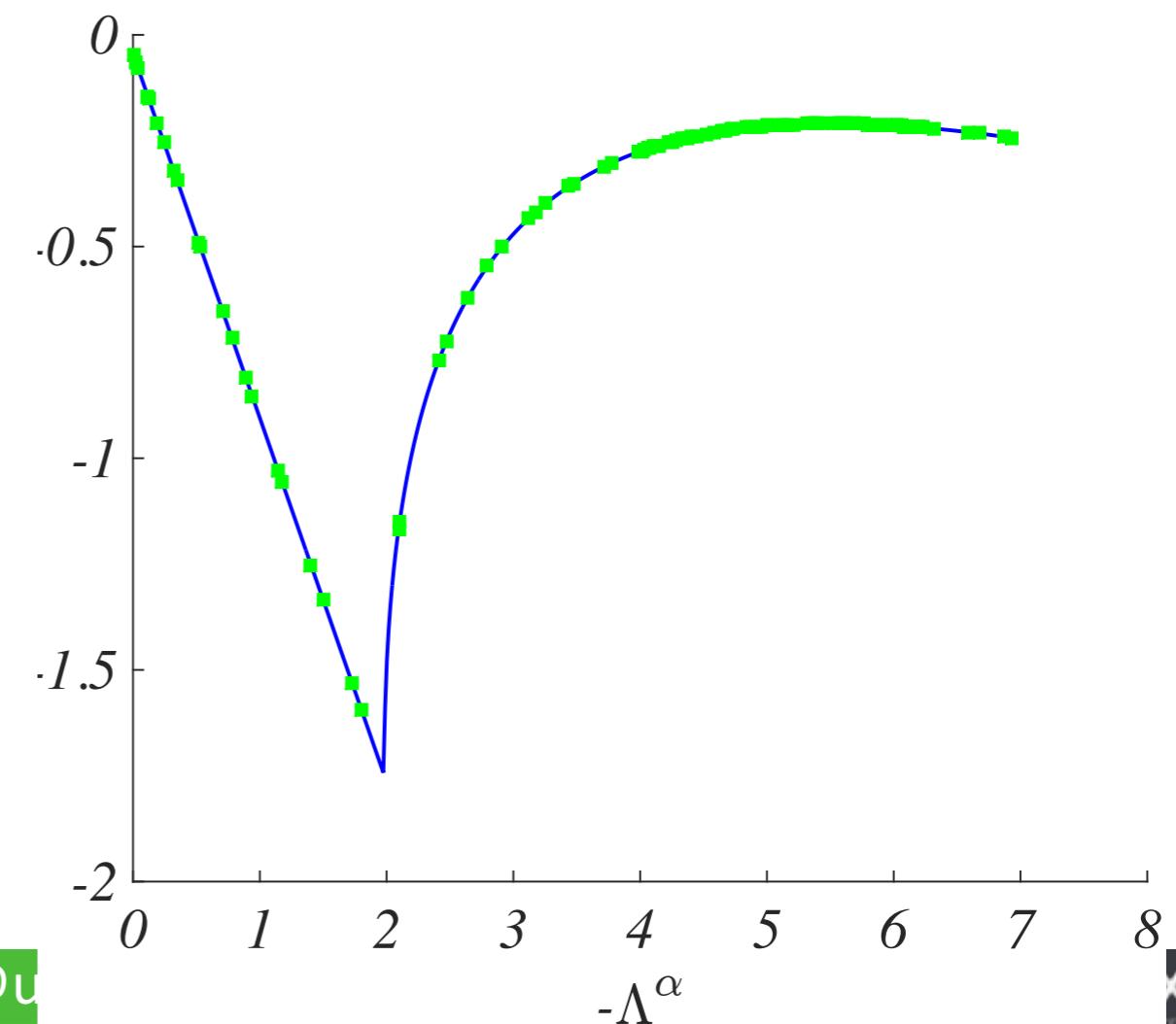
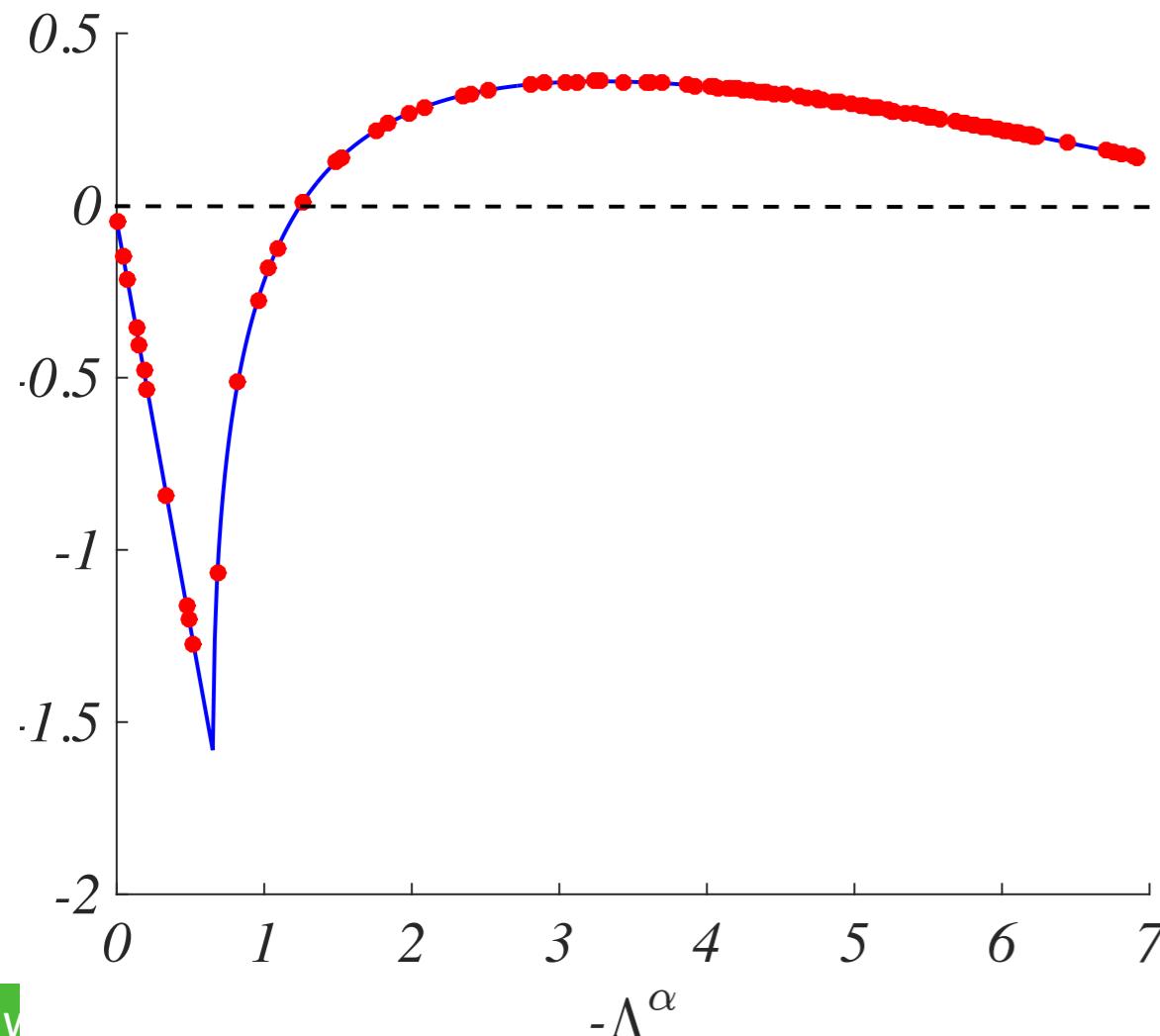
$$\delta u_i(t) = \sum_{\alpha=1}^n c_\alpha \phi_i^\alpha e^{\lambda_\alpha t}$$

## General strategy

iii)  $\lambda_\alpha$  (called relation dispersion) is solution of

$$\det \left[ \lambda_\alpha - \begin{pmatrix} f_u + D_u \Lambda^\alpha & f_v \\ g_u & g_v + D_v \Lambda^\alpha \end{pmatrix} \right] = 0$$

iv) if there exists  $\Lambda^{\alpha_c}$  such that  $\Re \lambda_{\alpha_c} > 0$  then Turing patterns do emerge.

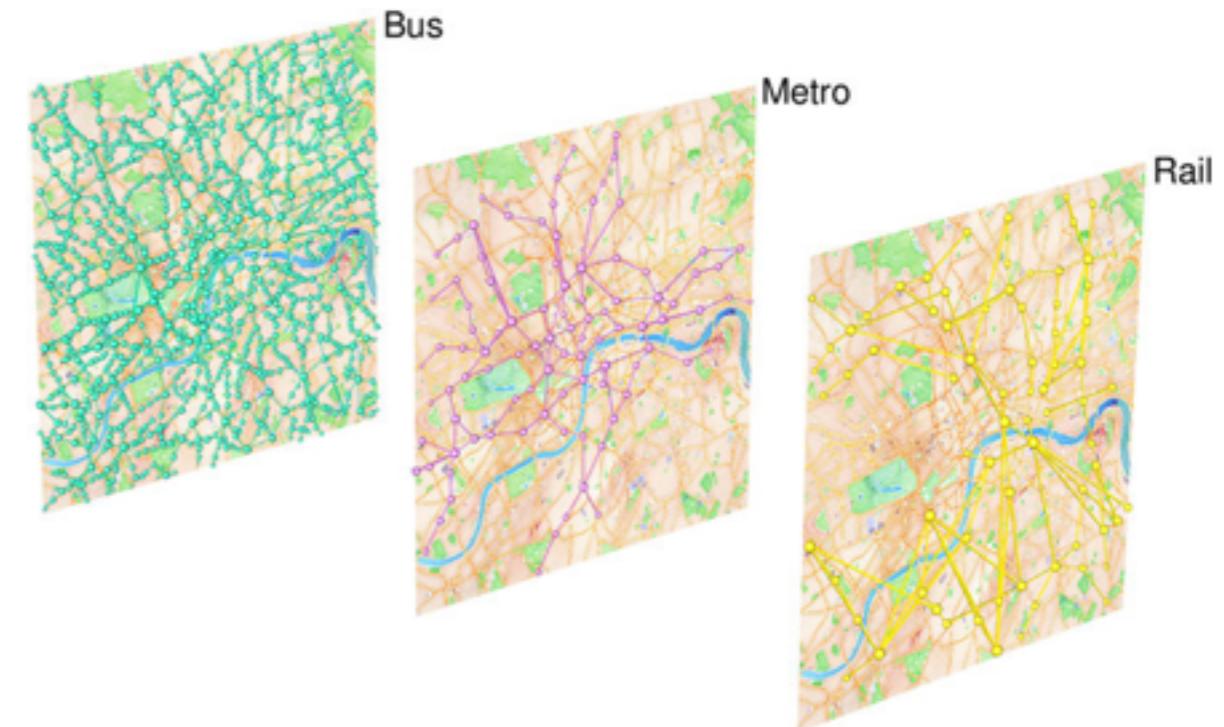
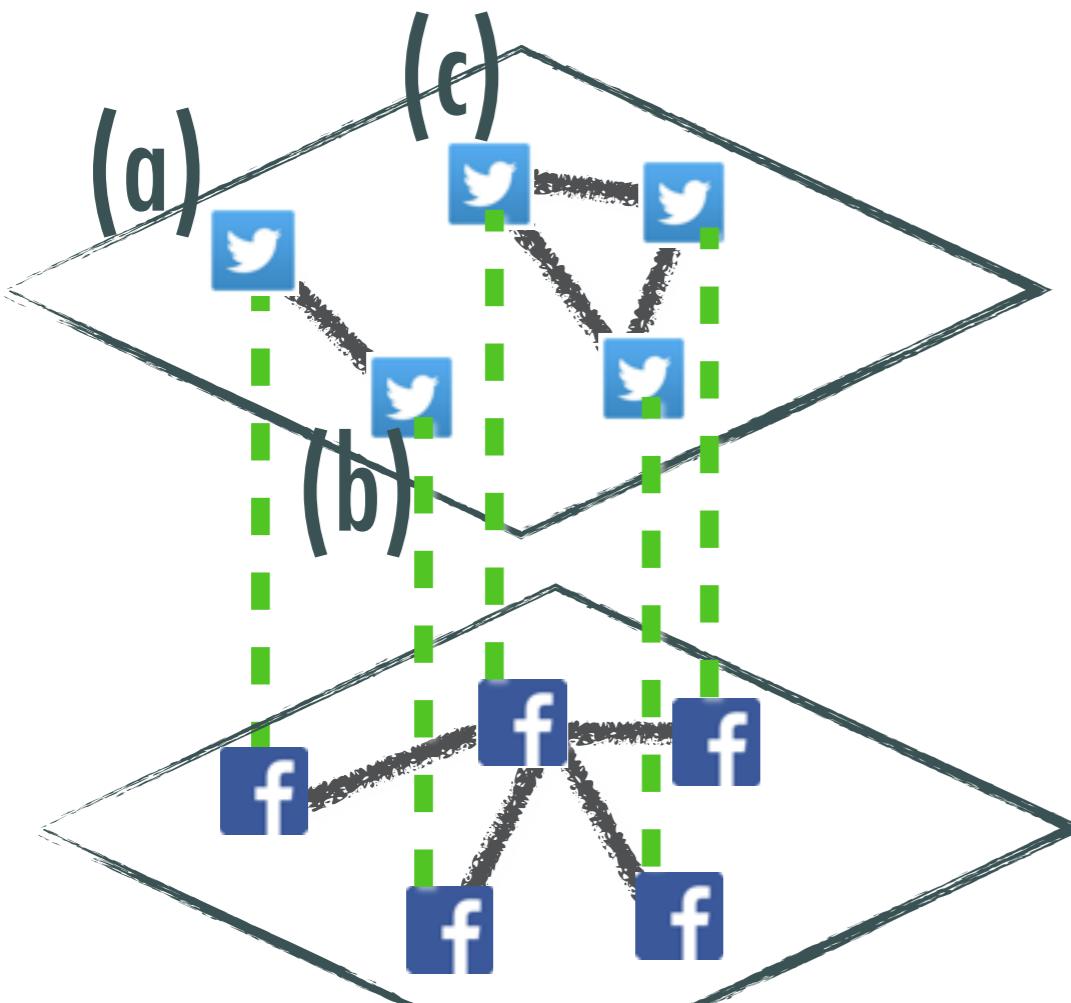


# Systems composed by layers of networks: Multiplexes

## Social networks

layers=different social networks

nodes=same agent in each SN



## Transportation networks

layers=different modalities

nodes=same spatial location

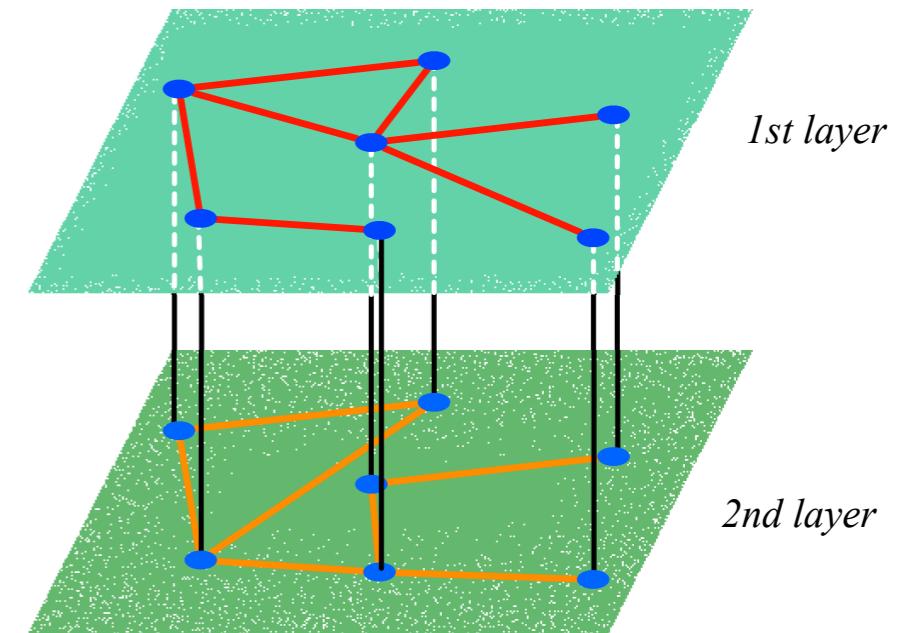
# Turing instabilities on multiplex networks

adjacency matrix of  
layer K

$$L_{ij}^K = A_{ij}^K - \delta_{ij} k_i^K$$

Laplacian matrix of  
layer K

degree of i-th note  
in layer K



The same  $\Omega$  nodes are present in each layer

$D_{u,v}^K$  **inter-layer diffusion coefficient**

$D_{u,v}^{12}$  **intra-layer diffusion coefficient**

$$\begin{cases} \dot{u}_i^K &= f(u_i^K, v_i^K) + D_u^K \sum_{j=1}^{\Omega} L_{ij}^K u_j^K + D_u^{12} (u_i^{K+1} - u_i^K) \\ \dot{v}_i^K &= g(u_i^K, v_i^K) + D_v^K \sum_{j=1}^{\Omega} L_{ij}^K v_j^K + D_v^{12} (v_i^{K+1} - v_i^K) \end{cases}$$

## General strategy

1) Assume there exists a spatially homogeneous solution:

$$(u_i^K, v_i^K) = (\hat{u}, \hat{v}) \quad \forall i = 1, \dots, n$$

2) Linearise around this solution

$$u_j^K = \hat{u} + \delta u_j^K$$

$$v_j^K = \hat{v} + \delta v_j^K$$

$$\begin{pmatrix} \dot{\delta u} \\ \dot{\delta v} \end{pmatrix} = \tilde{\mathcal{J}} \begin{pmatrix} \delta u \\ \delta v \end{pmatrix}$$

$$\tilde{\mathcal{J}} = \begin{pmatrix} f_u \mathbf{I}_{2\Omega} + \mathcal{L}_u + D_u^{12} \mathcal{I} & f_v \mathbf{I}_{2\Omega} \\ g_u \mathbf{I}_{2\Omega} & g_v \mathbf{I}_{2\Omega} + \mathcal{L}_v + D_v^{12} \mathcal{I} \end{pmatrix}$$

**Supra-Laplacian matrix**  $\mathcal{L}_u + D_u^{12} \mathcal{I}$

$$\mathcal{L}_u = \begin{pmatrix} D_u^1 \mathbf{L}^1 & \mathbf{0} \\ \mathbf{0} & D_u^2 \mathbf{L}^2 \end{pmatrix}$$

$$\mathcal{I} = \begin{pmatrix} -\mathbf{I}_\Omega & \mathbf{I}_\Omega \\ \mathbf{I}_\Omega & -\mathbf{I}_\Omega \end{pmatrix}$$

### 3) Study the spectrum of

$$\tilde{\mathcal{J}} = \begin{pmatrix} f_u \mathbf{I}_{2\Omega} + \mathcal{L}_u + D_u^{12} \mathcal{I} & f_v \mathbf{I}_{2\Omega} \\ g_u \mathbf{I}_{2\Omega} & g_v \mathbf{I}_{2\Omega} + \mathcal{L}_v + D_v^{12} \mathcal{I} \end{pmatrix}$$

to determine the existence of eigenvalues such that

$$\Re \lambda(D_{u,v}^{12}, D_{u,v}^K) > 0$$

Very hard for generic topologies, however ...

## Small intra-layer diffusion case

**Assume**  $D_v^{12} = \epsilon \ll 1$        $D_u^{12} = \mathcal{O}(\epsilon)$

$$\begin{aligned}\tilde{\mathcal{J}} &= \begin{pmatrix} f_u \mathbf{I}_{2\Omega} + \mathcal{L}_u & f_v \mathbf{I}_{2\Omega} \\ g_u \mathbf{I}_{2\Omega} & g_v \mathbf{I}_{2\Omega} + \mathcal{L}_v \end{pmatrix} + \epsilon \begin{pmatrix} \frac{D_u^{12}}{D_v^{12}} L^1 & 0 \\ 0 & L^2 \end{pmatrix} \\ &= \tilde{\mathcal{J}}_0 + \epsilon \mathcal{D}_0\end{aligned}$$

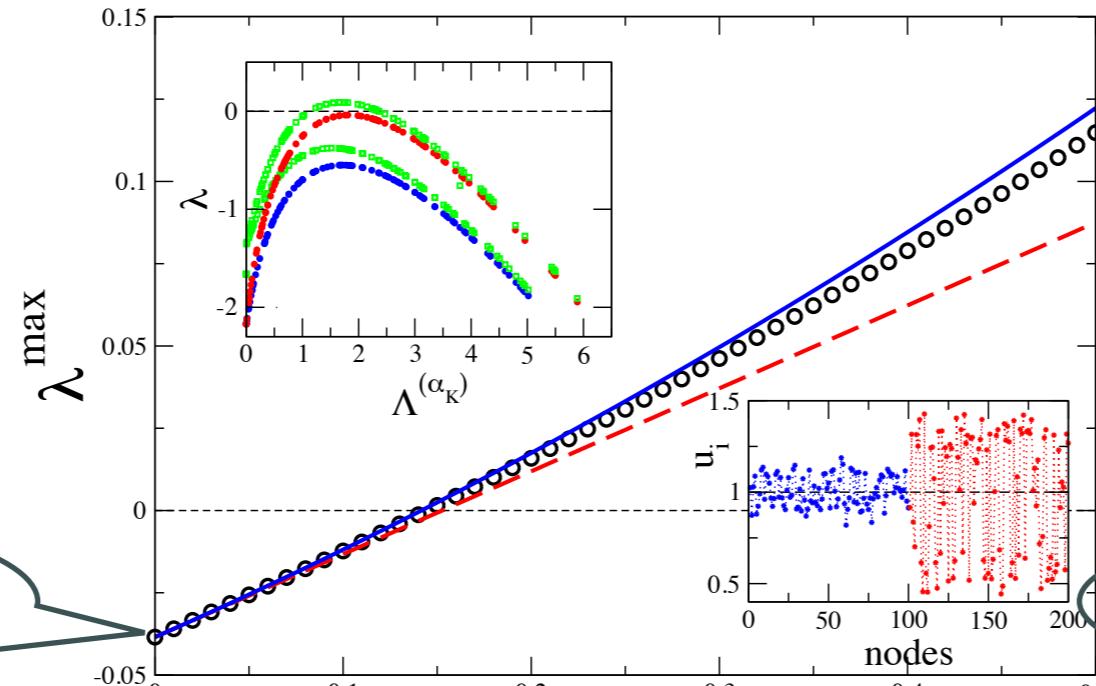
**Perturbative approach to compute the spectrum**

$$\lambda^{max}(\epsilon) = \lambda_0^{max} + \epsilon (U_0 \mathcal{D}_0 V_0)_{k_{max} k_{max}} + \mathcal{O}(\epsilon^2)$$

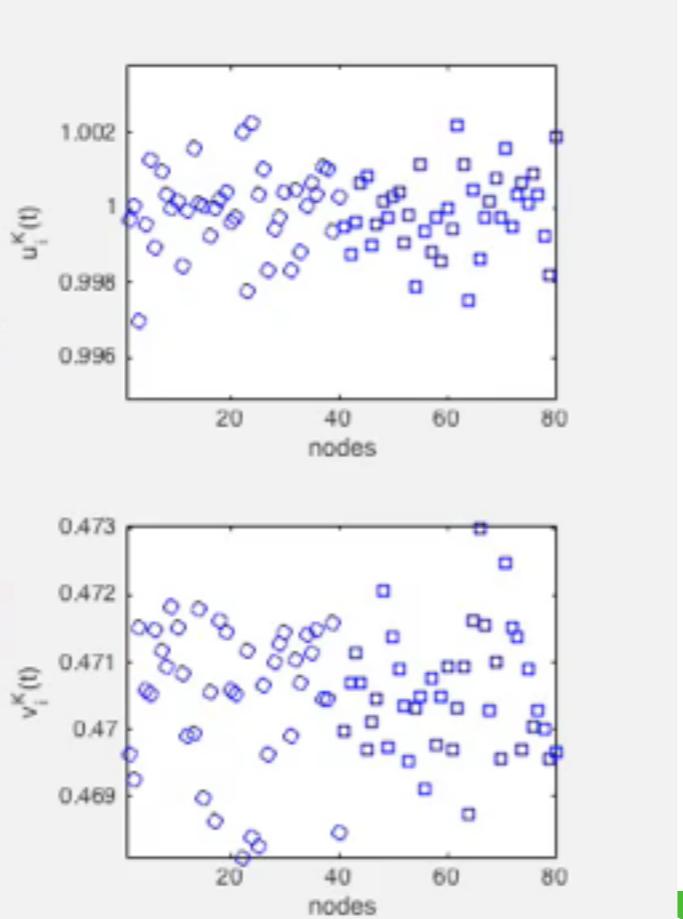
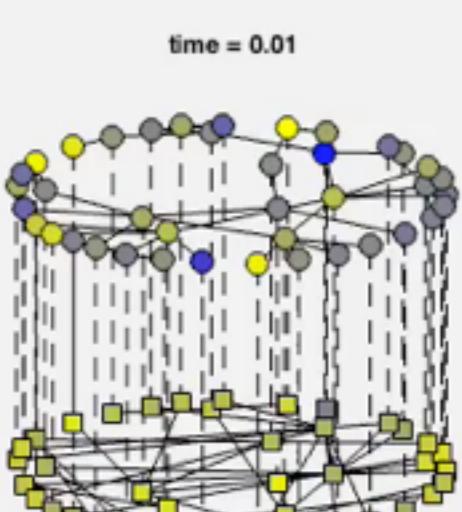
$$\lambda_0^{max} = \max \lambda_k(\epsilon = 0) \quad k_{max} = \arg \max \lambda_k(\epsilon = 0)$$

# Small intra-layer diffusion case: onset of patterns

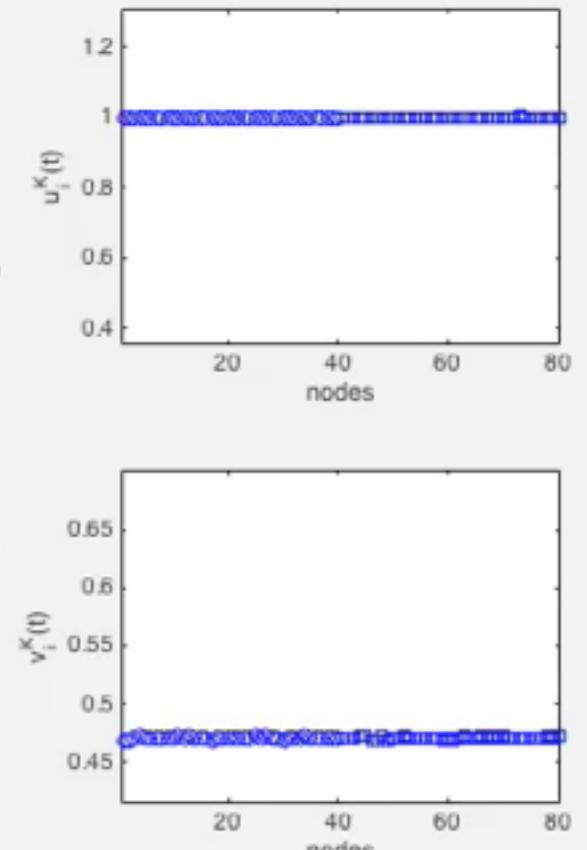
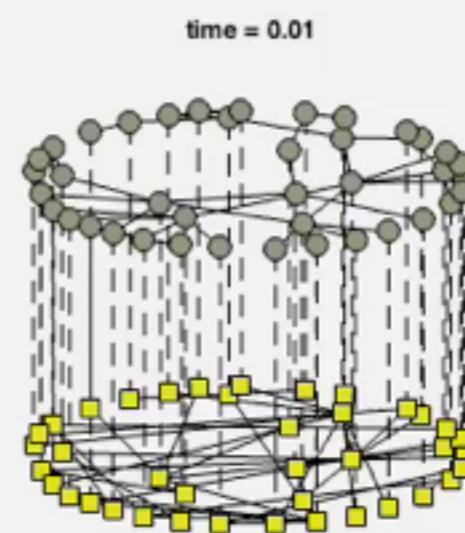
$$D_v^{12} = D_u^{12} = 0$$



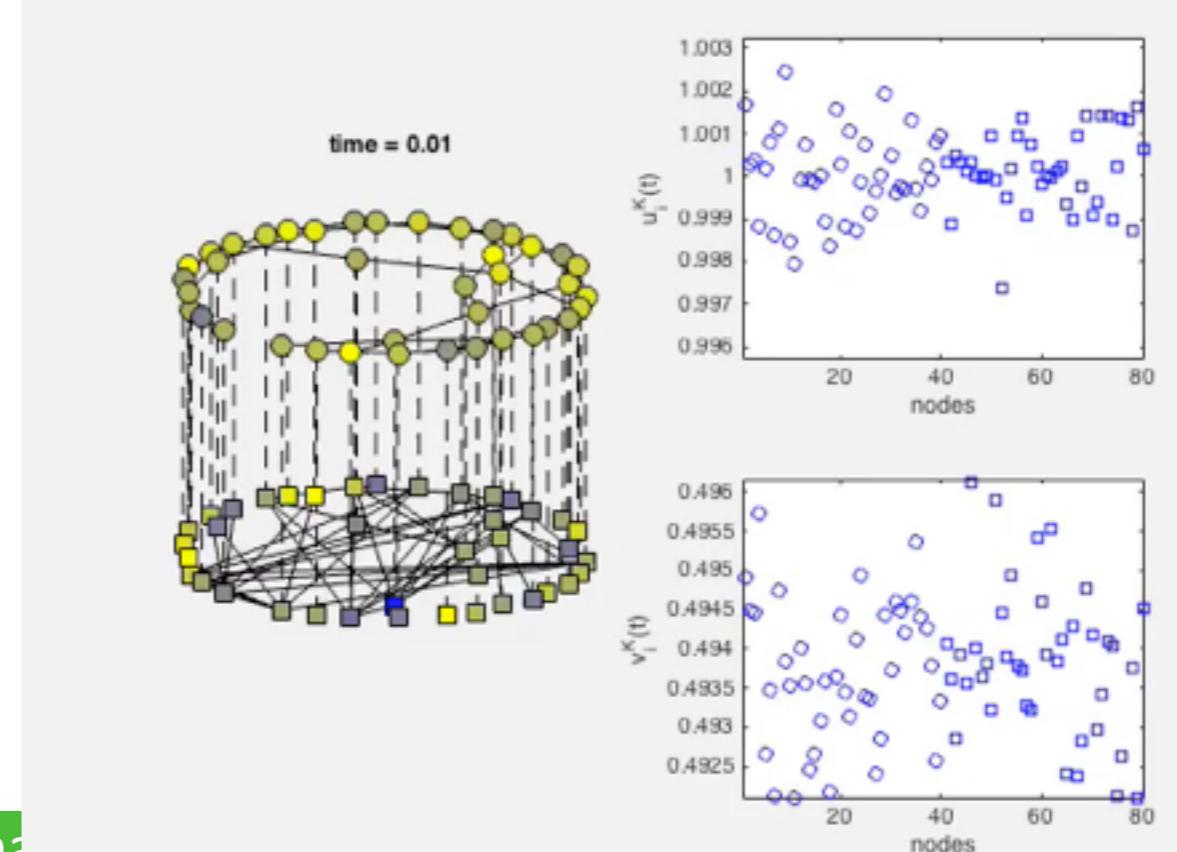
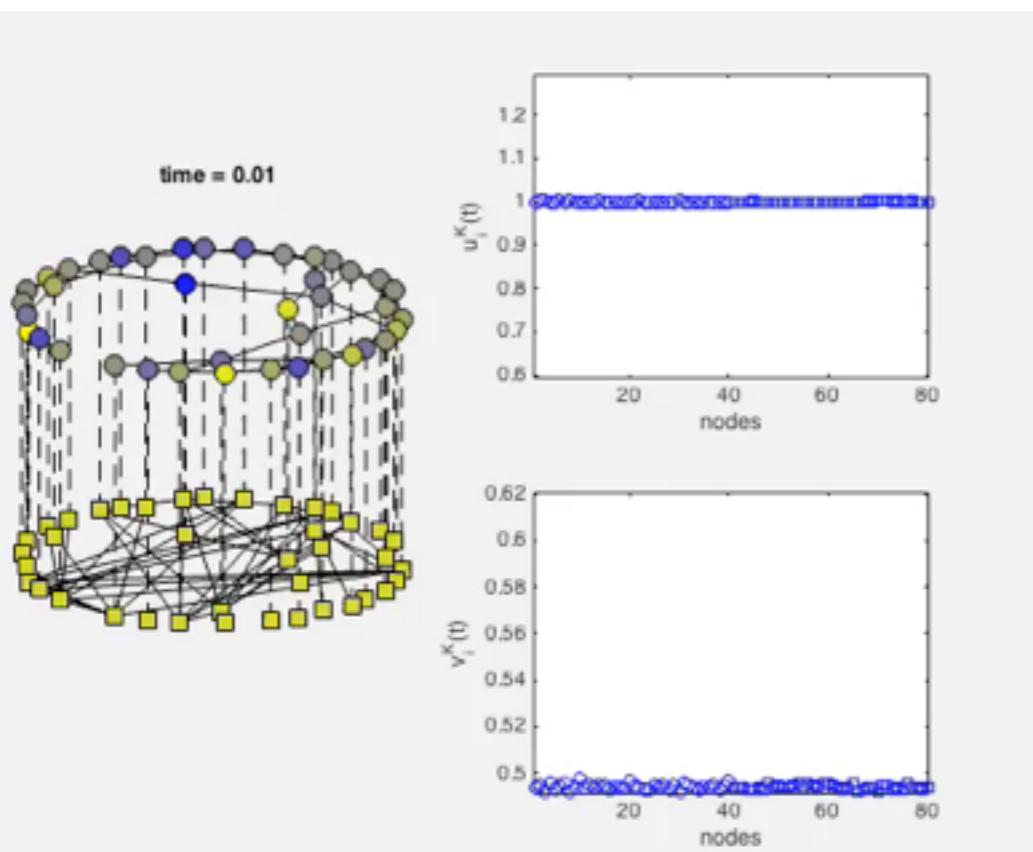
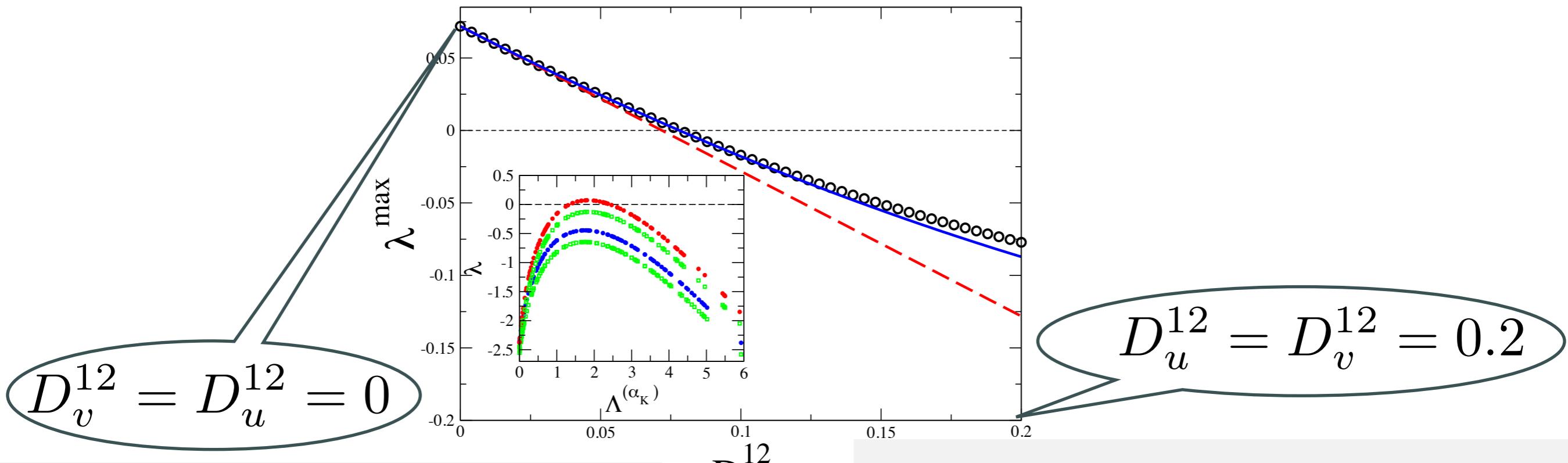
$$D_u^{12} = 0 \quad D_v^{12} = 0.5$$



$D_v^{12}$

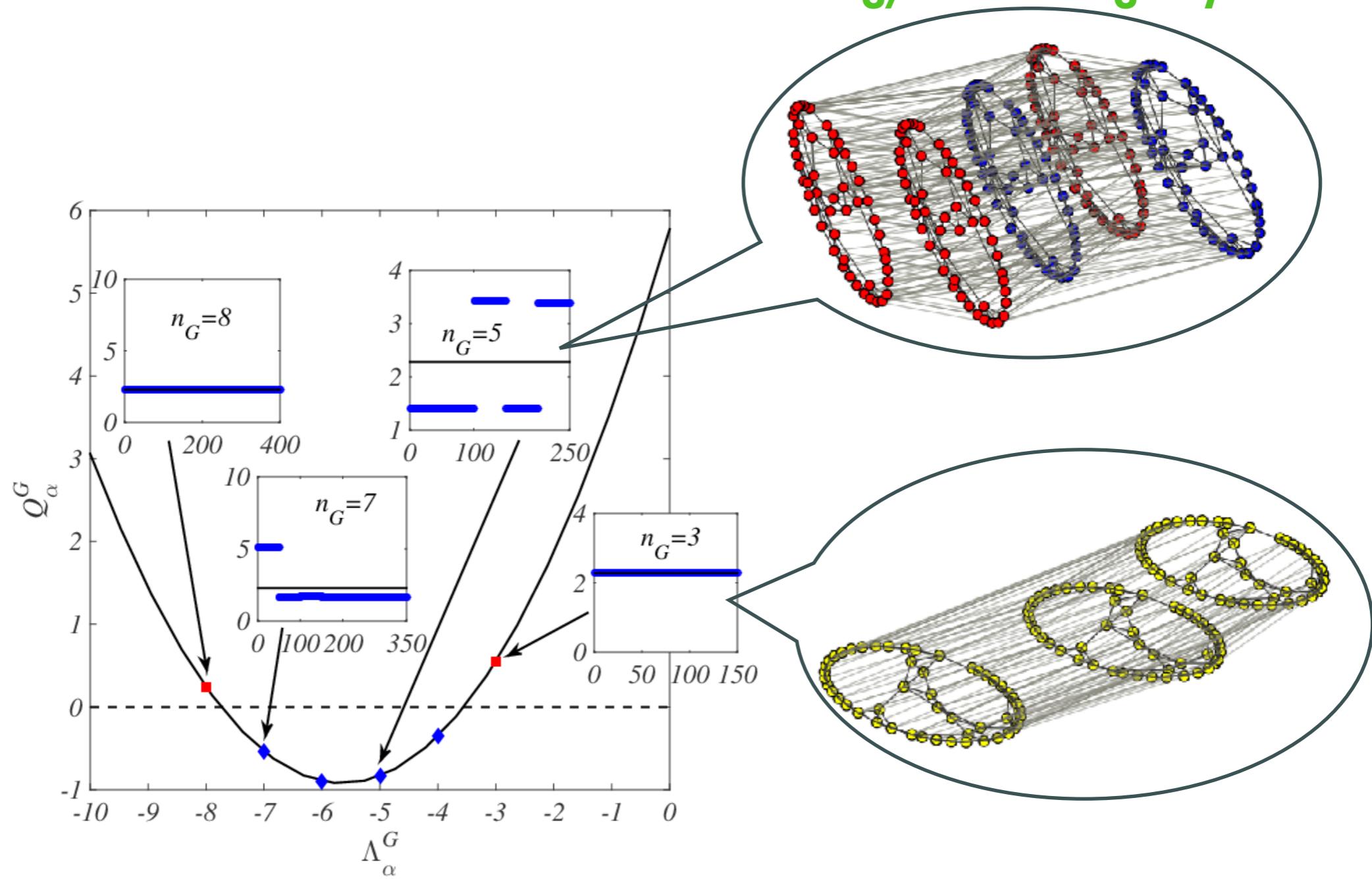


# Small intra-layer diffusion case: destruction of patterns



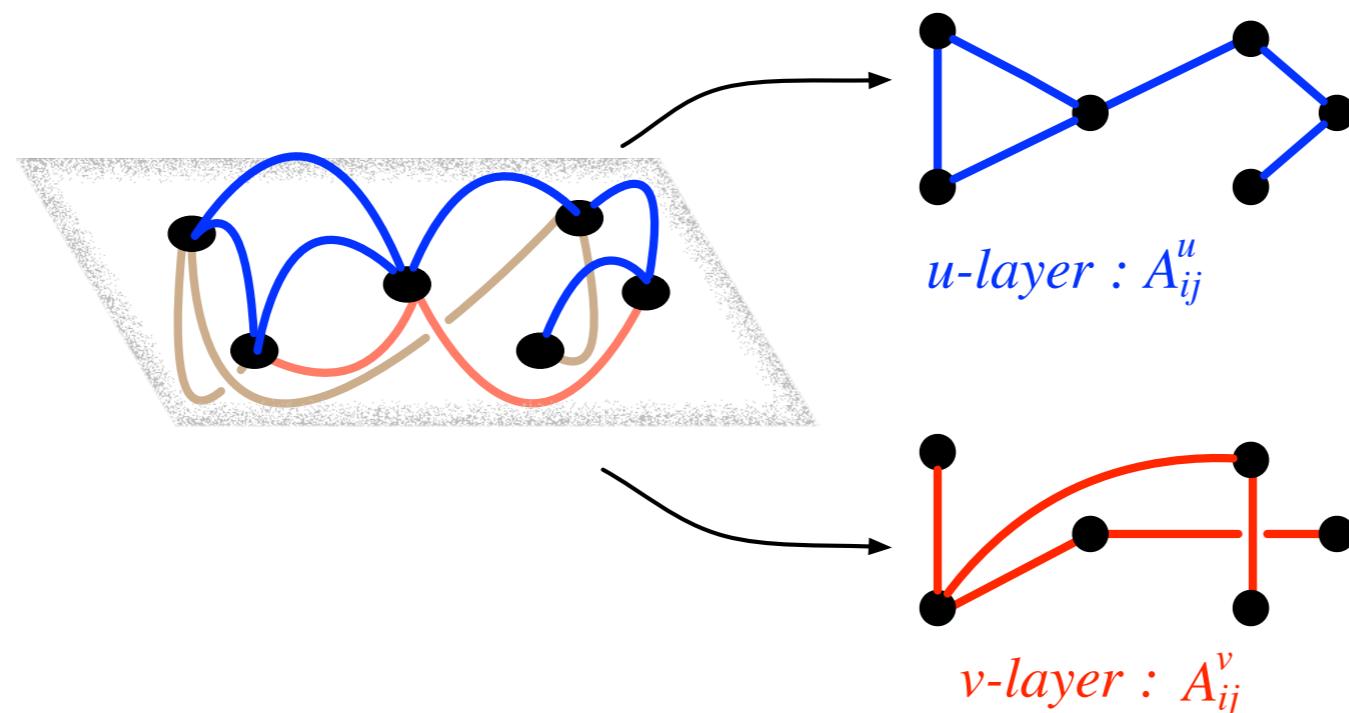
## Remarks

- 1) The large intra-layer diffusion - can be handled as well
- 2) One can be interested in the effect of adding/removing layers



# Can we control the topology to create (destroy) patterns?

Let us consider a multigraph, e.g. two nodes can be connected through different edges



$$\epsilon = 0$$

$$A^u(0) = A^0$$

$$A^v(0) = A^0$$

$$\epsilon = 1$$

$$A^u(\epsilon) = A^0 + \epsilon(A^1 - A^0)$$

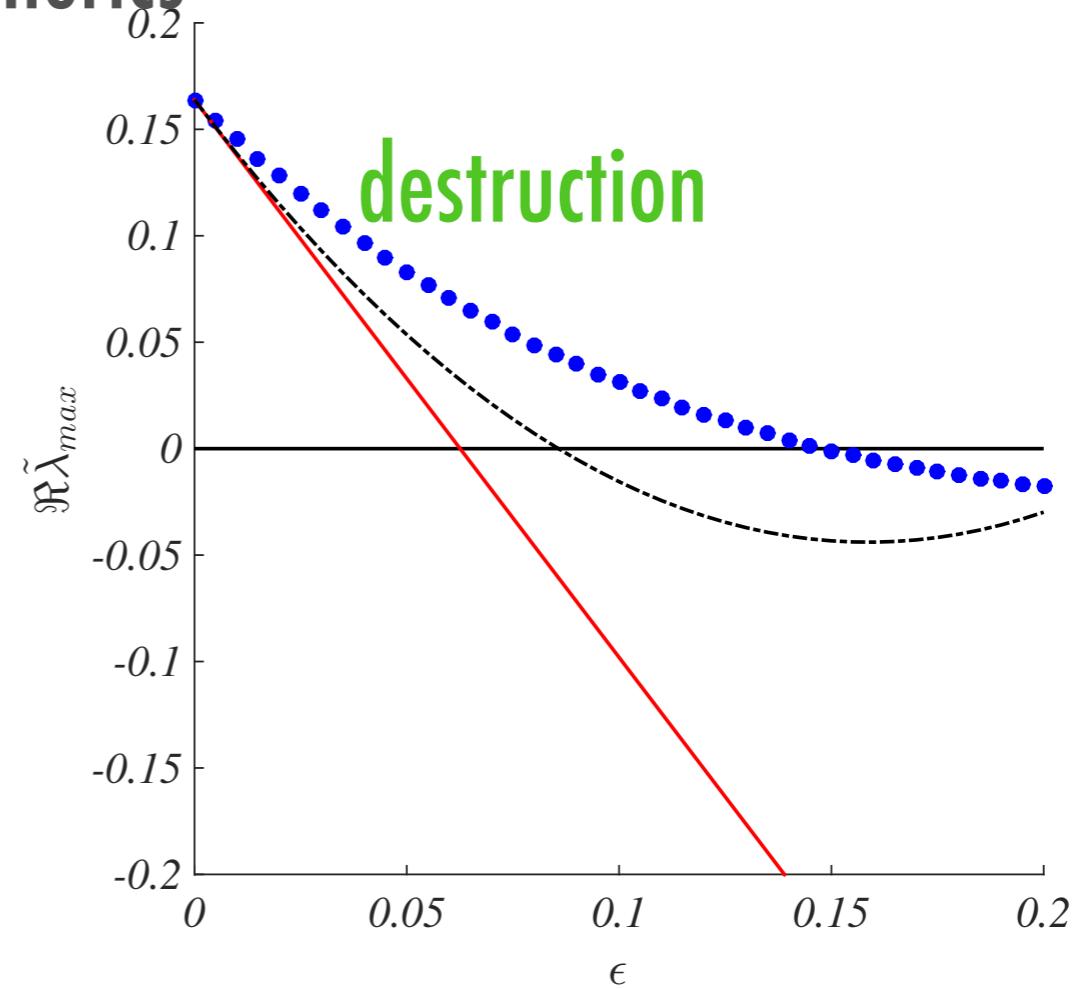
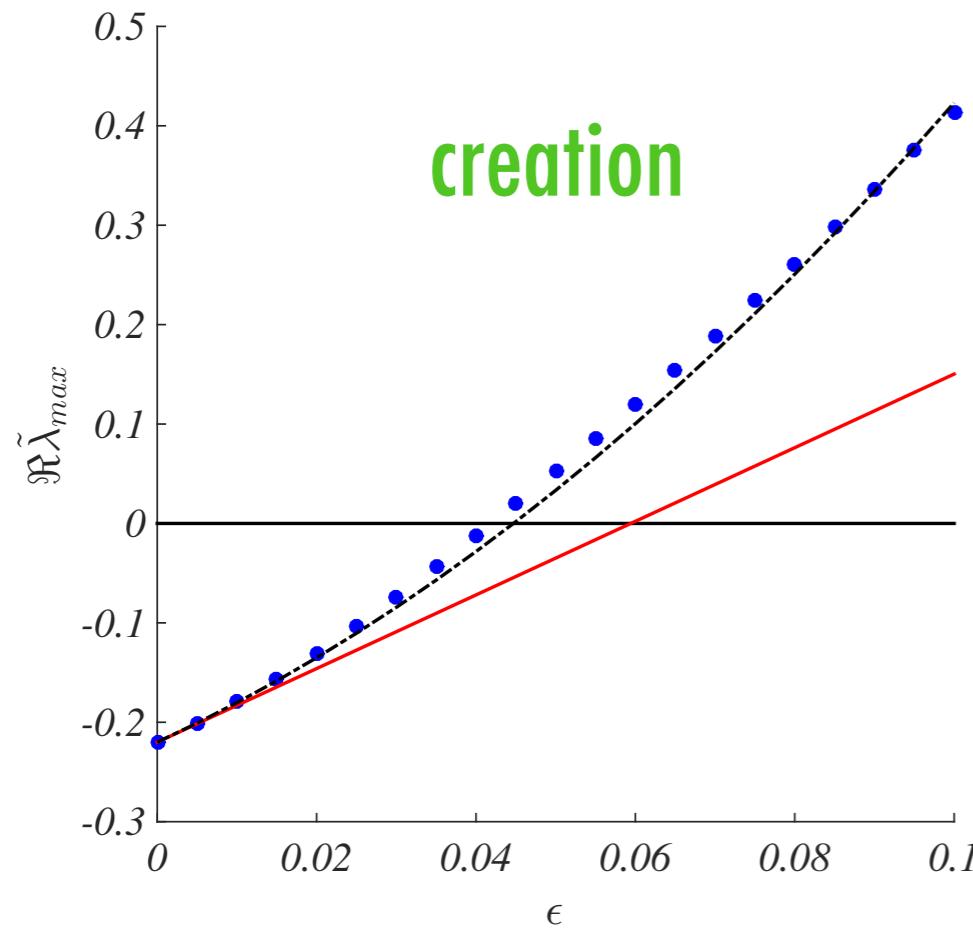
$$A^v(\epsilon) = A^0 + \epsilon(A^2 - A^0)$$

$$A^u(1) = A^1$$

$$A^v(1) = A^2$$

# Can we control the topology to create (destroy) patterns?

theory vs numerics



$$\epsilon = 0$$

$$A^u(0) = A^0$$

$$A^v(0) = A^0$$

$$A^u(\epsilon) = A^0 + \epsilon(A^1 - A^0)$$

$$A^v(\epsilon) = A^0 + \epsilon(A^2 - A^0)$$

$$\epsilon = 1$$

$$A^u(1) = A^1$$

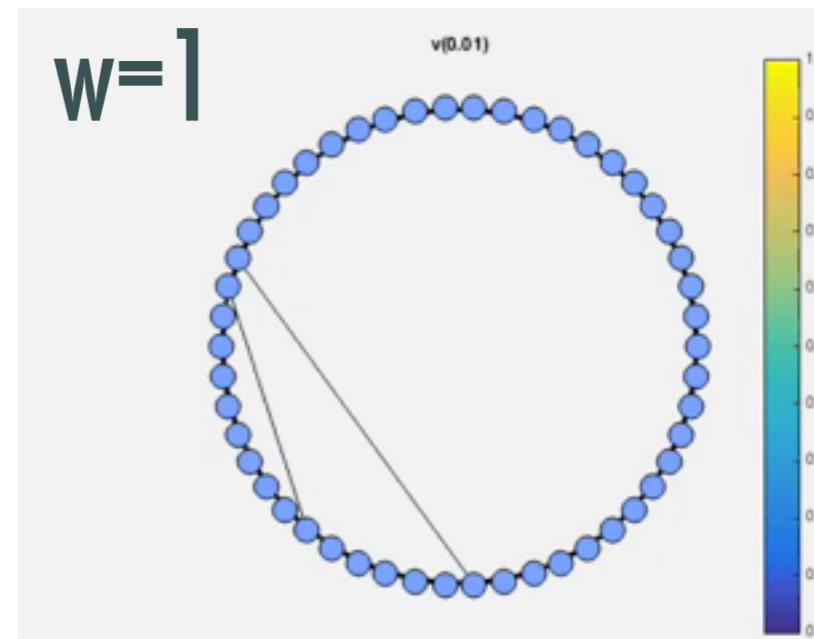
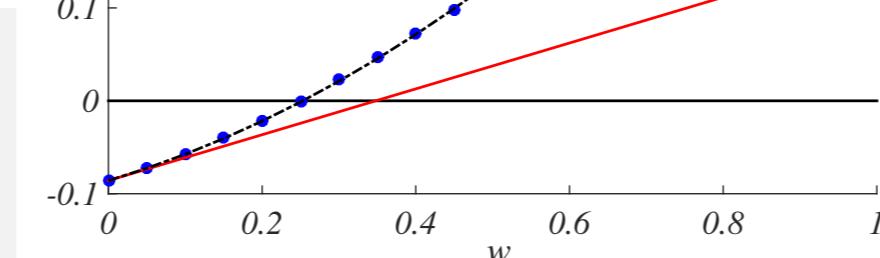
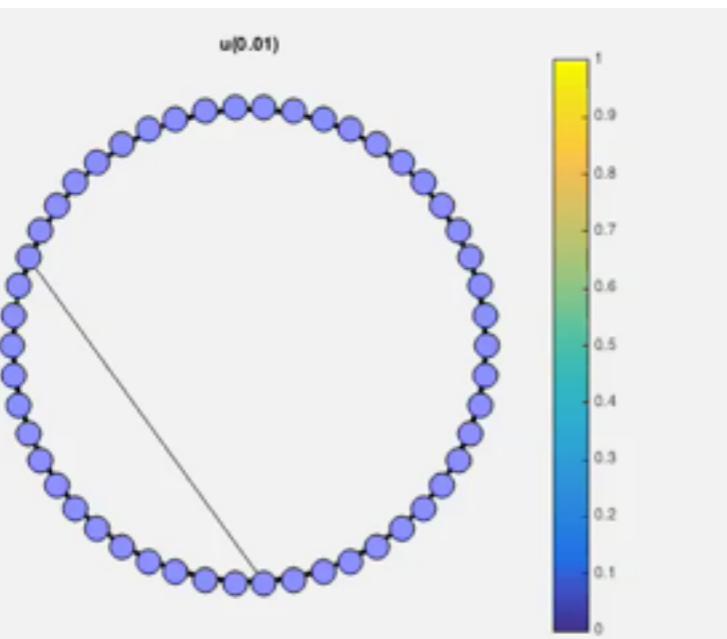
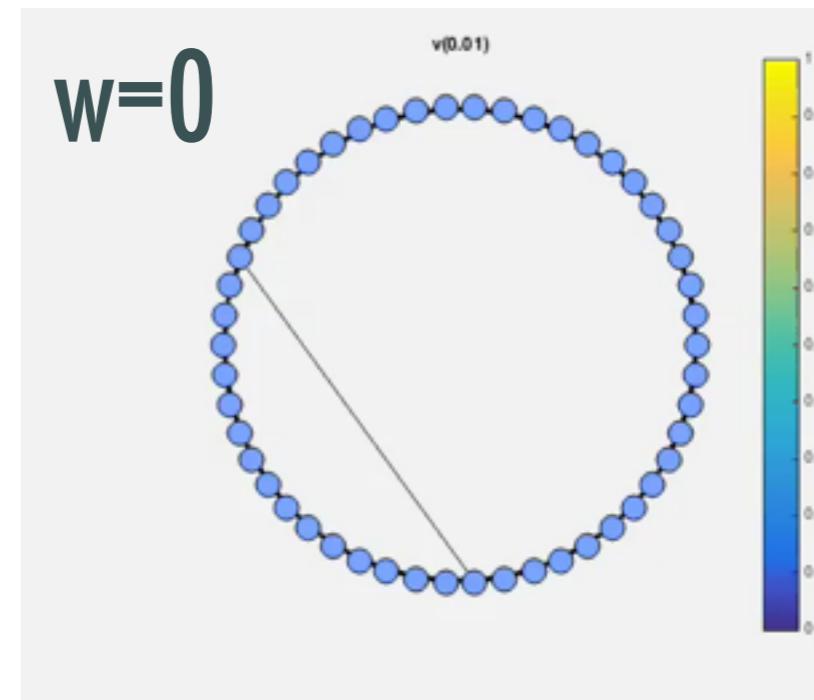
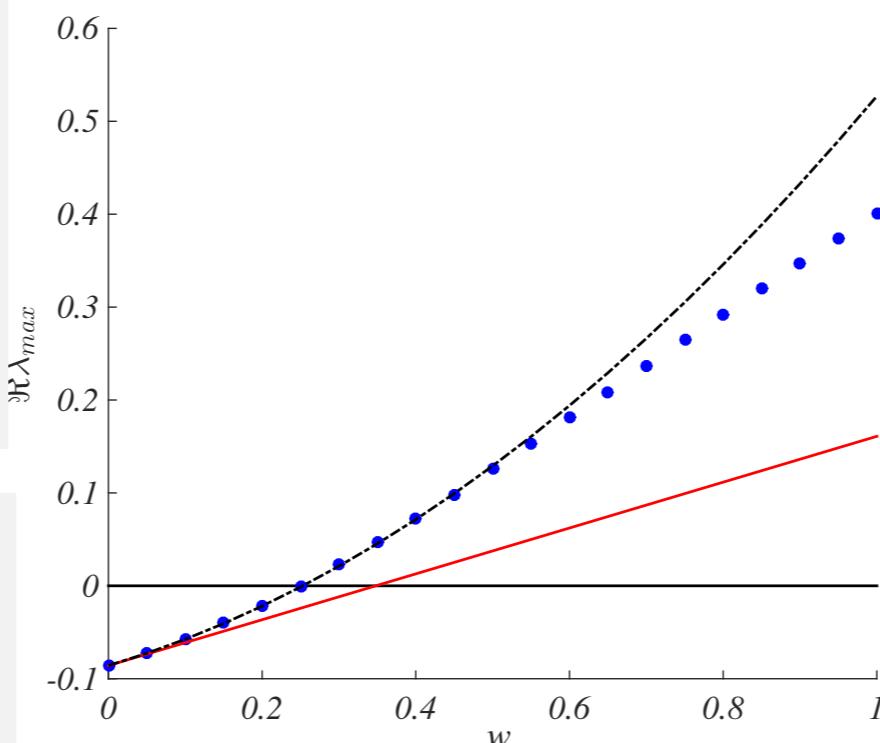
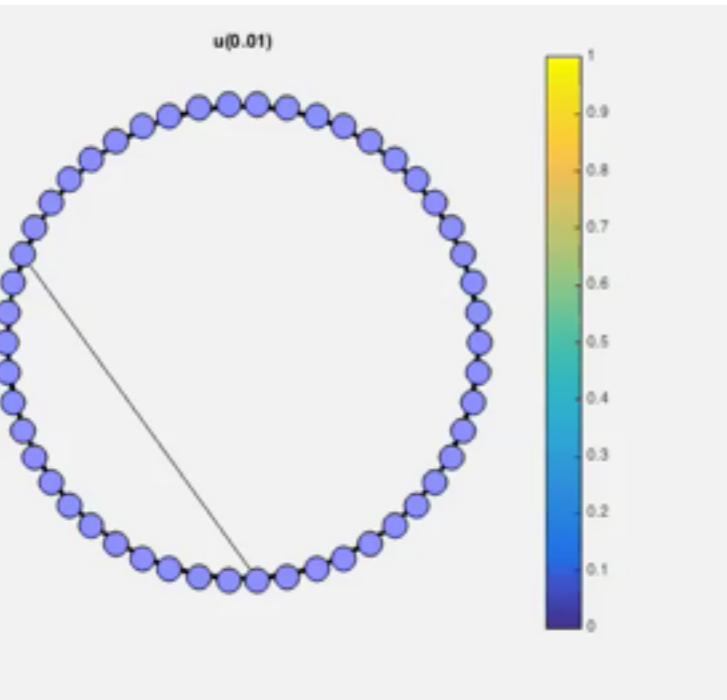
$$A^v(1) = A^2$$

# Can we control the topology to create (destroy) patterns?

Create patterns by adding a single (optimally chosen) link

$$A^u(w) = A^0$$

$$A^v(w) = A^0 + wT^{(ij)}$$



## Some papers

Benjamin-Feir instabilities on directed networks, F. Di Patti<sub>a,b,c</sub>, D. Fanelli<sub>a,b,c</sub>, F. Miele<sub>a</sub>, T. Carletti<sub>d</sub>, in press Chaos ( 2016)

Tune the topology to create or destroy patterns, M. Asllani, T. Carletti, D. Fanelli, in press EPJB (2016)

Pattern formation in a two-component reaction-diffusion system with delayed processes on a network, J. Petit, M. Asllani, D. Fanelli, B. Lauwens, T. Carletti, Physica A, **462**, pp. 230, ( 2016)

Delay induced Turing-like waves for one species reaction–diffusion model on a network, J. Petit, T. Carletti, M. Asllani, D. Fanelli, Europhysics Letters. **111**, 5, pp. 58002, (2015)

Turing instabilities on Cartesian product networks, M. Asllani, D.M. Busiello, T. Carletti, D. Fanelli, G. Planchon, Scientific Reports. **5**, pp. 12927, (2015)

Turing patterns in multiplex networks, M. Asllani, D.M. Busiello, T. Carletti, D. Fanelli, G. Planchon, Physical Review E , **90**, 4, pp. 042814, (2014)

November the 16th, 2016, ULB Brussels

# Timoteo Carletti

## A journey in the zoo of Turing patterns



# Patterns localisation

