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Homogeneous-per-layer patterns in multiplex networks

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Abstract – A new class of patterns for multiplex networks is studied, which consists in a collection of different homogeneous states each referred to a distinct layer. The associated stability diagram exhibits a tricritical point, as a function of the inter-layer diffusion coefficients. The patterns, made of alternating homogeneous layers of networks, are dynamically selected via non-homogeneous perturbations superposed to the underlying, globally homogeneous, fixed point and by properly modulating the coupling strength between layers. Furthermore, layer-homogeneous fixed points can turn unstable following a mechanism à la Turing, instigated by the intra-layer diffusion. This novel class of solutions enriches the spectrum of dynamical phenomena as displayed within the variegated realm of multiplex science.

Countless systems in Nature exhibit patterns and regularities. Chemistry [1,2], biology [3,4] and neuroscience [5] are just few examples of fields in which a macroscopic order spontaneously emerges from the microscopic interplay between many interacting agents.

A particular subset of processes driving the onset of patterns is represented by reaction-diffusion systems, i.e. systems made of at least two interacting species undergoing spatial diffusion. Introduced in the context of mammals pigmentation by Alan Turing [6], these systems obey an activator-inhibitor dynamics. Under suitable conditions, the diffusion drives an instability by amplifying a perturbation superposed to a homogeneous stable fixed point. The perturbation grows and, balanced by non-linear interactions, leads to spatially inhomogeneous steady states, termed in the literature Turing patterns.

Although regular lattices define a suitable framework to model physical reaction-diffusion systems, recently the theory has been extended so as to include complex networks [7], as the underlying medium where species are bound to diffuse. This approach is motivated by the fact that many real-world systems, ranging from ecology [8] to the brain structure [9], passing through the modeling of social communities [10], can be easily schematised by invoking the concept of graph [11]. In this context, numerous works have revealed a plethora of interesting phenomena, ascribing to the discrete nature of the embedding support a leading role [12,13].

However, the standard approach to network theory is not always able to encode for the high complexity of real-world systems, e.g., the human brain [14], the transportation network [15,16] and, in general, systems composed by multiple sub-layers. For this reason, a step forward has been taken in the modeling which led to the introduction of the concept of multiplex networks, i.e., interconnected multi-layered graph [17]. The properties of multiplex networks have been widely addressed in the literature. This includes investigating the non-trivial interplay between structure and dynamics [18–20]. A general theory for Turing patterns on multiplex networks has been developed in [21,22]. Interestingly, diffusion among adjacent layers can enhance or suppress the instability [21].

In this work we analyse further the zoology of phenomena that can emerge from a reaction-diffusion system defined on multiplex networks. In particular, we focus on a new class of instability-driven patterns, veritable attractor of the inspected system, which are homogeneous per layer [23], as depicted in fig. 1. These states can turn
Fig. 1: (Color online) Illustrative example of a layer-homogeneous multiplex network composed by \( M = 5 \) Watts-Strogatz layers [24], with probability of rewiring \( p = 0.5 \) and average connectivity ranging from 2 to 5. Each network is made of \( N = 100 \) nodes.

Here, the Brusselator model is assumed, with parameter \( b = 9, c = 30 \). The diffusion constant are set to the values \( D_u^{12} = D_v^{12} = D_u^{45} = D_v^{45} = 1 \) and \( D_u^{23} = D_v^{23} = D_u^{34} = D_v^{34} = 10 \). To facilitate visualization, only 30\% of the links in each layer and 40\% of the links among layers have been drawn.

unstable due to the injection of a non-homogeneous perturbation which may resonate with the intra-layer diffusion terms. Each layer of a multiplex network can be also thought as an individual node of a corresponding network of layers. In this setting, it is tempting to interpret the novel family of fixed points as coarse-grained patterns, which combines different macro-units so as to reflect the complexity of a multi-layers arrangement. We shall also prove that such coarse-grained patterns can be dynamical selected following a Turing-like instability of the global homogeneous equilibrium, the inter-layer diffusivity acting as the key control parameter.

Let us consider for the sake of simplicity a multiplex composed by two layers, but observe that the model can be readily extended to the case of \( M \)-independent layers; each layer is constituted by \( N \) nodes, and characterised by a \( N \times N \) adjacency matrix \( A_{ij}^K \), where the label \( K = 1, 2 \) denotes the layer of pertinence. By definition, \( A_{ij}^K = 1 \) if the nodes \( i \) and \( j \) are connected in the layer \( K \), \( A_{ij}^K = 0 \) otherwise. Let us observe that homologous nodes, i.e., the “same node” belonging to different layers, are, by definition, mutually connected. A two-species reaction-diffusion system can hence be cast in the following form [21]:

\[
\begin{align*}
\dot{u}_i^K &= f(u_i^K, v_i^K) + D_u^K \sum_{j=1}^N L_{ij}^K u_j^K + D_u^{12} (v_{i+1}^K - u_i^K), \\
\dot{v}_i^K &= g(u_i^K, v_i^K) + D_v^K \sum_{j=1}^N L_{ij}^K v_j^K + D_v^{12} (u_{i+1}^K - v_i^K),
\end{align*}
\]

assuming \( K = 1, 2 \) and \( K + 1 \) to be 1 for \( K = 2 \). Here \( u_i^K \) and \( v_i^K \) stand for the concentrations of the species on the node \( i \), as seen in layer \( K \). \( L_{ij}^K \) is the Laplacian matrix associated to the \( K \) layer, \( L_{ij}^K = A_{ij}^K - k^K \delta_{ij} \), where \( k^K = \sum_j A_{ij}^K \) refers to the connectivity of node \( i \) belonging to layer \( K \) (see footnote 1) and \( \delta_{ij} \) is Kroenecker’s delta. The matrix \( L_{ij}^K \) is nothing but the discrete version of a diffusion operator. \( D_u^K \) (respectively, \( D_v^K \)) is the intra-layer diffusion coefficient of species \( u \) (respectively, \( v \)); \( D_u^{12} \) (respectively, \( D_v^{12} \)) denotes the inter-layer diffusion coefficient associated to species \( u \) (respectively, \( v \)). Finally, the non-linear functions \( f(\cdot, \cdot) \) and \( g(\cdot, \cdot) \) encode for the local (on site) rule of interaction between the two considered species. In the following we shall assume that one species acts as an activator, by autocatalytically enhancing its own production, while the other behaves as an inhibitor, contrasting the activator growth.

The model in eq. (1) admits two classes of fixed points: i) the globally homogeneous (GH) fixed points, i.e. \( u^K = \hat{u} \) and \( v^K = \hat{v} \) for all \( i = 1, \ldots, N \) and for all \( K \), namely the equilibrium values are independent of the node and the layer; ii) the layer-homogeneous (LH) fixed points, defined as \( u_i^K = \hat{u}_K \) and \( v_i^K = \hat{v}_K \) for all \( i = 1, \ldots, N \). Note that we here emphasised the dependence of the equilibrium value on the layer, through the index \( K \).

To be concrete, let us consider a specific case study, the so-called Brusselator model for which the local reaction terms are given by \( f(u, v) = 1 - (b + 1)u + cu^2v \) and \( g(u, v) = bu - cu^2v \), depending on the parameters \( b \) and \( c \). A straightforward computation allows one to determine the GH fixed point \( \hat{u} = 1, \hat{v} = b/c \). Determining the LH fixed point proves more demanding and, to this end, we rely on numerical methods. In the following, the parameters of the model are assigned so that the corresponding GH fixed point is stable to external homogeneous perturbation. In other words, \((b, c)\) are selected in the region where the a-spatial version of the Brusselator (i.e., the model obtained when setting to zero all diffusion constants in eqs. (1)) returns a stable fixed point.

Note that the GH fixed point depends only on the model parameters \( b \) and \( c \), since both the intra-layer and the inter-layer diffusions vanish when all the nodes share the same concentration, independently of the layer that they are bound to occupy. On the other hand, LH fixed points are also functions of the inter-layer diffusion coefficients, \( D_u^{12} \) and \( D_v^{12} \), but do not depend on the intra-layer diffusion constants, because the species display an identical concentration on each layer, at equilibrium. Importantly, the existence of these two classes of fixed points is independent of the underlying topology of the layers composing the multiplex network.

To study the stability of the above-mentioned fixed points, we perform a standard linear stability analysis and monitor the time evolution of a perturbation assumed homogeneous per layer. In formulae, we shall

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1Notice that \( k_i^K \) does not account for inter-layers links.
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set $u^K_i = \hat{u} + \delta u^K_i$ and $v^K_i = \hat{v} + \delta v^K_i$, for $\forall i$, where $\hat{u}$ (respectively, $\hat{v}$) is either $\bar{u}$ (respectively, $\bar{v}$) or $\hat{u}^K$ (respectively, $\hat{v}^K$), and linearise eqs. (1), for $\delta u^K, \delta v^K$ small. The analysis materialises in an interesting picture, which can be efficaciously summarised in the plane $(D^u_{12}, D^v_{12})$, as reported in fig. 2. The parameters space is partitioned into two regions: in the lower portion of the plane LH solutions prove linearly stable. In the upper domain GH fixed points are stable equilibria. The two regions are separated by a transition line which we have determined analytically. The dashed line identifies a first-order transition: by monitoring $\delta u^K$, as a function of $D^u_{12}$, for $D^v_{12}$ frozen to a value that makes the crossing to happen where the transition is predicted discontinuous (horizontal, lower dash-dotted line), one obtains the typical bifurcation diagram as displayed in the lower inset of fig. 2. Conversely, when the transition is continuous (horizontal, upper dash-dotted line) one recovers the usual pitchfork bifurcation, as shown in the upper inset enclosed in fig. 2. First and second transition lines merge together at a tricritical point, the black circle in fig. 2. The bifurcation diagram follows a linear stability analysis. Formally, this implies computing the spectrum of the Jacobian matrix, evaluated at the fixed point. The transition lines correspond to the condition where the largest real part of the computed eigenvalues is found to be identically equal to zero. Since the fixed points belonging to the LH class cannot be expressed in a closed analytical form, the diagonalisation of the Jacobian is performed numerically. The prediction relies hence on analytical techniques, but to finalise the calculation numerical methods are needed. As an important remark, we recall that other families of critical points have been reported in spreading dynamics on multiplex networks [25–27].

Given the above scenario several interesting questions arise. When operating in the region where the LH fixed point is shown to be stable, can one seed a diffusion-driven instability à la Turing, triggered by a random non-homogeneous perturbation? And can one obtain the LH (coarse grained or homogeneous per layer) patterns, as following a symmetry breaking instability of a GH stable equilibrium? These are the questions that we set to answer in the following.

Taking inspiration from [21], we introduce a small perturbation $(\delta u^K, \delta v^K)$ to the fixed point $(\hat{u}^K, \hat{v}^K)$ and linearise eqs. (1) around it:

$$\frac{d}{dt} \begin{pmatrix} \delta u \\ \delta v \end{pmatrix} = \mathcal{J} \begin{pmatrix} \delta u \\ \delta v \end{pmatrix}$$

with

$$\mathcal{J} = \begin{pmatrix} f_u + \mathcal{L}_u + D^u_{12} \mathcal{I} & g_u \\ g_v + \mathcal{L}_v + D^v_{12} \mathcal{I} & f_v \end{pmatrix}.$$
The second-order transition in fig. 2 (similar conclusions can be reached when setting \( D \) vertical line located at \((D_{12}^{12})_c \)). For \( D_{12}^{12} > (D_{12}^{12})_c \), the GH is always unstable. Conversely, LH fixed points turn unstable, upon injection of a non-homogeneous perturbation, for a choice of the relevant parameters that position the examined system above the solid line, as obtained via the linear stability analysis. The dispersion relation, calculated as the largest real part of the eigenvalue of \( \mathcal{J} \), evaluated at the LH fixed point is plotted, in the region of LH instability, with an appropriate color code. The patterns obtained upon integration of the system are displayed in the inset: (a) and (b) refer to a choice of the parameters where GH and LH are simultaneously unstable (upper black circle in the main panel). Pattern (a) follows a perturbation imposed on the GH equilibrium, pattern (b) assumes LH as the background equilibrium. Coarse-grained patterns can emerge from a GH instability grows exponentially in the linear regime. Non-linear effects become eventually important: they compensate for the linear growth and consequently shape the final non-homogeneous stationary configuration.

From an analytical point of view [21], one cannot introduce a basis to expand the perturbations which diagonalise the global diffusion operators \( \mathcal{L}_u + D_{12}^{12} \mathcal{I} \) and \( \mathcal{L}_v + D_{12}^{12} \mathcal{I} \). We cannot hence simplify the \( 4N \times 4N \) eigenvalue problem, by projecting it into a reduced subspace, as it is instead possible when the dynamics is hosted on just one isolated layer [7]. Moreover, the spectrum of \( \mathcal{J} \) cannot be exactly related to the spectra of the homologous operators, obtained for the limiting setting when the two layers are formally decoupled \((D_{12}^{12} = D_{12}^{12} = 0)\).

It is, however, possible to determine numerically the region where the diffusion-driven instability is bound to occur, and then integrate the set of governing differential equations, so as to visualise the ensuing patterns. To this aim, we fix the reaction parameters of the inspected model \((b \text{ and } c, \text{ in the case of the Brusselator model})\), and one of the inter-layer diffusion coefficients, specifically \( D_{12}^{12} \). When modulating \( D_{12}^{12} \), at fixed \( D_{12}^{12} \), one operates an

\[
\begin{align*}
    x &= (x_1^1, \ldots, x_N^1, x_1^2, \ldots, x_N^2) \quad \text{for} \quad x = u, v, \text{ and } \\
    \mathcal{J} &= (-I_N \quad I_N) \quad \text{with} \quad I_N \text{ the } N \times N \text{ identity matrix.} \\
    \text{Laplace}\text{an for species } u \text{ reads } \mathcal{L}_u = (D_u^{12} L^1 \quad 0) \\
    \text{and, similarly for } f_u, g_u \text{ and } g_v. \text{ These additions constitute the main difference with respect to the standard Turing theory.} \\
\end{align*}
\]

By projecting it into a reduced subspace, as it is

\[
\begin{align*}
    f_u &= \left( \frac{\partial_u f|_{(g^1, g^1)}}{D_u^{12} L^1} - I_N \quad 0 \right) \\
    \text{and, similarly for } f_v, g_u \text{ and } g_v. \text{ These additions constitute the main difference with respect to the standard Turing theory.} \\
\end{align*}
\]

By projecting it into a reduced subspace, as it is
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Patterns are instigated by increasing the inter-layer diffusion rate, represented with an appropriate color code. Several complementary regions delineated hold however in general, irrespectively of the topology, the associated bifurcation diagram and the emergence relation, in the region where GH and LH are both unstable. Interestingly, by tuning $D^{12}$ one can drive unstable a system (in the two layers configuration), as following a symmetry breaking instability, seeded by the injection of an external perturbation. By evaluating $\Gamma$ at large times, i.e., when the system has eventually reached its asymptotic configuration, one obtains in fact three possible outcomes. When the system is confined in a globally homogeneous fixed point, $\Gamma = 0$, since both contributions entering the above definition vanish identically. On the contrary, if the system exhibits a disordered pattern, the first term is approximately equal to one, because in general $u_i^1 \neq u_i^2$, and the second term is equal to $1/2$, so yielding $\Gamma \simeq 1/2$. Finally, when the system lands in a layer-homogeneous fixed point, i.e., a coarse-grained pattern, $\Gamma = 1$, since the second term vanishes, being all the concentrations in the same layer equal. A similar indicator could be introduced to monitor the degree of large-scale organization of the competing species $v$.

In fig. 4 we report the value of $\Gamma$, as a function of the inter-layer diffusion coefficient $D^{12}$. The emerging patterns, classified in terms of $\Gamma$, are obtained by perturbing a GH fixed point, assumed stable in absence of diffusion. The results displayed in fig. 4 are computed by processing the patterns recorded via direct numerical integration of eqs. (1), and after averaging over different realizations of dynamics. The results refer $D^{12} = 2$ (second-order transition), being the other parameters frozen to the values declared in the caption of fig. 2 (qualitatively similar results are obtained when setting $D^{12}$ to a value which corresponds to a first-order transition). By eye inspection, it is immediate to conclude that the system exhibits disordered or coarse-grained (homogeneous per layer) patterns, depending on the value of the coupling strength between layers. The regions where different patterns are found, organise in adjacent blocks, as a function of the control parameter $D^{12}$. More specifically, for modest values of $D^{12}$, standard Turing patterns take place. By increasing $D^{12}$, coarse-grained patterns are instead established. The vertical (green and red) lines are respectively drawn in correspondence of the critical values of $D^{12}$, as identified in fig. 3, see (red and green) circles. These latter
provide a satisfying theoretical interpretation for the observed transitions.

Summing up, we have here developed a theory of pattern formation for a two-species reaction diffusion system on a two-layers multiplex network around a novel class of (layer-homogeneous) fixed points. The methodology can be readily generalised to the case of \( s \) interacting species upon \( K \)-independent layers. Indeed by allowing for more layers opens up several possibilities. Focus, for example, on the case of three adjacent layers. Three distinct LH fixed points can in principle appear. Alternatively, two networks could share an identical value of the concentration and then bifurcate to yield different homogeneous entries, in correspondence of a (second) bifurcation point, associated to a critical diffusion coefficient (as occurs for the logistic map).

Interestingly, the layer-homogeneous fixed points can be seen as coarse-grained patterns in a network of networks where each layer is replaced by a virtual supernode, bearing the concentration displayed by individual micro-nodes, belonging to the layer itself. This novel class of patterns can be dynamically selected by perturbing the micro-nodes, belonging to the layer itself. This novel class could share an identical value of the concentration and co-exist in systems defined on a stratified network and the diffusion act as trigger to resolve hidden microscopic structures. The potential interest of this observation embraces in principle a large set of applications. For example, ecological networks can be pictured as multi-layered graphs, in which the interactions on each layer influence the dynamics of the whole system [29]. Analogously, the complex topology of the brain can be sometimes represented as a multiplex network [30], the most glaring example being the barrel cortex [31], in which stratified patterns of activity may possibly take place. Exploring the applied relevance of the coarse-grained patterns here discussed remains however a challenge for future investigations.

### REFERENCES


