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Timoteo Carletti

Turing patterns on time varying networks

Namur Institute for Complex Systems

UNIVERSITÉ DE NAMUR

timoteo.carletti@unamur.be

Acknowledgements

Present and past collaborators

"Belgian" team:

M. Asllani, N. Kouvaris (post docs)

J. Petit (PhD)

A. Bellière, G. Planchon, R. Muolo (Master students)

Italian team:

D. Fanelli, D.M. Busiello, C. Cianci, M. Galanti, F. Miele, F. Di Patti

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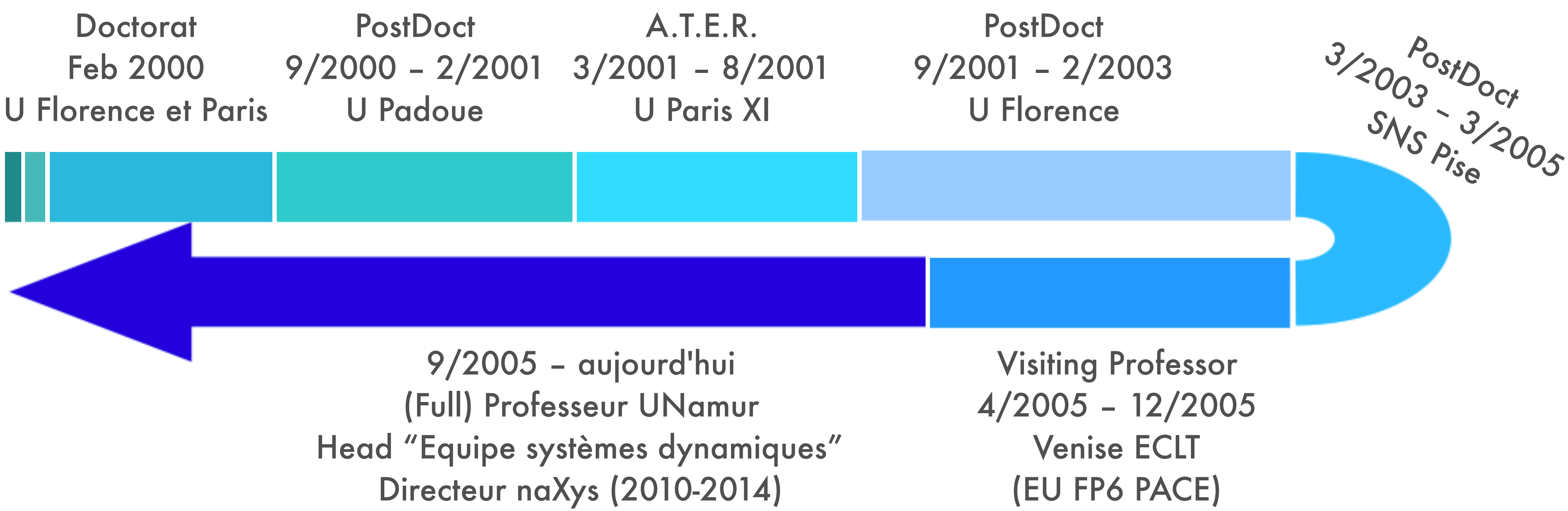
Snapshot of my career



Word cloud of the titles of my papers

Pattern ? [ref. Oxford dictionary]

pattern

Top 1000 frequently used words

Pronunciation: /ˈpat(ə)n/

NOUN

1 A repeated decorative design:

'a neat blue herringbone pattern'

− More example sentences

'Included are geometrics, florals and foliates, animals and nature motifs and other decorative repeat patterns.'

'These aspects then become ornamented with Islamic-inspired decorative patterns and Islamic cultural artifacts.'

'It featured exuberant decorative patterns, designs in the brickwork and wooden attachments.'

1 A repeated decorative design:

'a neat blue herringbone pattern'

+ More example sentences + Synonyms

1.1 An arrangement or design regularly found in comparable objects:

'the house had been built on the usual pattern'

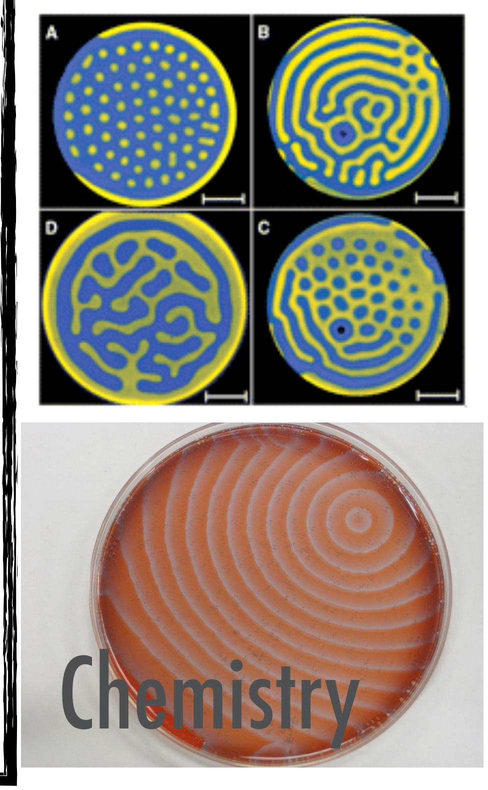
− More example sentences

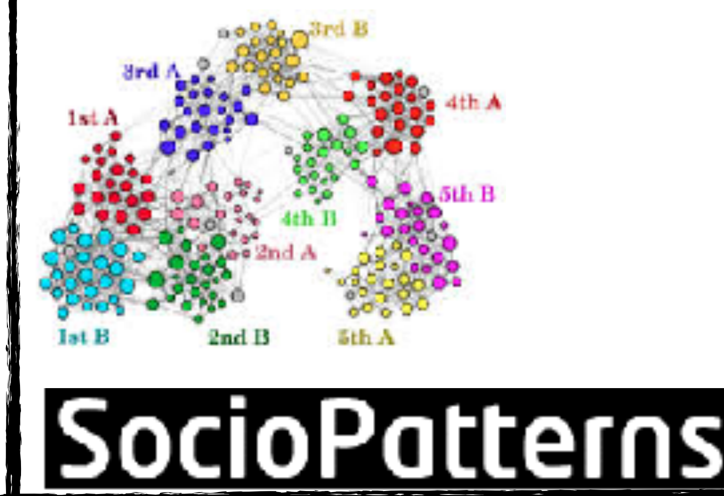
'Structurally, the tumor cells were arranged in a medullary pattern composed of polygonal tumor cells.'

'The hair-cells within the spiralling cochlear duct are arranged in a pattern like the bristles of a brush.'

'The fossils indicate the wings had feathers, arranged in a similar pattern to that of modern birds.'

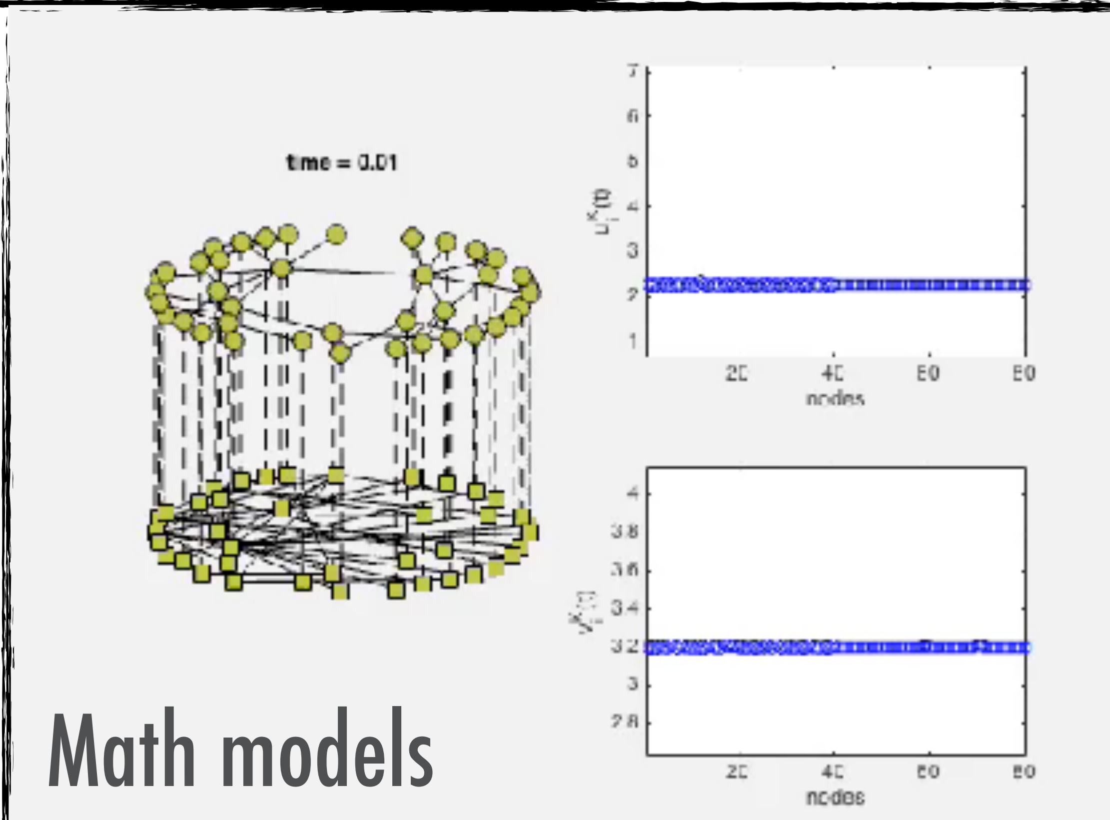
Patterns are ubiquitous





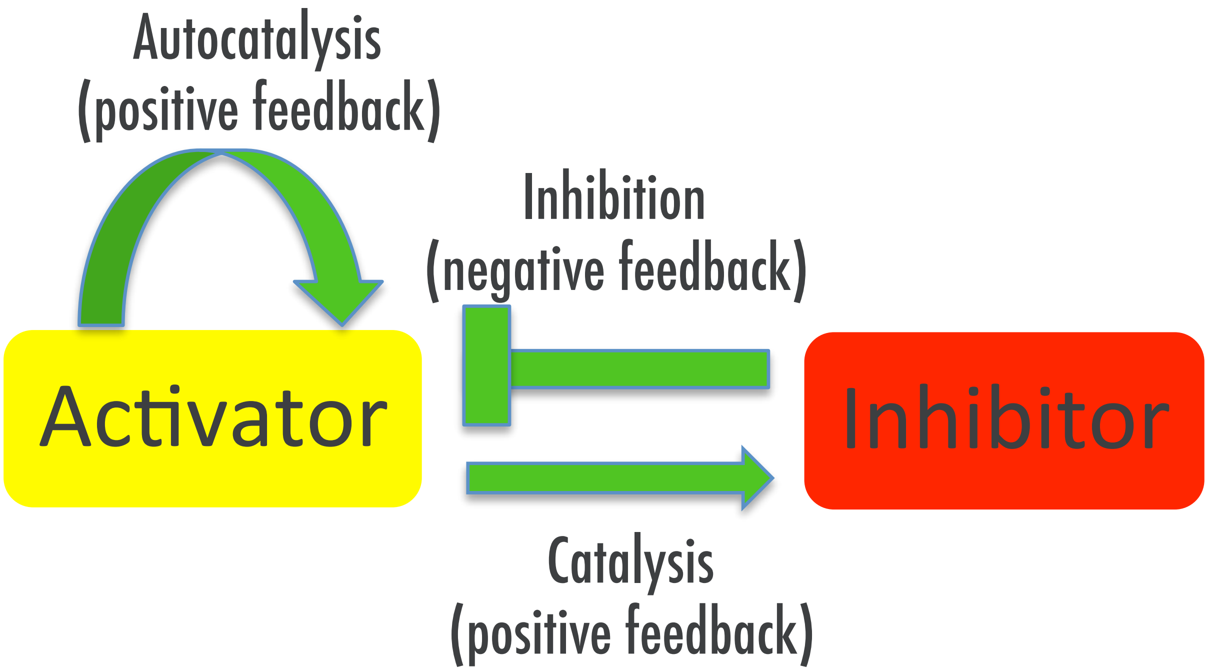
Internet

Twitter



One possible mechanism: Turing instability

Reaction



\+

Diffusion

$u(x,y,t)$: Amount of activator at time t and position (x,y)

$v(x,y,t)$: Amount of inhibitor at time t and position (x,y)

$$\begin{cases} \dfrac{\partial u}{\partial t} = f(u,v) + D_u \nabla^2 u \\ \dfrac{\partial v}{\partial t} = g(u,v) + D_v \nabla^2 v \end{cases}$$

$$(x,y) \in \Omega$$

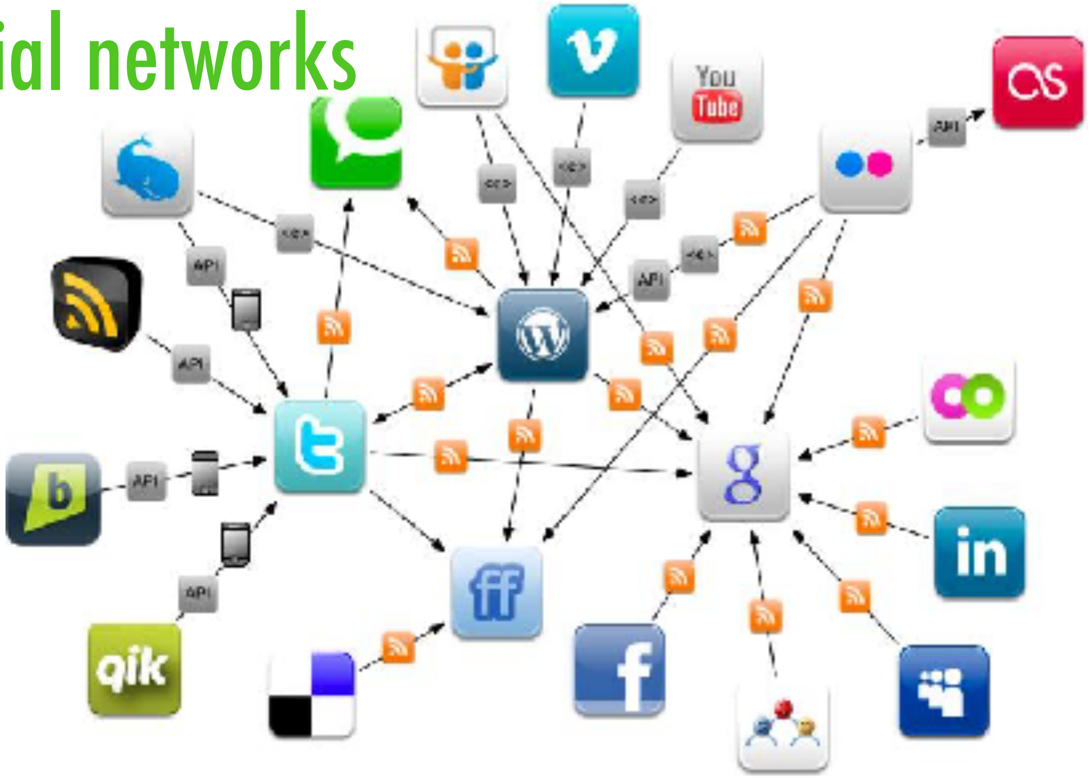
\+ boundary conditions
\+ initial condition

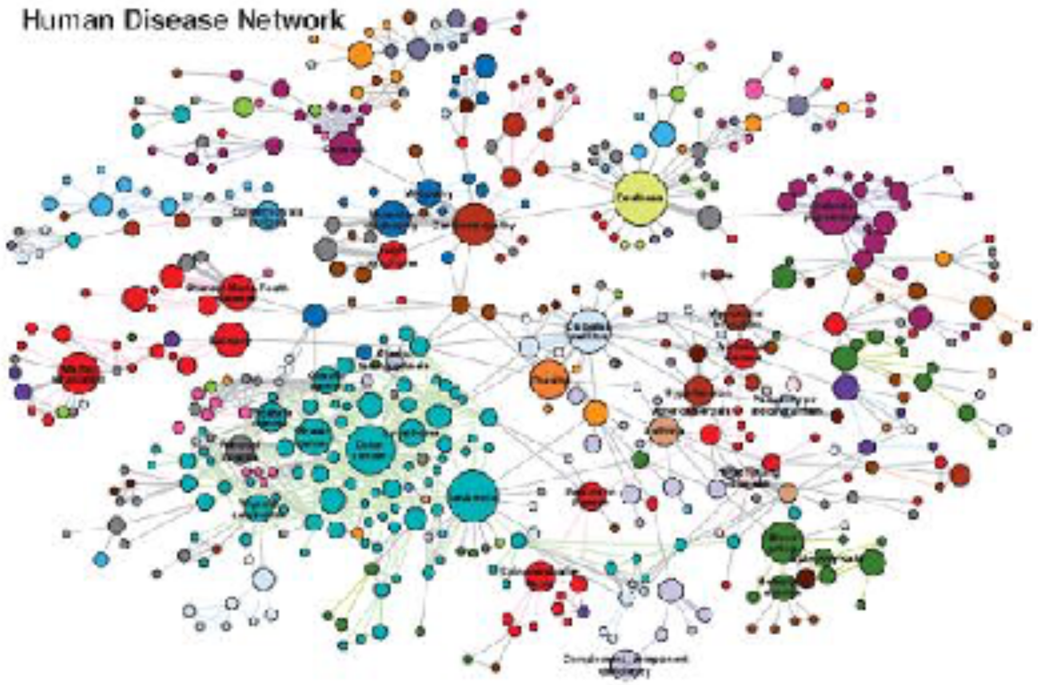
A.M.Turing, The chemical basis of morphogenesis, Phil. Trans. R Soc London B, 237, (1952), pp.37

Networks are everywhere

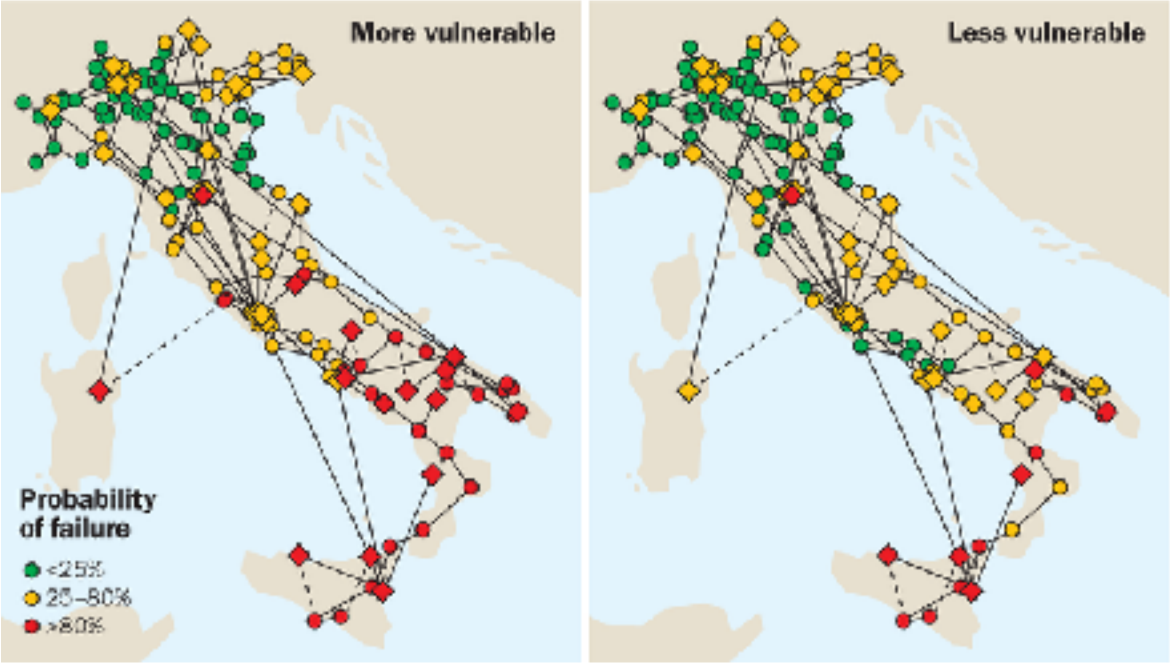
world flights map

social networks





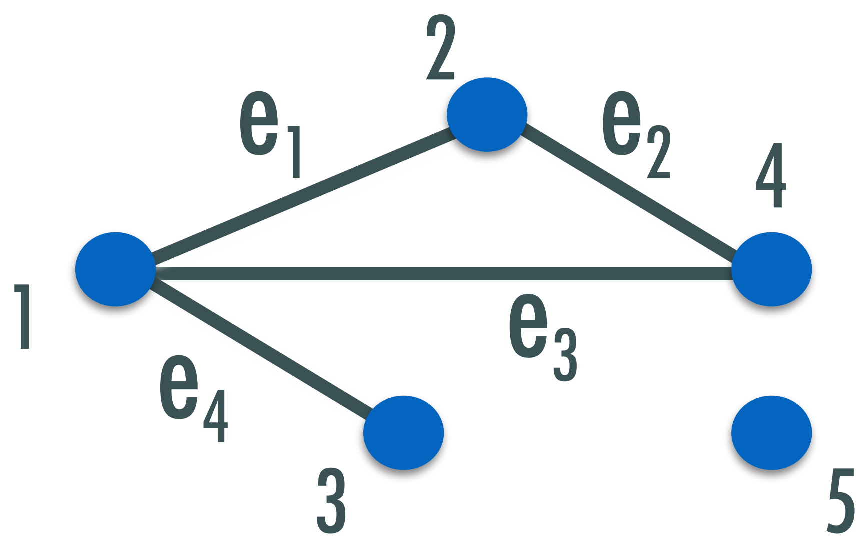
proteins networks



technological networks

(complex) Networks: some definitions

A network is a set of nodes connected by links (edges)



Ex.: 5 nodes and 4 edges (undirected)

Adjacency matrix

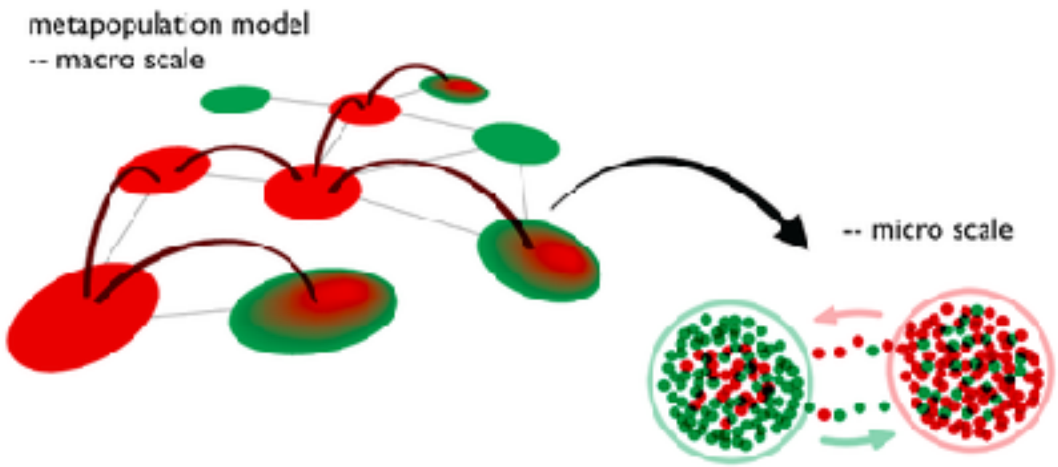
$$A_{ij} = \begin{cases} 1 & \text{if nodes } i \text{ and } j \text{ are linked} \\ 0 & \text{otherwise} \end{cases}$$

The number of links entering (going out) from each node is called in-degree (out-degree)

Ex.: degree node 1 = 3
degree nodes 2 & 4 = 2
degree node 3 = 1
degree node 5 = 0

A network is said to be complex if the degree distribution is not trivial, i.e. not constant (lattice) nor Poissonian (random, Erdős-Rényi)

Extension to networks



Metapopulation models
e.g. in the framework of ecology:

May R., Will a large complex system be stable? Nature, 238, pp. 413, (1972)

Reactions occur at each node. Diffusion occurs across edges.

Patterns
sets of nodes whose asymptotic state is far from the homogeneous equilibrium.

Reaction term:

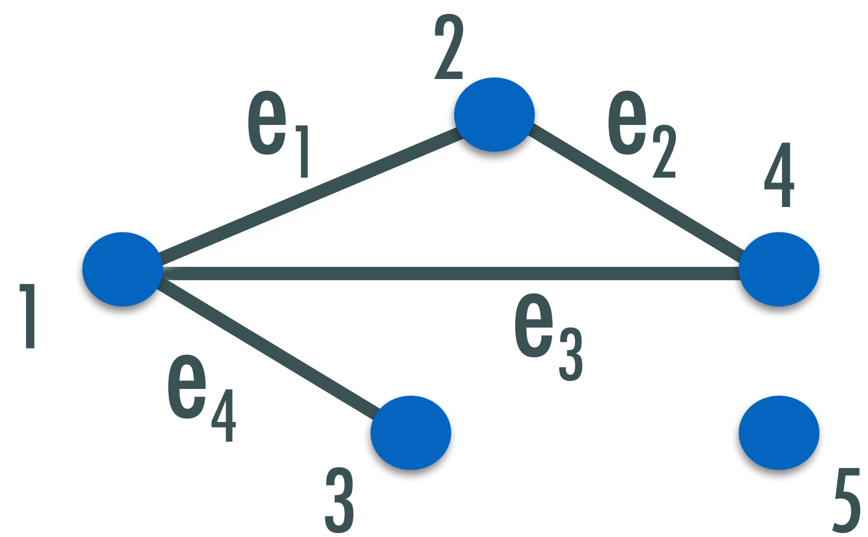
$$\begin{cases} \dot{u}_i(t) & = f(u_i(t), v_i(t)) \\ \dot{v}_i(t) & = g(u_i(t), v_i(t)) \quad \forall i = 1, \dots, n \text{ and } t > 0. \end{cases}$$

At each node i=1,…,n, "species" u and v react through some non-linear functions f and g depending on the quantities available at node i-th (metapopulation assumption)

Nakao H. and Mikhailov A. S.,
Turing patterns in network-organized activator–inhibitor systems, Nature Physics, 6, pp. 544 (2010)

Diffusion term:

Diffusive transport of species into a certain node i is given by the sum of incoming fluxes to node i from other connected nodes j, fluxes are proportional to the concentration difference between the nodes (Fick's law).



Ex.: consider the amount of u in node 1,
u can enter from 2, 3 and 4
u can leave 1 to go to 2, 3 and 4

$$u_2 + u_3 + u_4 - 3u_1 = \sum_j A_{1j} u_j - k_1 u_1 = \sum_j (A_{1j} - \delta_{1j} k_j) u_j := \sum_j L_{1j} u_j$$

L is called Laplacian matrix of the network

The model:

$$\begin{cases} \dot{u}_i(t) &= f(u_i(t), v_i(t)) + D_u \sum_{j=1}^n L_{ij} u_j(t) \\ \dot{v}_i(t) &= g(u_i(t), v_i(t)) + D_v \sum_{j=1}^n L_{ij} v_j(t) \quad \forall i = 1, \dots, n \text{ and } t > 0. \end{cases}$$

Du and Dv are the diffusion coefficients of species u and v

Observe that because the network is undirected, the matrices A and L are symmetric

General strategy for the network case

1) Assume there exists a spatially homogeneous solution:

$$(u_i, v_i) = (\hat{u}, \hat{v}) \quad \forall i = 1, \ldots, n$$

which moreover is stable when there is no diffusion:

$$D_u = D_v = 0$$

2) Linearize around this solution

$$\begin{aligned} u_i &= \hat{u} + \delta u_i \\ v_i &= \hat{v} + \delta v_i \end{aligned} \qquad \begin{pmatrix} \dot{\delta \boldsymbol{u}} \\ \dot{\delta \boldsymbol{v}} \end{pmatrix} = \tilde{\mathcal{J}} \begin{pmatrix} \delta \boldsymbol{u} \\ \delta \boldsymbol{v} \end{pmatrix}$$

$$\tilde{\mathcal{J}} = \begin{pmatrix} f_u \mathbf{I}_n + D_u \boldsymbol{L} & f_v \mathbf{I}_n \\ g_u \mathbf{I}_n & g_v \mathbf{I}_n + D_v \boldsymbol{L} \end{pmatrix}$$

General strategy for the network case

3) Prove that (possibly) the spatially homogeneous solution:

$$(u_i, v_i) = (\hat{u}, \hat{v}) \quad \forall i = 1, \ldots, n$$

turns out to be unstable once the diffusion is in action

$$D_u > 0 \text{ and } D_v > 0$$

Sketch of the proof

i) Let $L\vec{\phi}^\alpha = \Lambda^\alpha \vec{\phi}^\alpha,\ \alpha = 1, \ldots, n \qquad \vec{\phi}^\alpha = (\phi_1^\alpha, \ldots, \phi_n^\alpha)$

$$\sum_i \phi_i^\alpha \phi_i^\beta = \delta_{\alpha\beta} \qquad \Lambda^\alpha \leq 0$$

ii) decompose the solution on the eigenbasis and use the ansatz

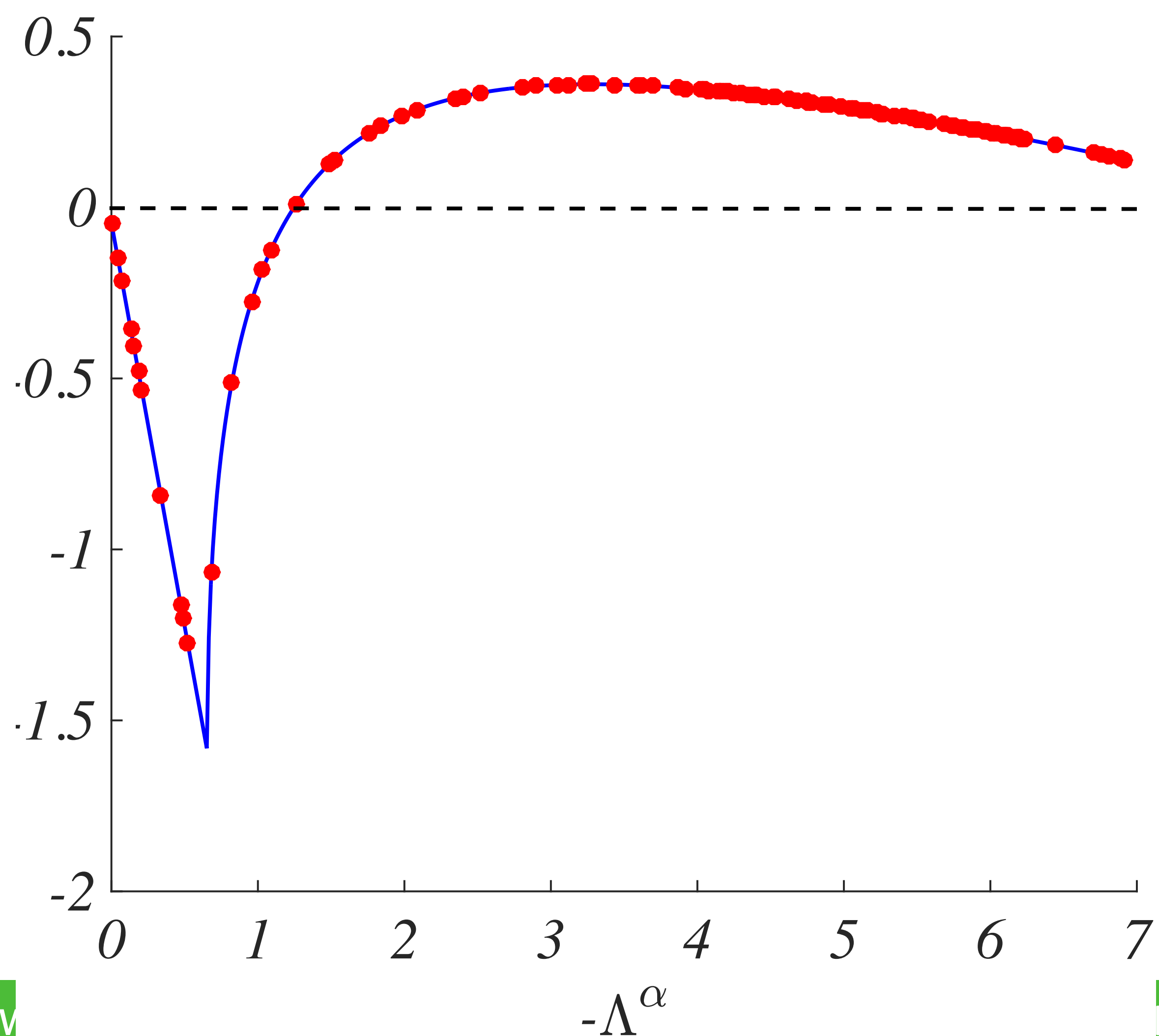
$$\delta u_i(t) = \sum_{\alpha=1}^{n} c_\alpha \phi_i^\alpha e^{\lambda_\alpha t}$$

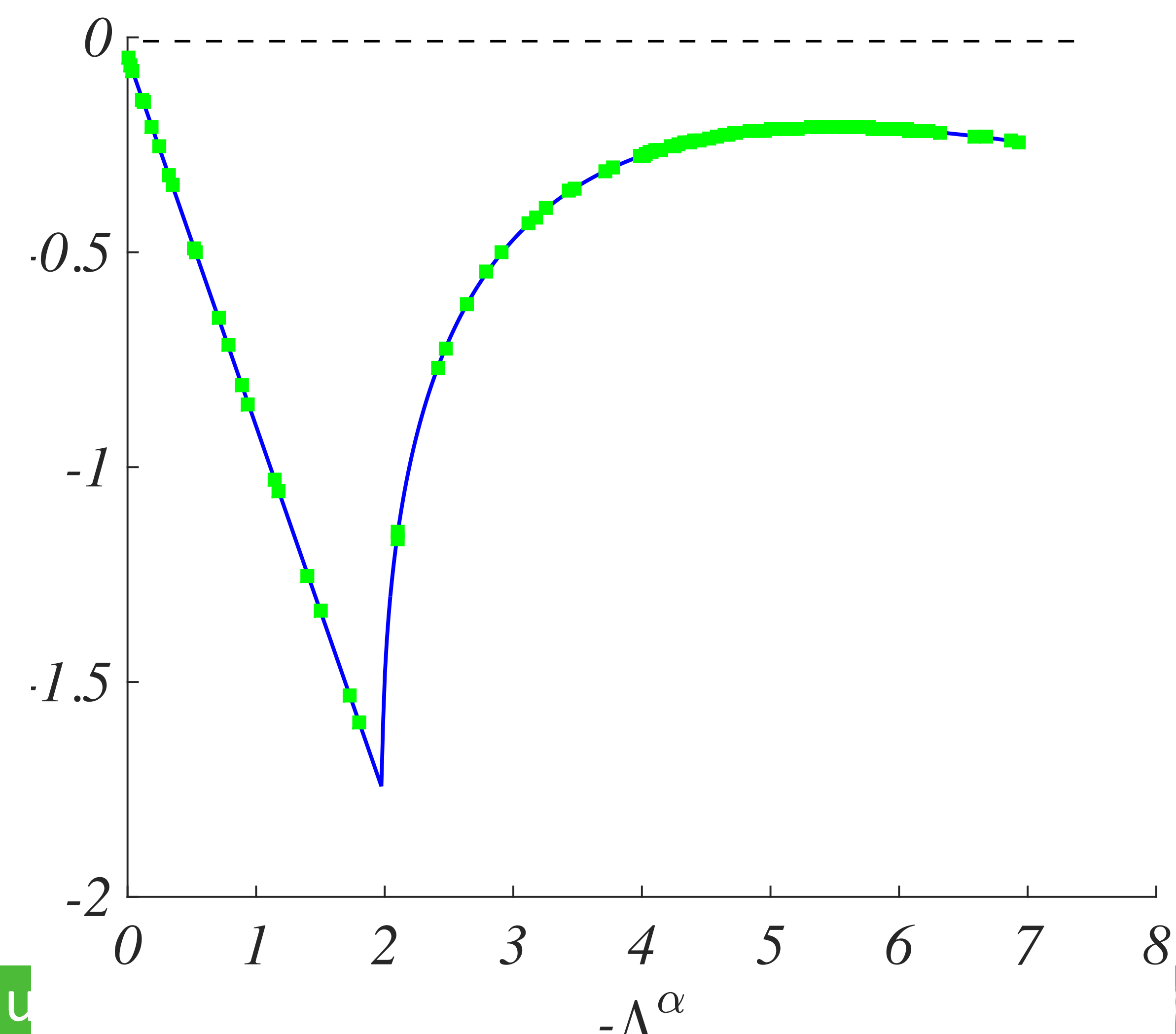
General strategy

iii) λ_α (called relation dispersion) is solution of

$$\det\left[\lambda_\alpha - \begin{pmatrix} f_u + D_u\Lambda^\alpha & f_v \\ g_u & g_v + D_v\Lambda^\alpha \end{pmatrix}\right] = 0$$

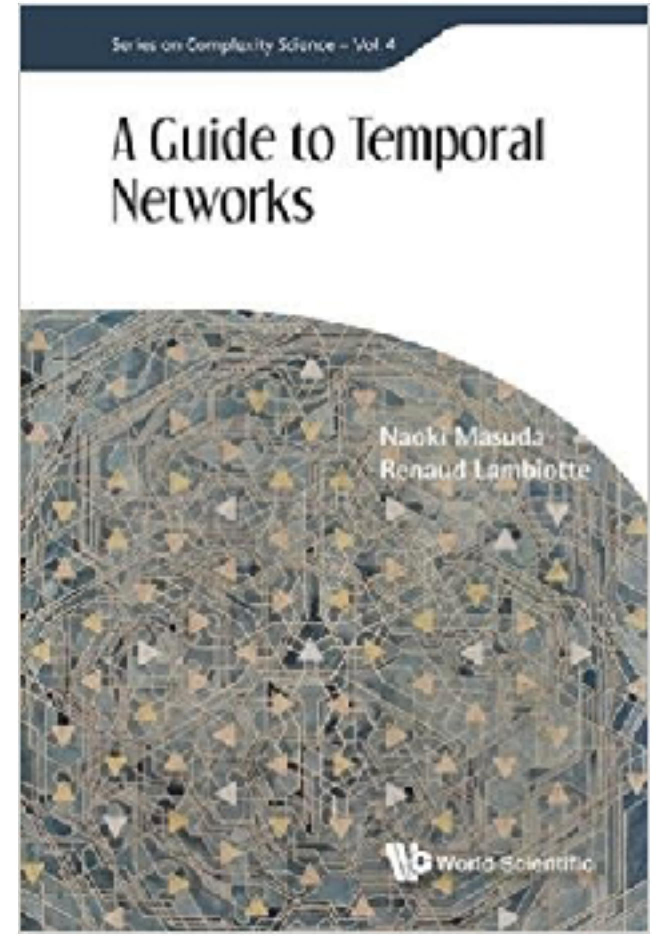
iv) if there exists Λ^{α_c} such that $\Re\lambda_{\alpha_c} > 0$ then Turing patterns do emerge.

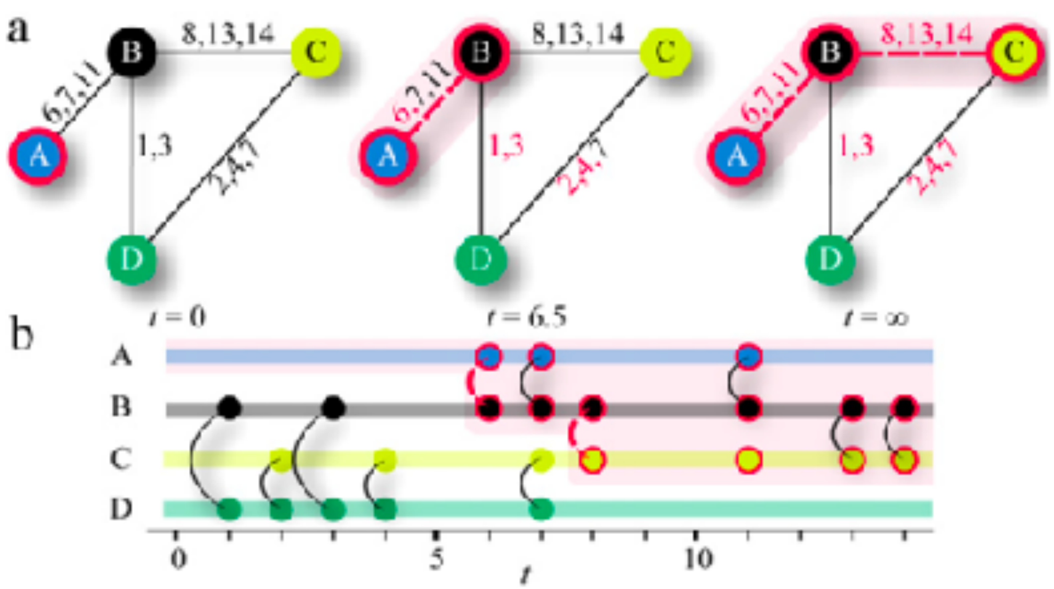




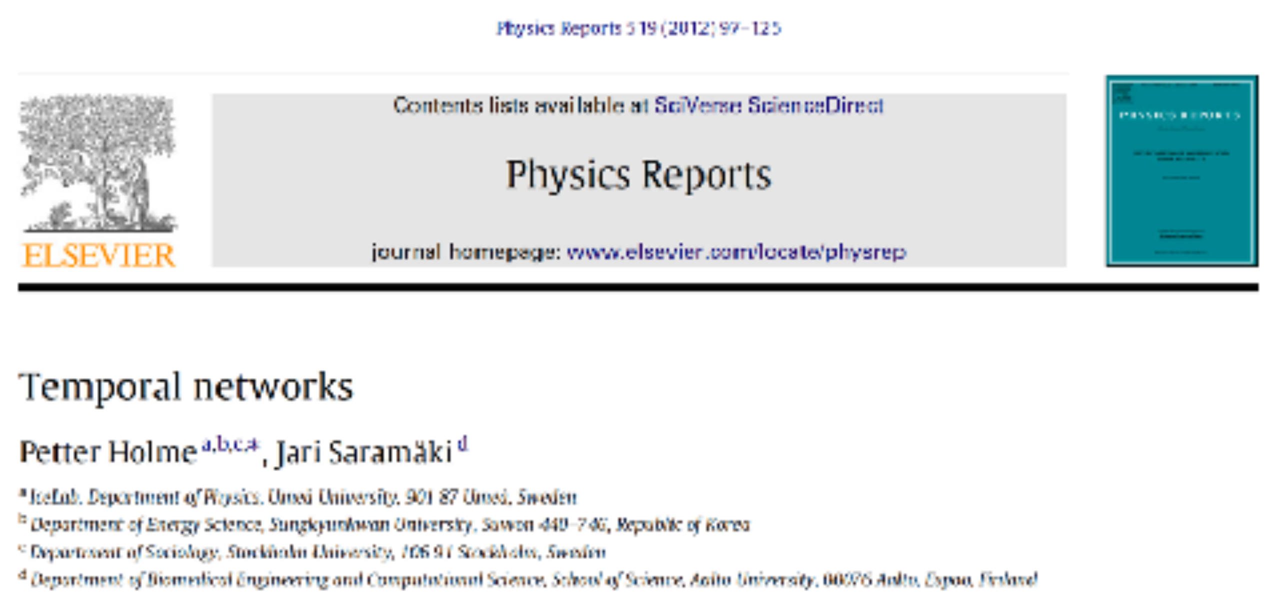
Networks are dynamical objects

Contact social networks





Phone calls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Temporal networks

Petter Holme [a,b,c,*], Jari Saramäki [d]
[a] IceLab, Department of Physics, Umeå University, 901 87 Umeå, Sweden
[b] Department of Energy Science, Sungkyunkwan University, Suwon 440-746, Republic of Korea
[c] Department of Sociology, Stockholm University, 106 91 Stockholm, Sweden
[d] Department of Biomedical Engineering and Computational Science, School of Science, Aalto University, 00076 Aalto, Espoo, Finland

The model: dynamical network

Links can change their weights, fade away, be created or be rewired:

$$A_{ij}(t) = \begin{cases} w_{ij}(t) > 0 & \text{if } i \text{ and } j \text{ are linked at time t} \\ 0 & \text{otherwise} \end{cases}$$

Species can relocate, as follows a standard diffusive mechanism, ruled by the (time dependent) Laplacian operator:

$$L_{ij}(t) = A_{ij}(t) - s_i(t)\delta_{ij}$$

$$s_i(t) = \sum_j A_{ij}(t)$$

The model: reaction-diffusion on dynamical network

possible different time scale

(1) $$\begin{cases} \dot{u}_i(t) &= f(u_i, v_i) + D_u \sum_{j=1}^N L_{ij}(t/\epsilon) u_j(t) \\ \dot{v}_i(t) &= g(u_i, v_i) + D_v \sum_{j=1}^N L_{ij}(t/\epsilon) v_j(t) \end{cases}$$

Can the system exhibit Turing patterns, once the static version doesn't?

Which is the role of network time scale?

Main result: periodic case

Assume the existence of the averaged Laplacian:

$$\langle \mathbf{L} \rangle = \frac{1}{T}\int_0^T \mathbf{L}(t)dt$$

And let us introduce the averaged reaction-diffusion system:

$$(2) \quad \begin{cases} \dot{u}_i(t) &= f(u_i, v_i) + D_u \sum_{j=1}^N \langle L_{ij} \rangle u_j \\ \dot{v}_i(t) &= g(u_i, v_i) + D_v \sum_{j=1}^N \langle L_{ij} \rangle v_j \end{cases}$$

Then, if (2) exhibits Turing patterns, there exists $\epsilon^* > 0$ such that (1) also exhibits Turing patterns $\forall \epsilon : \; 0 < \epsilon < \epsilon^*$

Sketch of the proof

a) Theorem of averaging

Theorem 1 (Averaging in the periodic case) *Let us consider the system*

$$\dot{x} = \epsilon f(t,x)\,, \quad x(0) = x_0 \in D \subset \mathbb{R}^n\,,$$

where f and $\partial_x f$ are defined, continuous and bounded in $[0,\infty) \times \mathbb{R}^n$ and assume $f(t,\cdot)$ to be T-periodic.
Let $\langle f\rangle(y)$ be the time average of $f(t,y)$, that is

$$\langle f\rangle(y) = \frac{1}{T}\int_0^T f(t,y)\,dt\,,$$

and let $y(t)$ be the solution of

$$\dot{y} = \epsilon\langle f\rangle(y)\,, \quad y(0) = x_0 \in D \subset \mathbb{R}^n\,.$$

Then $x(t) - y(t) = \mathcal{O}(\epsilon)$ for $t = \mathcal{O}(1/\epsilon)$.

Sketch of the proof

b) Rewrite 1 and 2 in compact form: $\vec{x} = (u_1, \dots, u_N, v_1, \dots, v_N)$

(b1) $\dot{\vec{x}}(t) = F(\vec{x}) + \mathcal{L}(t/\epsilon)\vec{x}\,,$ $\mathcal{L}(t)\vec{x} := \begin{pmatrix} D_u \mathbf{L}(t) & 0 \\ 0 & D_v \mathbf{L}(t) \end{pmatrix} \vec{x}$

(b2) $\dot{\vec{x}}(t) = F(\vec{x}) + \langle \mathcal{L} \rangle \vec{x}\,.$ $\langle \mathcal{L} \rangle = \begin{pmatrix} D_u \langle \mathbf{L} \rangle & 0 \\ 0 & D_v \langle \mathbf{L} \rangle \end{pmatrix}$

c) Rescale time: $\tau = t/\epsilon$

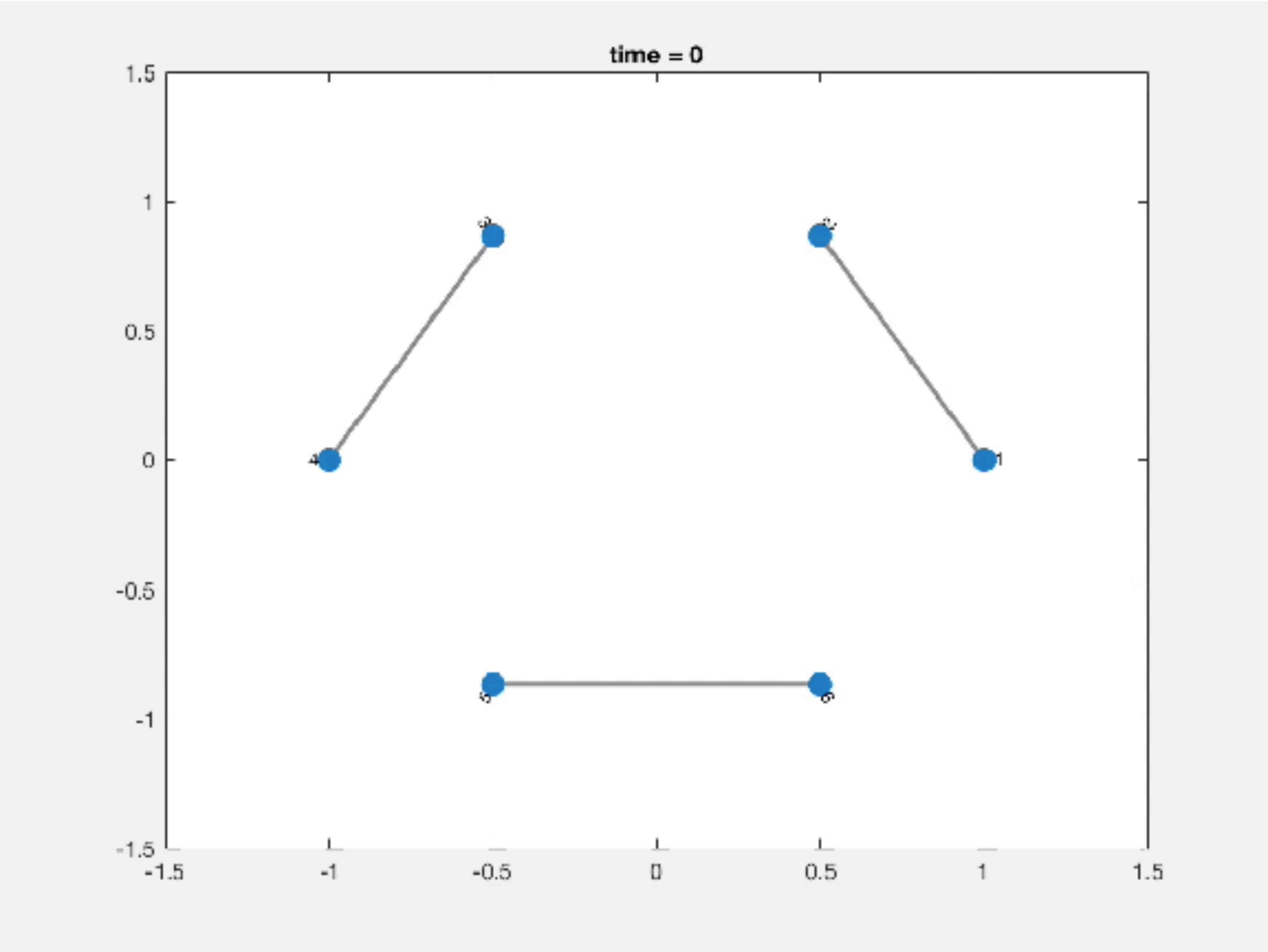
d) Conclude: $\exists\, \epsilon^* > 0$ $\forall \epsilon :\ 0 < \epsilon < \epsilon^*$

Solutions of (b1) & (b2) stay ϵ-close for $t = \mathcal{O}(1)$

e) Hence if the averaged system is unstable and exhibits Turing patterns, also the "accelerated" does.

An exemple: twins network

$$\mathbf{A}(t) = \begin{cases} \mathbf{A}_1, & \text{if } \{t/T\} \in [0, \gamma), \\ \mathbf{A}_2, & \text{if } \{t/T\} \in [\gamma, 1), \end{cases}$$



An exemple: twins network

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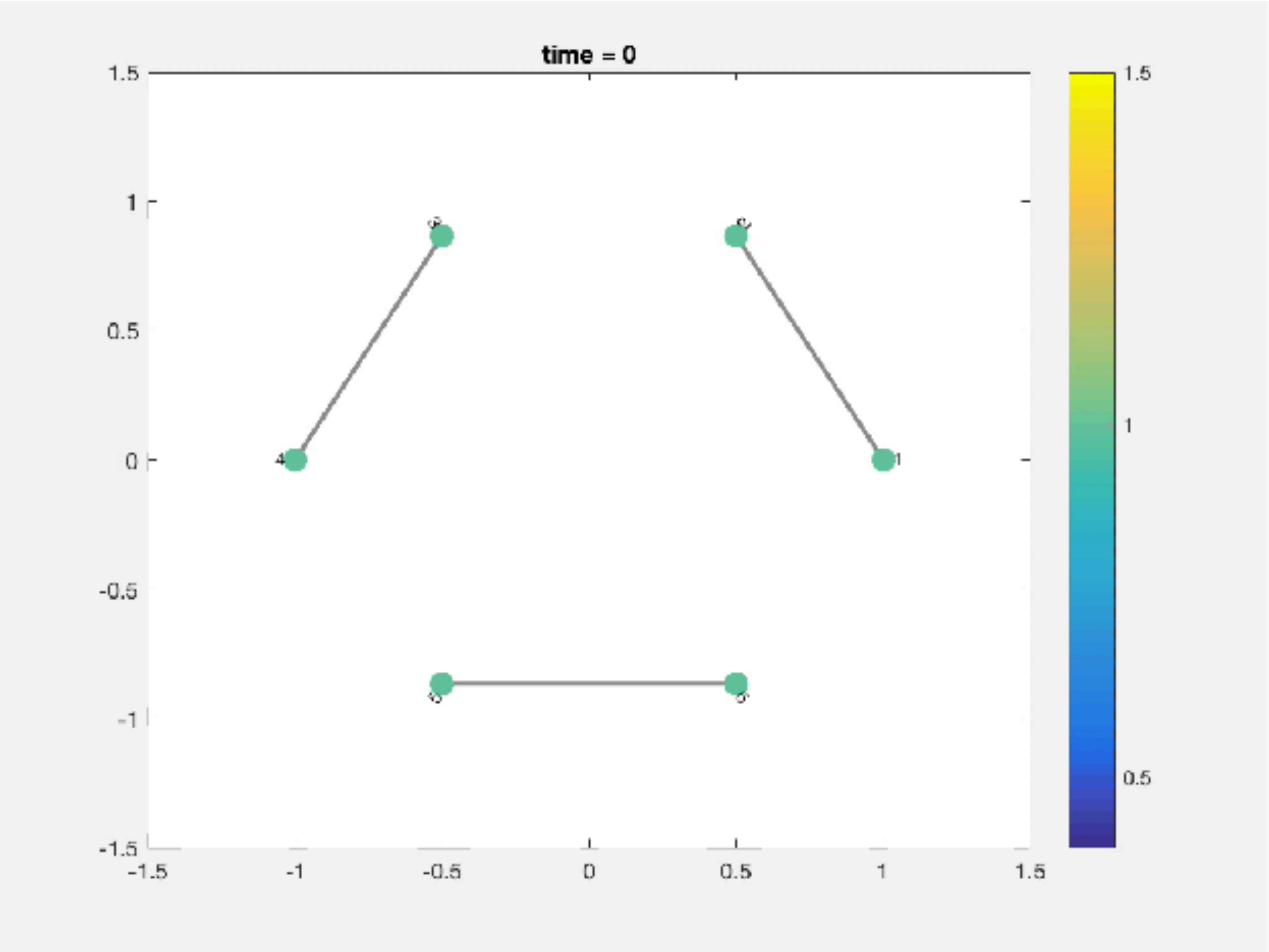
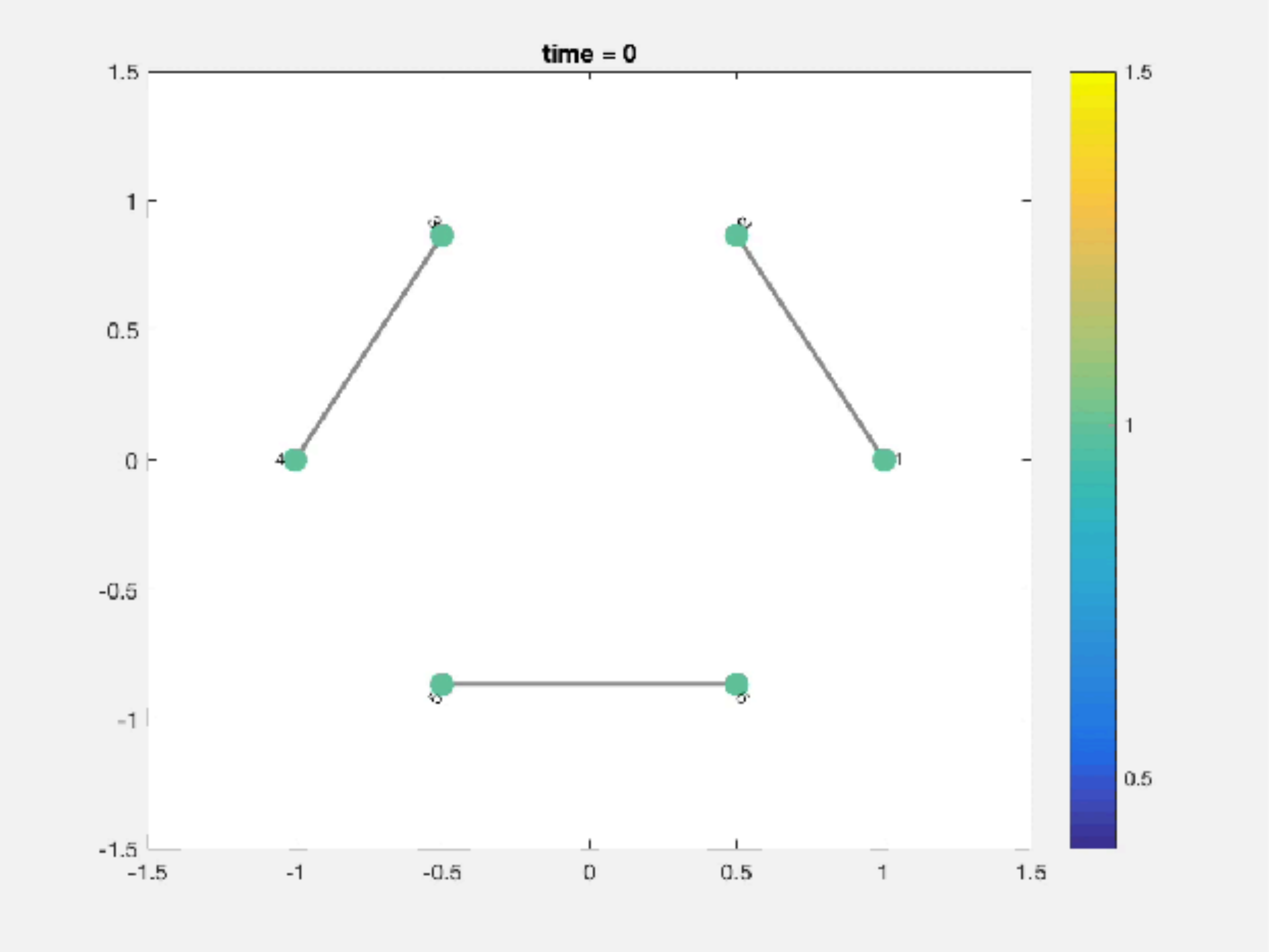
Dynamics on A1 (or A2) fixed



An exemple: twins network

$$\mathbf{A}(t) = \begin{cases} \mathbf{A}_1, & \text{if } \{t/T\} \in [0, \gamma), \\ \mathbf{A}_2, & \text{if } \{t/T\} \in [\gamma, 1), \end{cases}$$

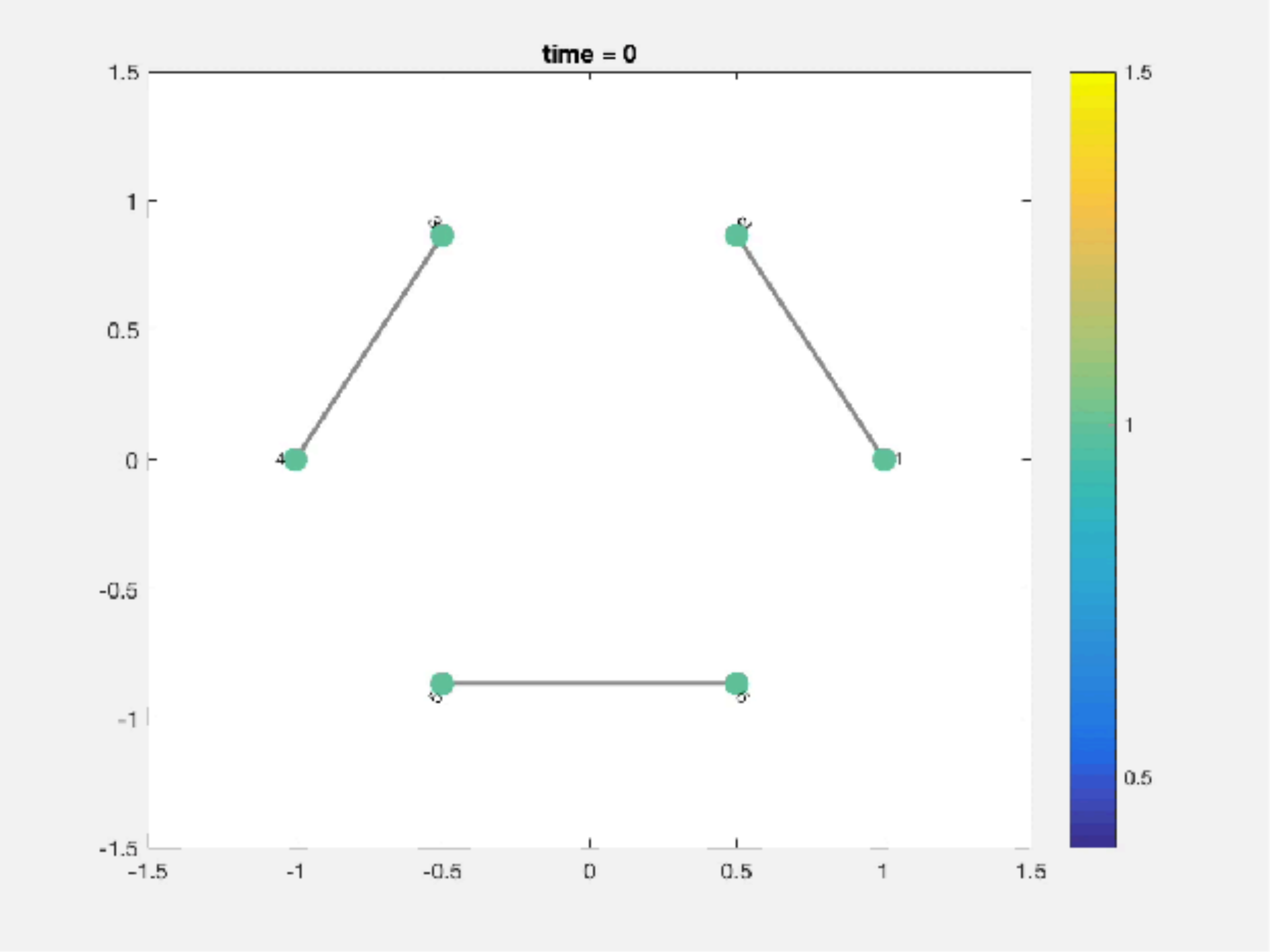
"fast" switch



An exemple: twins network

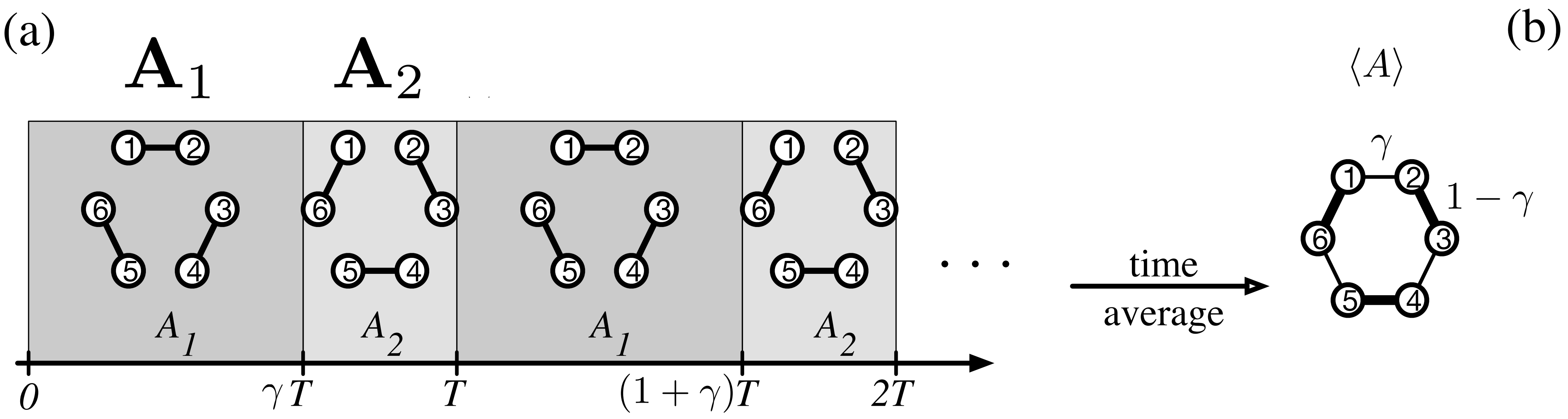
$$\mathbf{A}(t) = \begin{cases} \mathbf{A}_1, & \text{if } \{t/T\} \in [0, \gamma), \\ \mathbf{A}_2, & \text{if } \{t/T\} \in [\gamma, 1), \end{cases}$$

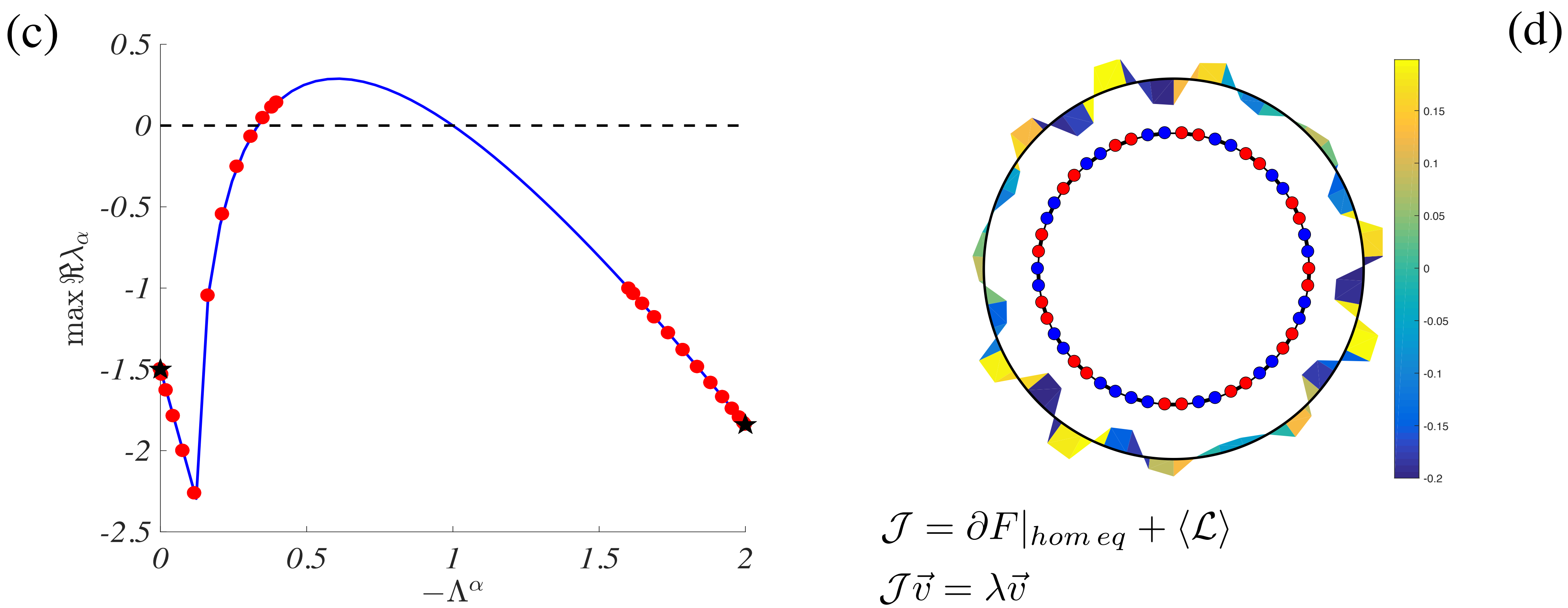
"slow" switch



An exemple: twins network

$$\mathbf{A}(t) = \begin{cases} \mathbf{A}_1, & \text{if } \{t/T\} \in [0, \gamma), \\ \mathbf{A}_2, & \text{if } \{t/T\} \in [\gamma, 1), \end{cases}$$





$$\mathcal{J} = \partial F|_{hom\, eq} + \langle \mathcal{L} \rangle$$

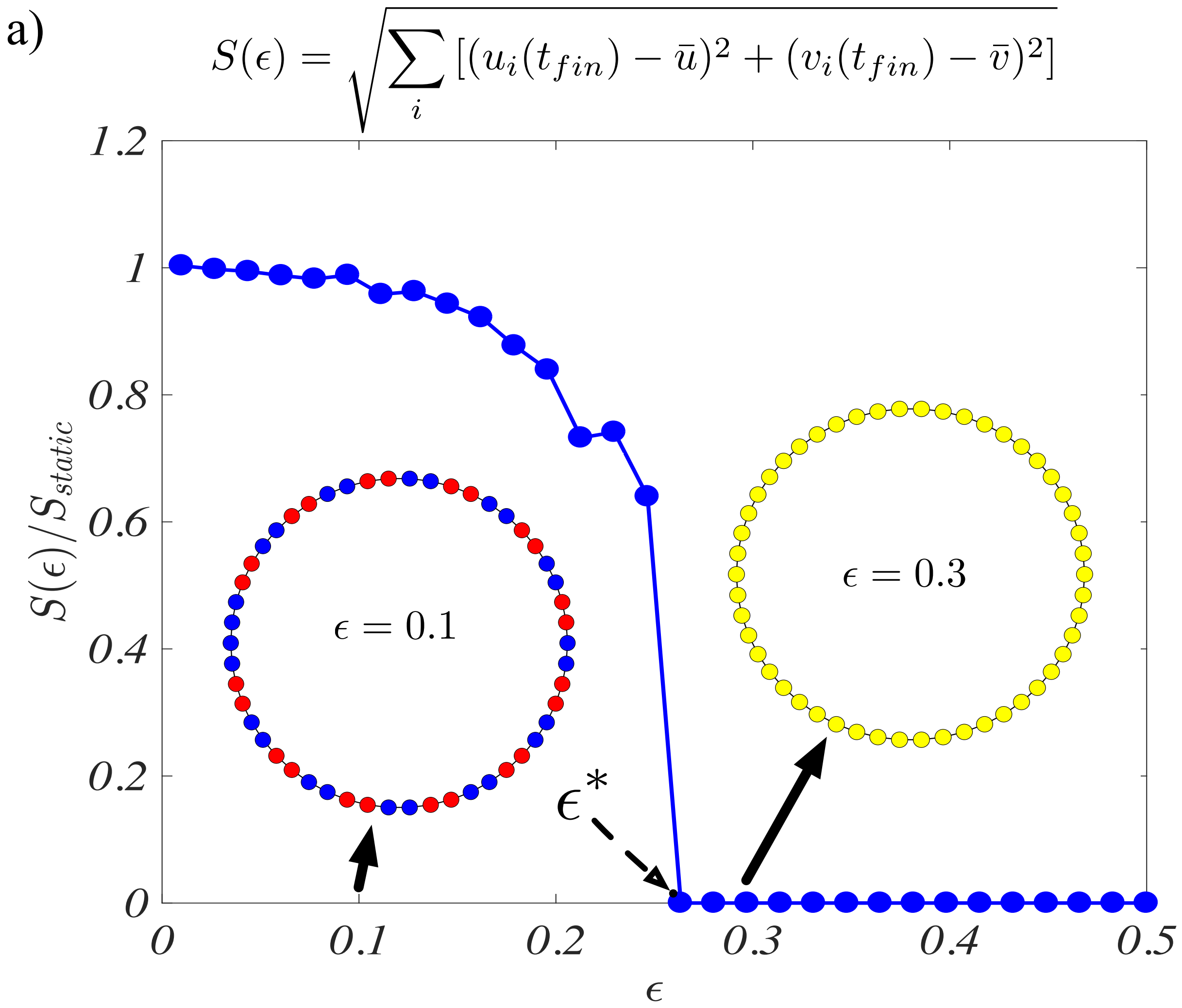
$$\mathcal{J}\vec{v} = \lambda\vec{v}$$

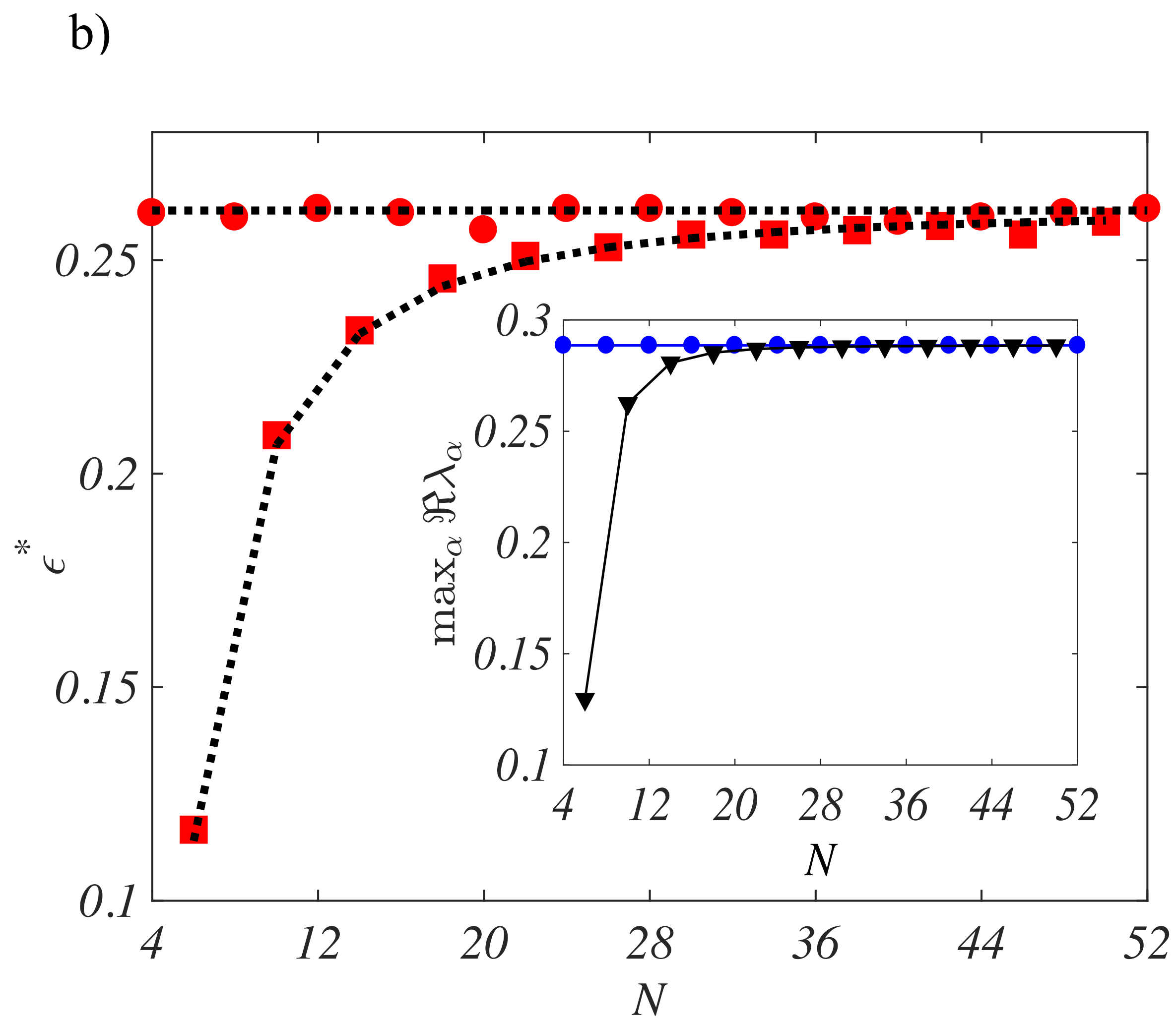
Critical threshold: twins network

$$\dot{\delta \vec{x}} = \mathbf{M}_c(t)\delta\vec{x} \qquad \text{with} \qquad \mathbf{M}_c(t) = \partial_x F(\bar{x}) + \mathcal{L}(t/\epsilon)$$

Monodromy matrix (acc. case): $\mathbf{Q}_\epsilon = e^{\epsilon \mathbf{M}_2 (1-\gamma) T} e^{\epsilon \mathbf{M}_1 \gamma T}$

$$\epsilon^* = \min\{\epsilon > 0: \quad \forall\ s \geq \epsilon : \rho(\mathbf{Q}_s) \leq 1\}$$





Generalizations

i) Continuous non-periodic case (use Floquet-Magnus expansion)

$$\langle \mathbf{L} \rangle = \lim_{T\to\infty} \frac{1}{T} \int_0^T \mathbf{L}(t)\, dt$$

ii) Discrete time random switching:

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \mathbf{L}(t) dt = \mathbb{E}\left[\mathbf{L}(t)\right] \quad \text{(almost surely)}$$

Oscillation death

$$\dot{x}_i = f(x_i, y_i) + D_x \sum_{j=1}^{N} L_{ij}(t/\epsilon) x_j,$$

$$\dot{y}_i = g(x_i, y_i) + D_y \sum_{j=1}^{N} L_{ij}(t/\epsilon) x_j,$$

Admits a T-periodic solution $(x_i, y_i) = (\bar{x}(t), \bar{y}(t))$ once $D_x = D_y = 0$

Let $D_x > 0$ and $D_y > 0$

The system <u>synchronizes</u>, if such solution is stable.

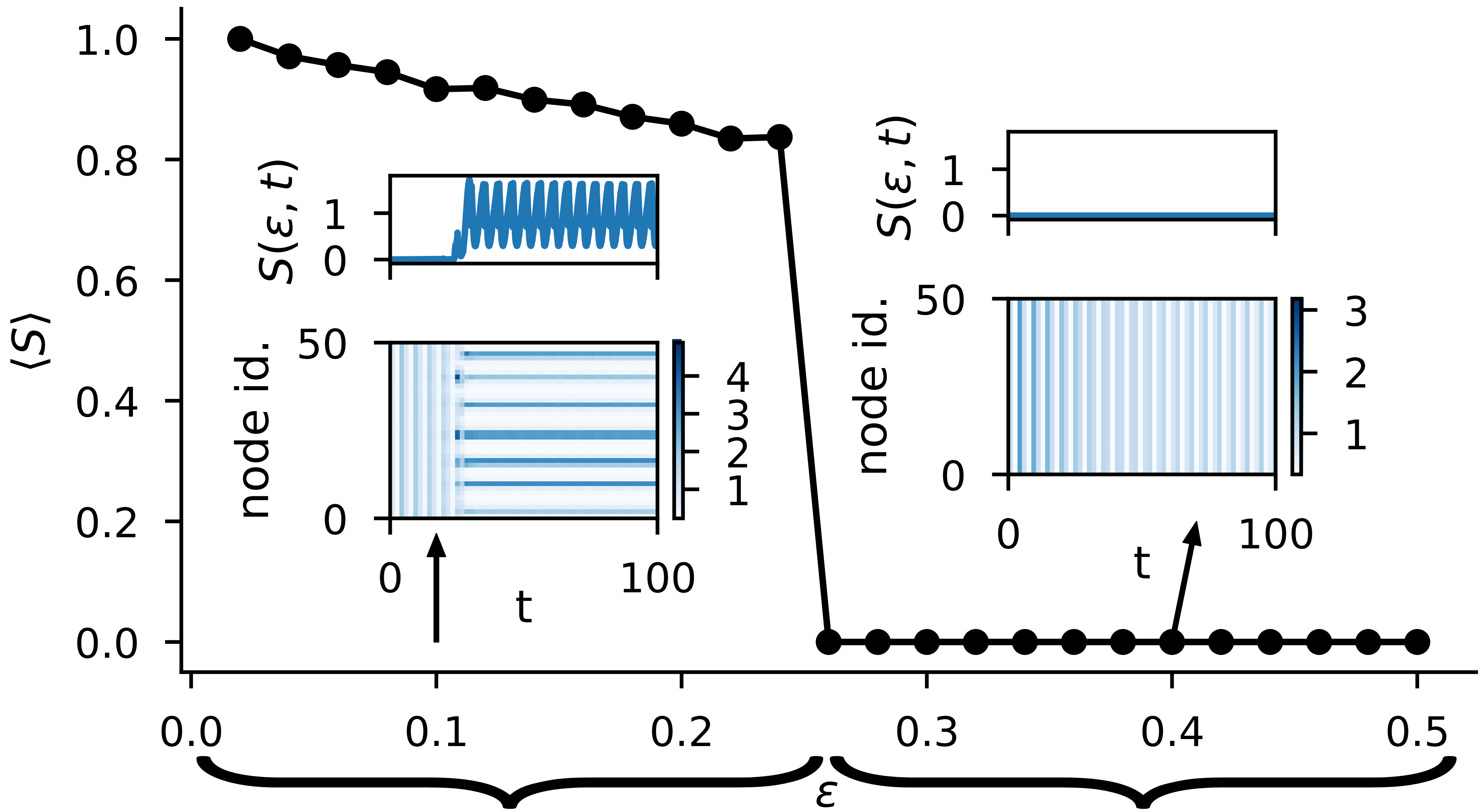
The <u>oscillations can die out</u>, if stability is lost.

Oscillation death induced by fast switching

(Twins network)

$$S(\epsilon, t) = \frac{1}{N}\|\mathbf{x}(t) - \bar{\mathbf{x}}(t)\|^2$$

$$\langle S\rangle = \frac{1}{T_s}\int_t^{t+T_s} S(\epsilon, u)\, du$$



Oscillation death

Synchronization is stable

Sketch of the proof

Use again the theorem of averaging but in two steps

$$\dot{\mathbf{x}} = \mathcal{F}(\mathbf{x}) + \mathcal{L}(t/\epsilon)\mathbf{x},$$

linearize close to T-periodic solution and change time scale

$$\delta\mathbf{x}' = \epsilon[(\mathcal{J}(\epsilon\tau) + \mathcal{L}(\tau)]\delta\mathbf{x}$$

1

$$\langle \mathbf{L} \rangle = \frac{1}{T_s}\int_0^{T_s} \mathbf{L}(t)dt$$

$$\dot{\mathbf{y}} = \mathcal{F}(\mathbf{y}) + \langle \mathcal{L} \rangle \mathbf{y}$$

2

partial averaging

Linearization and change time scale

$$\delta\mathbf{y}' = \epsilon\left[\mathcal{J}(\epsilon\tau) + \langle \mathcal{L} \rangle\right]\delta\mathbf{y}$$

Use Floquet and the eigenbasis of $\langle \mathcal{L} \rangle$

Papers for this work

Theory of Turing Patterns on Time Varying Networks, J. Petit, B. Lauwens, D. Fanelli, T. Carletti, Physical Review Letters, **119**, pp. 148301-1–5, (2017)

Oscillation death induced by time-varying network, M. Lucas, D. Fanelli, T. Carletti, J. Petit, arXiv:1802.06580, (2018)

Some papers

Tune the topology to create or destroy patterns, M. Asllani, T. Carletti, D. Fanelli, European Physical Journal B. **89**, pp. 260 (2016)

Pattern formation in a two-component reaction-diffusion system with delayed processes on a network, J. Petit, M. Asllani, D. Fanelli, B. Lauwens, T. Carletti, Physica A, **462**, pp.230, (2016)

Delay induced Turing-like waves for one species reaction–diffusion model on a network, J. Petit, T. Carletti, M. Asllani, D. Fanelli, Europhysics Letters. **111**, 5, pp. 58002, (2015)

Turing instabilities on Cartesian product networks, M. Asllani, D.M. Busiello, T. Carletti, D. Fanelli, G. Planchon, Scientific Reports. **5**, pp. 12927, (2015)

Turing patterns in multiplex networks, M. Asllani, D.M. Busiello, T. Carletti, D. Fanelli, G. Planchon, Physical Review E ,**90**, 4, pp. 042814, (2014)