December the 1st, 2016, DYSCO Study Day, Louvain La Neuve

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A journey in the zoo of Turing patterns

Namur Center for Complex Systems

UNIVERSITÉ DE NAMUR

Acknowledgements

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IAP VII/19 - DYSCO

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Pattern ? [Oxford dictionary]

pattern

Top 1000 frequently used words

Pronunciation: /ˈpat(ə)n/ ?

NOUN

1 A repeated decorative design:
'a neat blue herringbone pattern'

− More example sentences

'Included are geometrics, florals and foliates, animals and nature motifs and other decorative repeat patterns.'

'These aspects then become ornamented with Islamic-inspired decorative patterns and Islamic cultural artifacts.'

'It featured exuberant decorative patterns, designs in the brickwork and wooden attachments.'

1 A repeated decorative design:
'a neat blue herringbone pattern'

+ More example sentences + Synonyms

1.1 An arrangement or design regularly found in comparable objects:
'the house had been built on the usual pattern'

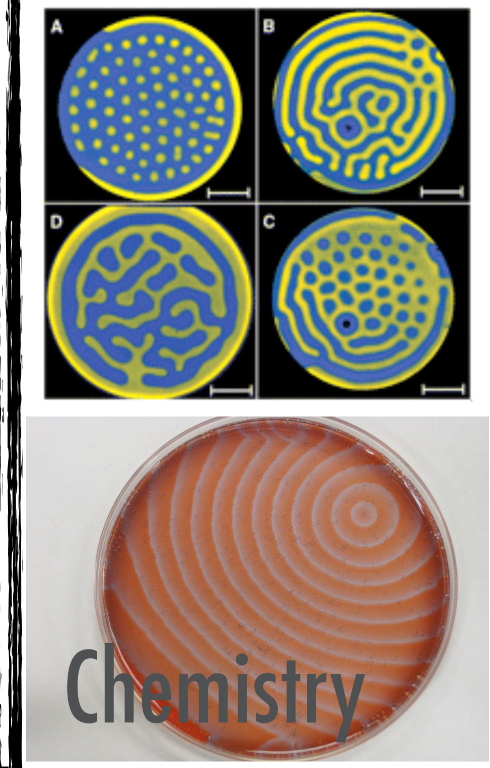
− More example sentences

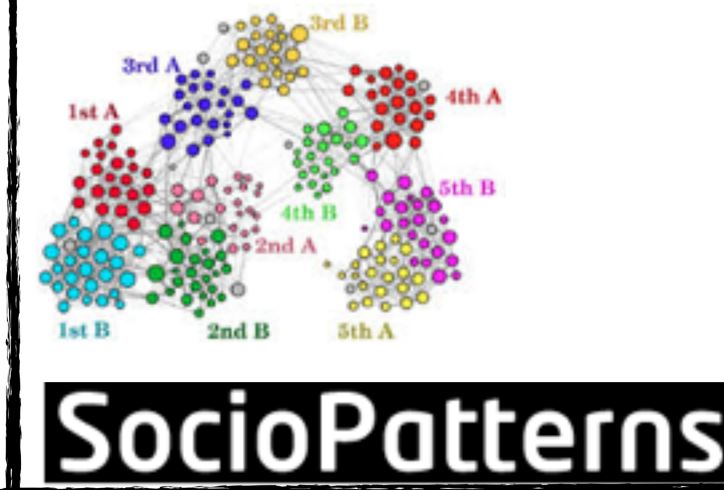
'Structurally, the tumor cells were arranged in a medullary pattern composed of polygonal tumor cells.'

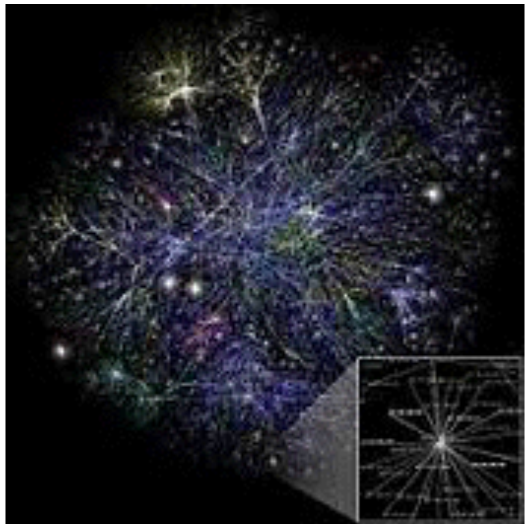
'The hair-cells within the spiralling cochlear duct are arranged in a pattern like the bristles of a brush.'

'The fossils indicate the wings had feathers, arranged in a similar pattern to that of modern birds.'

Patterns are ubiquitous

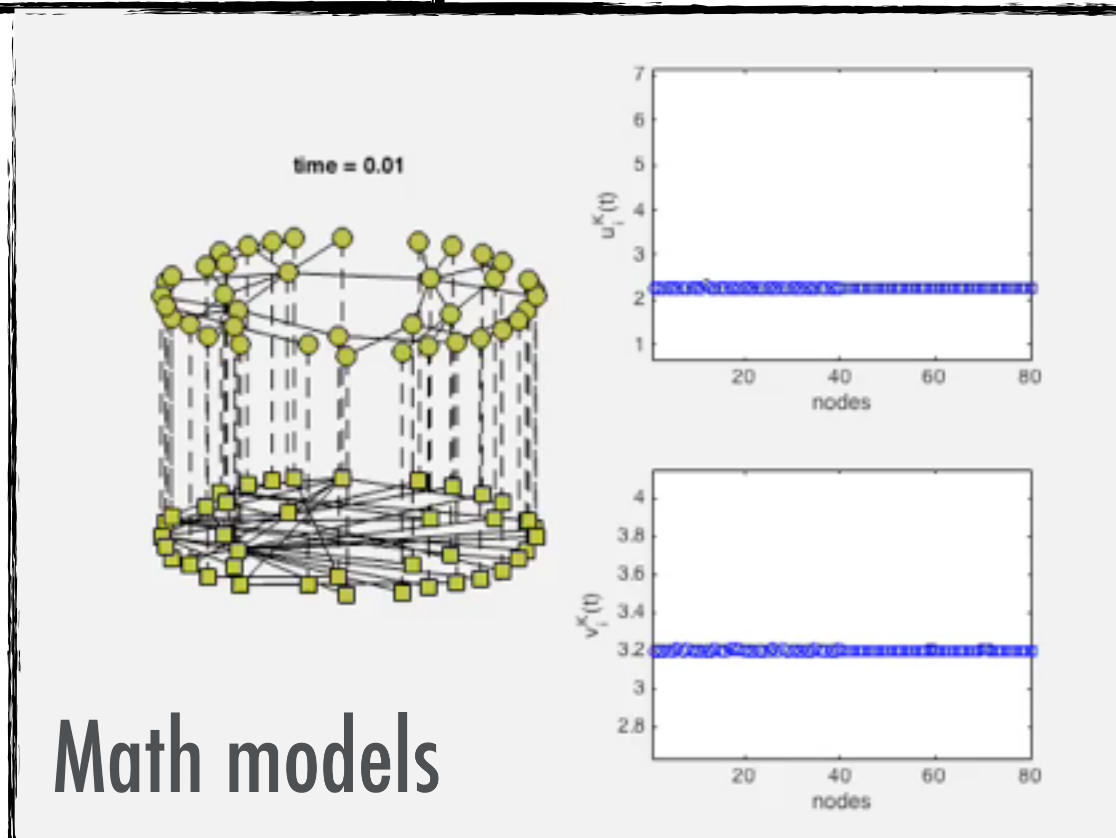






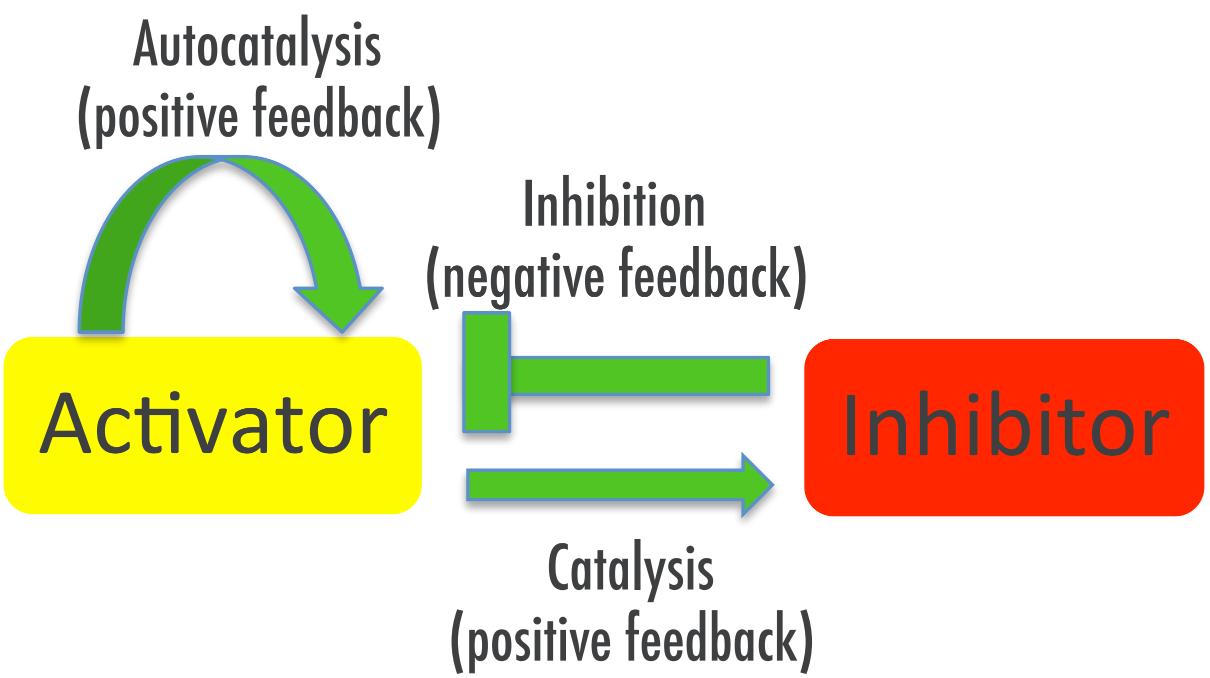
Internet

Twitter



One possible mechanism: Turing instability

Reaction



+

Diffusion

$u(x,y,t)$: Amount of activator at time t and position (x,y)

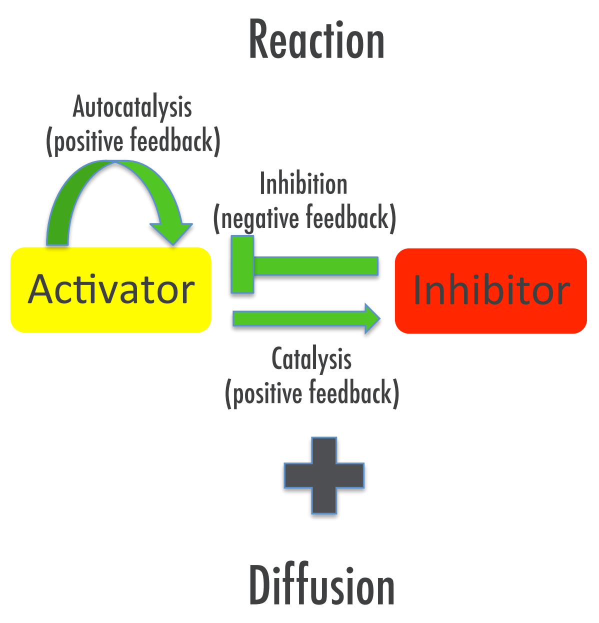
$v(x,y,t)$: Amount of inhibitor at time t and position (x,y)

$$\begin{cases} \dfrac{\partial u}{\partial t} = f(u,v) + D_u \nabla^2 u \\ \dfrac{\partial v}{\partial t} = g(u,v) + D_v \nabla^2 v \end{cases}$$

$$(x,y) \in \Omega$$

+ boundary conditions
+ initial condition

One possible mechanism: Turing instability



$u(x,y,t)$: Amount of activator at time t and position (x,y)

$v(x,y,t)$: Amount of inhibitor at time t and position (x,y)

$$\begin{cases} \frac{\partial u}{\partial t} = f(u,v) + D_u \nabla^2 u \\ \frac{\partial v}{\partial t} = g(u,v) + D_v \nabla^2 v \end{cases}$$

$$(x,y) \in \Omega$$

+ boundary conditions
+ initial condition

1

Diffusion can drive an instability by perturbing a homogeneous stable fixed point (in absence of diffusion)

2

Hence as the perturbation grows, non-linearities enter into the game yielding an asymptotic, spatially inhomogeneous, steady state (stationary pattern) or time varying one (wave like pattern).

A.M.Turing, *The chemical basis of morphogenesis*, Phil. Trans. R Soc London B, *237, (1952), pp.37*

Some mathematics for the Turing instability

$$\begin{cases} \frac{\partial u}{\partial t} = f(u,v) + D_u \nabla^2 u \\ \frac{\partial v}{\partial t} = g(u,v) + D_v \nabla^2 v \end{cases}$$

$$(x,y) \in \Omega$$

Assume Ω to be a square/rectangle and to use periodic boundary conditions

1) Assume there exists a spatially homogeneous solution:

$$u(x,y,t) = \hat{u} \text{ and } v(x,y,t) = \hat{v} \quad \forall (x,y) \in \Omega \text{ and } \forall t \geq 0$$

which moreover is stable when there is no diffusion:

$$D_u = D_v = 0$$

Some mathematics for the Turing instability

2) Linearise around this solution

$$\begin{cases} u(x,y,t) = \hat{u} + \delta u(x,y,t) \\ v(x,y,t) = \hat{v} + \delta v(x,y,t) \end{cases} \qquad \begin{pmatrix} \delta \dot{\boldsymbol{u}} \\ \delta \dot{\boldsymbol{v}} \end{pmatrix} = \tilde{\mathcal{J}} \begin{pmatrix} \delta \boldsymbol{u} \\ \delta \boldsymbol{v} \end{pmatrix}$$

$$\tilde{\mathcal{J}} = \begin{pmatrix} f_u + D_u \nabla^2 & f_v \\ g_u & g_v + D_v \nabla^2 \end{pmatrix}$$

3) Prove that the spatially homogeneous solution:

$$u(x,y,t) = \hat{u} \text{ and } v(x,y,t) = \hat{v}$$

turns out to be <u>unstable</u> once the diffusion is in action

$$D_u > 0 \text{ and } D_v > 0$$

Sketch of the proof

i) decompose the solution on the Fourier modes (Laplacian eigenbasis) and use the ansatz

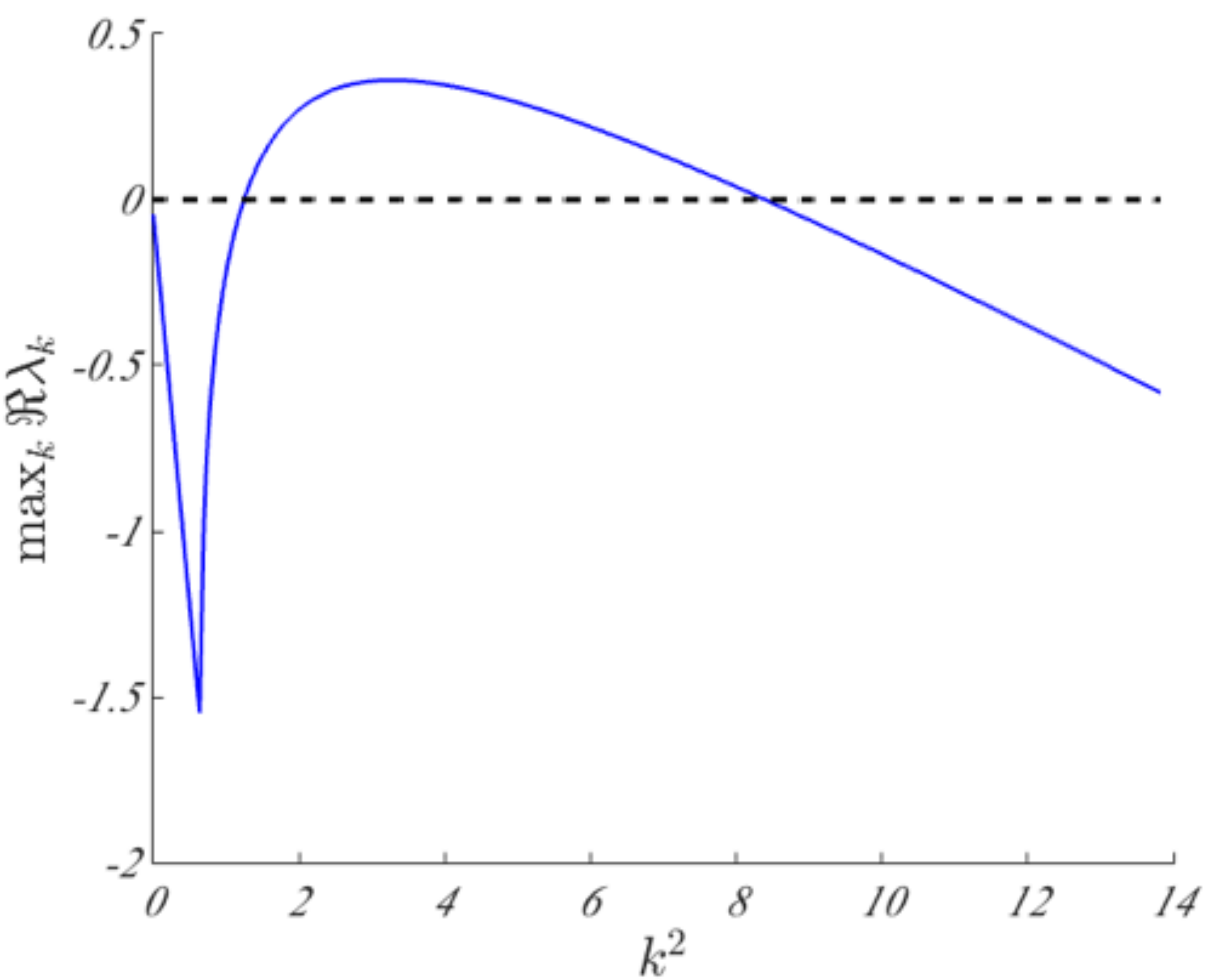
$$\delta u(x,y,t) = \sum_{k=(k_1,k_2)} c_k e^{2\pi i(k_1 x + k_2 y)} e^{\lambda_k t}$$

ii) λ_k is solution of

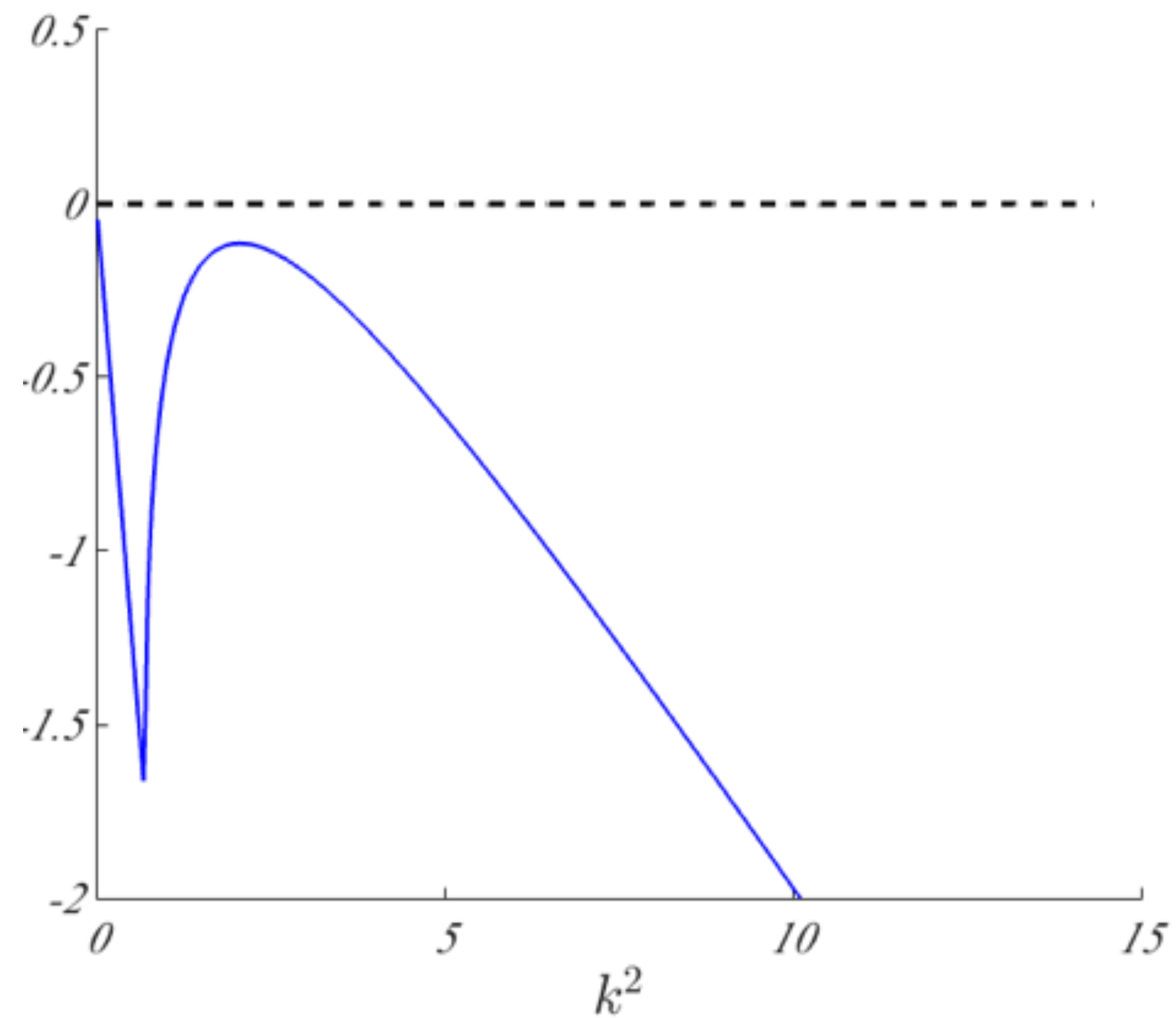
$$\det\left[\lambda_k - \begin{pmatrix} f_u - D_u k^2 & f_v \\ g_u & g_v - D_v k^2 \end{pmatrix}\right] = 0$$

Some mathematics for the Turing instability

iii) if there exists $\hat{k}^2 \in (k_-^2, k_+^2)$ such that $\Re\lambda_{\hat{k}} > 0$ then Turing patterns do emerge.



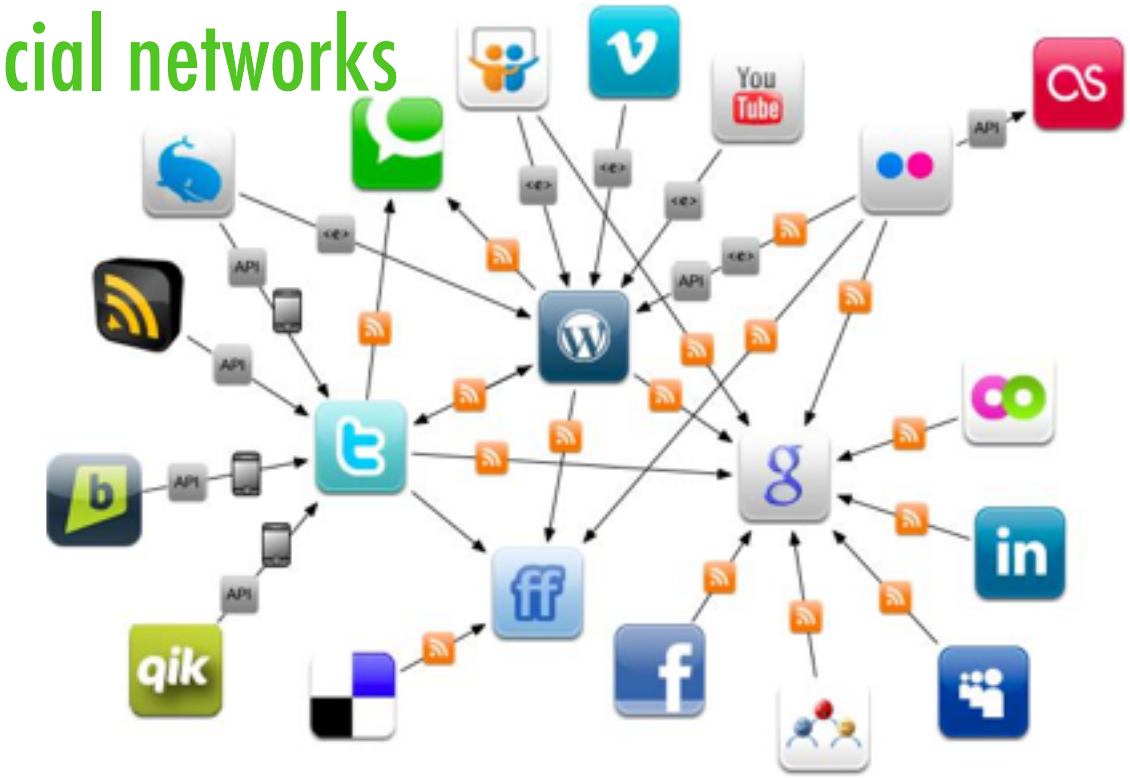
patterns do emerge



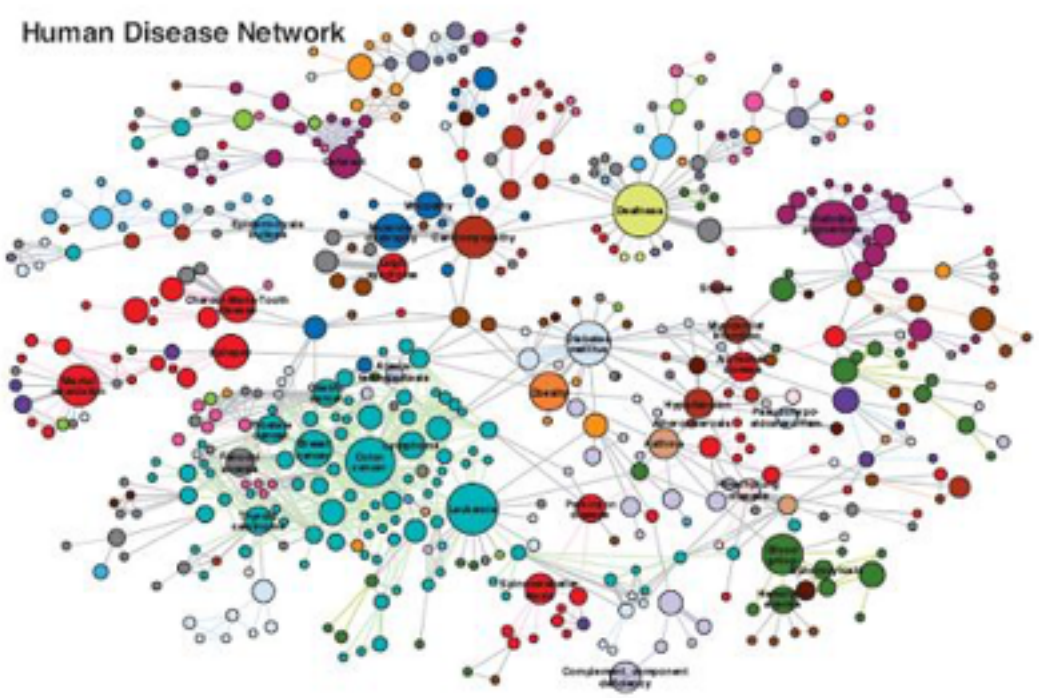
patterns do not emerge

Networks are everywhere

world flights map

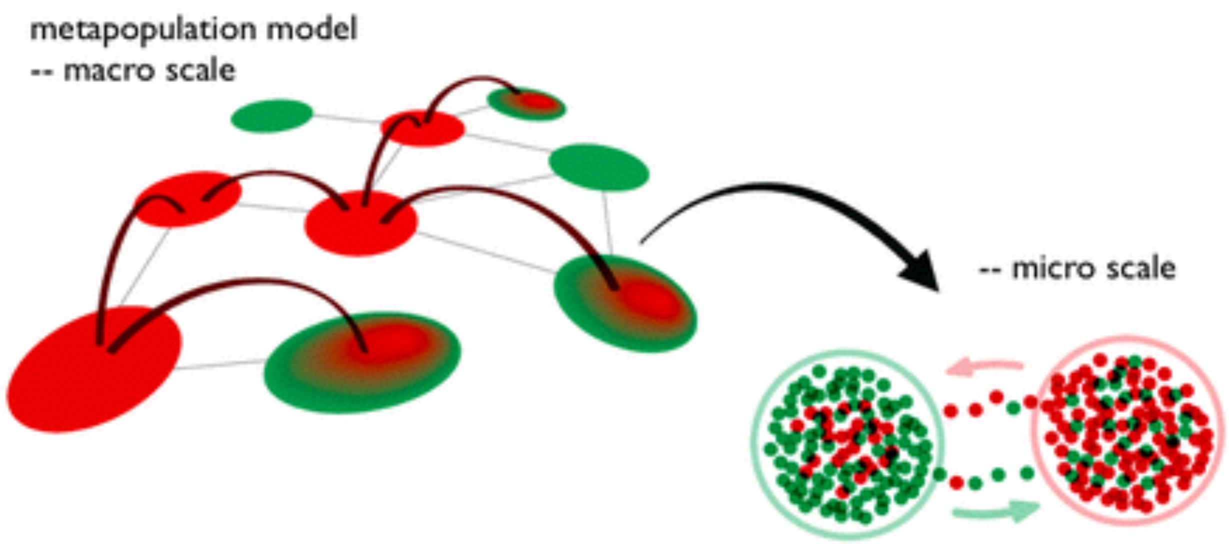


social networks



proteins networks

technological networks



Metapopulation models

e.g. in the framework of ecology:
May R.,
Will a large complex system be stable?
Nature, 238, pp. 413, (1972)

Reactions occur at each node. Diffusion occurs across edges.

Patterns :
Sets of nodes whose asymptotic state is far from the homogeneous equilibrium.

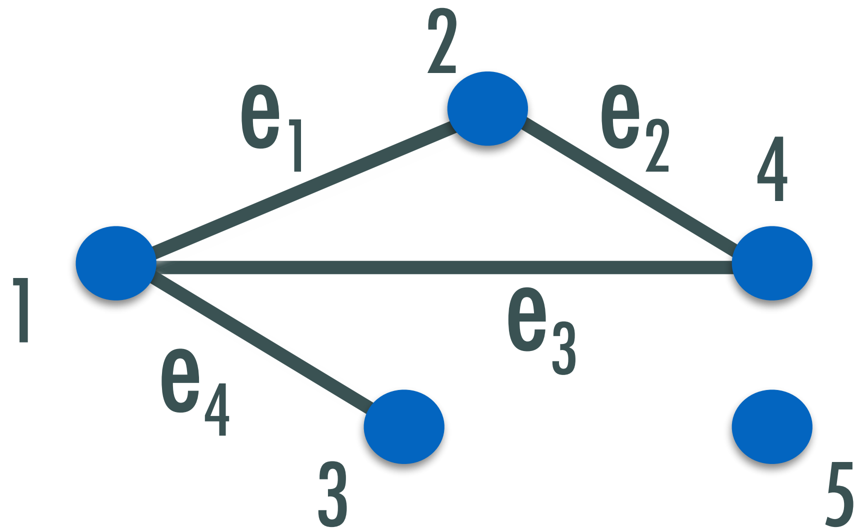
Reaction term:

$$\begin{cases} \dot{u}_i(t) &= f(u_i(t), v_i(t)) \\ \dot{v}_i(t) &= g(u_i(t), v_i(t)) \quad \forall i = 1, \dots, n \text{ and } t > 0. \end{cases}$$

At each node i=1,...,n, "species" u and v react through some non-linear functions f and g depending on the quantities available at the i-th node (metapopulation assumption)

Diffusion term:

Diffusive transport of species into a certain node i is given by the sum of incoming fluxes to node i from other connected nodes j, fluxes are proportional to the concentration difference between the nodes (Fick's law).



Ex.: consider the amount of u in node 1,
u can enter from 2, 3 and 4
u can leave 1 to go to 2, 3 and 4

$$u_2 + u_3 + u_4 - 3u_1 = \sum_j A_{1j} u_j - k_1 u_1 = \sum_j \left(A_{1j} - \delta_{1j} k_j\right) u_j := \sum_j L_{1j} u_j$$

L is called Laplacian matrix of the network

Nakao H. and Mikhailov A. S.,
Turing patterns in network-organized activator–inhibitor systems, Nature Physics, 6, pp. 544 (2010)

The model:

$$\begin{cases} \dot{u}_i(t) & = f(u_i(t), v_i(t)) + D_u \sum_{j=1}^n L_{ij} u_j(t) \\ \dot{v}_i(t) & = g(u_i(t), v_i(t)) + D_v \sum_{j=1}^n L_{ij} v_j(t) \quad \forall i = 1, \dots, n \text{ and } t > 0. \end{cases}$$

Du and Dv are the diffusion coefficients of species u and v

Observe that because the network is undirected, the matrices A and L are symmetric

General strategy for the network case

1) Assume there exists a spatially homogeneous solution:

$$(u_i, v_i) = (\hat{u}, \hat{v}) \quad \forall i = 1, \dots, n$$

which moreover is stable when there is no diffusion:

$$D_u = D_v = 0$$

2) Linearize around this solution

$$\begin{aligned} u_i &= \hat{u} + \delta u_i \\ v_i &= \hat{v} + \delta v_i \end{aligned}$$

$$\begin{pmatrix} \dot{\delta \boldsymbol{u}} \\ \dot{\delta \boldsymbol{v}} \end{pmatrix} = \tilde{\mathcal{J}} \begin{pmatrix} \delta \boldsymbol{u} \\ \delta \boldsymbol{v} \end{pmatrix}$$

$$\tilde{\mathcal{J}} = \begin{pmatrix} f_u \mathbf{I}_n + D_u \boldsymbol{L} & f_v \mathbf{I}_n \\ g_u \mathbf{I}_n & g_v \mathbf{I}_n + D_v \boldsymbol{L} \end{pmatrix}$$

General strategy for the network case

3) Prove that (possibly) the spatially homogeneous solution:

$$(u_i, v_i) = (\hat{u}, \hat{v}) \quad \forall i = 1, \dots, n$$

turns out to be unstable once the diffusion is in action

$$D_u > 0 \text{ and } D_v > 0$$

Sketch of the proof

i) Let $\quad L\vec{\phi}^\alpha = \Lambda^\alpha \vec{\phi}^\alpha,\ \alpha = 1, \dots, n \qquad \vec{\phi}^\alpha = (\phi_1^\alpha, \dots, \phi_n^\alpha)$

$$\sum_i \phi_i^\alpha \phi_i^\beta = \delta_{\alpha\beta} \qquad \Lambda^\alpha \leq 0$$

ii) decompose the solution on the eigenbasis and use the ansatz

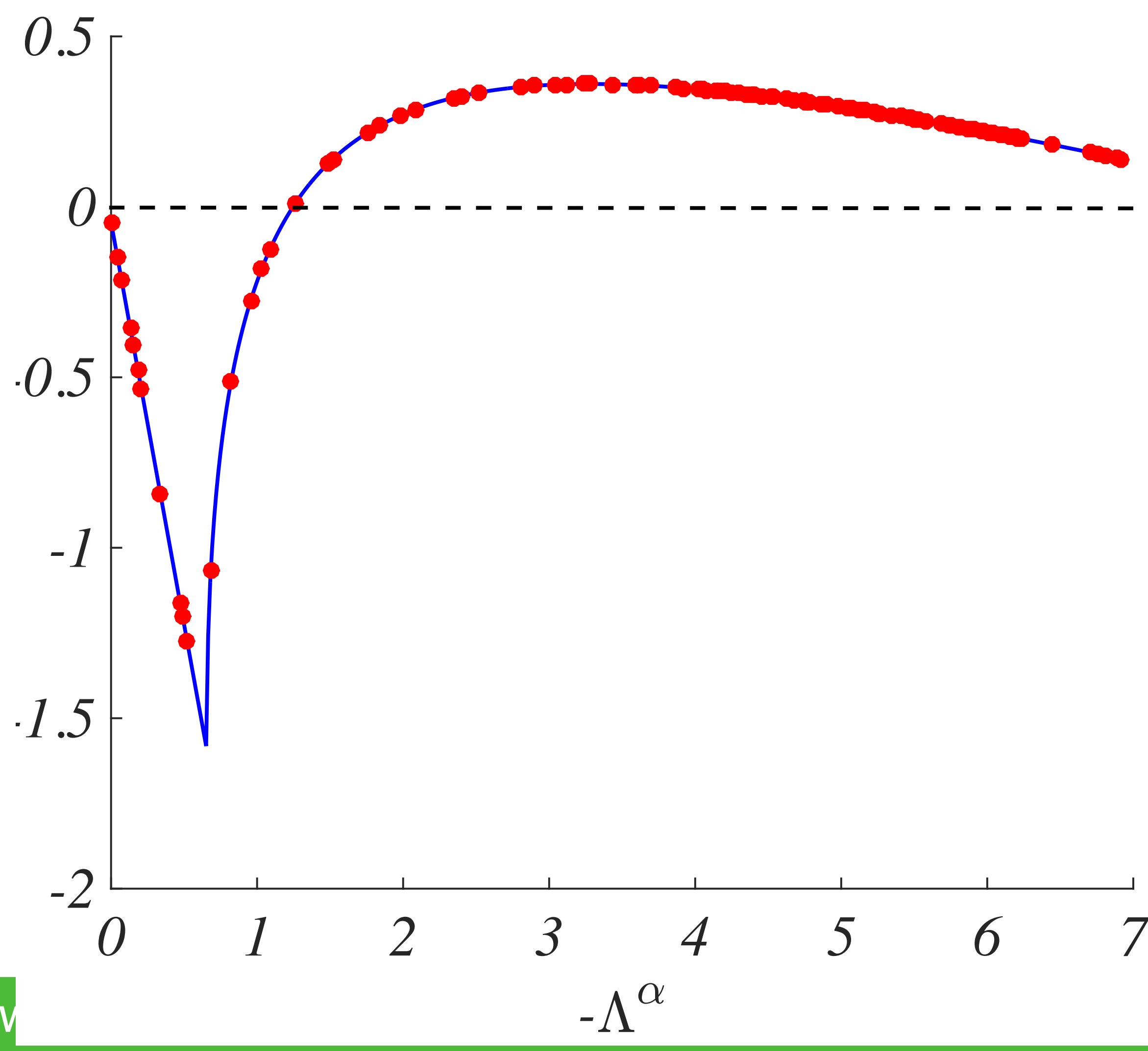
$$\delta u_i(t) = \sum_{\alpha=1}^{n} c_\alpha \phi_i^\alpha e^{\lambda_\alpha t}$$

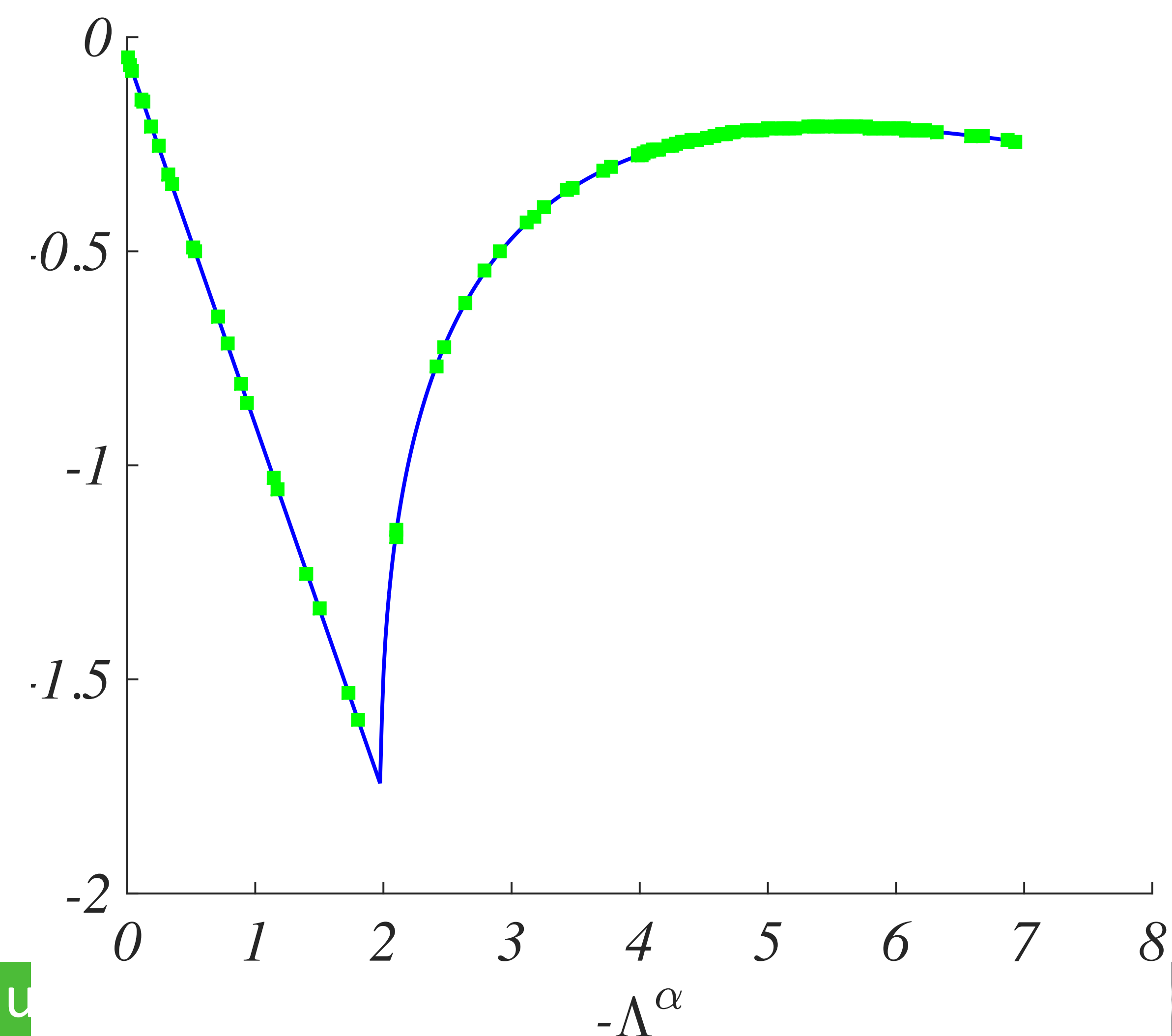
General strategy

iii) λ_α (called relation dispersion) is solution of

$$\det\left[\lambda_\alpha - \begin{pmatrix} f_u + D_u\Lambda^\alpha & f_v \\ g_u & g_v + D_v\Lambda^\alpha \end{pmatrix}\right] = 0$$

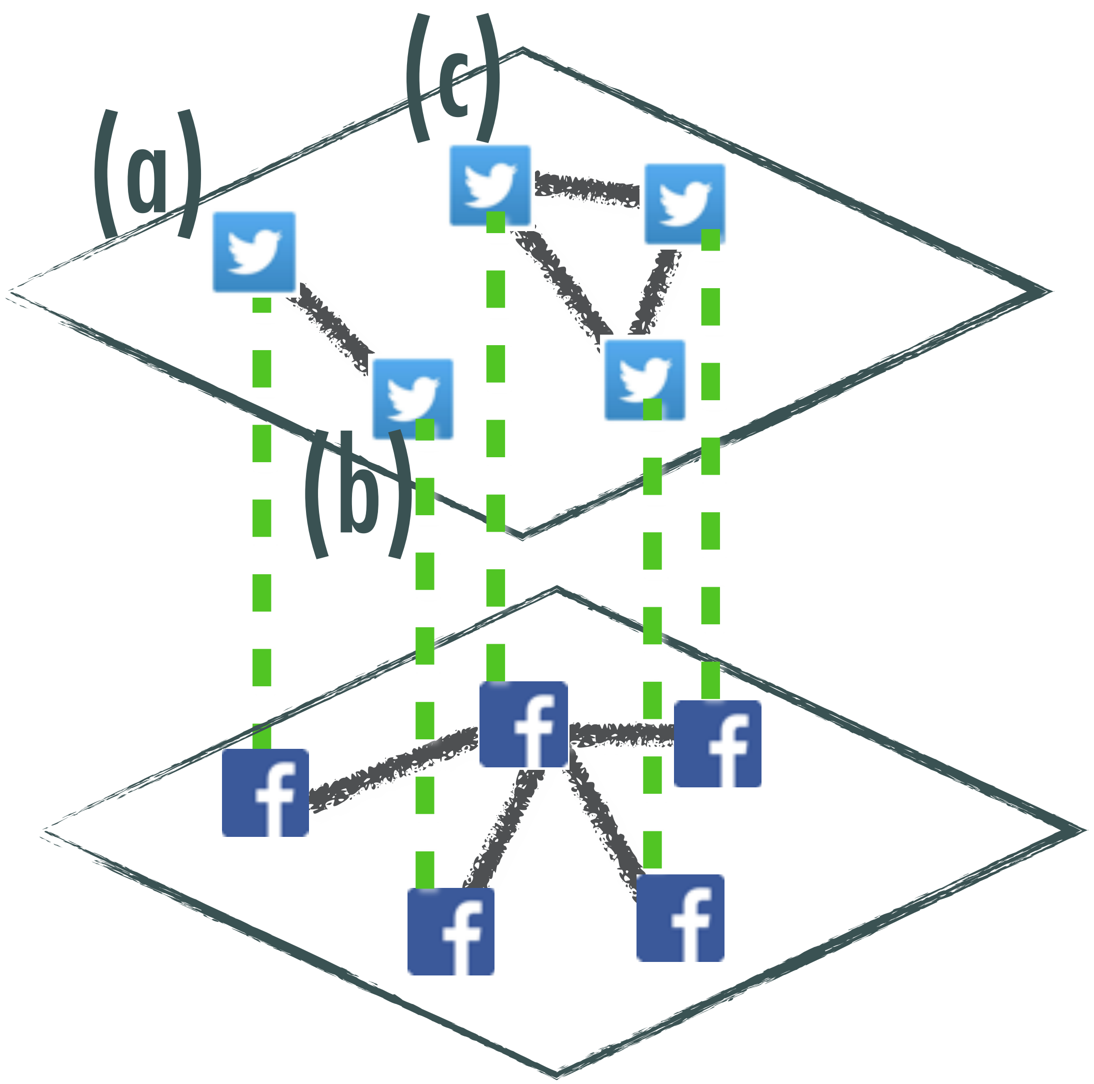
iv) if there exists Λ^{α_c} such that $\Re\lambda_{\alpha_c} > 0$ then Turing patterns do emerge.

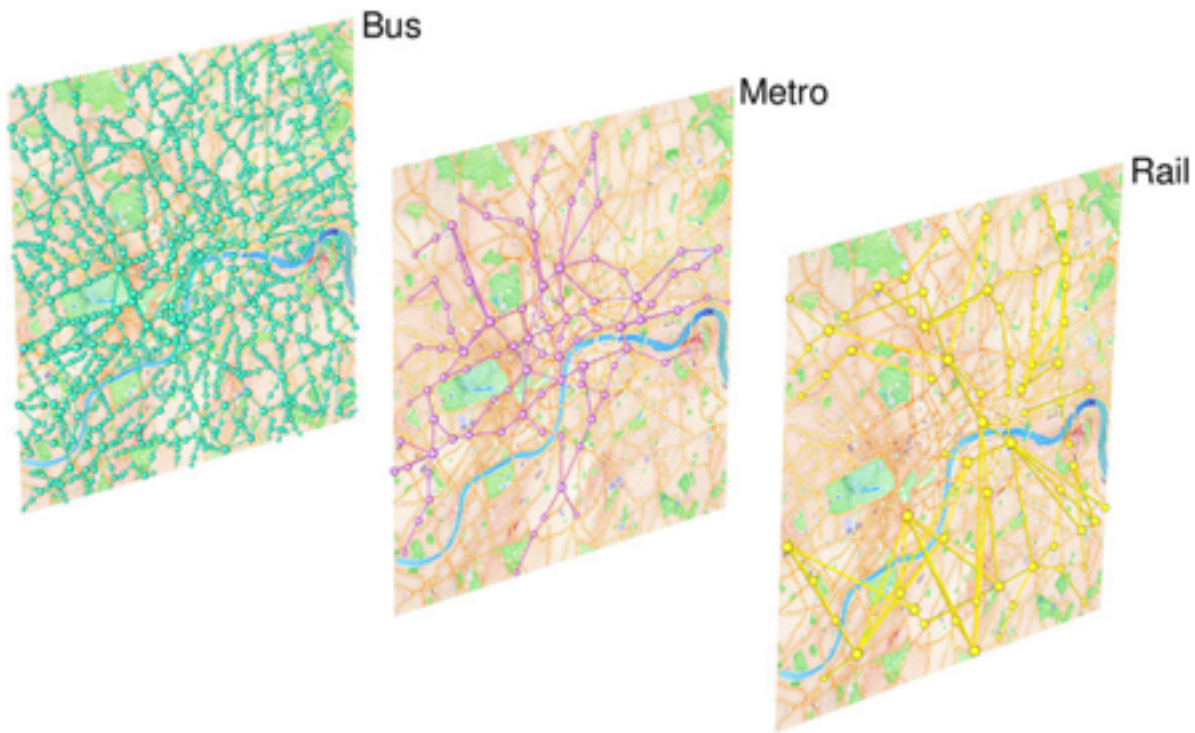




Systems composed by layers of networks: Multiplexes

Social networks
layers=different social networks
nodes=same agent in each SN





Transportation networks
layers=different modalities
nodes=same spatial location

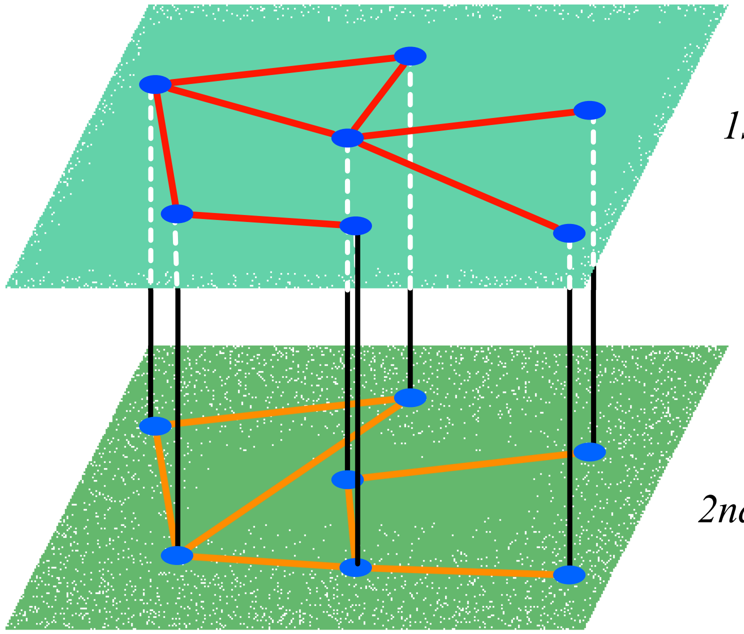
Turing instabilities on multiplex networks

adjacency matrix of layer K

degree of i-th note in layer K

$$L_{ij}^K = A_{ij}^K - \delta_{ij} k_i^K$$

Laplacian matrix of layer K



The same Ω nodes are present in each layer

$D_{u,v}^K$ **inter**-layer diffusion coefficient

$D_{u,v}^{12}$ **intra**-layer diffusion coefficient

$$\begin{cases} \dot{u}_i^K &= f(u_i^K, v_i^K) + D_u^K \sum_{j=1}^{\Omega} L_{ij}^K u_j^K + D_u^{12}\left(u_i^{K+1} - u_i^K\right) \\ \dot{v}_i^K &= g(u_i^K, v_i^K) + D_v^K \sum_{j=1}^{\Omega} L_{ij}^K v_j^K + D_v^{12}\left(v_i^{K+1} - v_i^K\right) \end{cases}$$

General strategy

1) Assume there exists a spatially homogeneous solution:

$$(u_i^K, v_i^K) = (\hat{u}, \hat{v}) \quad \forall i = 1, \ldots, n$$

2) Linearise around this solution

$$u_j^K = \hat{u} + \delta u_j^K$$
$$v_j^K = \hat{v} + \delta v_j^K$$

$$\begin{pmatrix} \delta \dot{\boldsymbol{u}} \\ \delta \dot{\boldsymbol{v}} \end{pmatrix} = \tilde{\boldsymbol{\mathcal{J}}} \begin{pmatrix} \delta \boldsymbol{u} \\ \delta \boldsymbol{v} \end{pmatrix}$$

$$\tilde{\boldsymbol{\mathcal{J}}} = \begin{pmatrix} f_u \mathbf{I}_{2\Omega} + \boldsymbol{\mathcal{L}}_u + D_u^{12} \boldsymbol{\mathcal{I}} & f_v \mathbf{I}_{2\Omega} \\ g_u \mathbf{I}_{2\Omega} & g_v \mathbf{I}_{2\Omega} + \boldsymbol{\mathcal{L}}_v + D_v^{12} \boldsymbol{\mathcal{I}} \end{pmatrix}$$

Supra-Laplacian matrix $\boldsymbol{\mathcal{L}}_u + D_u^{12} \boldsymbol{\mathcal{I}}$

$$\boldsymbol{\mathcal{L}}_u = \begin{pmatrix} D_u^1 \mathbf{L}^1 & \mathbf{0} \\ \mathbf{0} & D_u^2 \mathbf{L}^2 \end{pmatrix} \qquad \boldsymbol{\mathcal{I}} = \begin{pmatrix} -\mathbf{I}_\Omega & \mathbf{I}_\Omega \\ \mathbf{I}_\Omega & -\mathbf{I}_\Omega \end{pmatrix}$$

General strategy

3) Study the spectrum of

$$\tilde{\mathcal{J}} = \begin{pmatrix} f_u \mathbf{I}_{2\Omega} + \boldsymbol{\mathcal{L}}_u + D_u^{12} \boldsymbol{\mathcal{I}} & f_v \mathbf{I}_{2\Omega} \\ g_u \mathbf{I}_{2\Omega} & g_v \mathbf{I}_{2\Omega} + \boldsymbol{\mathcal{L}}_v + D_v^{12} \boldsymbol{\mathcal{I}} \end{pmatrix}$$

to determine the existence of eigenvalues such that

$$\Re\lambda(D^{12}_{u,v}, D^{K}_{u,v}) > 0$$

Very hard for generic topologies, however ...

Small intra-layer diffusion case

Assume $D_v^{12} = \epsilon << 1 \qquad D_u^{12} = \mathcal{O}(\epsilon)$

$$\tilde{\mathcal{J}} = \begin{pmatrix} f_u \mathbf{I}_{2\Omega} + \boldsymbol{\mathcal{L}}_u & f_v \mathbf{I}_{2\Omega} \\ g_u \mathbf{I}_{2\Omega} & g_v \mathbf{I}_{2\Omega} + \boldsymbol{\mathcal{L}}_v \end{pmatrix} + \epsilon \begin{pmatrix} \frac{D_u^{12}}{D_v^{12}} \boldsymbol{L}^1 & \mathbf{0} \\ \mathbf{0} & \boldsymbol{L}^2 \end{pmatrix}$$

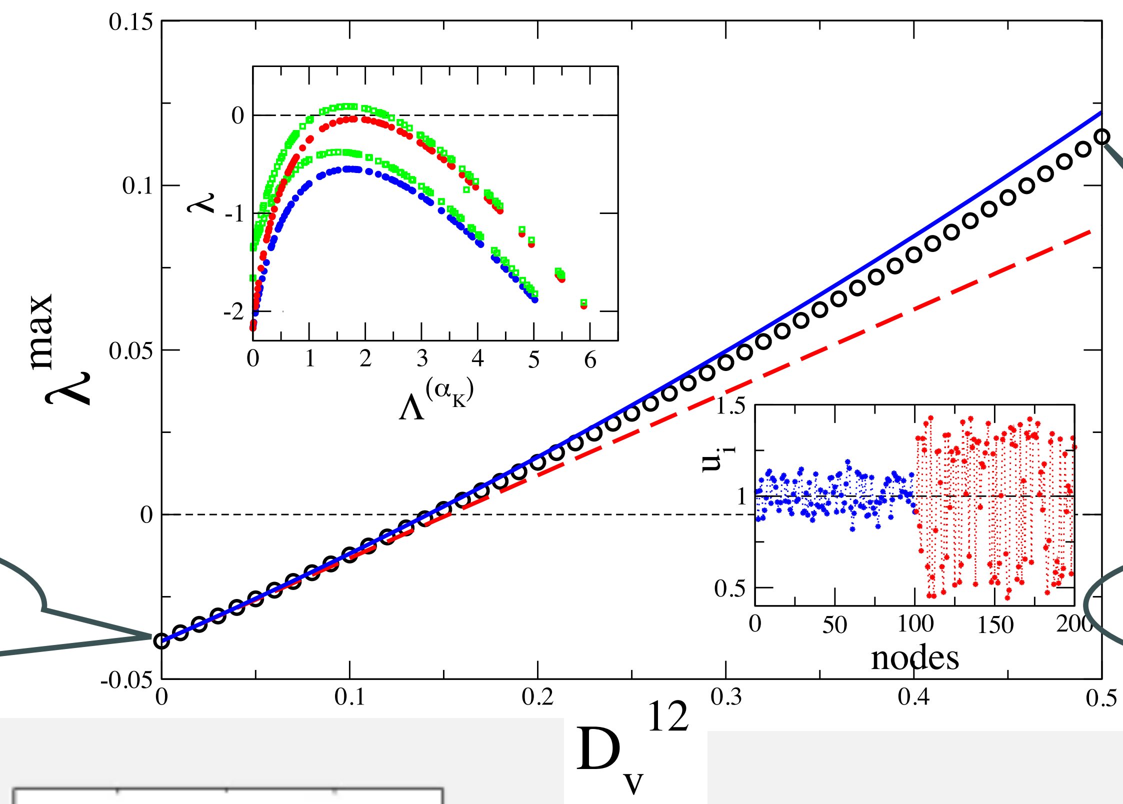
$$= \tilde{\mathcal{J}}_0 + \epsilon \boldsymbol{\mathcal{D}}_0$$

Perturbative approach to compute the spectrum

$$\lambda^{max}(\epsilon) = \lambda_0^{max} + \epsilon (U_0 \boldsymbol{\mathcal{D}}_0 V_0)_{k_{max} k_{max}} + \mathcal{O}(\epsilon^2)$$

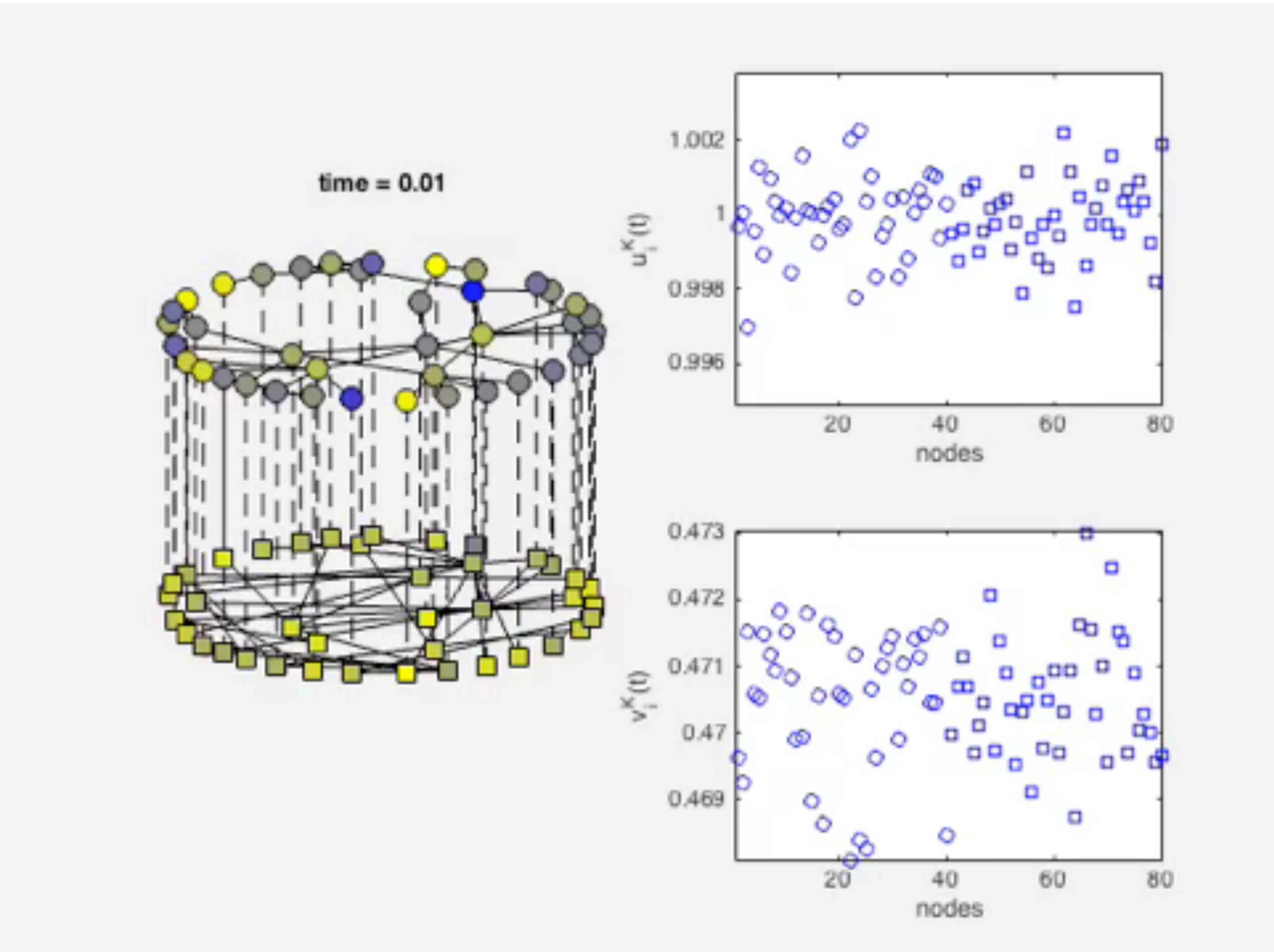
$$\lambda_0^{max} = \max \lambda_k(\epsilon = 0) \qquad k_{max} = \arg\max \lambda_k(\epsilon = 0)$$

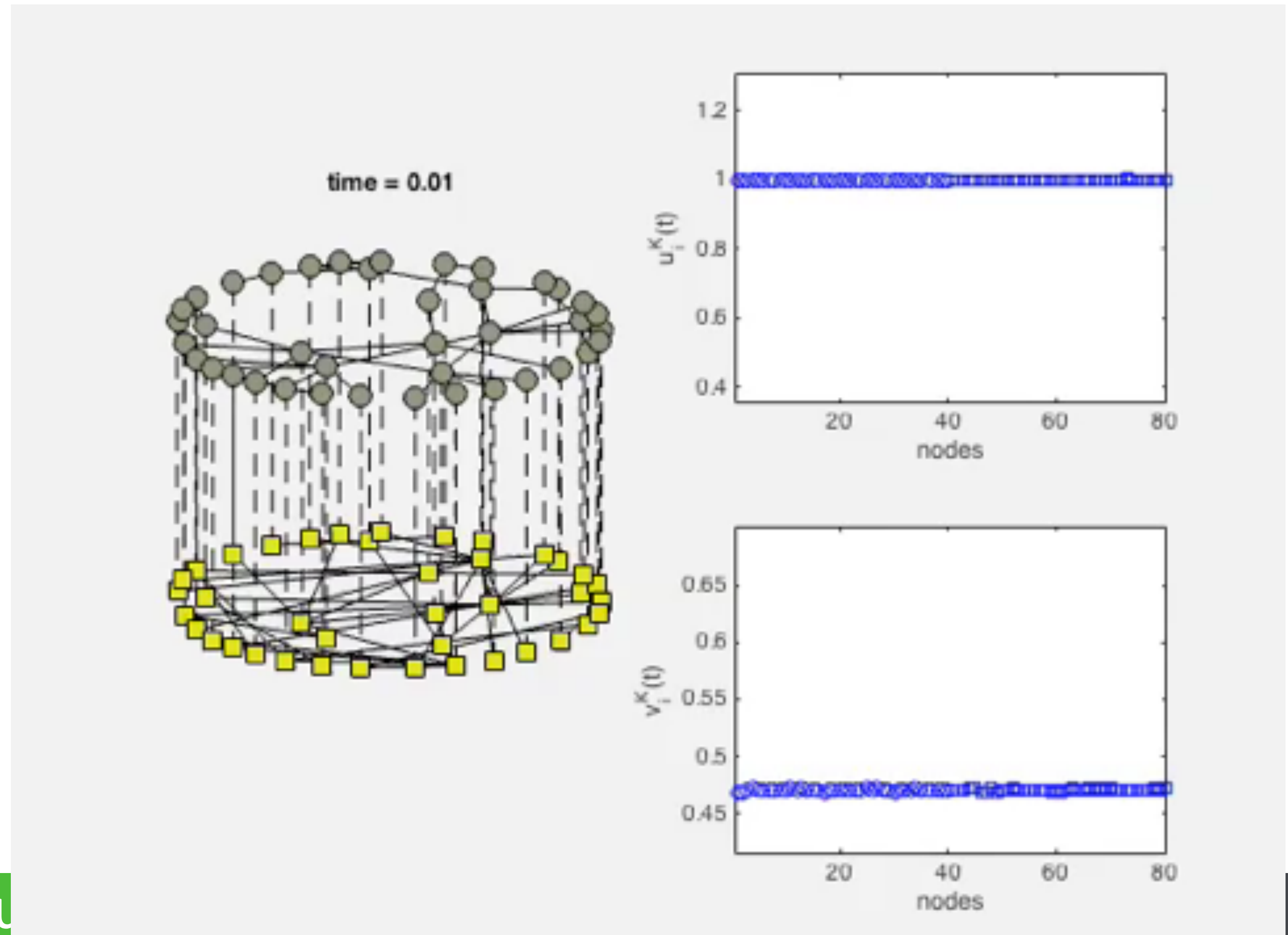
Small intra-layer diffusion case: onset of patterns



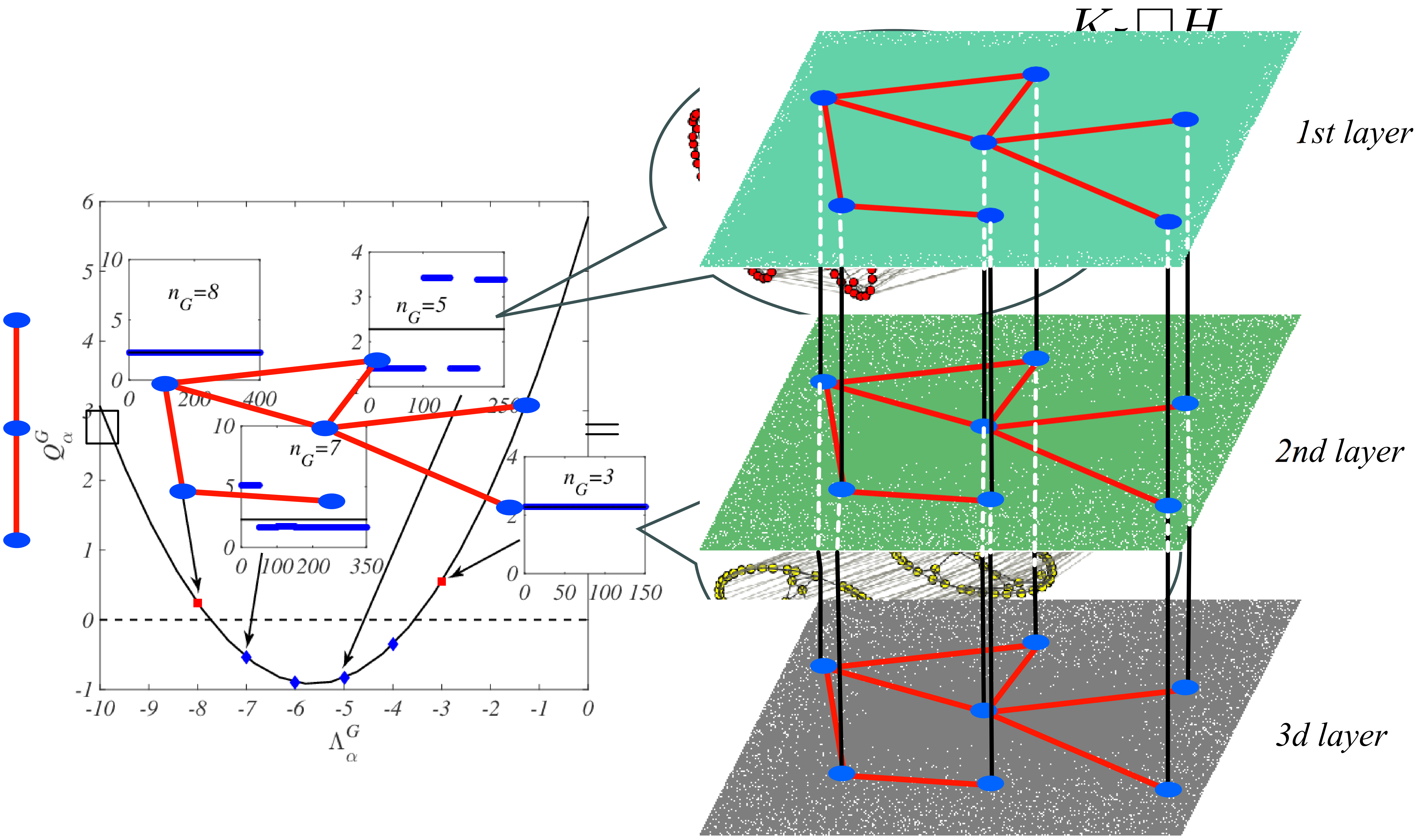
$$D_v^{12} = D_u^{12} = 0$$

$$D_u^{12} = 0 \quad D_v^{12} = 0.5$$



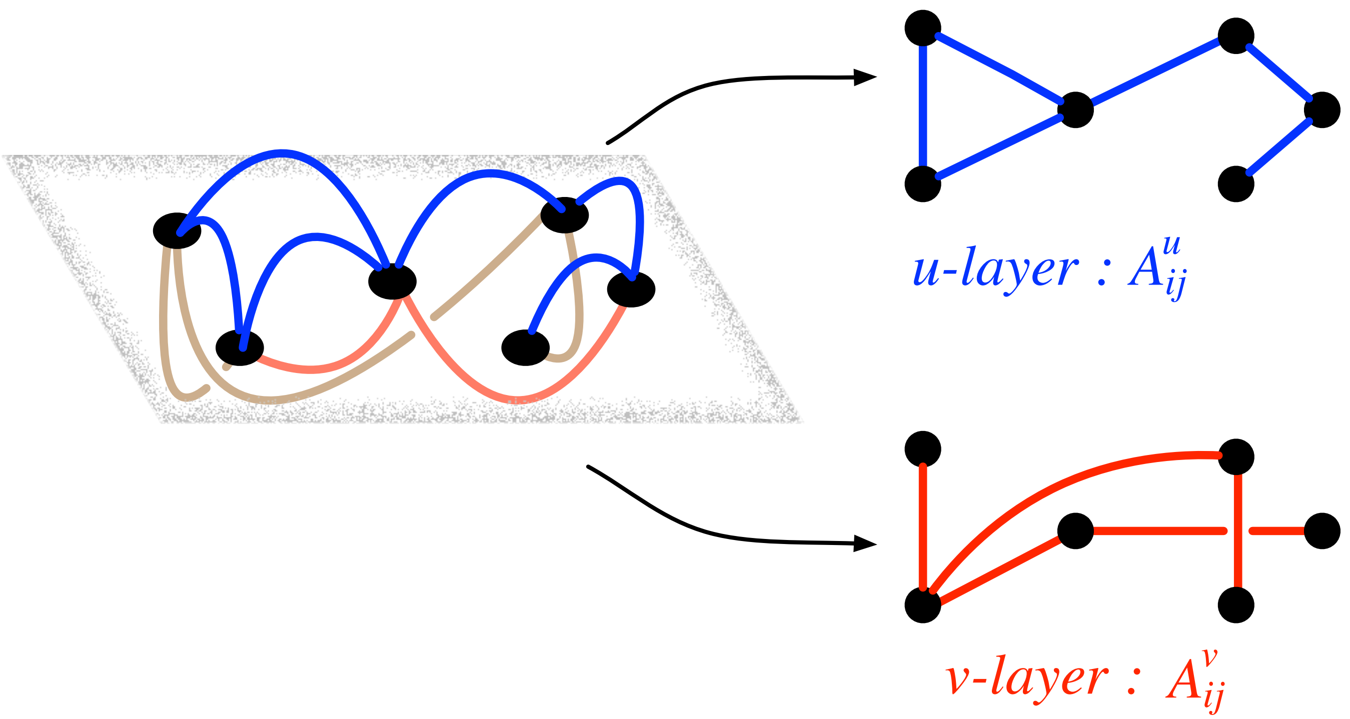


Add-Remove one layer: Cartesian product networks



Can we control the topology to create (destroy) patterns?

Let us consider a multigraph, e.g. two nodes can be connected through different edges



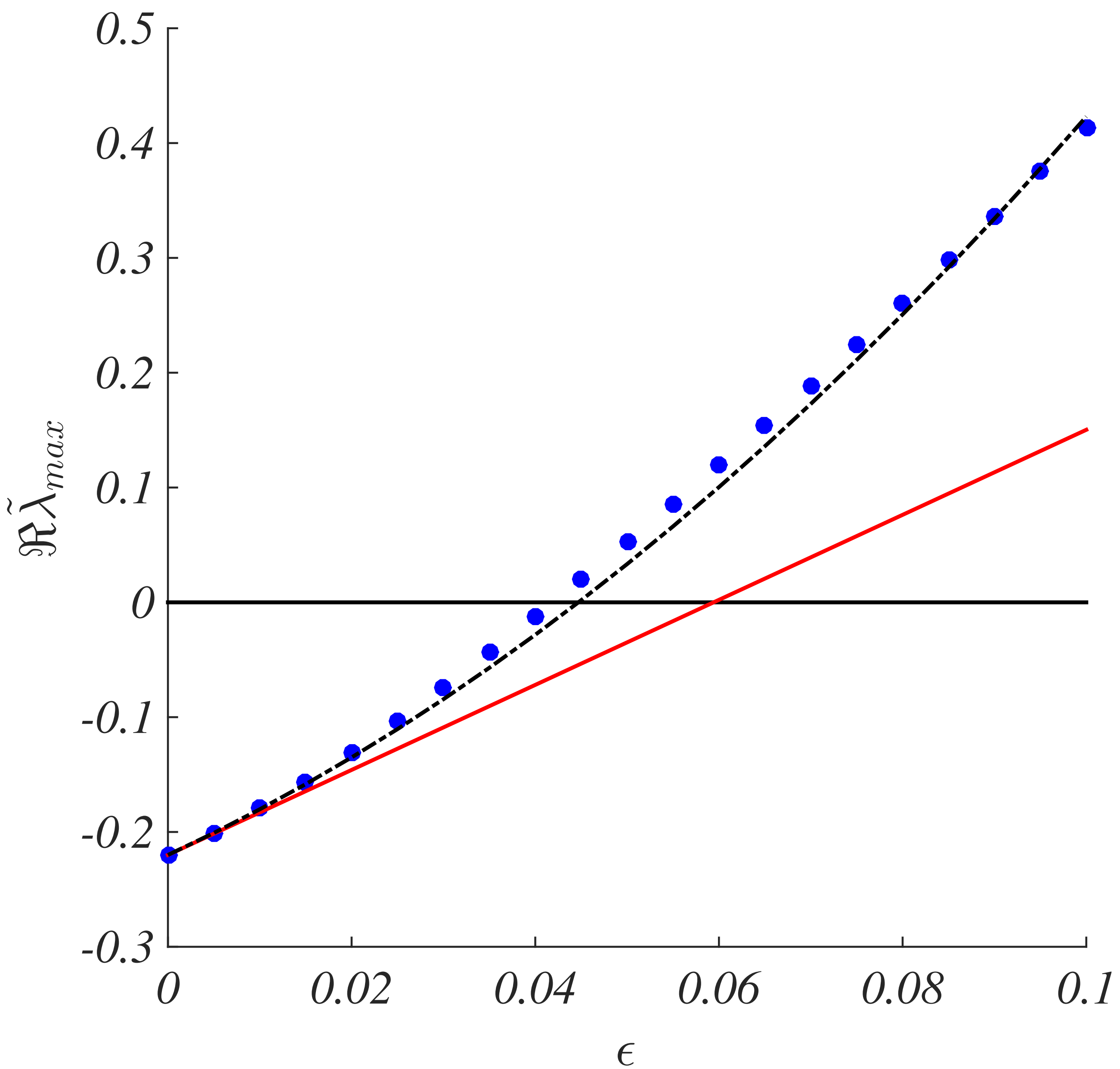
$\epsilon = 0$ $\qquad$ $\epsilon = 1$

$$A^u(0) = A^0 \qquad A^u(\epsilon) = A^0 + \epsilon(A^1 - A^0) \qquad A^u(1) = A^1$$

$$A^v(0) = A^0 \qquad A^v(\epsilon) = A^0 + \epsilon(A^2 - A^0) \qquad A^v(1) = A^2$$

Can we control the topology to create (destroy) patterns?

creation of patterns



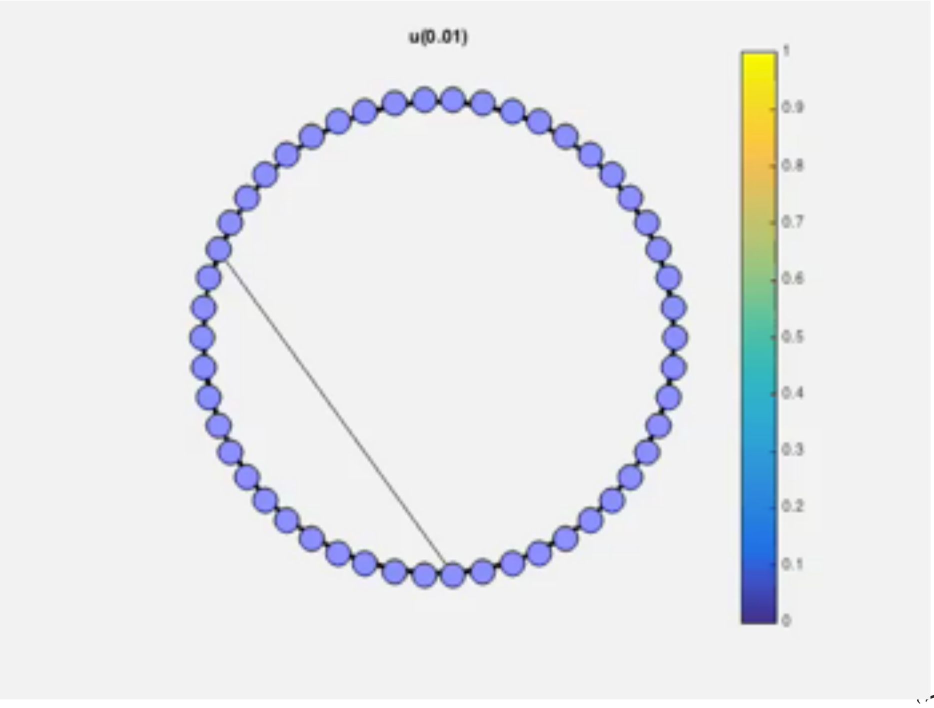
$\epsilon = 0$ $\epsilon = 1$

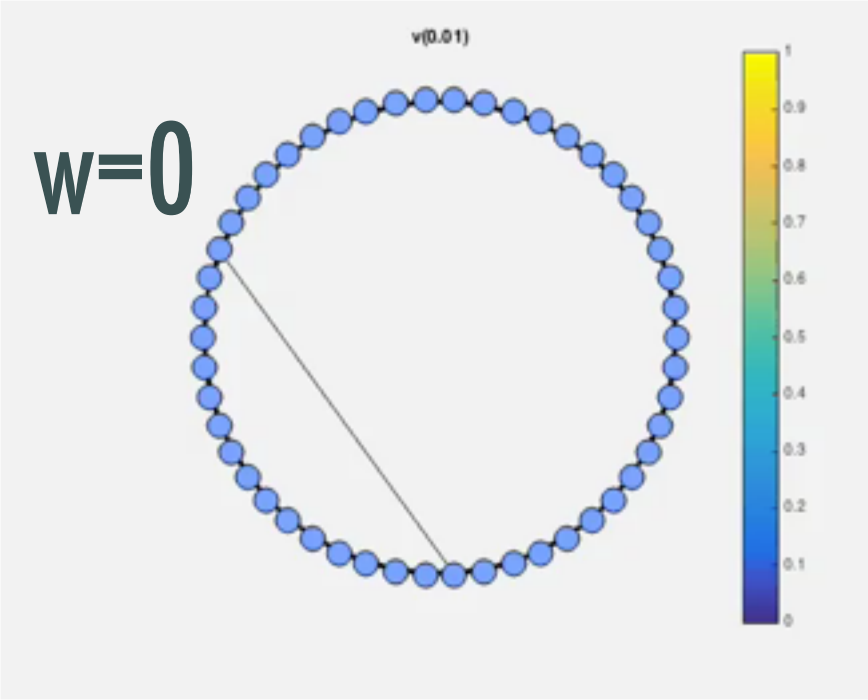
$$A^u(0) = A^0 \qquad A^u(\epsilon) = A^0 + \epsilon(A^1 - A^0) \qquad A^u(1) = A^1$$

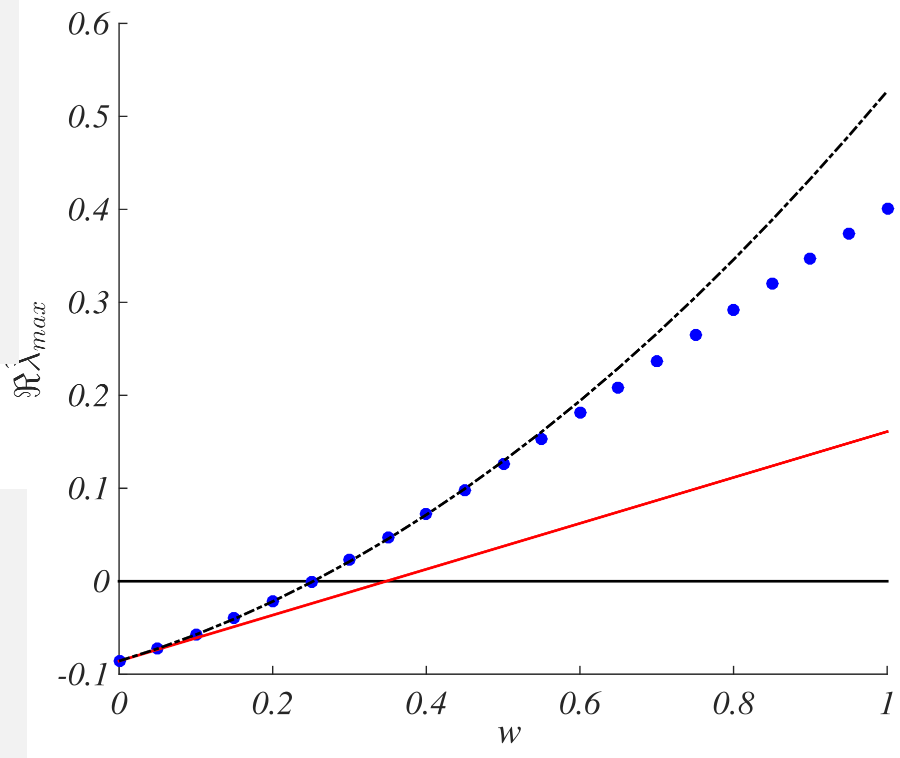
$$A^v(0) = A^0 \qquad A^v(\epsilon) = A^0 + \epsilon(A^2 - A^0) \qquad A^v(1) = A^2$$

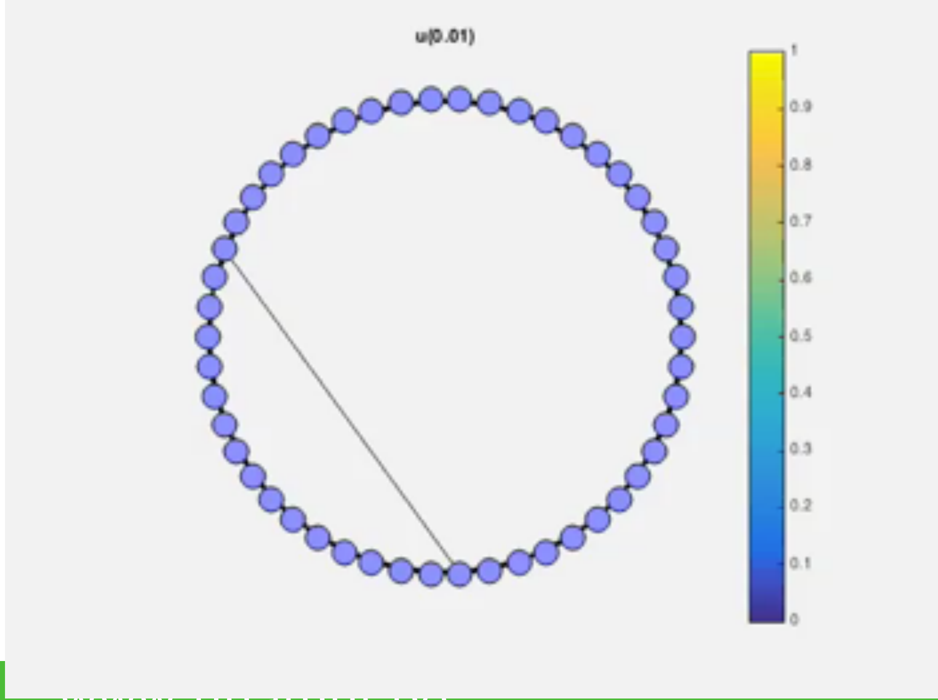
Create patterns by adding a single (optimally chosen) link

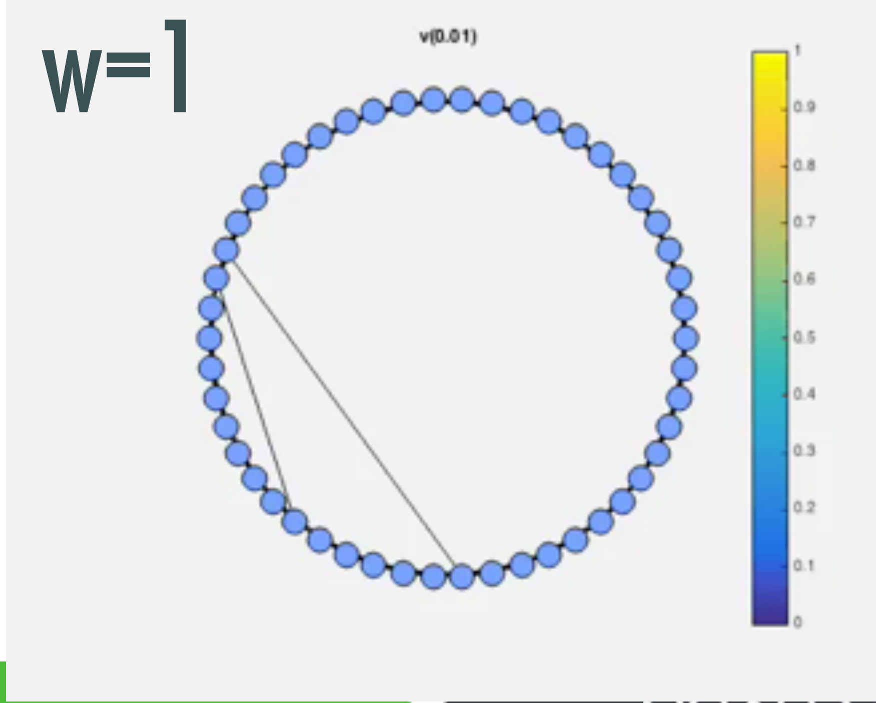
$$A^u(w) = A^0 \qquad A^v(w) = A^0 + wT^{(ij)}$$











Turing patterns in time varying networks

$$\dot{u}_i = f(u_i, v_i) + D_u \sum_{j=1}^{\Omega} L_{ij}(t) u_j$$

$$\dot{v}_i = g(u_i, v_i) + D_v \sum_{j=1}^{\Omega} L_{ij}(t) v_j$$

time dependent Laplacian matrix

$$\langle \mathbf{L} \rangle = \frac{1}{T} \int_0^T \mathbf{L}(t)\, dt$$

averaged Laplacian matrix

Theorem

If the time averaged system exhibits Turing patterns, then also the time varying does provided it is "fast enough accelerated".

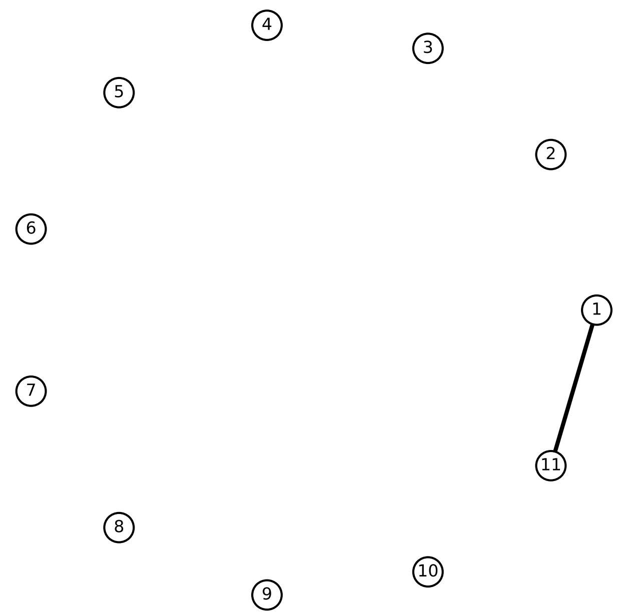
$$0 < \epsilon < \epsilon^*$$

$$\dot{u}_i(t) = f(u_i, v_i) + D_u \sum_{j=1}^{\Omega} L_{ij}(t/\epsilon) u_j(t)$$

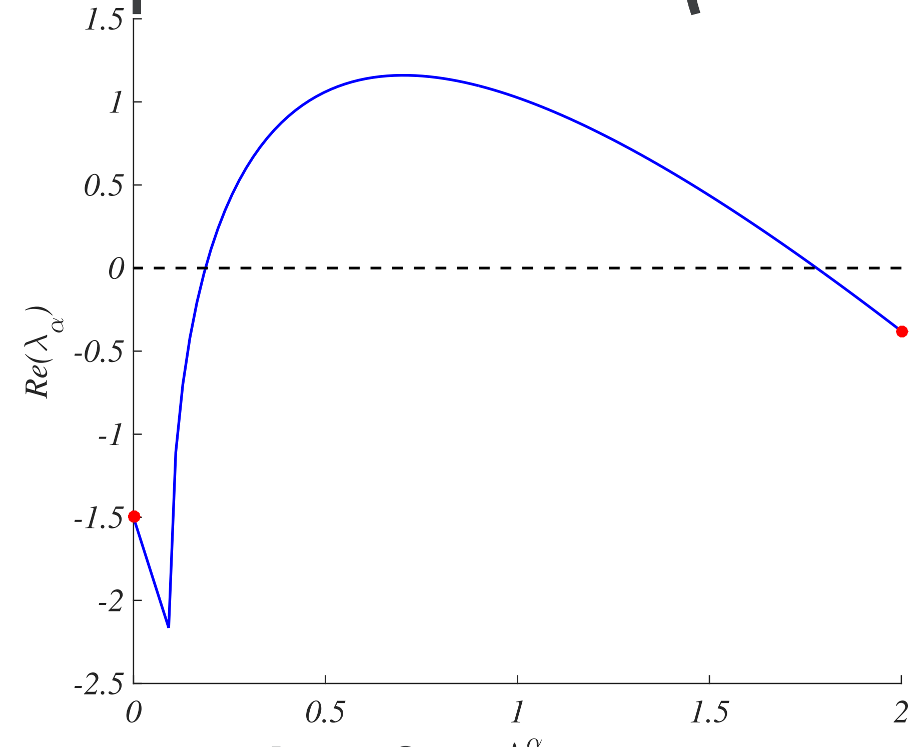
$$\dot{v}_i(t) = g(u_i, v_i) + D_v \sum_{j=1}^{\Omega} L_{ij}(t/\epsilon) v_j(t)$$

Turing patterns in time varying networks

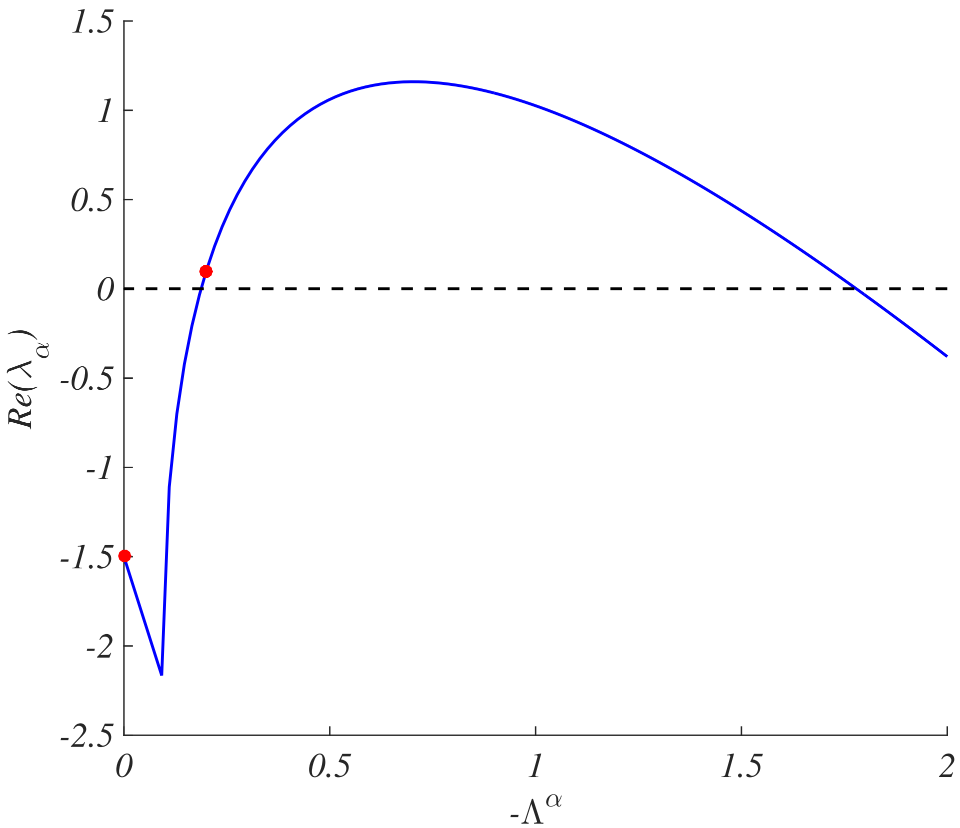
Blinking network : 1 link per time



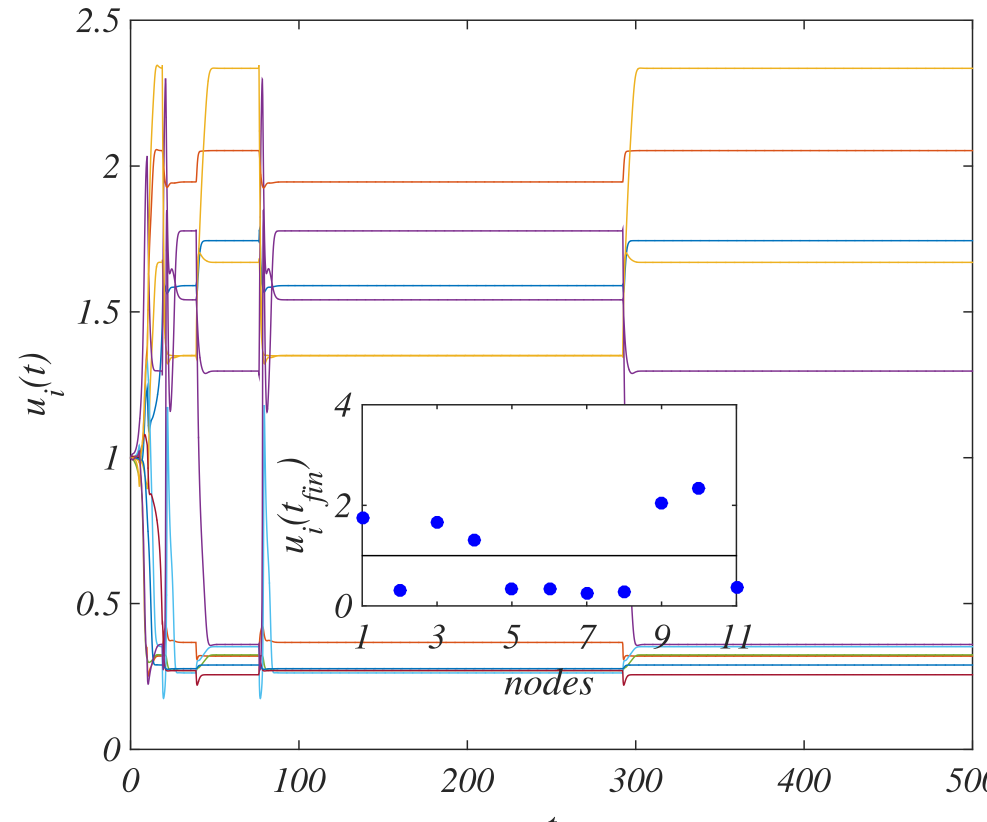
No patterns for fixed t (1 link network)



Patterns in the averaged network



Patterns in the “fast” time varying network



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Some papers

Benjamin-Feir instabilities on directed networks, F. Di Patti[a,b,c], D. Fanelli[a,b,c], F. Miele[a], T. Carletti[d], in press Chaos (2016)

Tune the topology to create or destroy patterns, M. Asllani, T. Carletti, D. Fanelli, in press EPJB (2016)

Pattern formation in a two-component reaction-diffusion system with delayed processes on a network, J. Petit, M. Asllani, D. Fanelli, B. Lauwens, T. Carletti, Physica A, **462**, pp. 230, (2016)

Delay induced Turing-like waves for one species reaction–diffusion model on a network, J. Petit, T. Carletti, M. Asllani, D. Fanelli, Europhysics Letters. **111**, 5, pp. 58002, (2015)

Turing instabilities on Cartesian product networks, M. Asllani, D.M. Busiello, T. Carletti, D. Fanelli, G. Planchon, Scientific Reports. **5**, pp. 12927, (2015)

Turing patterns in multiplex networks, M. Asllani, D.M. Busiello, T. Carletti, D. Fanelli, G. Planchon, Physical Review E ,**90**, 4, pp. 042814, (2014) December the 1st, 2016, DYSCO Study Day, Louvain La Neuve

Timoteo Carletti

A journey in the zoo of Turing patterns

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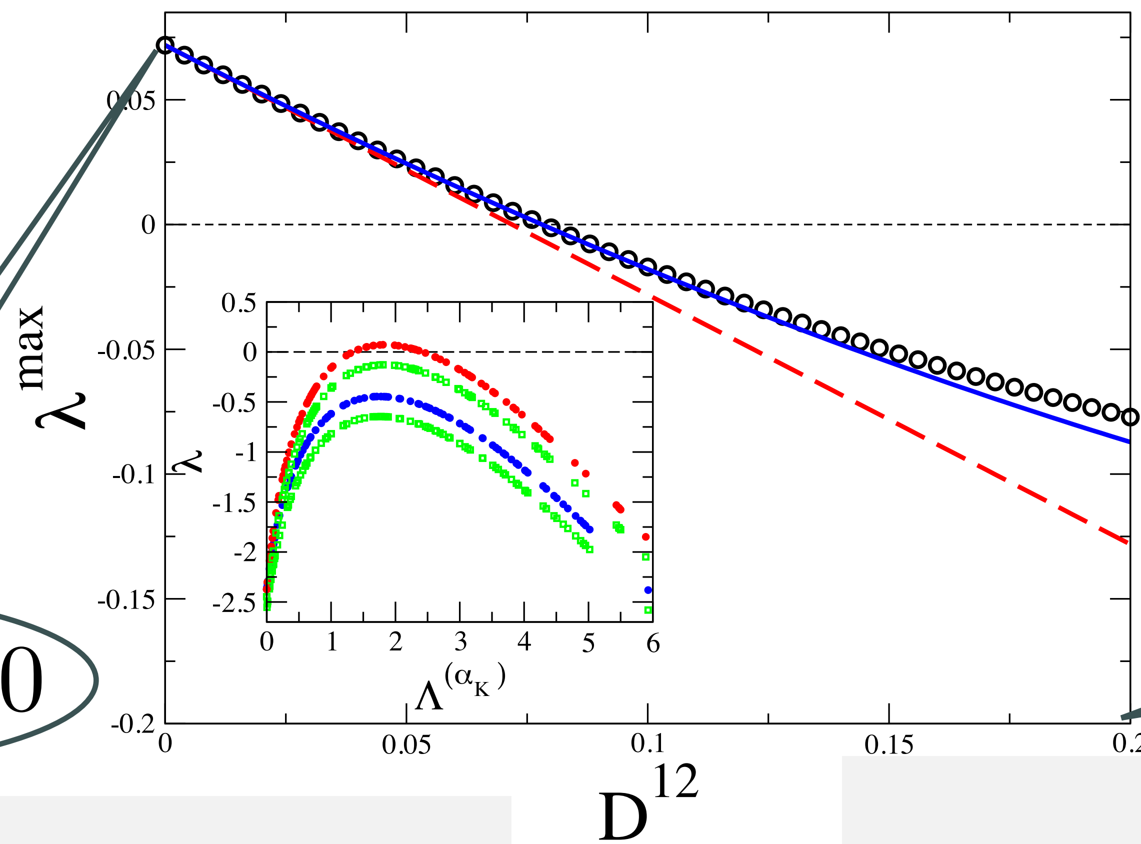
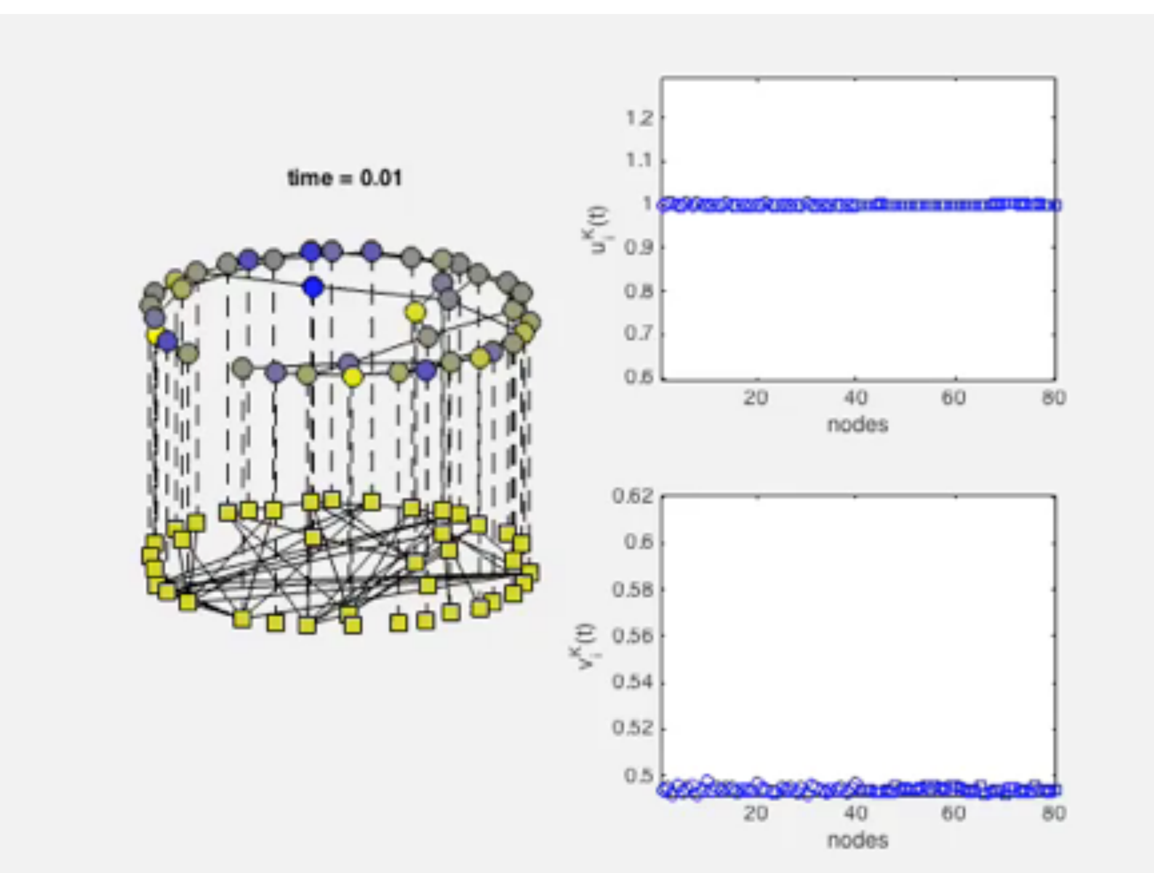
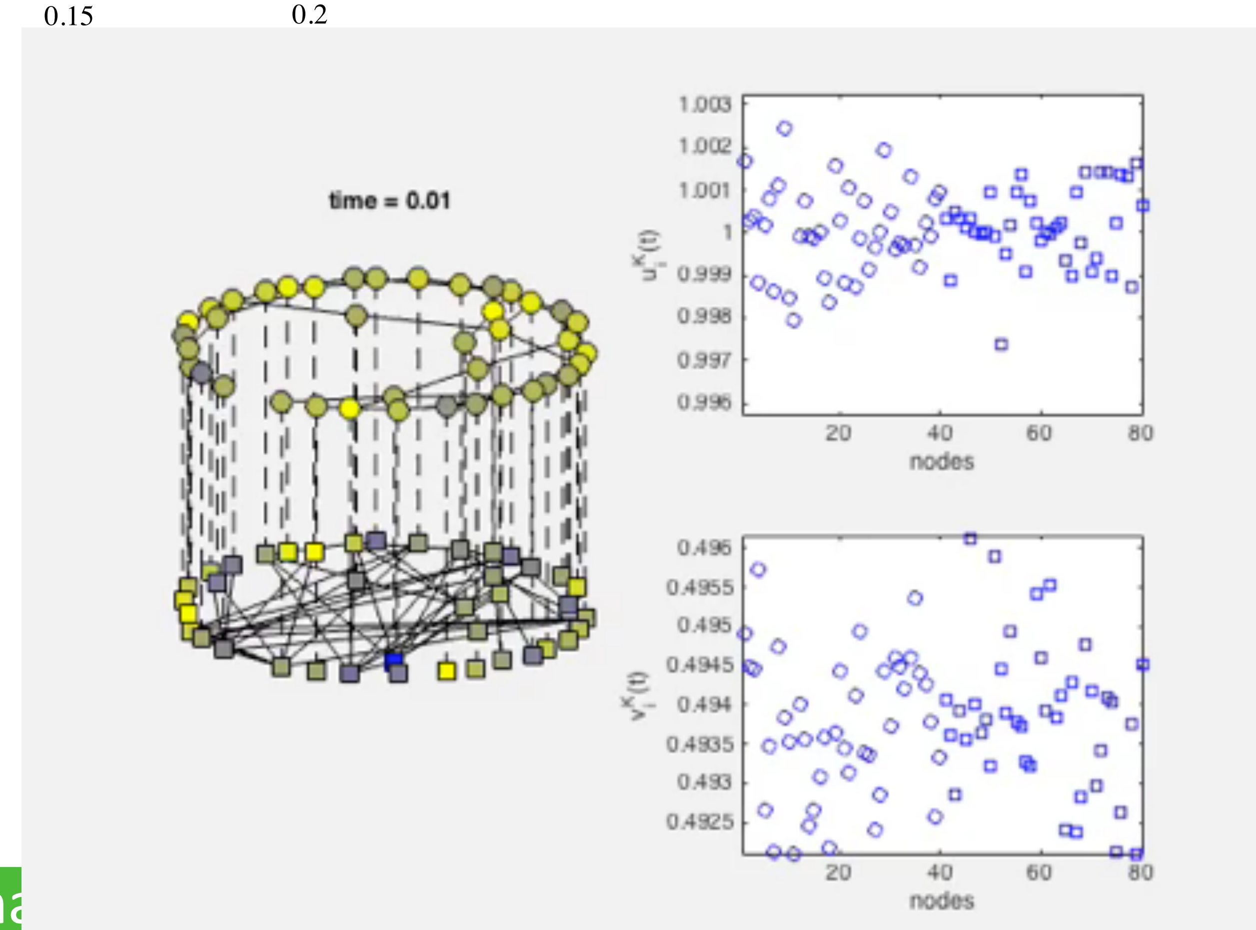
Small intra-layer diffusion case: destruction of patterns

$$D_v^{12} = D_u^{12} = 0$$

$$D_u^{12} = D_v^{12} = 0.2$$







Patterns localisation