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THE JUNGLE UNIVERSE AND ITS TWISTING SPECIES

by J. Perez[1], A. Füzfa[2], T. Carletti[2], L. Mélot[3], L. Guedezounme[4]

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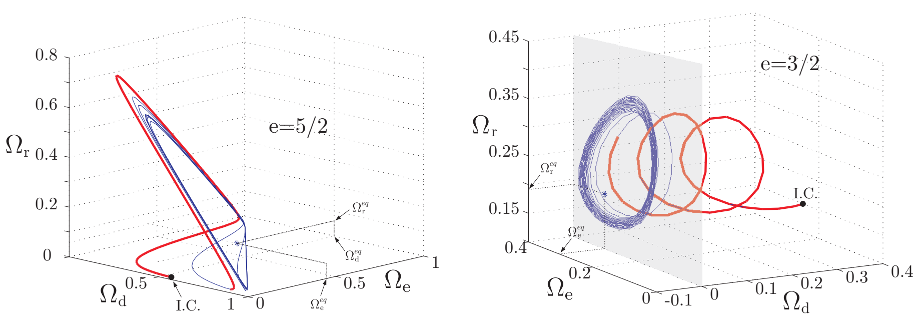

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The Jungle Universe and its Twisting Species

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Abstract. In this paper, we show that the dynamics of homogeneous and isotropic Friedmann-Lematre universes can be studied with population dynamics, in particular with the generalized Lotka-Volterra equation where the competitive species are the barotropic fluids filling the Universe. Without coupling between those fluids, Lotka-Volterra formulation offers a pedagogical way to interpret usual Friedmann-Lematre cosmological dynamics. When the cosmological fluids are coupled however, we establish new dynamical properties of Friedmann-Lematre universes consisting of cycles between its interacting components. This provides a new asymptotic behavior for cosmic expansion in presence of coupled species, beyond the standard de Sitter, Einstein-de Sitter and Milne cosmologies. Finally, we conjecture that chaos can appear for at least four interacting fluids.

1. Introduction

According to the cosmological principle, the geometry of the Universe is supposed to be homogeneous and isotropic at any given time, giving rise to the famous Friedmann-Lematre (FL) cosmology. In this picture, cosmological properties are subject to evolution and the whole cosmos is therefore the biggest dynamical system one can study. In particular, one of the questions that can be handled is the one of the fate of the Universe. The various scenari for the fate of the Universe in FL cosmologies have been popularized as Big Chill - when cosmic expansion is endless, Big Crunch - a final singularity of same nature than the Big Bang for spatially closed cosmologies with vanishing or small cosmological constant, or more recently Big Rip [1], when Universe's scale factor become infinite at a finite time in the future. Dynamical systems tools have allowed some important results in the question of future asymptotic behavior of cosmic expansion, for instance by demonstrating the existence of attracting regimes and scaling solutions in quintessence models [2, 3]. Solutions to cosmological dynamics consists of time evolution of density parameters associated to the barotropic fluids usually invoked to model matter contents of the universe. The fate of the Universe is completely related to its matter content. For example, Big-Rip singularity occurs when Universe contains the so-called "Phantom dark energy" associated to a barotropic fluid with equation of state $p = \omega\rho$ where the barotropic index $\omega < -1$.

In this paper, we present for the first time FL cosmological dynamics in the terms of interacting population dynamics and Lotka-Volterra system. When there are no interactions between constitutive fluids, this formulation allows to interpret those dynamics in a pedagogical way through one intuitive and simple formulation. The cosmological dynamics can then be seen as a competition between several species, each associated to one of the fluids filling the universe. Those species all compete for feeding upon the same resource which is spatial curvature. The usual asymptotic states of FL dynamics, de Sitter, Einstein-de Sitter and Milne universes, can all be seen as a particular equilibrium between cosmic species. This is the picture of the *Jungle Universe*. In addition this analogy with population dynamics allowed us to complete the set of possibilities for the fate of the Universe. This is achieved when we apply the general techniques of Lotka-Volterra systems to the case of directly coupled fluids. It appears a new asymptotic state for cosmological dynamics in which the coupled species endlessly alternate to dominate the energy content, leading for instance to transient acceleration phases. Those cycles explain the name of *twisting species*, as they take the cosmic expansion into an eternal dance.

The paper is structured as follow : in section 2 we show how to present FL cosmological dynamics as population dynamics; in section 3 we obtain the general formulation of the Jungle Universe dynamics; in section 4, we introduce direct interaction between two fluids and show under which conditions they can behave as twisting species; in section 5, we generalize the formulation to N directly coupled species and focus in particular to triads ($N = 3$) and quartets ($N = 4$); finally, we draw some conclusions in section 6.

Notation

In what follows, vectors are written bold faced (e.g. $\mathbf{r} \in \mathbb{R}^n$) and the associated coordinates in the canonical basis are denoted by the italic corresponding letters with an index (e.g.

$\mathbf{r} = (r_1, \cdots, r_n)^\top$).

2. Friedmann-Lemaître cosmology as generalized Lotka-Volterra dynamical systems

Taking into account a cosmological constant Λ, Einstein's equations of general relativity write

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = \chi T_{\mu\nu}$$

where $g_{\mu\nu}$ and $R_{\mu\nu}$ are respectively the metric and the Ricci tensors, R is the scalar curvature (contraction of the Ricci), $T_{\mu\nu}$ is the stress-energy tensor and $\chi = 8\pi G c^{-4}$. The general paradigm of standard cosmology consists of imposing Friedmann-Lemaître-Robertson-Walker metric as an isotropic and homogeneous description of the universe i.e.

$$\mathrm{d}s^2 = c^2 \mathrm{d}t^2 - a^2(t) \left[\frac{\mathrm{d}r^2}{1 - kr^2} + r^2 \left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2 \right) \right]$$

where $a(t)$ and k are respectively the scale factor and the curvature parameter, t and (r, θ, ϕ) being the synchronous time and usual spherical coordinates, respectively . If one assumes that this universe is filled by a perfect fluid of density ρ, pressure p and quadri-velocity field u_μ for which $T_{\mu\nu} = (\rho + c^{-2} p) u_\mu u_\nu - p g_{\mu\nu}$, it is well known that the dynamics of the universe are governed by Friedmann-Lemaître and conservation equations :

$$H^2 = \frac{8\pi G}{3} \rho + \frac{\Lambda c^2}{3} - \frac{kc^2}{3} \tag{1}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3} \left(\rho + 3\frac{p}{c^2} \right) + \frac{\Lambda c^2}{3} \tag{2}$$

$$\dot{\rho} = -3H \left(\rho + \frac{p}{c^2} \right) \tag{3}$$

where $H(t) = \dfrac{\dot{a}}{a}$ is the Hubble parameter and a dot over a quantity indicates a derivation with respect to the synchronous time t, the independent variable of the cosmological differential system. Both parameters k and Λ might be seen as fixing the spatial and intrinsic curvature of the geometry‡. Among the three above equations, only two are independent since all are related through the second Bianchi identities. The remaining two equations still include three unknown functions: $\rho(t)$, $p(t)$ and $a(t)$. This under-determination can be raised by introducing an equation of state for the matter fluids. For example, barotropic fluids are such that $p = \omega\rho$ where the constant ω is called the barotropic index. In a general physical way, this index ranges from $\omega_{\min} = -1$ for scalar field frozen in unstable vacuum to $\omega_{\max} = +1$ for stiff matter (e.g. free scalar field) where sound velocity equals to speed of light. In this paper we restrict our analysis to such barotropic physical fluids.

Following standard procedure, we rewrite the above equations in terms of density parameters for matter $\Omega_m = \frac{8\pi G\rho}{3H^2}$, cosmological constant $\Omega_\Lambda = \frac{\Lambda}{3H^2}$, curvature $\Omega_k = -\frac{k}{3a^2H^2}$ and deceleration parameter $q = -\frac{\ddot{a}a}{\dot{a}^2}$. Friedmann-Lemaître equations and energy conservation write for barotropic

‡ If one interprets the cosmological constant as the curvature associated to vacuum.

fluids therefore become

$$\begin{cases} 1 = \Omega_m + \Omega_\Lambda + \Omega_k \\ q = \frac{1}{2}\Omega_m \left(1 + 3\omega\right) - \Omega_\Lambda \\ \dot{\rho} = -3H\rho\left(1 + \omega\right) \end{cases}$$

Please note that the latter equation can be directly integrated for constant equation of state to give $\rho \sim a^{-3(1+\omega)}$.

Finally, we rewrite the above equations by changing the independent variable to the number of efoldinds $\lambda = \log(a)$ and noting $'$ for $\lambda-$derivatives, one gets

$$\begin{cases} 1 = \Omega_m + \Omega_\Lambda + \Omega_k \\ \Omega_m' = \Omega_m \left[-\left(1 + 3\omega\right) + \left(1 + 3\omega\right)\Omega_m - 2\Omega_\Lambda\right] \\ \Omega_\Lambda' = \Omega_\Lambda \left[2 + \left(1 + 3\omega\right)\Omega_m - 2\Omega_\Lambda\right] \end{cases}$$

The dynamics of the Friedmann-Lemaître universe is contained in the two last equations which form a differential system of generalized Lotka-Volterra [12, 13, 14] equations well known in population dynamics. As a matter or fact, introducing the dynamical vector $\mathbf{x} = \left(\Omega_m, \Omega_\Lambda\right)^\top$and the capacity vector $\mathbf{r} = \left(-\left(1 + 3\omega\right), 2\right)^\top$, for $i = 1, 2$ we have $x_i' = x_i\ f\left(x_i\right)$ where the vector function $\mathbf{f}(\mathbf{x}) = \mathbf{r} + A\mathbf{x}$ is linear in the variables x_i, the community matrix A being defined by

$$A = \begin{bmatrix} (1 + 3\omega) & -2 \\ (1 + 3\omega) & -2 \end{bmatrix}$$

This formulation allows us to assimilate the dynamics of Friedmann-Lemaître universes to those of a competition between species, represented by Ω_m and Ω_Λ, for the resources in Ω_k. This point of view is not anecdotal and will reveal a lot of benefit : such equations are very well known to the dynamical system specialist, it allows a lot of intuitive non trivial results, establish an analogy that will help us deriving new cosmological behavior for coupled models besides of providing a pedagogic and interesting insight on cosmic expansion.

First of all, it is easy to see that orbits cannot cross the $\Omega_m = 0$ or $\Omega_\Lambda = 0$ axes which are orbits themselves. As the matrix A is clearly not invertible, equilibrium points must lie on axis. In particular as denoted by [4] or [5] using a slightly different dynamical system, there exists 3 equilibria which are Milne universe $\mathbf{x}_0 = (0, 0)$, Einstein-de Sitter universe $\mathbf{x}_1 = (1, 0)$ and de Sitter universe $\mathbf{x}_2 = (0, 1)$. Using the large knowledge of such systems from bio-mathematics (e.g. [6],[7]) the $\mathbf{r}$ vector contains the intrinsic birth or death rates of the species. The dynamics of competitive Lotka-Volterra systems with such a degenerate matrix is well known:

- If the initial condition is located in the positive quadrant $Q^+ = \{\Omega_m > 0\} \times \{\Omega_\Lambda > 0\}$ then $\mathbf{x} \to \mathbf{x}_2$ when t or λ goes to infinity, the reason of this attractive character of the de Sitter universe is uniquely contained in the fact that $r_2 \geq r_1$ for all physical values of the barotropic index ω. If we extend values of ω considering phantom dark energy instead of pressureless matter by letting $\omega < -1$ the attractor become the (phantom DE-dominated) Einstein-de Sitter universe ($\mathbf{x}_1$) simply because in this case $r_1 \geq r_2$. This is obvious since in this case the energy density of the phantom DE grows like a power-law with the scale factor ($\rho_{DE} \sim a^{-3(1+\omega)}$ where $\omega < -1$), therefore asymptotically dominating the constant density associated to the cosmological term.

- If the initial condition lies on the Ω_m axis the attractor is the Einstein-de Sitter universe if $\omega < -\frac{1}{3}$ and Milne universe ($\mathbf{x}_0$) if $\omega \geq -\frac{1}{3}$. This is obvious since, in the absence of a cosmological constant ($\Omega_\Lambda = 0$), the competition is left between matter and curvature energy densities, the latter decreasing as a^{-2}. Therefore, asymptotic dominance of matter is only possible when $\omega < -1/3$, so that the related density can eventually dominate (since it scales as $\rho_m \sim a^{-3(1+\omega)}$).
- If the initial condition lies on the Ω_Λ axis the attractor is the de Sitter universe for any values of ω. Once again, this is obvious since asymptotically the constant energy density of the cosmological term will dominate the decreasing energy density related to the curvature.

These results are well known and presented in a slightly different manner in [4] or [5]. The new point here is the dynamical population formulation of the problem and interesting results will be derived through usual techniques in dynamical system theory. We will also present new cosmological consequences on coupled models which are directly inspired by the analogy with evolution of populations in competition. One possibility consists of investigating how far the natural cyclic orbits appearing usually in population dynamics could appear in standard cosmology. This is the object of the next section.

3. Multi-components Friedman-Lemaître universes : jungle universes.

In the latter section we have presented the generalized Lokta-Volterra formulation for the dynamics of usual Friedmann-Lemaître universe with non-vanishing cosmological constant. In particular we have only considered one simple barotropic fluid characterized by a given value of ω. We can generalize this situation to the more complicated yet realistic case where the universe is filled by several kinds of barotropic fluids without any direct interactions. In this section, we consider for example baryonic matter (b−indiced and for which $\omega_\mathrm{b} = 0$) and radiation (r−indiced and for which $\omega_\mathrm{r} = \frac{1}{3}$). It is well known that the repulsive feature obtained with a positive cosmological constant can also advantageously be obtained through some dark energy fluid component (e−indiced) associated to a barotropic index $\omega_\mathrm{e} \in [-1, -1/3]$; the cosmological constant term could then be obtained taking $\omega_\mathrm{e} = -1$. In the following, roman indexes refer to the fluid component considered.

The cosmological term in Friedmann-Lemaître equations can therefore be removed, introducing the densities $\Omega_\mathrm{x} = \frac{8\pi G\rho_\mathrm{x}}{3H^2}$ for $\mathrm{x} = \mathrm{b}, \mathrm{r}$ and e including the conservation of each kind of fluids they write

$$
\begin{aligned}
&1 = \Omega_\mathrm{b} + \Omega_\mathrm{r} + \Omega_\mathrm{e} + \Omega_\mathrm{k} \\
&2q = \Omega_\mathrm{b} + 2\Omega_\mathrm{r} + (1 + 3\omega_\mathrm{e})\,\Omega_\mathrm{e} \\
&(\ln \rho_\mathrm{x})' = -3(1 + \omega_\mathrm{x}) \quad \text{for } \mathrm{x} = \mathrm{b}, \mathrm{r} \text{ and e;}
\end{aligned}
$$

A basic calculus shows that $(\ln H)' = -q - 1$ hence Friedmann-Lemaître equations write

$$
\frac{\Omega_\mathrm{x}'}{\Omega_\mathrm{x}} = (\ln \Omega_\mathrm{x})' = \Omega_\mathrm{b} + 2\Omega_\mathrm{r} + (1 + 3\omega_\mathrm{e})\,\Omega_\mathrm{e} - 3\omega_\mathrm{x} - 1 \quad \text{for } \mathrm{x} = \mathrm{b}, \mathrm{r} \text{ and e}
$$

This three dimensional differential system is again of Lotka-Volterra form with, this time however, a fully degenerate community matrix. The dynamics is then always governed by the capacity vector $\mathbf{r} = [-1, -2, -3\omega_\mathrm{e} - 1]$ which actually rules the asymptotic behavior. Besides of the origin, there is now one additional equilibrium on each axis and if $\mathbf{r}$ possesses a component which is greater than all others, the corresponding equilibrium with this component maximal is globally stable over the positive orthant. This smart result is sufficient to claim that dark energy (for which $\omega_{\rm e} \in [-1, -1/3]$) correspond to this $\mathbf{r}$ maximal components and then the universe such that $\Omega_{\rm b} = \Omega_{\rm r} = 0$ and $\Omega_{\rm e} = 1$ is globally stable out from axis $\Omega_{\rm b} = 0$ and $\Omega_{\rm r} = 0$.

This three dimensional situation is readily generalizable to any number of non interacting fluids each governed by a separated conservation equation. The dynamical behavior is asymptotically always the same : the system evolves like a competitive one in which all species (predators) are fed by the same prey (which is curvature...). Asymptotically and out of axis, only one species survives, the one which possesses the greater value of $-3\omega_{\rm x} - 1$. This species is always the dark energy fluid in our physical hypotheses $\omega \in [-1, 1]$. Once the Universe is filled with even a small amount of dark energy, there is no way it cannot dominate forever the fate of the cosmos. This is Jungle Law for a jungle universe. Fortunately, this will cease to be true, as we shall see in the next section, if dark energy is not so dark, but exchanges energy with the matter component.

4. Cooperative Universes

4.1. General dynamics with coupling

In the last sections we have presented a new way to express the dynamics of Friedmann-Lemaître universes using generalized Lotka-Volterra differential system theory. This also offers new perspectives in determining cosmological analogues of specific cases in competitive dynamics. It is well known that the generic dynamics of such systems contains limit cycles or periodic orbits. We will describe in this section how direct coupling can be used to bring such a behavior in the context of cosmology.

When the fluids filling the universe are not interacting with each other, the community matrix of the generalized Lotka-Volterra system must have the same rows and then must be fully degenerated. In order to make its rank greater than one, we must introduce coupling between species, i.e. interactions between cosmological fluids. On the other hand, this kind of interactions is broadly used in cosmology, with the coupling between inflaton and radiation during reheating (e.g., [8]) or the one between dark matter and dark energy (e.g., [2, 9, 10]), or even the decay of heavy matter particles like WIMPS into light relativistic particles (e.g., [11]). Modern cosmology make strong use of coupled fluids for a variety of purposes, therefore making this study of coupled models in terms of Lotka-Volterra systems of first heuristic interest.

In order to show the phenomenon we will present in this section the situation where the universe contains radiation, baryonic matter, dark matter (d−indiced)§, dark energy and we suppose a coupling between the two dark components. This constitutes a coupled quintessence scenario [10]. On one hand, it is necessary to preserve the global energy conservation as imposed by Noether theorem and Poincare invariance, energy transfer must compensate in the global energy balance. Hence, at each time, the part of the energy taken by the first component must be given to the other to which it couples. To achieve this, conservation equations for two coupled

§ Although both are pressureless with $\omega = 0$, we split both to allow for different couplings.

dark fluids must be of the following form:

$$\begin{cases} \dot{\rho}_{\rm d} = -3H\rho_{\rm d}\left(1+\omega_{\rm d}\right) + \mathcal{Q} \\ \dot{\rho}_{\rm e} = -3H\rho_{\rm e}\left(1+\omega_{\rm e}\right) - \mathcal{Q} \end{cases}$$

where $\mathcal{Q}$ represents the energy transfer. This coupling leaves unchanged the global energy-momentum conservation, it is then invisible in standard general relativity and it glimpses at (micro-)physics describing dark components of the universe. In literature, one usually finds that this energy transfer is arbitrarily expressed as a linear combination of the dark sector densities:

$$\mathcal{Q} = A_{\rm d}\rho_{\rm d} + A_e\rho_{\rm e}$$

where the coefficients are either proportional to Hubble parameter H either constant (see [2, 9, 10]). In this paper, we introduce a new non-linear parametrization of the energy transfer that allows us matching the coupled model to a general Lotka-Volterra system. This ansatz is given by

$$\mathcal{Q} = \frac{8\pi G}{3H}\varepsilon\rho_{\rm e}\rho_{\rm d} \tag{4}$$

where the coupling parameter ε is a positive constant.

Since the Raychaudhuri equation (2) and consequently $(\ln H)'$ are left unchanged by the introduction of such couplings‖, but we have now

$$\begin{aligned} (\ln\Omega_{\rm d})' &= (\ln\rho_{\rm d})' + 2q + 2 \\ &= \Omega_{\rm b} + (1+3\omega_{\rm d})\,\Omega_{\rm d} + 2\Omega_{\rm r} + (\varepsilon + 1 + 3\omega_{\rm e})\,\Omega_{\rm e} - (3\omega_{\rm d+1}) \end{aligned}$$

and

$$\begin{aligned} (\ln\Omega_{\rm e})' &= (\ln\rho_{\rm e})' + 2q + 2 \\ &= \Omega_{\rm b} + (1+3\omega_{\rm d} - \varepsilon)\,\Omega_{\rm d} + 2\Omega_{\rm r} + (1+3\Omega_{\rm d})\,\Omega_{\rm d} - (3\omega_{\rm e+1})\ . \end{aligned}$$

The other two remaining equations for $\Omega_{\rm b}$ and $\Omega_{\rm d}$ are not affected by the dark coupling. In order to reduce the number of parameters we will place in the case where the non baryonic dark matter is non-relativistic and pressureless, i.e. $\omega_{\rm d} = 0$, and the dark energy is a cosmological constant, i.e. $\omega_{\rm e} = -1$. We therefore focus on late periods of cosmological history, and not on radiation-dominated era. It is important to notice that all the dynamical properties of the solution that we are going to exhibit will be independent of these hypotheses provided that $\omega_{\rm e}$ stay lesser than $-1/3$ and $\omega_{\rm d}$ greater than $-1/3$. With this coupling and under these last hypotheses the generalized Lotka-Volterra equations associated to isotropic, homogeneous and barotropic fluid filled universe for the dynamical variable $\mathbf{x} = (\Omega_{\rm b}, \Omega_{\rm d}, \Omega_{\rm r}, \Omega_{\rm e})^\top$ are defined by a capacity vector $\mathbf{r}$ and a community matrix A such that

$$A = \begin{bmatrix} 1 & 1 & 2 & -2 \\ 1 & 1 & 2 & \varepsilon - 2 \\ 1 & 1 & 2 & -2 \\ 1 & 1-\varepsilon & 2 & -2 \end{bmatrix} \quad \text{and } \mathbf{r} = \begin{bmatrix} -1 \\ -1 \\ -2 \\ 2 \end{bmatrix} \tag{5}$$

‖ This is so since gravity is still minimally coupled to matter fluids.

As desired this matrix is not fully degenerate but has a rank 2. This dynamic is characterized by five equilibria $\tilde{\mathbf{x}}^0 = (0,0,0,0)^\top$, $\tilde{\mathbf{x}}^1 = (0,0,1,0)^\top$, $\tilde{\mathbf{x}}^2 = (0,0,0,1)^\top$, $\tilde{\mathbf{x}}^3 = (1-\alpha,\alpha,0,0)^\top$ with $\alpha \in \,]0,1]$ and $\tilde{\mathbf{x}}^4 = (0,\varepsilon^{-1},0,2\varepsilon^{-1})^\top$ the first four being globally unstable while the last is by far the most interesting. When the coupling is low (precisely $\varepsilon \in \,]0,3]$), $\tilde{\mathbf{x}}^4$ is generally unstable¶, nevertheless when the coupling become stronger ($\varepsilon > 3$), the equilibrium point $\tilde{\mathbf{x}}^4$ is no more hyperbolic and two complex eigenvalues with no real part occur in the spectrum of the linearized dynamics around $\tilde{\mathbf{x}}^4$: a precise analysis of the dynamical behavior of the system is then required. In order to do this we have decomposed the job into two parts : in a first step we have restricted the analysis to the dark plane $(\Omega_\mathrm{d},\Omega_\mathrm{e})$ where we have found cycles, and, in a second step we have shown that this dark plane is attractive for all orbits whose initial conditions belong to the hyper-tetrahedron

$$T_4 = \{\Omega_\mathrm{b} > 0\} \cup \{\Omega_\mathrm{d} > 0\} \cup \{\Omega_\mathrm{r} > 0\} \cup \{\Omega_\mathrm{e} > 0\} \cup \{\Omega_\mathrm{b} + \Omega_\mathrm{d} + \Omega_\mathrm{r} + \Omega_\mathrm{e} < 1\}\,. \quad (6)$$

Finally, we will propose some numerical analysis of the whole dynamics in order to test these general properties.

4.2. Cyclicity of orbits in the dark plane

In an pedagogical objective and as we deal with a generalizable example in which $\omega_\mathrm{d} = 0$ and $\omega_\mathrm{e} = -1$, we will present the details of the construction of the orbits properties. In the so-called dark plane ($\Omega_\mathrm{b} = \Omega_\mathrm{r} = 0$) let us define shorter notations $x = \Omega_\mathrm{d}$ and $y = \Omega_\mathrm{e}$. The dynamics is then governed by the system

$$\begin{cases} x' = x\left[-1 + x + (\varepsilon - 2)\, y\right] \\ y' = y\left[2 + (1-\varepsilon)\, x - 2y\right] \end{cases}$$

hence

$$\mathbf{r} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & \varepsilon - 2 \\ 1 - \varepsilon & -2 \end{bmatrix}$$

As mentioned below there is a unique equilibrium in the positive quadrant, namely $(\tilde{x},\tilde{y}) = (\varepsilon^{-1}, 2\varepsilon^{-1})$. Using a bit of intuition and inspired by dynamical population analysis (e.g. [6]) one can use the function $V_\varepsilon(x,y) = x^\alpha y^\beta (a + bx + cy) + V_0$ where α and β are functions of ε; a, b, c and V_0 are four constants, all being determined in order that V_ε becomes a Lyapunov function. As A is now invertible choosing $(\alpha,\beta)^\top = A^{-1}\mathbf{r}$ i.e. $\alpha = 2(\varepsilon-3)^{-1}$ and $\beta = (\varepsilon-3)^{-1}$, it is easy to check that

$$V_\varepsilon' = x^\alpha y^\beta \left[xy\varepsilon\,(b-c) - (a+b)\,x + 2\,(a+c)\,y\right]$$

Hence, choosing finally $a = -c$, $b = c$ and $V_0 = -\tilde{x}^\alpha \tilde{y}^\beta (a + b\tilde{x} + c\tilde{y})$ one can verify that

- The function V_ε vanishes when $x = \tilde{x}$ and $y = \tilde{y}$;
- If $x \neq \tilde{x}$ and $y \neq \tilde{y}$ then $V_\varepsilon(x,y) > 0$;
- The derivative V_ε' is vanishing for any value of x and y.

¶ For the particular case we have chosen this instability develops thanks to the radiative components of the fluid. But it is clearly a particular case of a general behavior.

Hence the function

$$V_{\varepsilon}(x,y)=x^{\frac{2}{\varepsilon-3}}y^{\frac{1}{\varepsilon-3}}(x+y-1)+\left(\frac{2^{2/3}}{\varepsilon}\right)^{\frac{3}{\varepsilon-3}}\left(1-\frac{3}{\varepsilon}\right)$$

is a Lyapunov function for this dynamics and orbits are confined on curves $V_{\varepsilon}(x,y)=\mu$ where μ is any positive constant, that is V_{ε} is a first integral of the system. Such curves are plotted on figure 1 for the generic values^{+} $\varepsilon=4$.

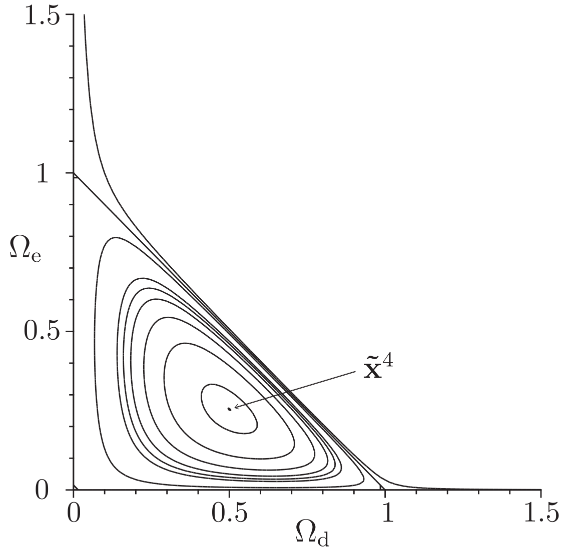

Figure 1. Contour levels of $V_4(x,y)$

It must be noted that when $\Omega_{\rm d}+\Omega_{\rm e}<1$, the dynamics in the dark plane is periodic as all the contour levels of V_{ε} are closed and all solutions are maximal. The corresponding cosmological solution correspond to endless oscillations of the density parameters $(\Omega_{\rm d},\Omega_{\rm e})$ who forever compete with each other for ruling the curvature parameter. Cosmic expansion is in this case an eternal sequence of transient acceleration (when DE dominates) and deceleration (when DM dominates) phases.

Solutions such that $\Omega_{\rm d}+\Omega_{\rm e}>1$ are unbounded. They correspond to spatially closed universes, since $\Omega_k<0$, in which cosmic expansion can reverse into contraction at some stage, leading to $H=0$ and consequent singularities in all density parameters. The present formalism with monotonically growing $\lambda=\ln(a)$ cannot extrapolate beyond in vanishing H toward cosmic contraction $H<0$, since this would imply decreasing λ.

4.3. Attractiveness of the dark plane

We now turn our attention to the behavior of orbits whose initial conditions are not in the dark plane but have non vanishing components in $\Omega_{\rm b}$ and/or $\Omega_{\rm r}$. Intuitively one could claim that these components are going to vanish because the eigenvalues associated to them are negative, but as the two others, associated to the dark components, are purely imaginary, the equilibrium

$^{+}$ Let us observe that for $\varepsilon>3$ each ε-family of contour levels of V_{ε} are topologically equivalent.

is no longer a hyperbolic one hence Hartmann-Groβman theorem says that the linear analysis is not sufficient to have a complete description of the system behavior. However, even if the invariant manifold methods cannot be straightforwardly used, because the centre manifold is infinitely flat at $\tilde{\mathbf{x}}^4$, we are able, using dynamical systems tools, to prove the attractiveness of the dark plane, for all orbits whose initial conditions belong to the hyper-tetrahedron (6). A detailed proof of the latter statement will be provided in the appendix A.

4.4. Numerical illustration

As we have obtained a general proof of the attractiveness of the dark plane, we give only a simple numerical illustration of this fact. We have numerically solved the dynamical system $(\ln(\mathrm{x}))' = \mathrm{r} + A\mathrm{x}$ with $\mathbf{x} = (\Omega_\mathrm{b}, \Omega_\mathrm{d}, \Omega_\mathrm{r}, \Omega_\mathrm{e})^\top$, the community matrix and the capacity vector defined in (5) with $\varepsilon = 4$. Considering various initial conditions $\mathbf{x}_0$ we always recover an exponential convergence to the dark plane when $\mathbf{x}_0$ has non vanishing first and third components. The figure 2 illustrate such a behavior: from the initial condition $\mathbf{x}_0 = (0.3008, 0.2683, 0.0418, 0.2983)^\top$, which belongs to the stable hyper-tetrahedron, we have 3D-plotted the dynamical evolution of the vector $(\Omega_\mathrm{d}, \Omega_\mathrm{e}, \Omega_\mathrm{b} + \Omega_\mathrm{r})^\top$. As expected the third component vanishes and the two others are catched by a contour level of V_4. View from the top in the right part of the figure 2 is particularly explicit about this last fact.

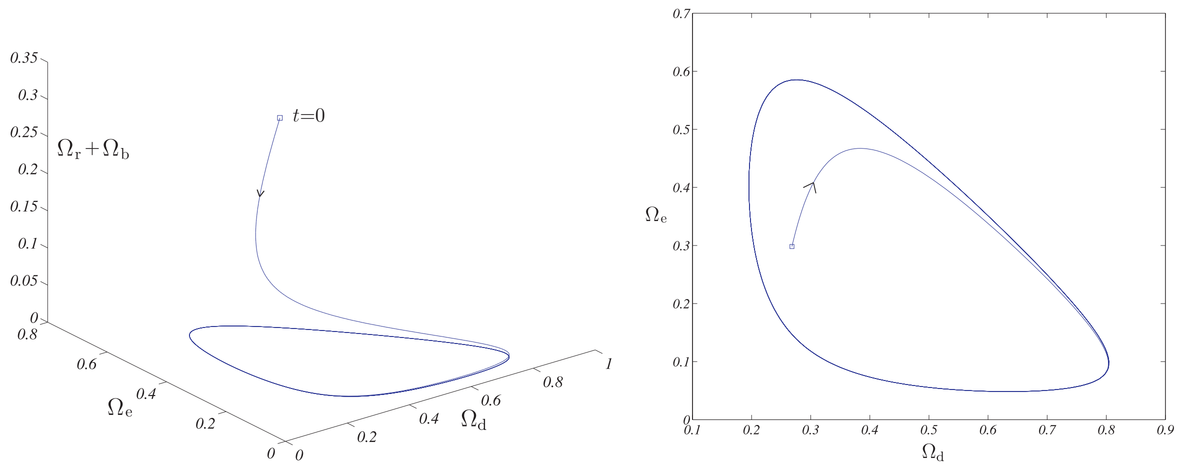

Figure 2. Time evolution of the orbit inside T_4. Left panel: 3D plot of the orbit (the vertical axis is $\Omega_\mathrm{r}(t) + \Omega_\mathrm{b}(t)$). Right panel: 2D projection on the $(\Omega_\mathrm{d}, \Omega_\mathrm{e})$ plane. Parameter and initial conditions: $\varepsilon = 4$, $\Omega_\mathrm{d}(0) = 0.2683$, $\Omega_\mathrm{e}(0) = 0.2983$, $\Omega_\mathrm{r}(0) = 0.0418$ and $\Omega_\mathrm{b}(0) = 0.3008$.

5. General correspondence between coupled models and Lotka-Volterra competitive dynamics

In the previous section, we have shown that the coupling between two components of the universe can make appear a new kind of dynamics of FL universes. We propose to call such cosmological components twisting species since the special example proposed in the last section represent an eternal dance between dark energy and dark matter. We will shown now that such a behavior can be generalized introducing more couplings.

In this section, we extend the previous discussion to a set of N inter-coupled cosmological species and establish the correspondence with general formulation of competitive Lotka-Volterra models. The goal here is therefore to rewrite the evolution, with the variable $\lambda = \ln(a)$, of cosmological density parameters of interacting fluids under the following Lotka-Volterra form :

$$\mathbf{x}' = \mathrm{diag}(\mathbf{x})\mathbf{f}(\mathbf{x}) \quad \text{with}\, \mathbf{x} \in \mathbb{R}^n \tag{7}$$

where diag($\mathbf{x}$) is the diagonal matrix with $\mathbf{x}$ on its diagonal, the ith component of the vector $\mathbf{x}$ denotes the population of the ith species, $\mathbf{f}(\mathbf{x}) = \mathbf{r} + A\mathbf{x}$ is the previously defined linear function which combines the capacity vector $\mathbf{r}$ and the community matrix A. Each coupled fluid characterized by energy density ρ_i, equation of state parameter ω_i and obey the following modified conservation equation:

$$\dot{\rho}_i + 3H\rho_i(1+\omega_i) = \mathcal{Q}_i \; ; \; i = 1, \cdots, N \tag{8}$$

with the energy balance condition imposing that

$$\sum_{i=1}^{N} \mathcal{Q}_i = 0 \tag{9}$$

where the interaction terms $\mathcal{Q}_i$ take the form of a combination of the involved energy densities:

$$\mathcal{Q}_i = \sum_{j=1}^{N} \beta_{ij}\rho_j. \tag{10}$$

Defining the density parameters $\Omega_i = \frac{8\pi G\rho_i}{3H^2}$, and recalling that the deceleration parameter can be written as

$$q = -\frac{\ddot{a}a}{\dot{a}^2} = \frac{1}{2}\sum_{i=1}^{N}\Omega_i(1+3\omega_i)$$

then Eq. (8) becomes

$$\dot{\Omega}_i = \frac{8\pi G\mathcal{Q}_i}{3H^2} + H\Omega_i\left(2 - 3(1+\omega_i) + \sum_{j=1}^{N}\Omega_j(1+3\omega_j)\right). \tag{11}$$

To rewrite the above equation under Lotka-Volterra form, it is then mandatory to set

$$\mathcal{Q}_i = \sum_{j=1}^{N}\beta_{ij}\rho_j \equiv H\Omega_i\sum_{j=1}^{N}\varepsilon_{ij}\rho_j \tag{12}$$

or, equivalently that the coefficients β_{ij} are no longer constant but are given by

$$\beta_{ij} = H\Omega_i\varepsilon_{ij}$$

with ε_{ij} arbitrary parameters to be specified further. Lotka-Volterra dynamics therefore requires non-linear interaction terms. Given Eq. (12), one can directly rewrite Eq.(11) under Lotka-Volterra form (7) with the following glossary:

$$\begin{aligned}
x_i &= \Omega_i \\
(\cdot)' &= \frac{d(\cdot)}{d\ln(a)} \\
r_i &= -(1+3\omega_i) \\
A_{ij} &= 1 + 3\omega_j + \varepsilon_{ij}
\end{aligned} \tag{13}$$

The energy balance constraint Eq.(9) with the hypothesis (12) now reduces to

$$\sum_{i=1}^{N} \Omega_i \left(\sum_{j=1}^{N} \varepsilon_{ij} \Omega_j \right) = 0 \tag{14}$$

which imposes that the interaction parameters ε_{ij} are *antisymmetric*:

$$\varepsilon_{ij} = -\varepsilon_{ji} \ ; \ \varepsilon_{ii} = 0\cdot$$

In the context of cosmology we find solution of this ODE system in the hyper-tetraedron

$$\mathcal{T} = \left\{ 1 > \sum_{i=1}^{N} x_1 \right\} \bigcap_{i=1}^{N} \{x_i > 0\}$$

The generalization of the results obtained in the previous section show that ODE system (7) has generically a lot of equilibria but we are interested only by the ones who haven't vanishing component (i.e. the ones not lying on an axis). These "interesting" equilibria are $\tilde{\mathbf{x}}$ such that $A\tilde{\mathbf{x}}+\mathbf{r} = \mathbf{0}$. We can now apply this general formulation to the case of several interacting species.

5.1. Two species in interaction

This case $N = 2$ has been treated in details in section 3 for specific values of the equation of state parameters $(\omega_1, \omega_2) = (0, -1)$ and serves here as a validation of the glossary (13). Setting $\varepsilon_{12} = -\varepsilon_{21} \equiv \varepsilon$ the unique non-vanishing component of the interaction tensor ε_{ij}, we obtain after some computation the following equilibria of the cosmological Lotka-Volterra system

$$\Omega_1^{eq} = -\frac{3\omega_2 + 1}{\varepsilon} \tag{15}$$

$$\Omega_2^{eq} = +\frac{3\omega_1 + 1}{\varepsilon} \tag{16}$$

which reduces to the equilibria $(2/\varepsilon, 1/\varepsilon)$ of section 3 when $\omega_1 = 0$ and $\omega_2 = -1$. These equilibria are density parameters in open universes $(\Omega_k < 1)$ and then must satisfy $0 < \Omega_i^{eq} < 1$. This condition constrains the choice of ε once the choice of the nature of the interacting fluids has been chosen by fixing ω_2 and ω_1.

5.2. The interplay between three coupled species : Jungle triads

Let us set $\varepsilon_{12} = e_1$, $\varepsilon_{13} = e_2$ and $\varepsilon_{23} = e_3$ and compute the corresponding equilibria to find

$$\begin{aligned} \Omega_1^{eq} &= +\frac{e_3 - 3\omega_2 + 3\omega_3}{e_1 - e_2 + e_3} \\ \Omega_2^{eq} &= -\frac{e_2 - 3\omega_1 + 3\omega_3}{e_1 - e_2 + e_3} \\ \Omega_3^{eq} &= +\frac{e_1 - 3\omega_1 + 3\omega_2}{e_1 - e_2 + e_3} \end{aligned} \tag{17}$$

Let us remark that in all cases of fluids and coupling we have $\sum_{i=1}^{3} \Omega_i^{eq} = 1$. This fact seems generic for odd values of the number of interacting fluids. If we now impose the fact that the density parameters are comprised between 0 and 1 $(0 < \Omega_i^{eq} < 1)$ the constraint on interaction parameters e_1, e_2, e_3 is very complicated, but allows a lot of possibilities. Let us illustrate this case with an example. We consider that the three fluids are made of (1) non-relativistic matter $\omega_1 = 0$, $(x_1 = \Omega_{\rm d})$; (2) dark energy $\omega_2 = -1$, $(x_2 = \Omega_{\rm e})$ and (3) some relativistic particles $\omega_3 = 1/3$, $(x_3 = \Omega_{\rm r})$ all coupled with interaction parameters $e_1 = e_2 = e$ and $e_3 = \varepsilon$. The corresponding equilibria are

$$\Omega_{\rm d}^{eq} = \frac{4+\varepsilon}{\varepsilon}, \quad \Omega_{\rm e}^{eq} = -\frac{1+e}{\varepsilon} \quad \text{and } \Omega_{\rm r}^{eq} = \frac{e-3}{\varepsilon}$$

Providing $\varepsilon < -4$ and $e \in [-1, 3]$ equilibria are cosmologically acceptable. Choosing for example $\varepsilon = -8$, the spectrum of the jacobian matrix near the equilibrium is composed by a real number $\lambda = 1 - \frac{e}{2}$ and two purely imaginary and complex conjugated numbers $\lambda_\pm = \pm\frac{i}{2}\sqrt{2\,|(e+1)(e-3)|}$. When $e \in [-1, 2]$, as $\lambda > 0$ the system twists outward $(0, \Omega_{\rm e}^{eq}, \Omega_{\rm r}^{eq})$ staying in the corresponding 3-tetraedron, collapsing on the $\Omega_{\rm d} = 0$ plane. When $e \in [2, 3]$, as $\lambda < 0$ the system twists toward a limit cycle contained in a plane of non vanishing density and including the equilibrium. These results are illustrated on the figure 3.

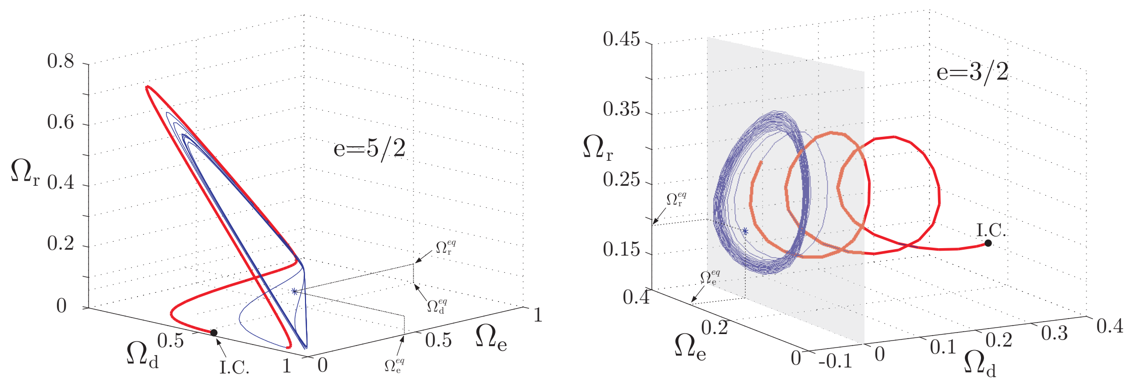

Figure 3. Evolution of the three coupled density parameters, in the 3D phase space. The beginning of the orbit is overlined. Initial condition is indicated by a black dot. Relevant equilibria are indicated by a star.

5.3. Jungle quartets

With $N = 4$, the number of free parameters in the scheme (10 in total with 6 for interactions and 4 for equations of state) is too high to be fully constrained by requirements of positiveness and boundedness of density parameters for instance. As for $N = 2$, the positions of the equilibria once again depend on all parameters. If we set $\varepsilon_{12} = e_1$, $\varepsilon_{13} = e_2$, $\varepsilon_{14} = e_3$, $\varepsilon_{23} = e_4$, $\varepsilon_{24} = e_5$ and $\varepsilon_{34} = e_6$, we find that the positions of the equilibria are given by

$$\begin{aligned}
\Omega_1^{eq} &= -\frac{e_4 - e_5 + e_6 + 3(e_4\omega_4 - e_5\omega_3 + e_6\omega_2)}{e_1e_6 - e_2e_5 + e_4e_3} \\
\Omega_2^{eq} &= +\frac{e_2 - e_3 + e_6 + 3(e_2\omega_4 - \omega_3e_3 + e_6\omega_1)}{e_1e_6 - e_2e_5 + e_4e_3} \\
\Omega_3^{eq} &= -\frac{e_1 - e_3 + e_5 + 3(e_1\omega_4 + \omega_1e_5 - \omega_2e_3)}{e_1e_6 - e_2e_5 + e_4e_3} \\
\Omega_4^{eq} &= +\frac{e_1 - e_2 + e_4 + 3(e_1\omega_3 - e_2\omega_2 + e_4\omega_1)}{e_1e_6 - e_2e_5 + e_4e_3}
\end{aligned} \tag{18}$$

Since this system of 4 cosmological coupled species is equivalent to 4D Lotka-Volterra system, chaos can emerge [15] for specific choices of parameters in a so-called normal system where all r_i are positive, which means among cosmological fluids with $\omega_i < -1/3$. As an illustration we propose a double twist in a universe filled by two kinds of dark energy and two kinds of dark matter all interacting. We choose $\omega_1 = -1$, $(x_1 = \Omega_{\mathrm{e},1})$; $\omega_2 = 0$, $(x_2 = \Omega_{\mathrm{d},1})$; $\omega_3 = 0$, $(x_3 = \Omega_{\mathrm{d},2})$ and $\omega_4 = -1$, $(x_4 = \Omega_{\mathrm{e},2})$ for the fluid components, and $e_1 = -4$, $e_2 = 1$, $e_3 = -2$, $e_4 = -1/2$, $e_5 = 1$ and $e_6 = \varepsilon$, we get the following equilibria

$$\Omega^{eq}_{\mathrm{e},1} = \frac{1}{4},\ \Omega^{eq}_{\mathrm{d},1} = \frac{1}{2},\ \Omega^{eq}_{\mathrm{d},2} = \frac{2}{\varepsilon},\ \Omega^{eq}_{\mathrm{e},2} = \frac{1}{\varepsilon}$$

The condition on the density parameters then gives $\varepsilon > 12$. Taking $\varepsilon = 16$ we get four complicated but, purely imaginary and conjugated eigenvalues for the Jacobian matrix around the equilibrium:

$$\lambda_1^{\pm} = \pm i\frac{\sqrt{51134 + 6\sqrt{69956601}}}{262} \quad \text{and} \quad \lambda_2^{\pm} = \pm i\frac{\sqrt{51134 - 6\sqrt{69956601}}}{262}$$

The corresponding dynamics is the double twist plotted on figure 4.

6. Conclusion

Let us summarize the main points obtained in this paper :

- We have formulated the classical dynamics of Friedmann Universes in the context of population dynamics. Without coupling, this formulation allows a very simple and pedagogic interpretation of the evolution of the universe. Varying parameters describing the nature of the fluids one can easily understand the corresponding behavior of the so-called Jungle universes.
- Introducing a coupling between two fluids components namely dark matter and dark energy, we have obtained a Lyapunov function of the dynamics. We have also proven that such a dark fluid coupled universe has a dynamics which possess a limit cycle. This is the simplest way to obtain a so-called twisting species.
- Introducing a coupling between the N barotropic fluids filling the Universe we have obtained the general formulation of the Friedmann dynamics in the context of the generalized Lotka-Volterra equation. This offers the amusing possibility to translate in cosmological terms biological systems.
- In the case of 3 or 4 interacting batrotropic fluids, we have found particular solutions which illustrate the general properties of twisting species : an expanding twist for $N = 3$ and a double twist for $N = 4$.
- Following the results of the population dynamics, we conjecture that chaos occurs as a rule for the dynamics of universes filled by more than 3 interacting fluids.

We conclude by claiming that the presented analogy with Lotka-Volterra dynamical systems has offered new unexpected and interesting applications to coupled models in cosmology. Twisting species naturally produce transient phenomena in cosmic expansion, an original feature that could make cosmic coincidence a non unique and therefore less problematic feature.

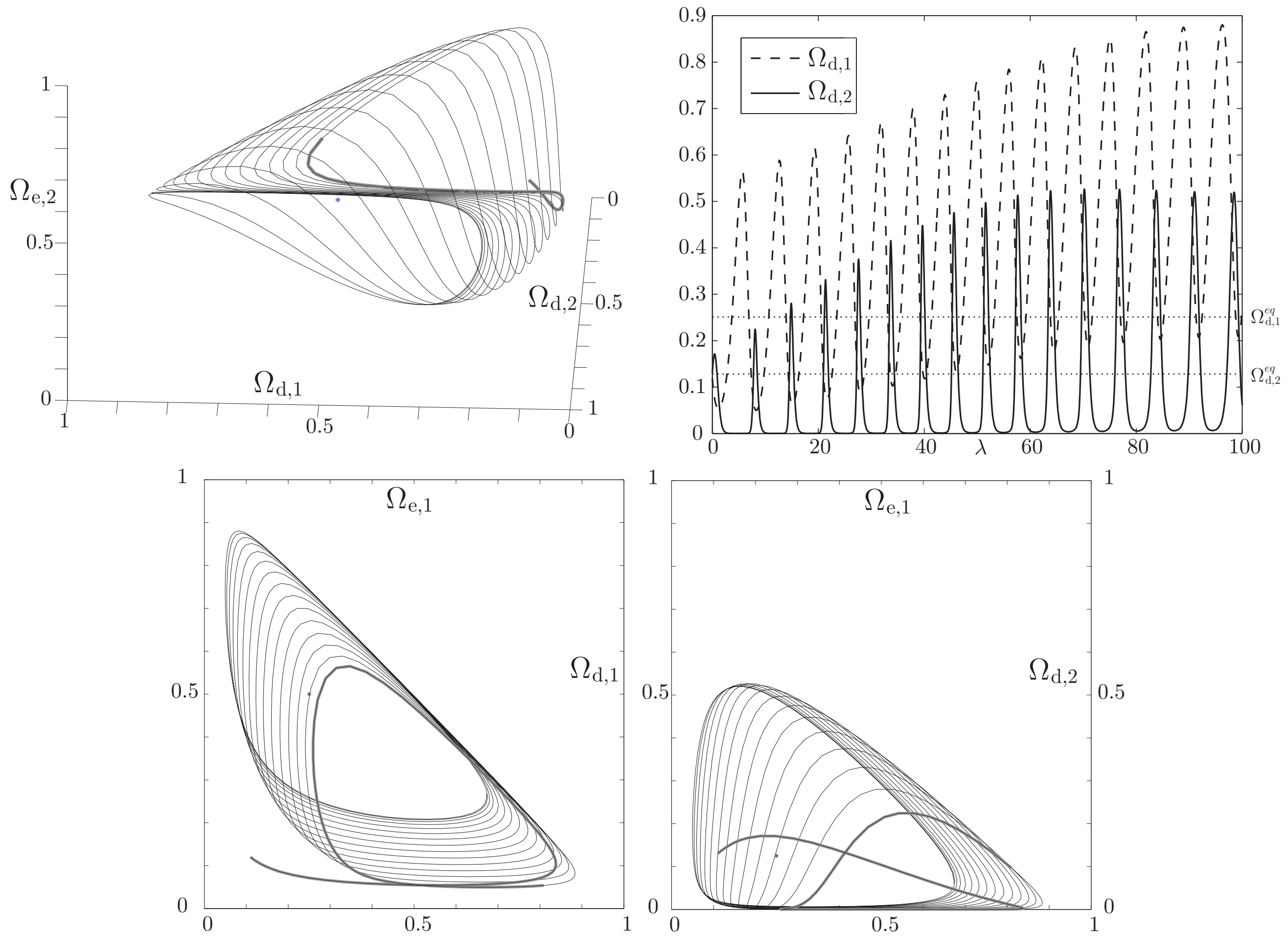

Figure 4. Jungle quartet : The left top panel is a 3D section of the 4D phase space, the three other panels are 2D sections on the 4D phase space. The beginning of the orbit is overlined, the relevant equilibrium is indicated by a star. Initial conditions for the numerical integration are $x_1(0) = 0.11$, $x_2(0) = 0.12$, $x_3(0) = 0.13$ and $x_4(0) = 0.14$

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Appendix A. Proof of the stability $\Omega_{\rm d} - \Omega_{\rm e}$ plane

The aim of this section is to provide a simple proof of the attractiveness of the $\Omega_{\rm d} - \Omega_{\rm e}$ plane for all orbits whose initial conditions belong to the hyper-tetrahedron:

$$T_4 = \{\Omega_{\rm d} > 0\} \cup \{\Omega_{\rm e} > 0\} \cup \{\Omega_{\rm r} > 0\} \cup \{\Omega_{\rm b} > 0\} \cup \{\Omega_{\rm d} + \Omega_{\rm e} + \Omega_{\rm r} + \Omega_{\rm b} < 1\}\,.$$

Let us recall the ODE system describing the equations of motion:

$$\left\{\begin{array}{l} \dot\Omega_{\rm d} = \Omega_{\rm d}(\Omega_{\rm d} + (\varepsilon - 2)\Omega_{\rm e} + 2\Omega_{\rm r} + \Omega_{\rm b} - 1) \\ \dot\Omega_{\rm e} = \Omega_{\rm e}((1-\varepsilon)\Omega_{\rm d} - 2\Omega_{\rm e} + 2\Omega_{\rm r} + \Omega_{\rm b} + 2) \\ \dot\Omega_{\rm r} = \Omega_{\rm r}(\Omega_{\rm d} - 2\Omega_{\rm e} + 2\Omega_{\rm r} + \Omega_{\rm b} - 2) \\ \dot\Omega_{\rm b} = \Omega_{\rm b}(\Omega_{\rm d} - 2\Omega_{\rm e} + 2\Omega_{\rm r} + \Omega_{\rm b} - 1)\,. \end{array}\right. \tag{A.1}$$

To prove our claim we need to prove first the invariance with respect to the flow of (A.1) of the hyper-tetrahedron T_4. The invariance of each coordinates hyperplanes is trivial and follows straightforwardly from (A.1). For instance any solution such that $\Omega_d(0) = 0$ will have $\Omega_d(t) = 0$ for all t, then using the uniqueness of the Cauchy problem we can ensure that any solution with $\Omega_d(0) > 0$ will never cross the hyperplane $\Omega_d = 0$. A very similar analysis can be performed for the remaining cases.

Let us now consider the remaining piece of the boundary of T_4, that is the hyperplane$\{\Omega_d + \Omega_e + \Omega_r + \Omega_b = 1\}$. A straightforward computation gives:

$$\frac{d}{dt}(\Omega_d + \Omega_e + \Omega_r + \Omega_b) = (\Omega_d - 2\Omega_e + 2\Omega_r + \Omega_b)(\Omega_d + \Omega_e + \Omega_r + \Omega_b - 1),$$

thus any solution with initial conditions

$$\Omega_d(0) + \Omega_e(0) + \Omega_r(0) + \Omega_b(0) = 1,$$

will always satisfies the constraint

$$\Omega_d(t) + \Omega_e(t) + \Omega_r(t) + \Omega_b(t) = 1 \quad \forall t.$$

Thus once again the uniqueness result of the Cauchy problem implies that any solution such that $\Omega_d(0)+\Omega_e(0)+\Omega_r(0)+\Omega_b(0) < 1$, will never reach the hyperplane $\Omega_d+\Omega_e+\Omega_r+\Omega_b = 1$.

Finally putting together the above partial results, we can conclude that any orbit with initial condition inside T_4 will never leave it.

A byproduct of the invariance of the tetrahedron is that orbits inside T_4 will always have positive projections on the axes. This allows us to compute the distance from the plane Ω_d-Ω_e using the linear function $F(\Omega_r, \Omega_b) = \Omega_r + \Omega_b$, which is zero if and only if $\Omega_r = \Omega_b = 0$, that is the point belongs to the plane Ω_d-Ω_e.

We can then compute the Lie derivative of F and prove that its restriction to T_4 is strictly negative, hence $F(\Omega_r(t), \Omega_b(t)) \to 0$ for $t \to +\infty$ and because of the positiveness of $\Omega_r(t)$ and $\Omega_b(t)$ we can conclude that both $\Omega_r(t)$ and $\Omega_b(t)$ goes asymptotically to zero.

To prove the latter claim let us compute the derivative of F along the flow of (A.1):

$$\frac{d}{dt}F|_{flow} = (\Omega_d - 2\Omega_e + 2\Omega_r + \Omega_b - 1)(\Omega_r + \Omega_b) - \Omega_r,$$

because of our previous result $\Omega_d(t) + \Omega_b(t) - 1 < -\Omega_e(t) - \Omega_r(t)$ for all t, we get

$$\begin{aligned}\tfrac{d}{dt}F &= (\Omega_d - 2\Omega_e + 2\Omega_r + \Omega_b - 1)(\Omega_r + \Omega_b) - \Omega_r < (-3\Omega_e + \Omega_r)(\Omega_r + \Omega_b) - \Omega_r \\ &= -3\Omega_e(\Omega_r + \Omega_b) + \Omega_r(\Omega_r + \Omega_b - 1),\end{aligned}$$

let us observe that the right hand side is strictly negative, in fact

$$-3\Omega_e(\Omega_r + \Omega_b) < 0$$

and

$$\Omega_r + \Omega_b - 1 < \Omega_d + \Omega_e + \Omega_r + \Omega_b - 1 < 0.$$

This concludes our proof of the attractiveness of the dark plane for all orbits whose initial conditions belong to T_4.

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