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Turing patterns on time varying networks



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naxys
Namur Institute for Complex Systems

Acknowledgements

Main collaborators working on “Turing patterns”

“Belgian” team:

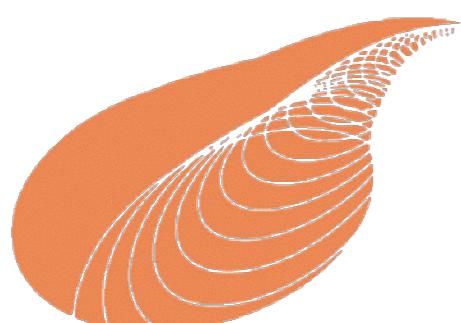
M. Asllani, N. Kouvaris (post docs)

J. Petit (PhD)

A. Bellière, G. Planchon, R. Muolo (Master students)

Italian team:

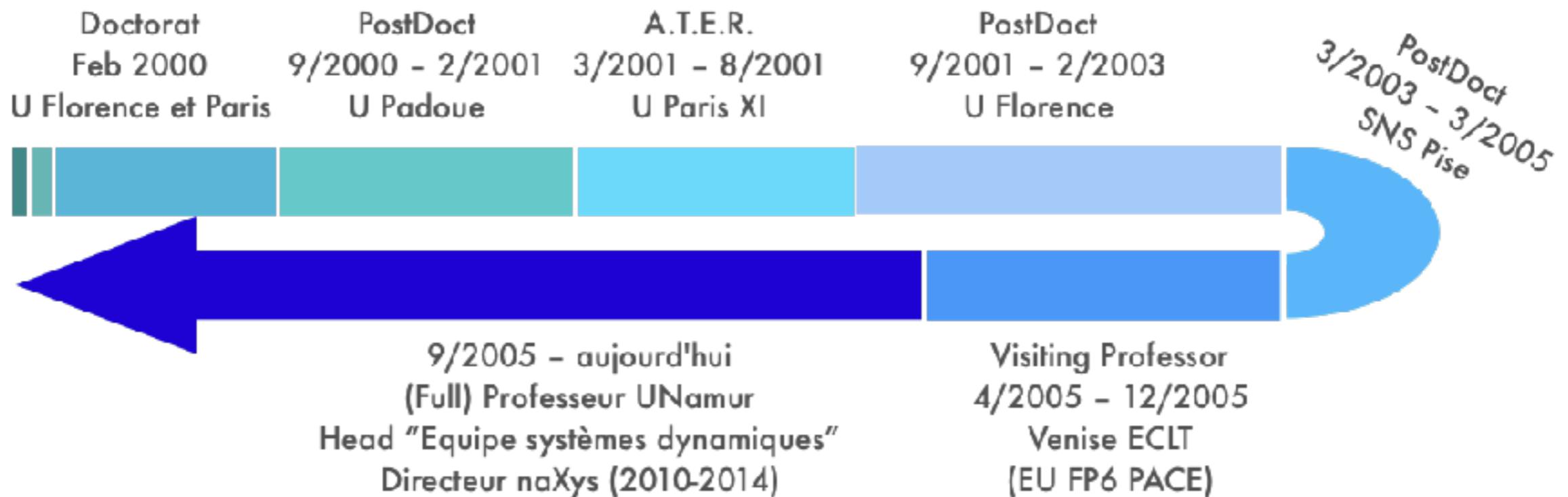
D. Fanelli, D.M. Busiello, C. Cianci, M. Galanti, F. Miele,
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IAP VII/19 - DYSCO



Snapshot of my career



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naxys
Network Center for Complex Systems

Word cloud of the titles of my papers



My research in a nutshell

Abstract models:

- Simple/simplified models, with relatively few variables and parameters, relatively under control.
- Generic questions and answers model independent.
- Ex.: networked systems, chaos/stability, emergent behaviours ...

Data driven models:

- Model created from data, usually a large number of parameters and variables.
- Specific questions and model dependent.
- Ex.: researchers careers, synthetic populations, circadian rhythms in wikipedia ...

Algorithms & tools

- Heuristic optimization algorithms.
- Neural networks.
- Numerical solvers, ...

Abstract models:

Simple/simplified models, with relatively few variables and parameters, relatively under control.

- ~~Generic questions and answers model independent.~~
- Ex.: networked systems, chaos/stability, emergent behaviours ...

Patterns formation in networked systems

Outlook:

- **Introduction**
- **Networked systems**
- **Turing patterns (static networks)**
- **TP in time varying networks**
- **Two examples**
- **Desynchronization (if time allows)**
- **Conclusions**

Pattern ? [ref. Oxford dictionary]

pattern

★ Top 1000 frequently used words

Pronunciation: /'pat(ə)n/ (?)

NOUN

1 A repeated decorative design:
'a neat blue herringbone pattern'

– More example sentences

'Included are geometrics, florals and foliates, animals and nature motifs and other decorative repeat patterns.'

'These aspects then become ornamented with Islamic-inspired decorative patterns and Islamic cultural artifacts.'

'It featured exuberant decorative patterns, designs in the brickwork and wooden attachments.'

1 A repeated decorative design:
'a neat blue herringbone pattern'

+ More example sentences

+ Synonyms

1.1 An arrangement or design regularly found in comparable objects:
'the house had been built on the usual pattern'

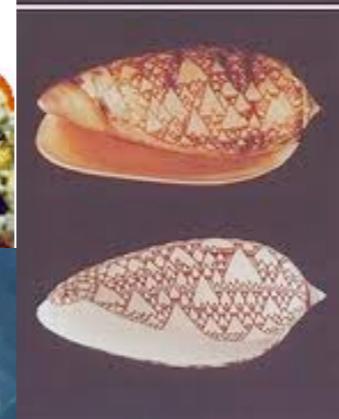
– More example sentences

'Structurally, the tumor cells were arranged in a medullary pattern composed of polygonal tumor cells.'

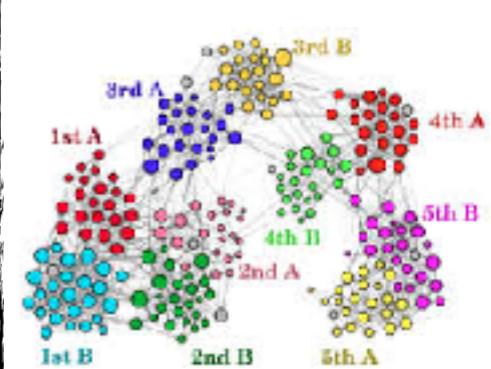
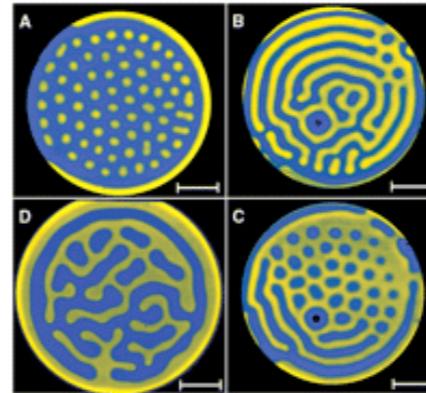
'The hair-cells within the spiralling cochlear duct are arranged in a pattern like the bristles of a brush.'

'The fossils indicate the wings had feathers, arranged in a similar pattern to that of modern birds.'

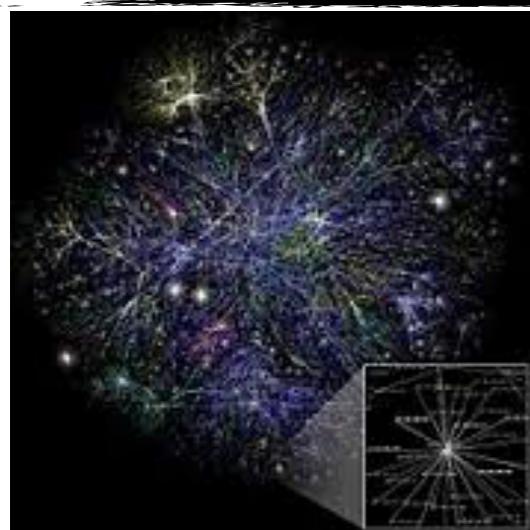
Patterns are ubiquitous



Animal kingdom



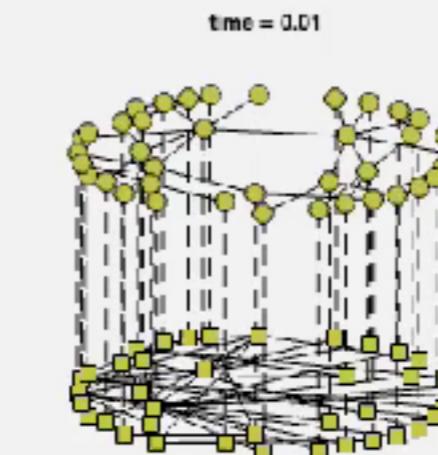
SocioPatterns



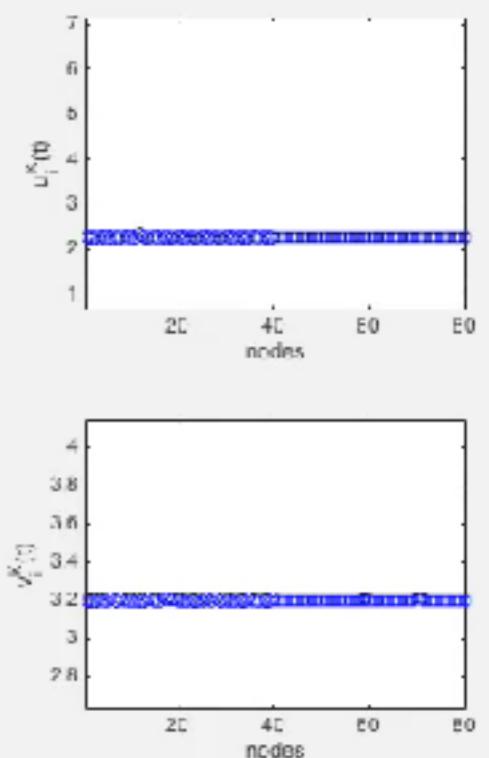
Internet



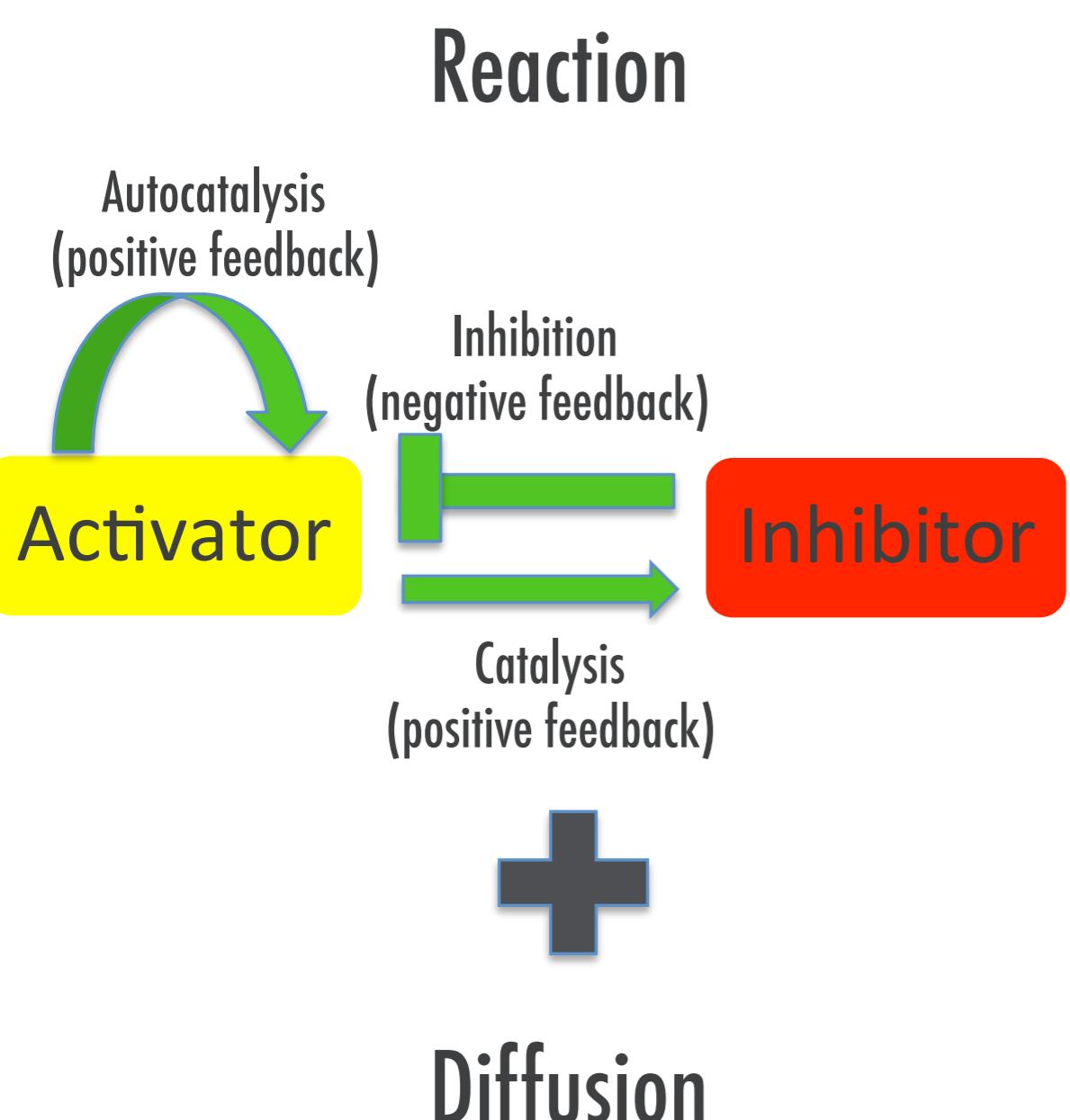
Twitter



Math models



One possible mechanism: Turing instability



$u(x, y, t)$: Amount of activator at time t and position (x, y)

$v(x, y, t)$: Amount of inhibitor at time t and position (x, y)

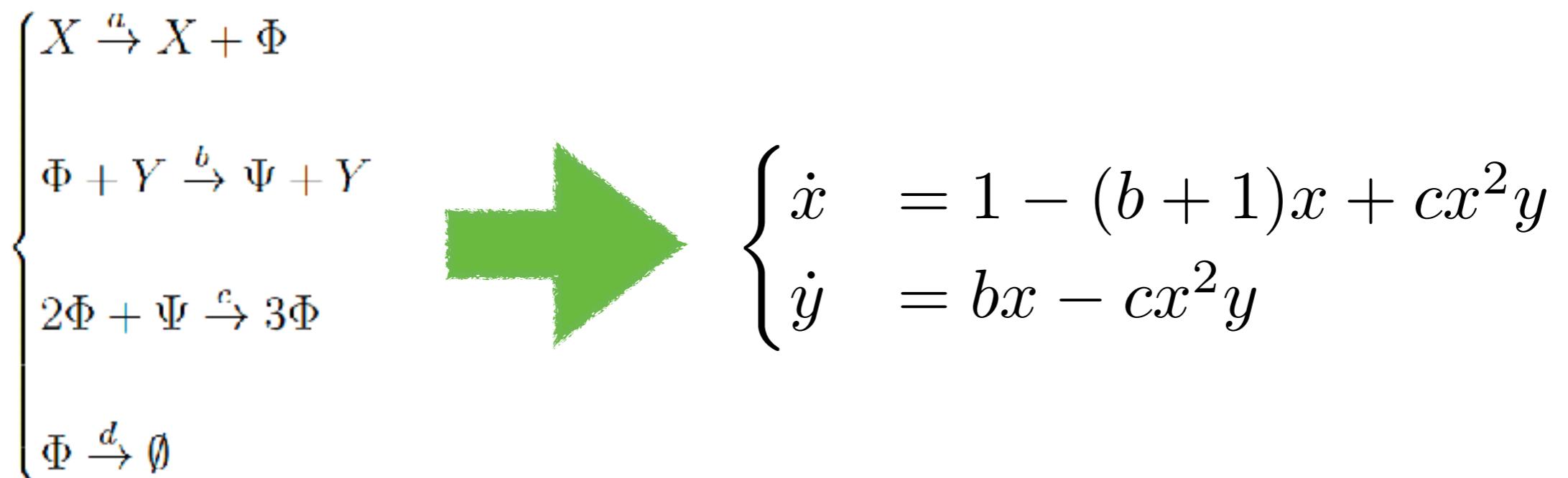
$$\begin{cases} \frac{\partial u}{\partial t} = f(u, v) + D_u \nabla^2 u \\ \frac{\partial v}{\partial t} = g(u, v) + D_v \nabla^2 v \end{cases}$$

$$(x, y) \in \Omega$$

+ boundary conditions
+ initial condition

A.M.Turing, The chemical basis of morphogenesis, Phil. Trans. R Soc London B, 237, (1952), pp.37

The Brusselator model



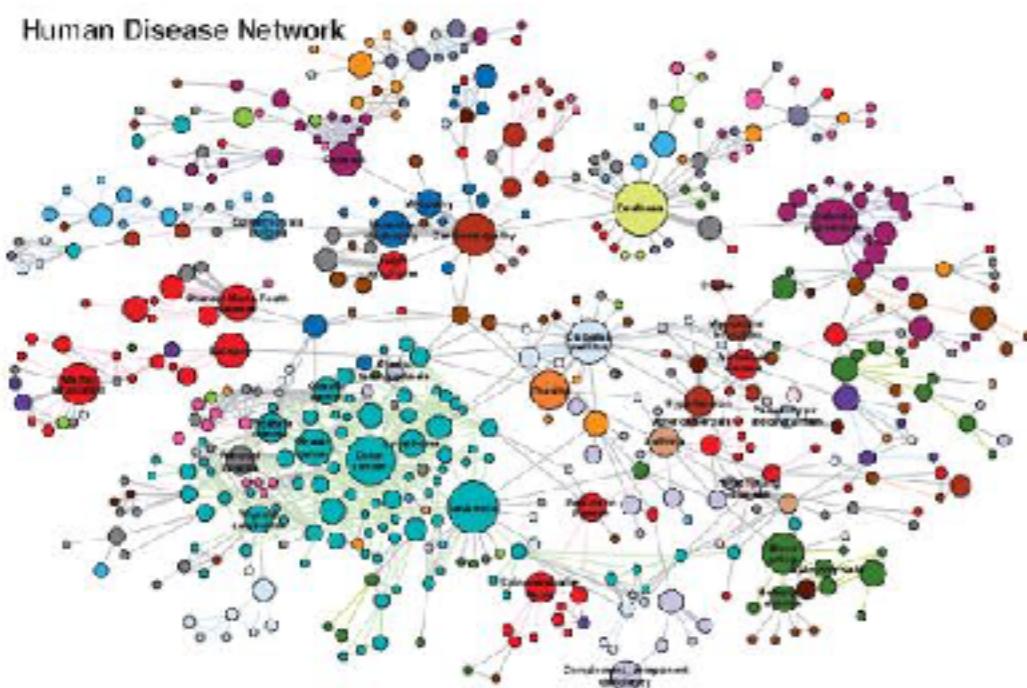
Prigogine I. & Nicolis G., Symmetry Breaking Instabilities in Dissipative Systems, Journal of Chemical Physics (1967)

Prigogine I. & Lefever R., Symmetry Breaking Instabilities in Dissipative Systems. II, Journal of Chemical Physics (1968)

Networks are everywhere

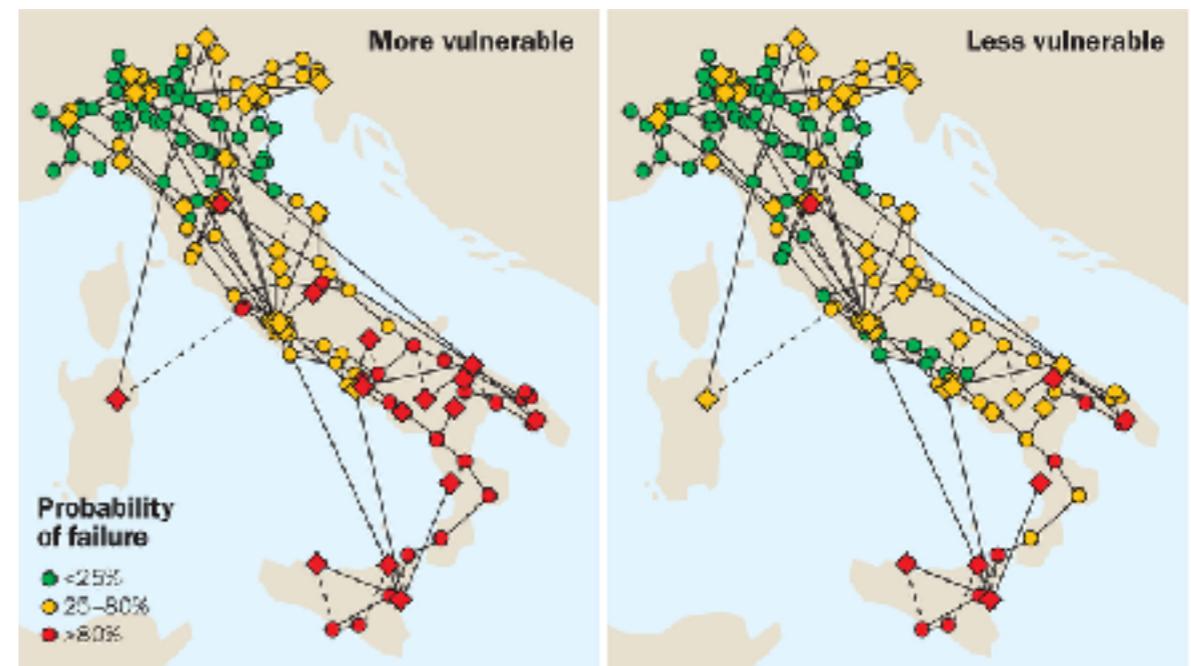
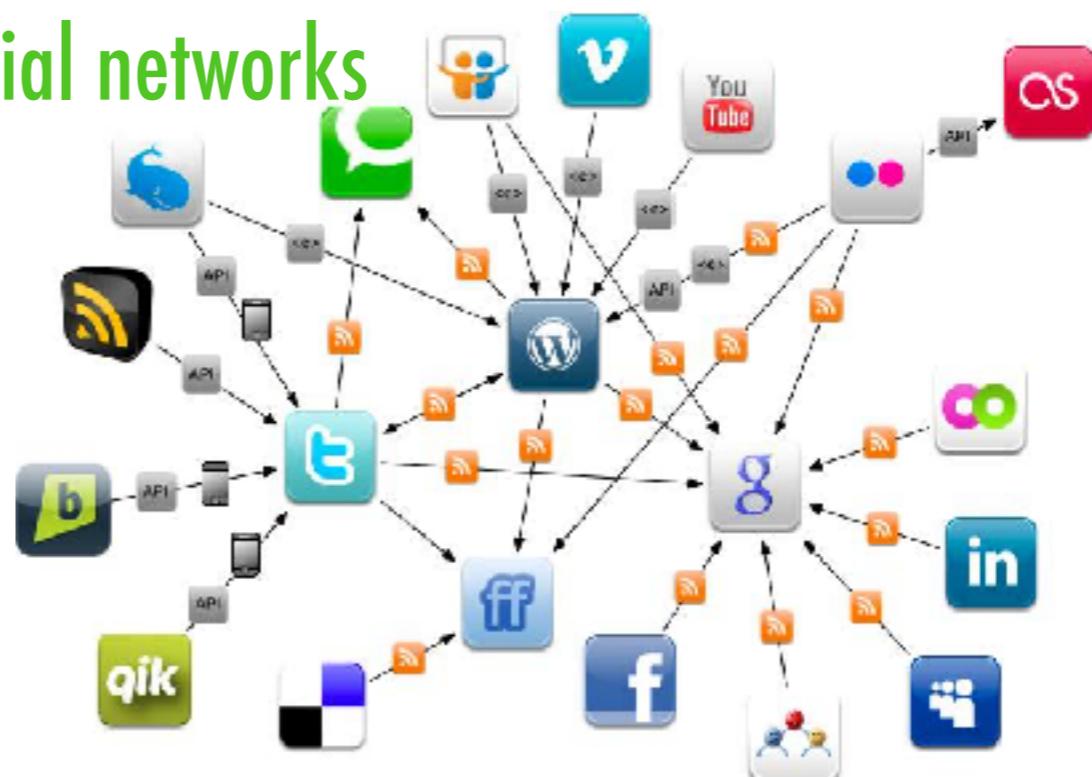


world flights map



proteins networks

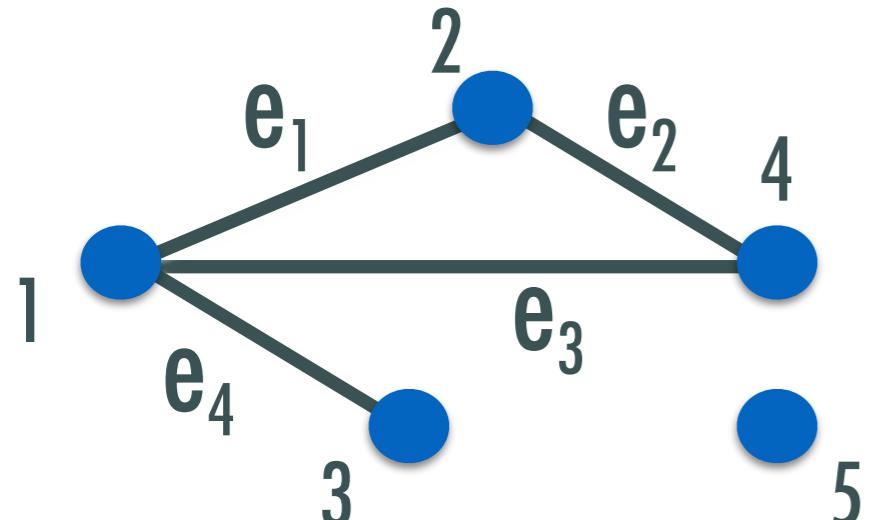
social networks



technological networks

(complex) Networks: some definitions

A network is a set of nodes connected by links (edges)



Ex.: 5 nodes and 4 edges (undirected)

Adjacency matrix

$$A_{ij} = \begin{cases} 1 & \text{if nodes } i \text{ and } j \text{ are linked} \\ 0 & \text{otherwise} \end{cases}$$

The number of links entering (going out) from each node is called in-degree (out-degree)

Ex.: degree node 1 = 3

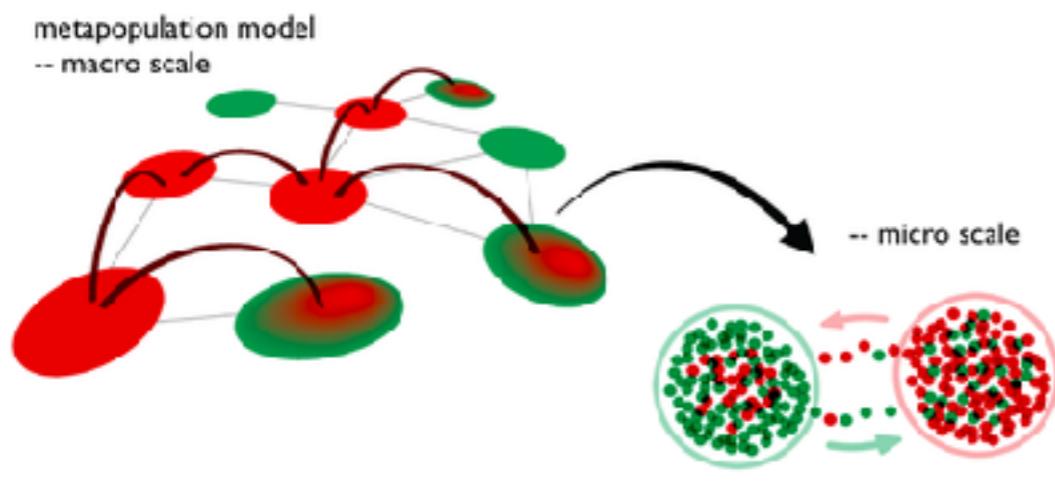
degree nodes 2 & 4 = 2

degree node 3 = 1

degree node 5 = 0

A network is said to be complex if the degree distribution is not trivial, i.e. not constant (lattice) nor Poissonian (random, Erdős-Rényi)

Extension to networks



Metapopulation models
e.g. in the framework of ecology:

May R., Will a large complex system be stable?
Nature, 238, pp. 413, (1972)

Reactions occur at each node. Diffusion occurs across edges.

Patterns

sets of nodes whose asymptotic state is far from the homogeneous equilibrium.

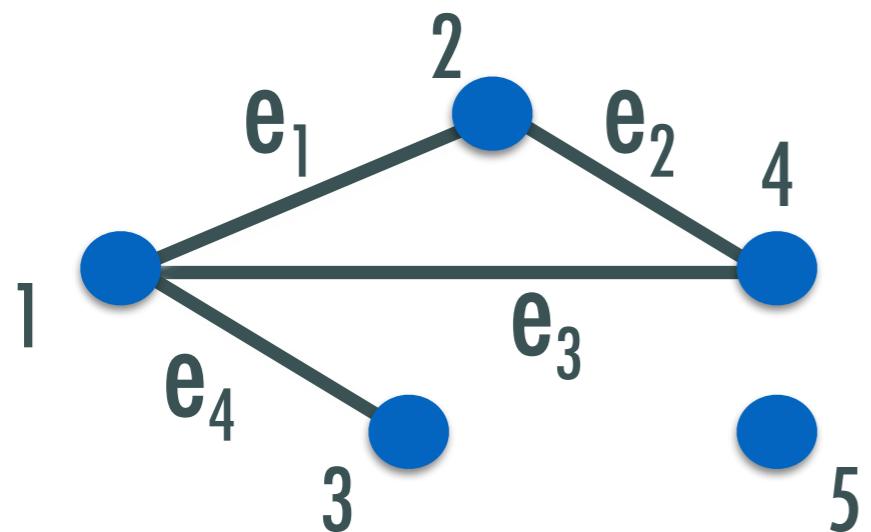
Reaction term:

$$\begin{cases} \dot{u}_i(t) = f(u_i(t), v_i(t)) \\ \dot{v}_i(t) = g(u_i(t), v_i(t)) \end{cases} \quad \forall i = 1, \dots, n \text{ and } t > 0.$$

At each node $i=1,\dots,n$, “species” u and v react through some non-linear functions f and g depending on the quantities available at node i -th
(metapopulation assumption)

Diffusion term:

Diffusive transport of species into a certain node i is given by the sum of incoming fluxes to node i from other connected nodes j , fluxes are proportional to the concentration difference between the nodes (Fick's law).



Ex.: consider the amount of u in node 1,
 u can enter from 2, 3 and 4
 u can leave 1 to go to 2, 3 and 4

$$u_2 + u_3 + u_4 - 3u_1 = \sum_j A_{1j}u_j - k_1u_1 = \sum_j (A_{1j} - \delta_{1j}k_j) u_j := \sum_j L_{1j}u_j$$

L is called Laplacian matrix of the network

The model:

$$\begin{cases} \dot{u}_i(t) &= f(u_i(t), v_i(t)) + D_u \sum_{j=1}^n L_{ij} u_j(t) \\ \dot{v}_i(t) &= g(u_i(t), v_i(t)) + D_v \sum_{j=1}^n L_{ij} v_j(t) \end{cases} \quad \forall i = 1, \dots, n \text{ and } t > 0.$$

D_u and D_v are the diffusion coefficients of species u and v

Observe that because the network is undirected, the matrices A and L are symmetric

General strategy for the network case

1) Assume there exists a spatially homogeneous solution:

$$(u_i, v_i) = (\hat{u}, \hat{v}) \quad \forall i = 1, \dots, n$$

which moreover is stable when there is no diffusion:

$$D_u = D_v = 0$$

2) Linearize around this solution

$$\begin{aligned} u_i &= \hat{u} + \delta u_i \\ v_i &= \hat{v} + \delta v_i \end{aligned} \quad \left(\begin{array}{c} \dot{\delta u} \\ \dot{\delta v} \end{array} \right) = \tilde{\mathcal{J}} \left(\begin{array}{c} \delta u \\ \delta v \end{array} \right)$$

$$\tilde{\mathcal{J}} = \begin{pmatrix} f_u \mathbf{I}_n + D_u L & f_v \mathbf{I}_n \\ g_u \mathbf{I}_n & g_v \mathbf{I}_n + D_v L \end{pmatrix}$$

General strategy for the network case

3) Prove that (possibly) the spatially homogeneous solution:

$$(u_i, v_i) = (\hat{u}, \hat{v}) \quad \forall i = 1, \dots, n$$

turns out to be unstable once the diffusion is in action

$$D_u > 0 \text{ and } D_v > 0$$

Sketch of the proof

i) Let $L\vec{\phi}^\alpha = \Lambda^\alpha \vec{\phi}^\alpha$, $\alpha = 1, \dots, n$ $\vec{\phi}^\alpha = (\phi_1^\alpha, \dots, \phi_n^\alpha)$

$$\sum_i \phi_i^\alpha \phi_i^\beta = \delta_{\alpha\beta} \quad \Lambda^\alpha \leq 0$$

ii) decompose the solution on the eigenbasis and use the ansatz

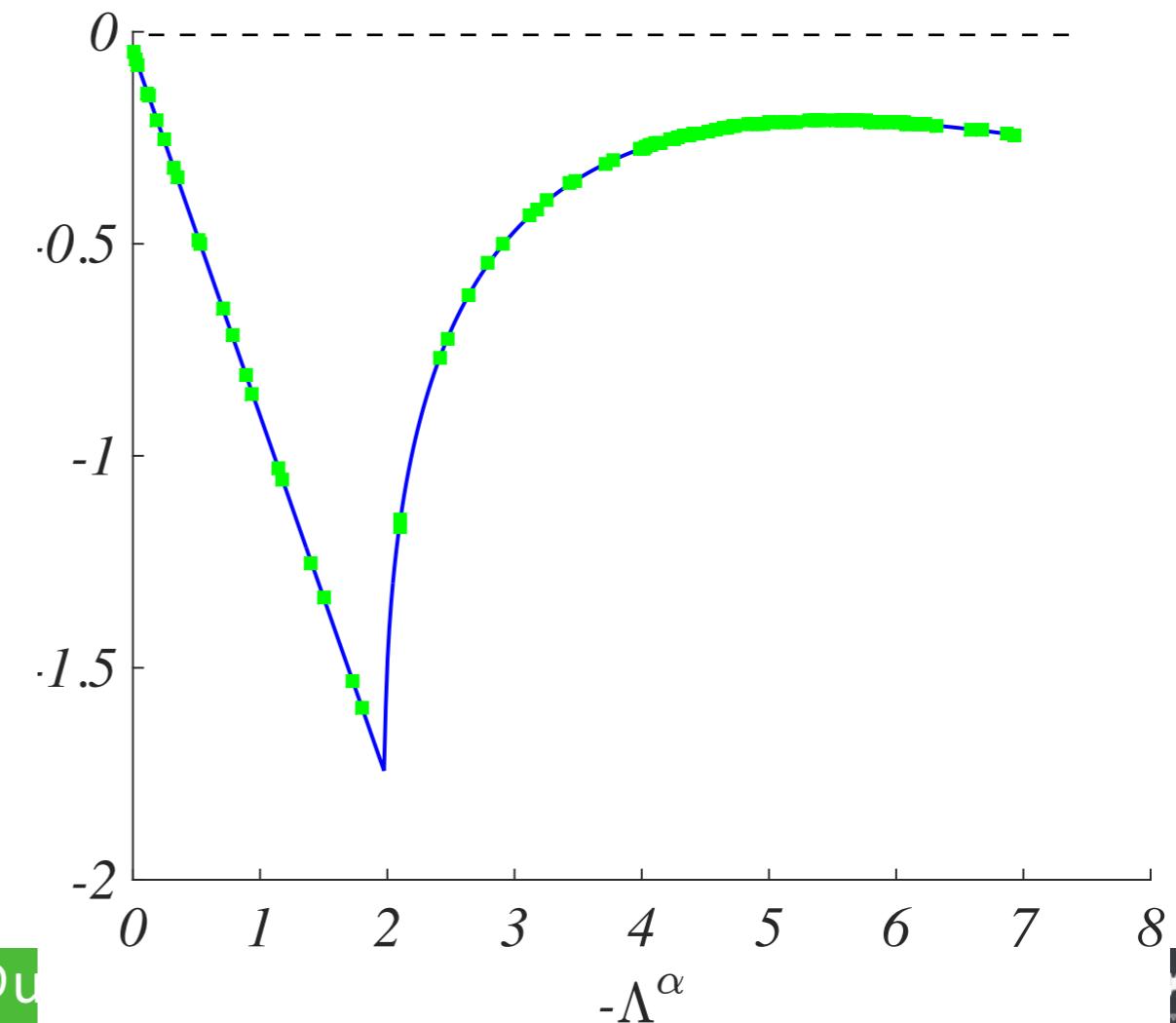
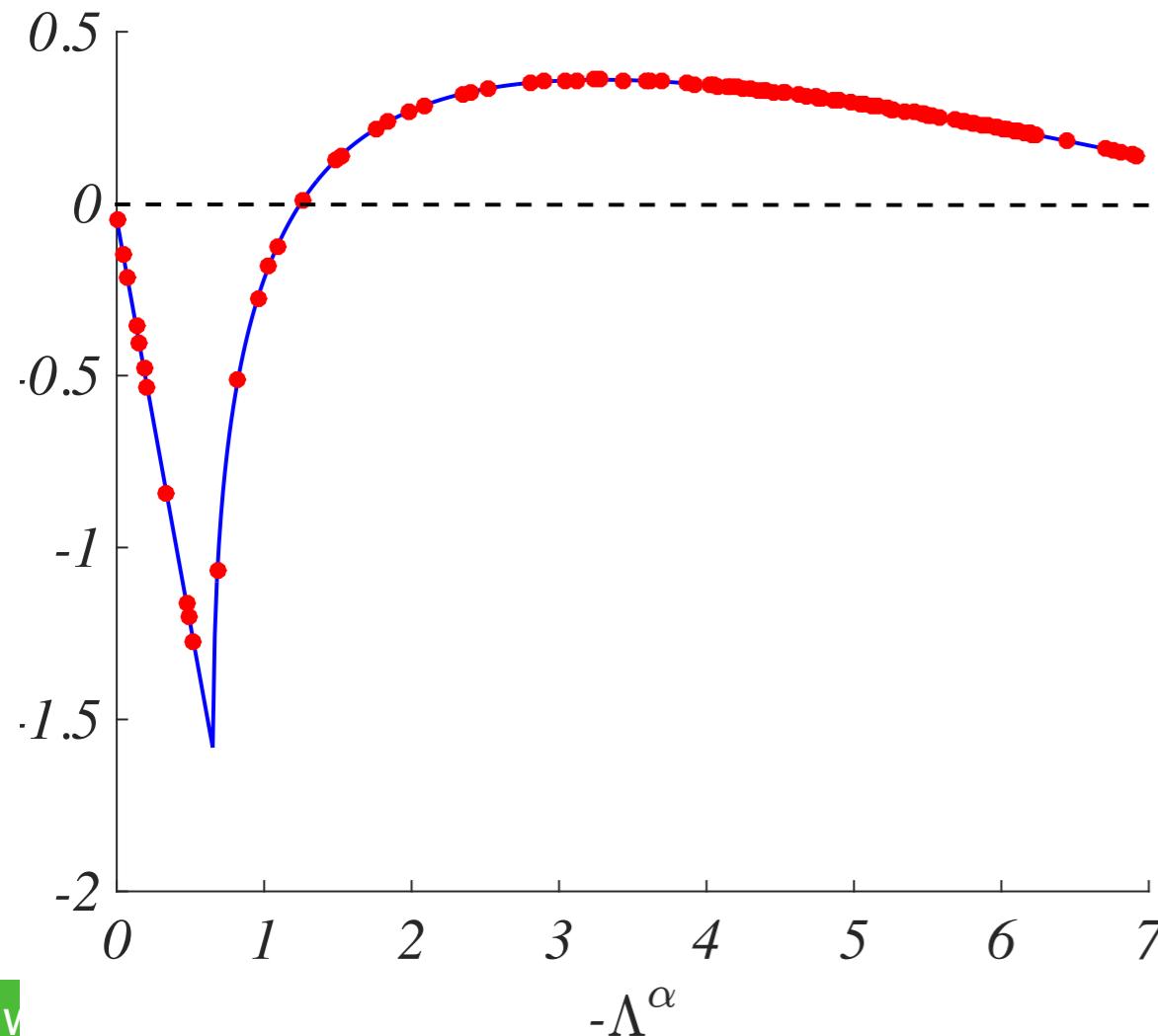
$$\delta u_i(t) = \sum_{\alpha=1}^n c_\alpha \phi_i^\alpha e^{\lambda_\alpha t}$$

General strategy

iii) λ_α (whose real part is called **relation dispersion**) is solution of

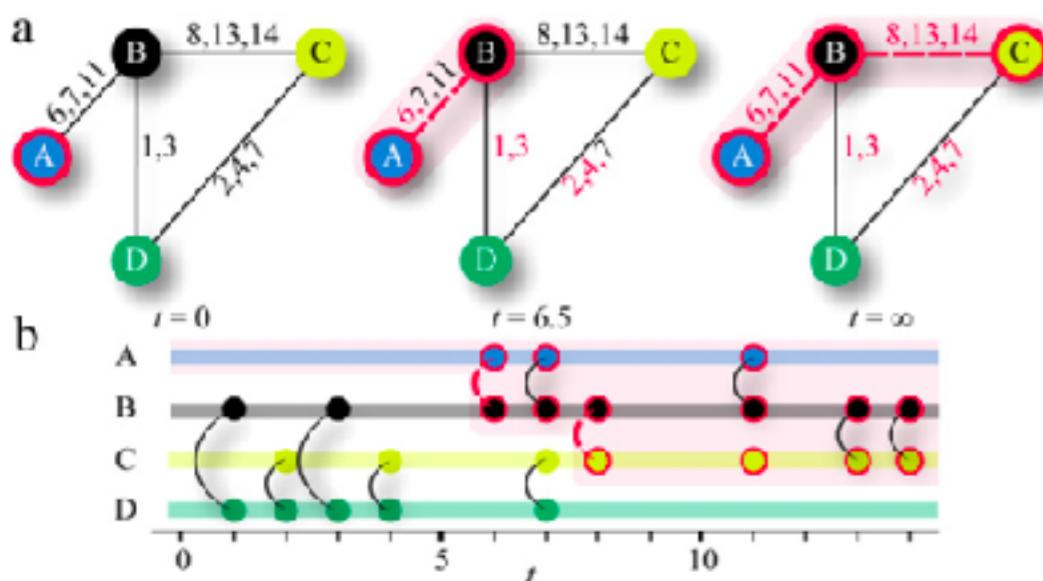
$$\det \left[\lambda_\alpha - \begin{pmatrix} f_u + D_u \Lambda^\alpha & f_v \\ g_u & g_v + D_v \Lambda^\alpha \end{pmatrix} \right] = 0$$

iv) if there exists Λ^{α_c} such that $\Re \lambda_{\alpha_c} > 0$ then Turing patterns do emerge.

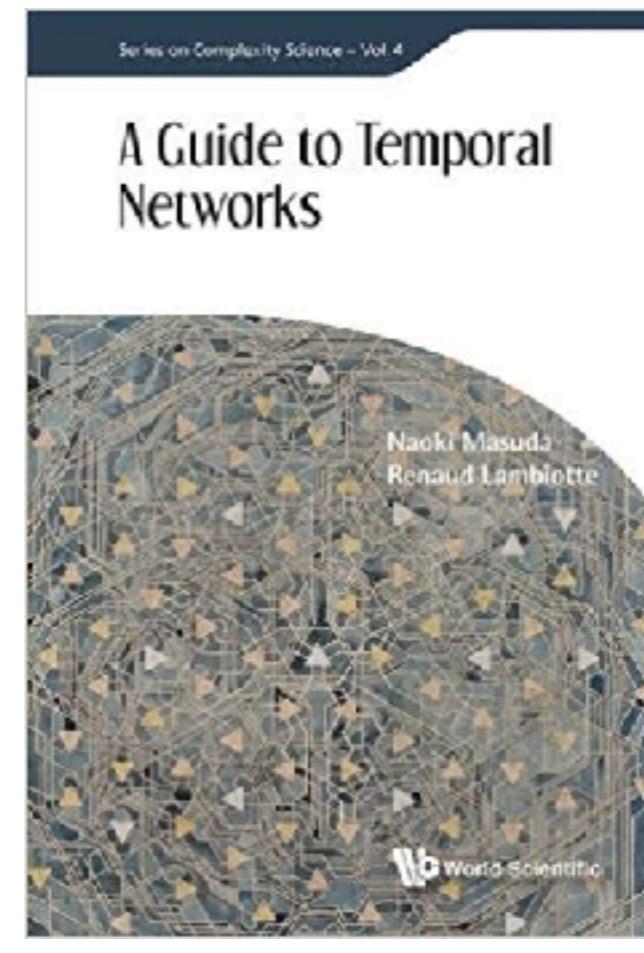


Networks are dynamical objects

Contact social networks



Phone calls



Physics Reports 519 (2012) 99–125



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Physics Reports

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Temporal networks

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The model: dynamical network

Links can change their weights, fade away, be created or be rewired:

$$A_{ij}(t) = \begin{cases} w_{ij}(t) > 0 & \text{if } i \text{ and } j \text{ are linked at time } t \\ 0 & \text{otherwise} \end{cases}$$

Species can relocate, as follows a standard diffusive mechanism, ruled by the (time dependent) Laplacian operator:

$$L_{ij}(t) = A_{ij}(t) - s_i(t)\delta_{ij}$$

$$s_i(t) = \sum_j A_{ij}(t)$$

The model: reaction-diffusion on dynamical network

possible different time scale

$$(1) \quad \begin{cases} \dot{u}_i(t) &= f(u_i, v_i) + D_u \sum_{j=1}^N L_{ij}(t/\epsilon) u_j(t) \\ \dot{v}_i(t) &= g(u_i, v_i) + D_v \sum_{j=1}^N L_{ij}(t/\epsilon) v_j(t) \end{cases}$$

Can the system exhibit Turing patterns,
once the static version doesn't?

Which is the role of network time scale?

Main result: periodic case

Assume the existence of the averaged Laplacian:

$$\langle \mathbf{L} \rangle = \frac{1}{T} \int_0^T \mathbf{L}(t) dt$$

And let us introduce the averaged reaction-diffusion system:

$$(2) \quad \begin{cases} \dot{u}_i(t) &= f(u_i, v_i) + D_u \sum_{j=1}^N \langle L_{ij} \rangle u_j \\ \dot{v}_i(t) &= g(u_i, v_i) + D_v \sum_{j=1}^N \langle L_{ij} \rangle v_j \end{cases}$$

Then, if (2) exhibits Turing patterns, there exists $\epsilon^* > 0$ such

that (1) also exhibits Turing patterns $\forall \epsilon : 0 < \epsilon < \epsilon^*$

Sketch of the proof

a) Theorem of averaging

Theorem 1 (Averaging in the periodic case) *Let us consider the system*

$$\dot{x} = \epsilon f(t, x), \quad x(0) = x_0 \in D \subset \mathbb{R}^n,$$

where f and $\partial_x f$ are defined, continuous and bounded in $[0, \infty) \times \mathbb{R}^n$ and assume $f(t, \cdot)$ to be T -periodic.

Let $\langle f \rangle(y)$ be the time average of $f(t, y)$, that is

$$\langle f \rangle(y) = \frac{1}{T} \int_0^T f(t, y) dt,$$

and let $y(t)$ be the solution of

$$\dot{y} = \epsilon \langle f \rangle(y), \quad y(0) = x_0 \in D \subset \mathbb{R}^n.$$

Then $x(t) - y(t) = \mathcal{O}(\epsilon)$ for $t = \mathcal{O}(1/\epsilon)$.

Sketch of the proof

b) Rewrite 1 and 2 in compact form: $\vec{x} = (u_1, \dots, u_N, v_1, \dots, v_N)$

$$(b1) \quad \dot{\vec{x}}(t) = F(\vec{x}) + \mathcal{L}(t/\epsilon)\vec{x}, \quad \mathcal{L}(t)\vec{x} := \begin{pmatrix} D_u \mathbf{L}(t) & 0 \\ 0 & D_v \mathbf{L}(t) \end{pmatrix} \vec{x}.$$

$$(b2) \quad \dot{\vec{x}}(t) = F(\vec{x}) + \langle \mathcal{L} \rangle \vec{x}, \quad \langle \mathcal{L} \rangle = \begin{pmatrix} D_u \langle \mathbf{L} \rangle & 0 \\ 0 & D_v \langle \mathbf{L} \rangle \end{pmatrix}$$

c) Rescale time: $\tau = t/\epsilon$

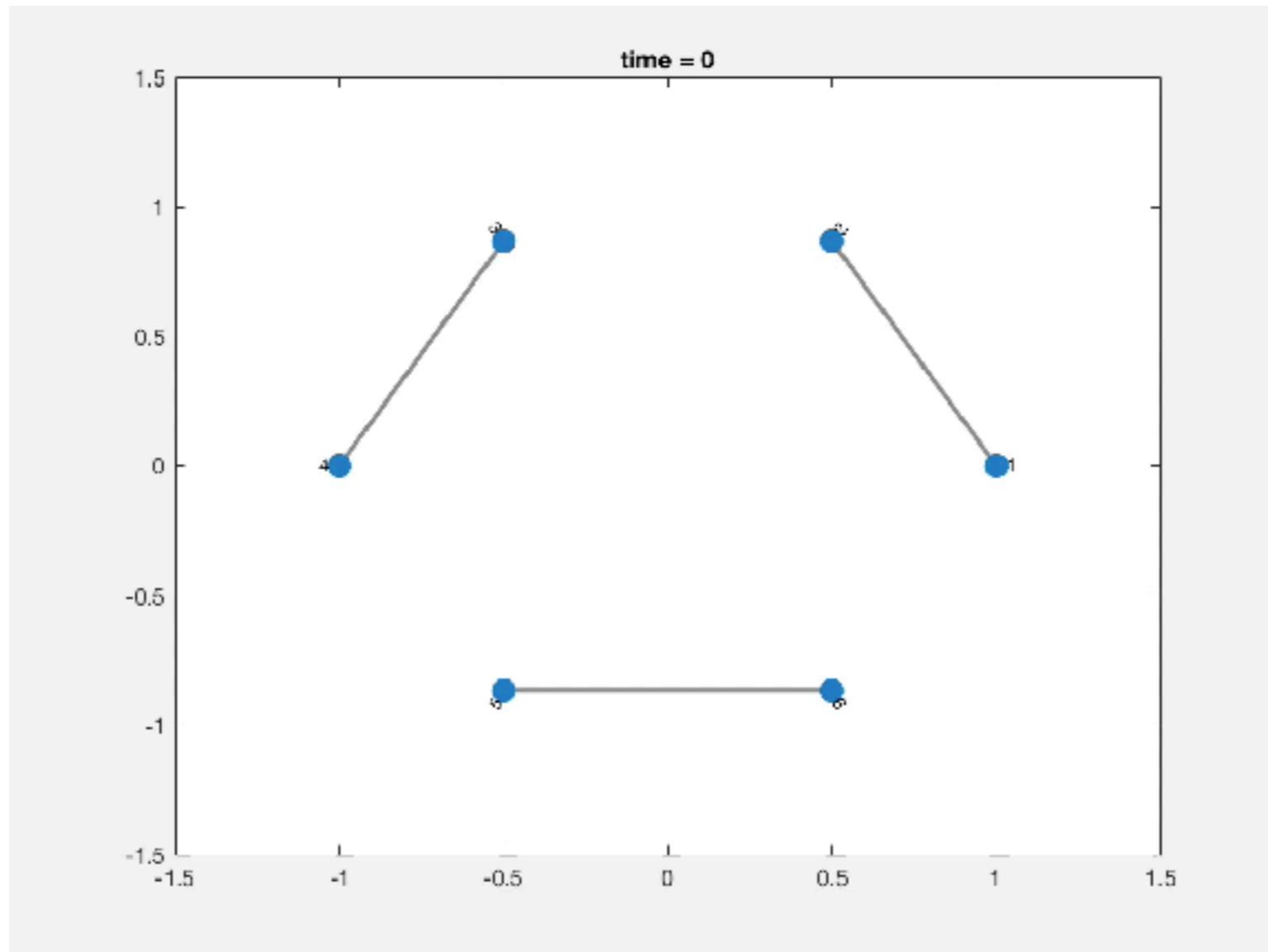
d) Conclude: $\exists \epsilon^* > 0 \quad \forall \epsilon : 0 < \epsilon < \epsilon^*$

Solutions of (b1) & (b2) stay ϵ -close for $t = \mathcal{O}(1)$

e) Hence if the averaged system is unstable and exhibits Turing patterns, also the “accelerated” does.

An exemple: twins network

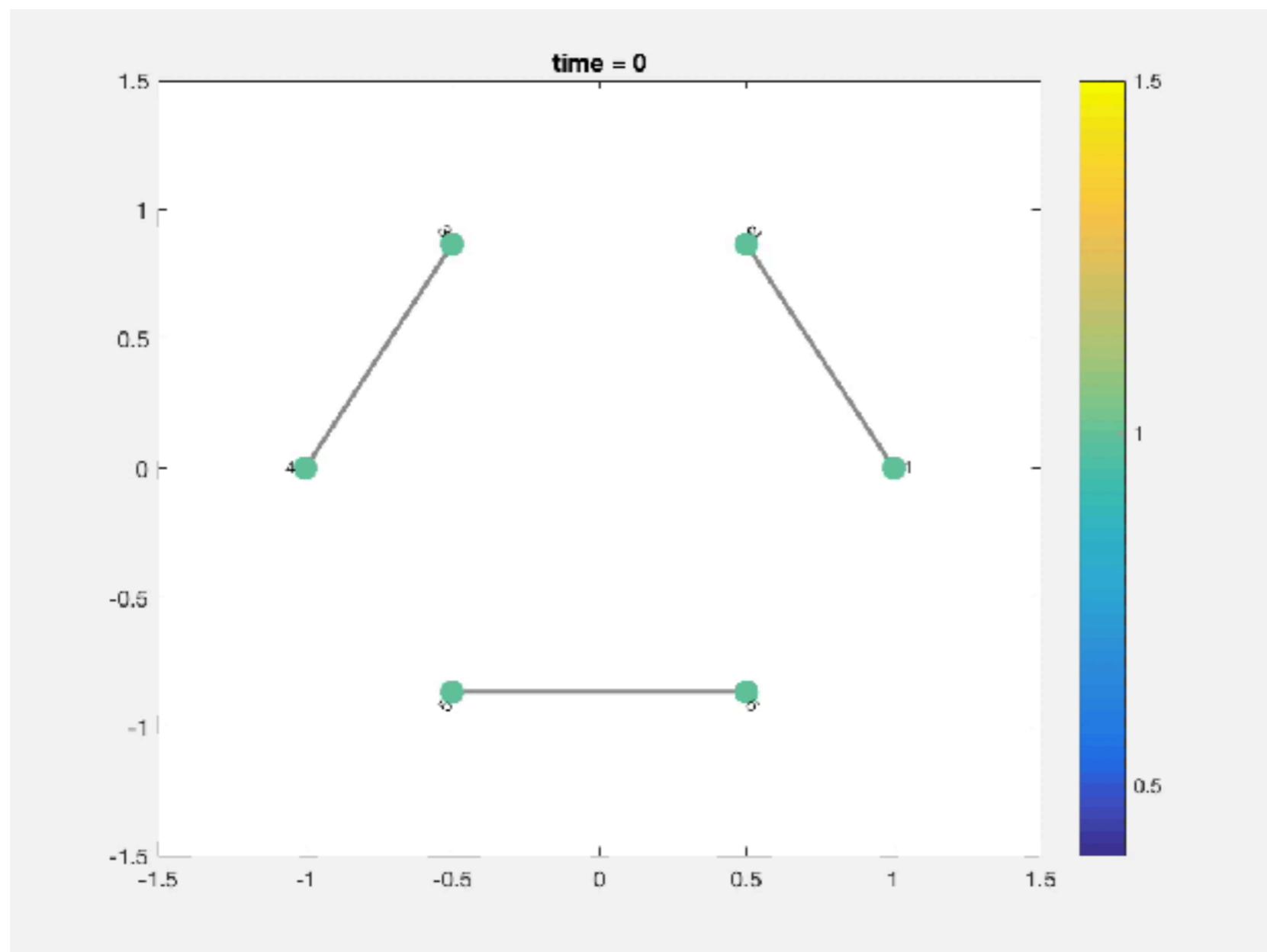
$$\mathbf{A}(t) = \begin{cases} \mathbf{A}_1, & \text{if } \{t/T\} \in [0, \gamma), \\ \mathbf{A}_2, & \text{if } \{t/T\} \in [\gamma, 1), \end{cases}$$



An exemple: twins network

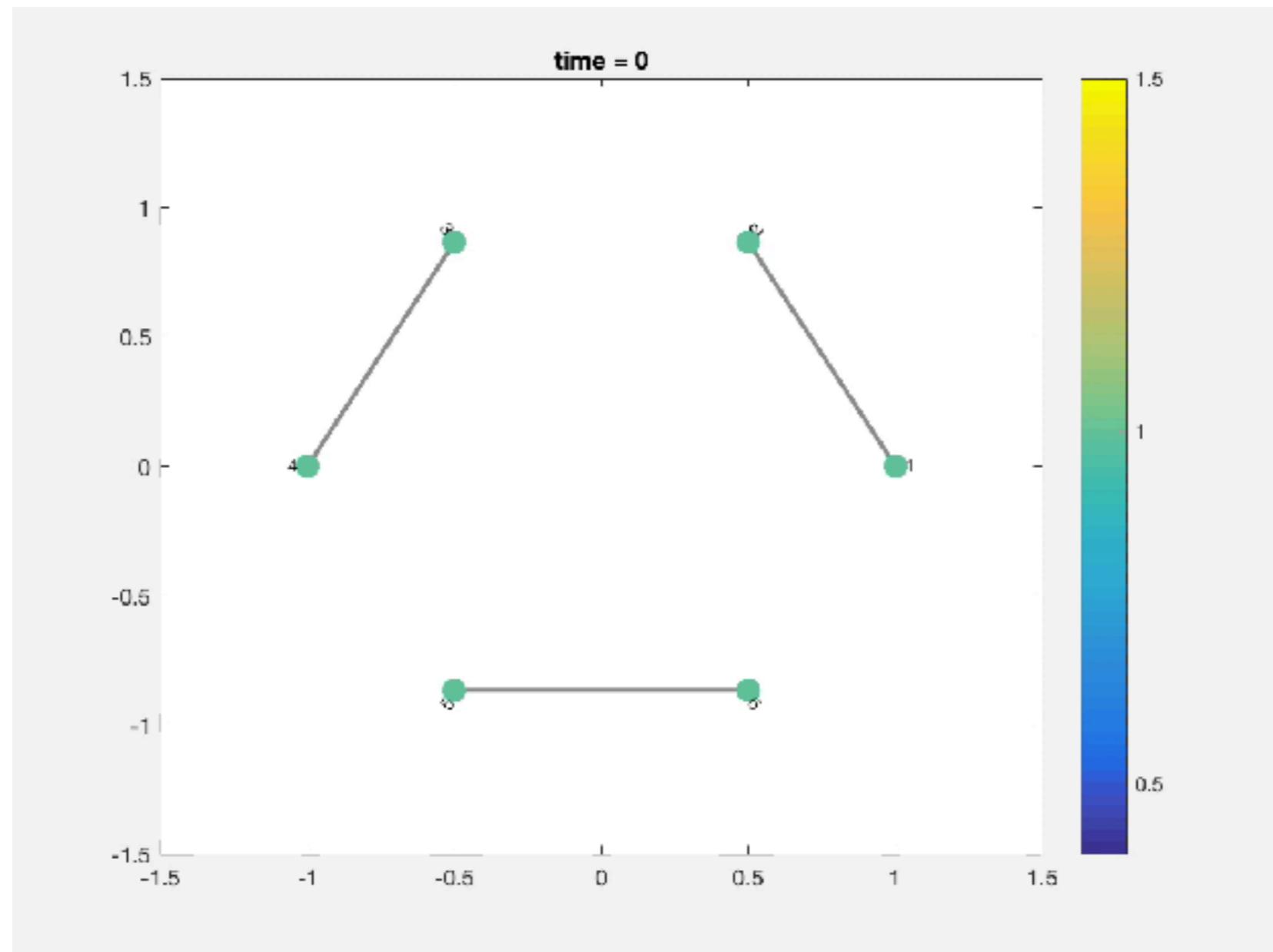
$$\mathbf{A}(t) = \begin{cases} \mathbf{A}_1, & \text{if } \{t/T\} \in [0, \gamma), \\ \mathbf{A}_2, & \text{if } \{t/T\} \in [\gamma, 1), \end{cases}$$

Dynamics on A1 (or A2) fixed



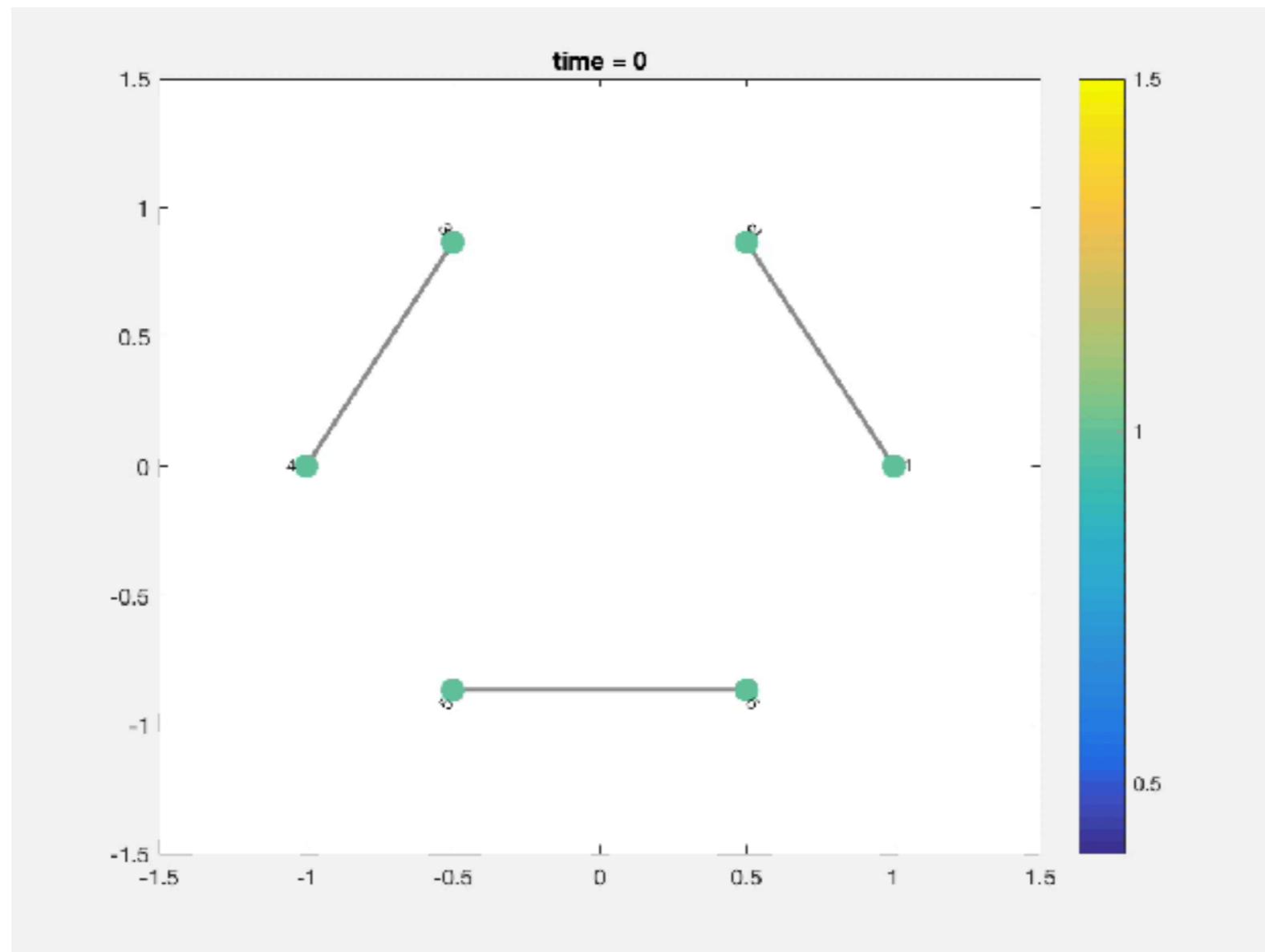
An exemple: twins network

$$\mathbf{A}(t) = \begin{cases} \mathbf{A}_1, & \text{if } \{t/T\} \in [0, \gamma), \\ \mathbf{A}_2, & \text{if } \{t/T\} \in [\gamma, 1), \end{cases} \quad \text{"fast" switch}$$



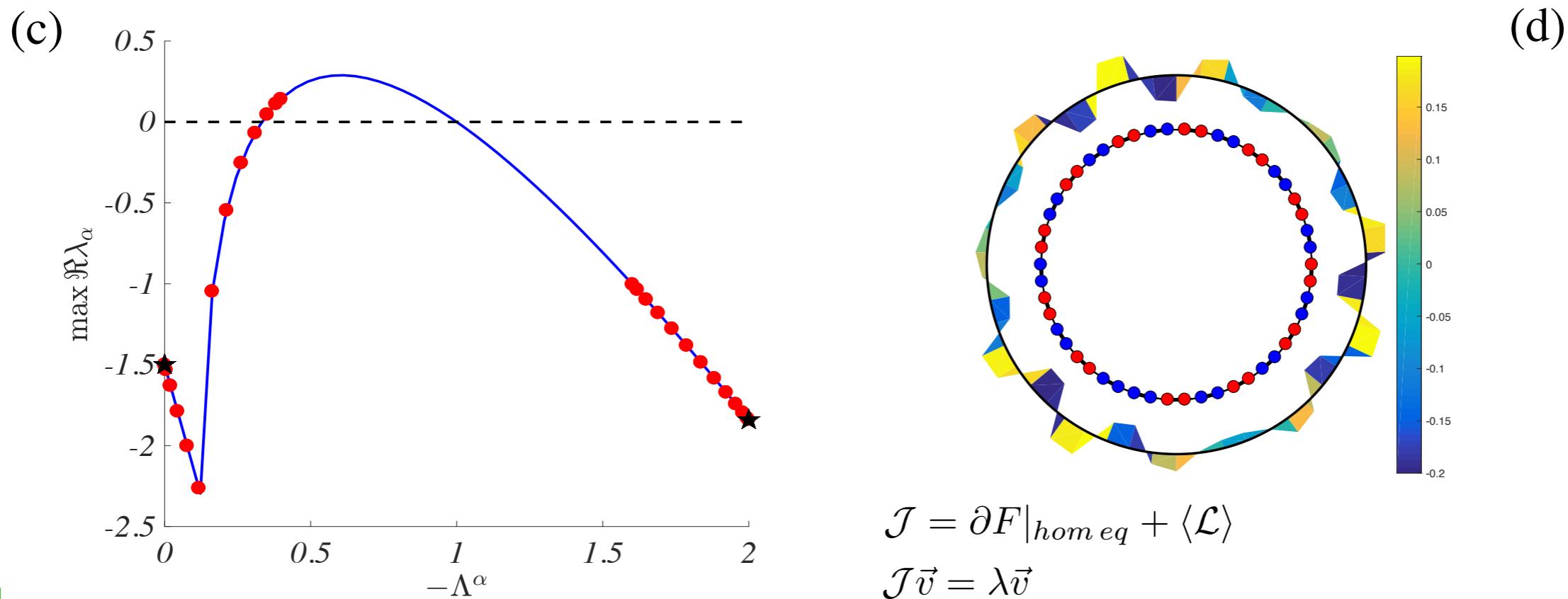
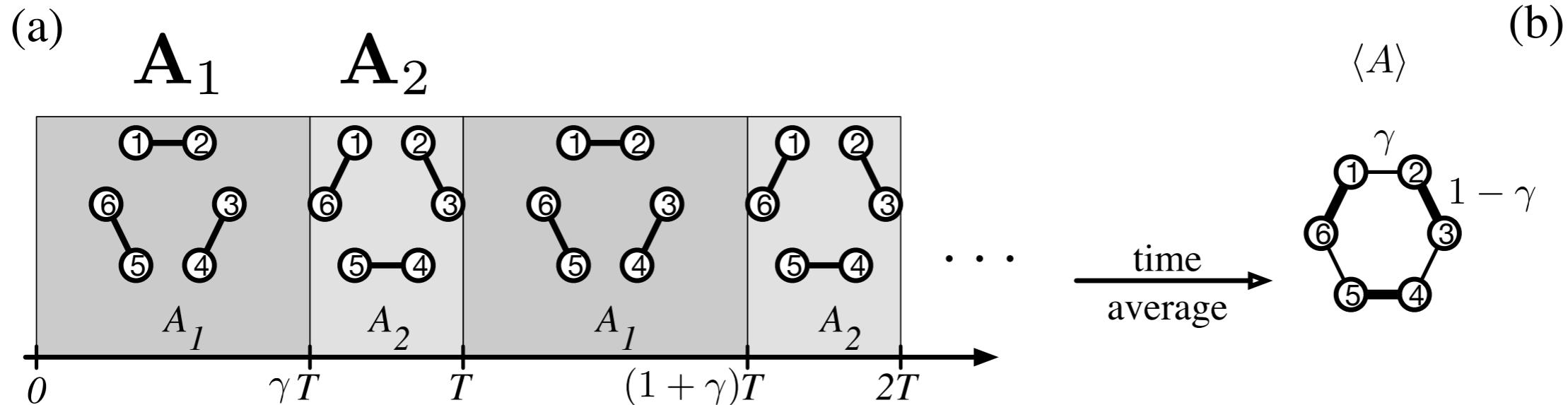
An exemple: twins network

$$\mathbf{A}(t) = \begin{cases} \mathbf{A}_1, & \text{if } \{t/T\} \in [0, \gamma), \\ \mathbf{A}_2, & \text{if } \{t/T\} \in [\gamma, 1), \end{cases} \quad \text{"slow" switch}$$



An exemple: twins network

$$\mathbf{A}(t) = \begin{cases} \mathbf{A}_1, & \text{if } \{t/T\} \in [0, \gamma), \\ \mathbf{A}_2, & \text{if } \{t/T\} \in [\gamma, 1), \end{cases}$$



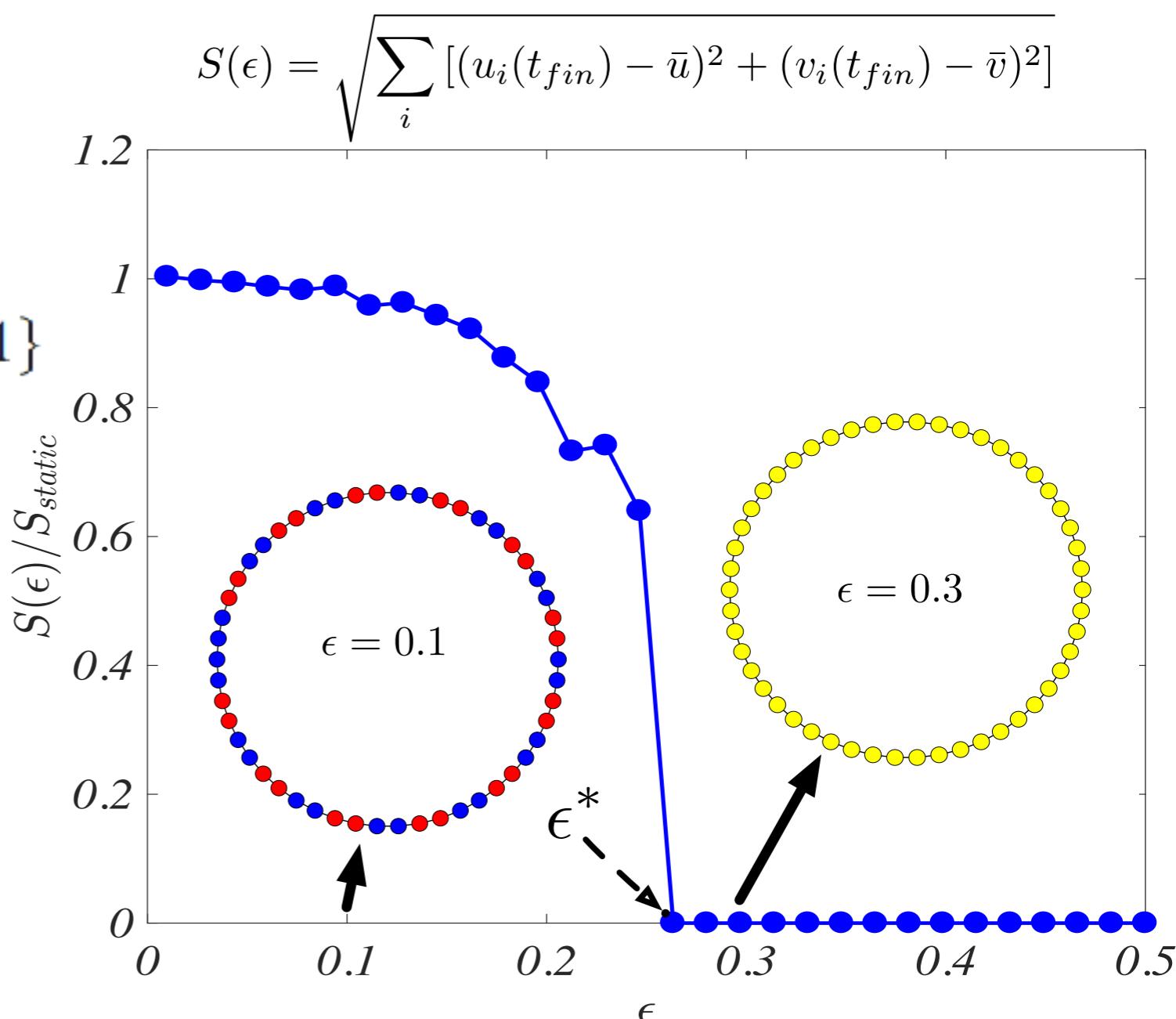
Critical threshold: twins network

$$\dot{\delta\vec{x}} = \mathbf{M}_c(t)\delta\vec{x} \quad \text{with} \quad \mathbf{M}_c(t) = \partial_x F(\bar{x}) + \mathcal{L}(t/\epsilon)$$

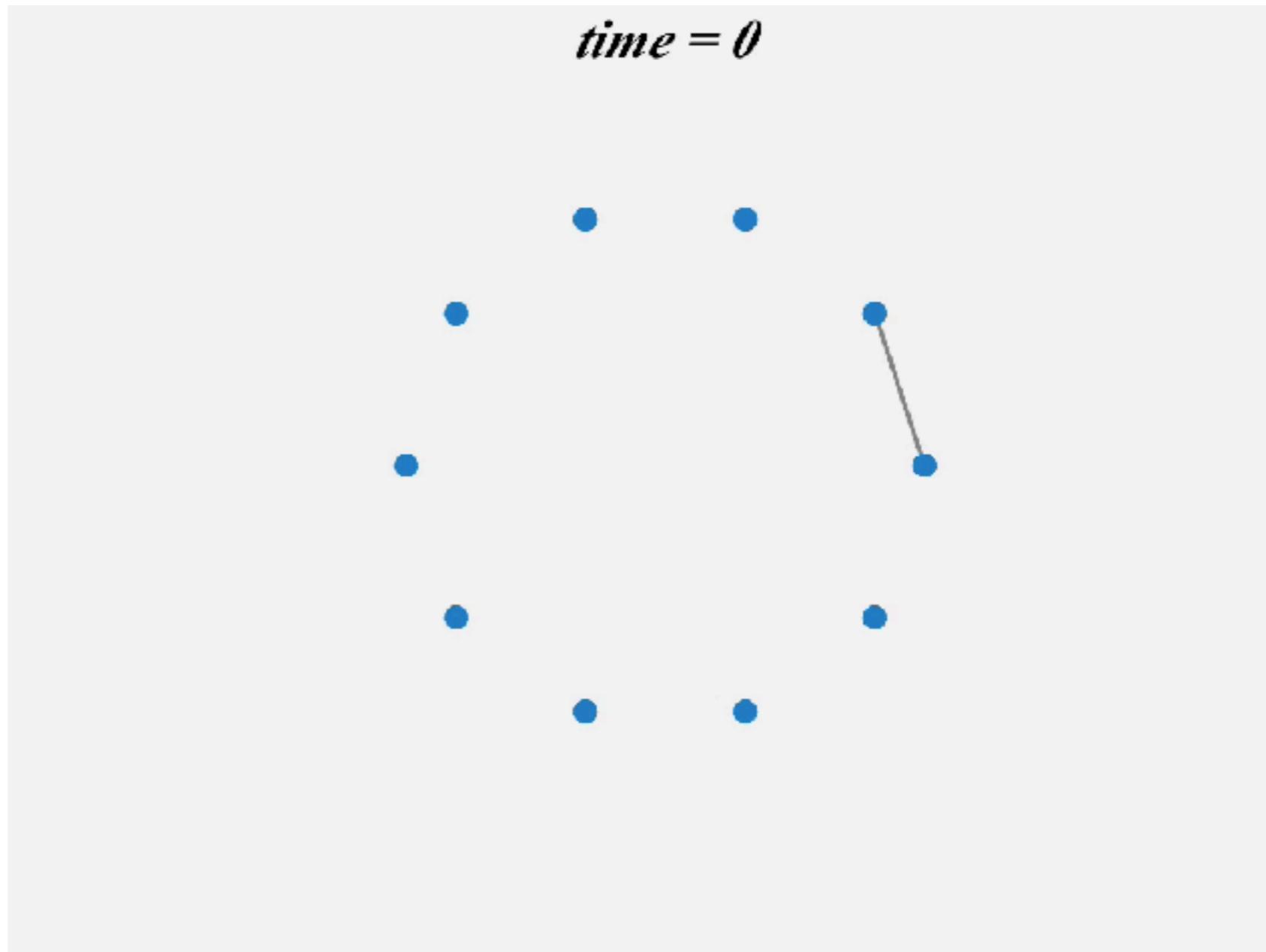
Monodromy matrix (acc. case):

$$\mathbf{Q}_\epsilon = e^{\epsilon \mathbf{M}_2(1-\gamma)^T} e^{\epsilon \mathbf{M}_1 \gamma^T}$$

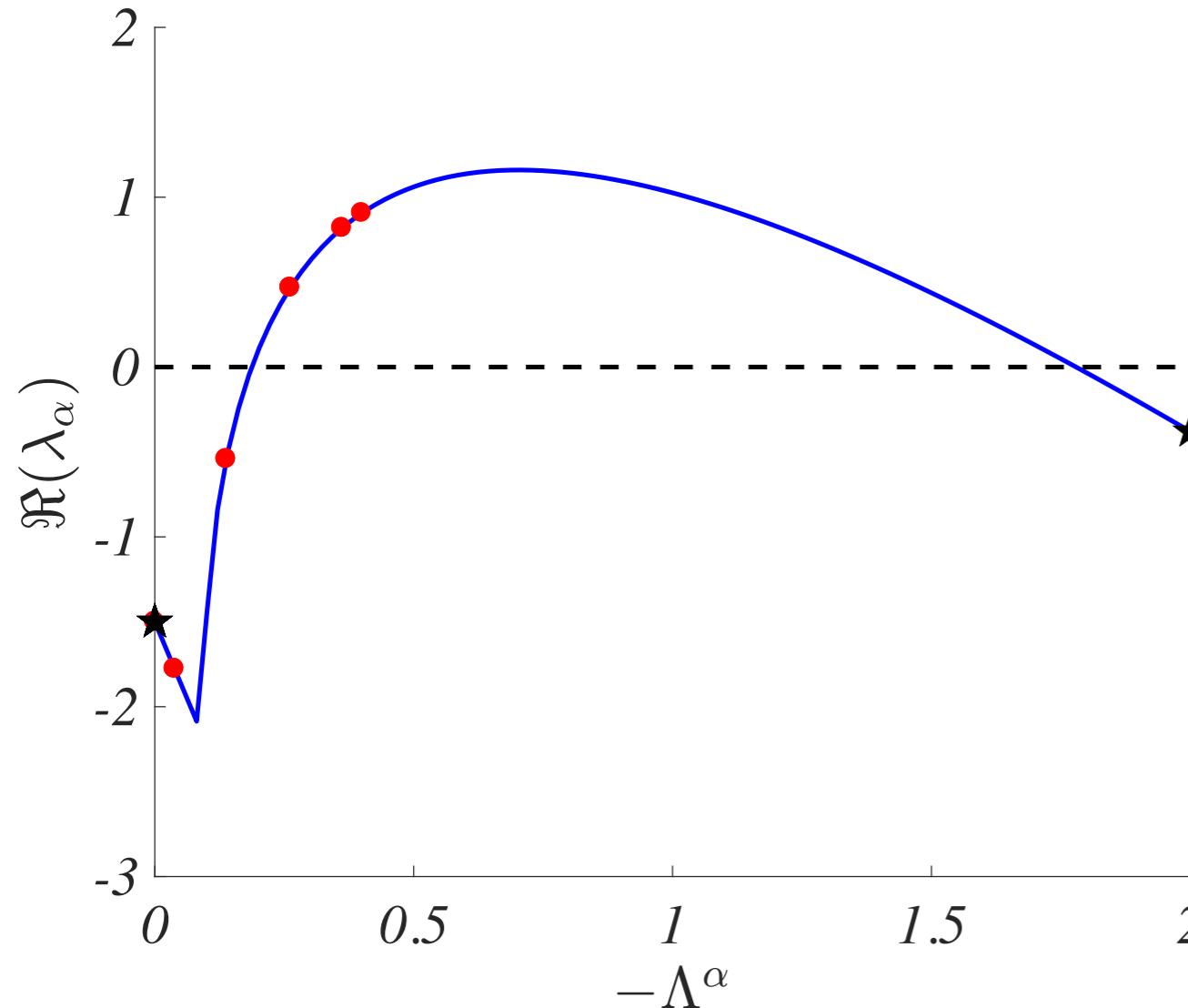
$$\epsilon^* = \min\{\epsilon > 0: \forall s \geq \epsilon: \rho(\mathbf{Q}_s) \leq 1\}$$



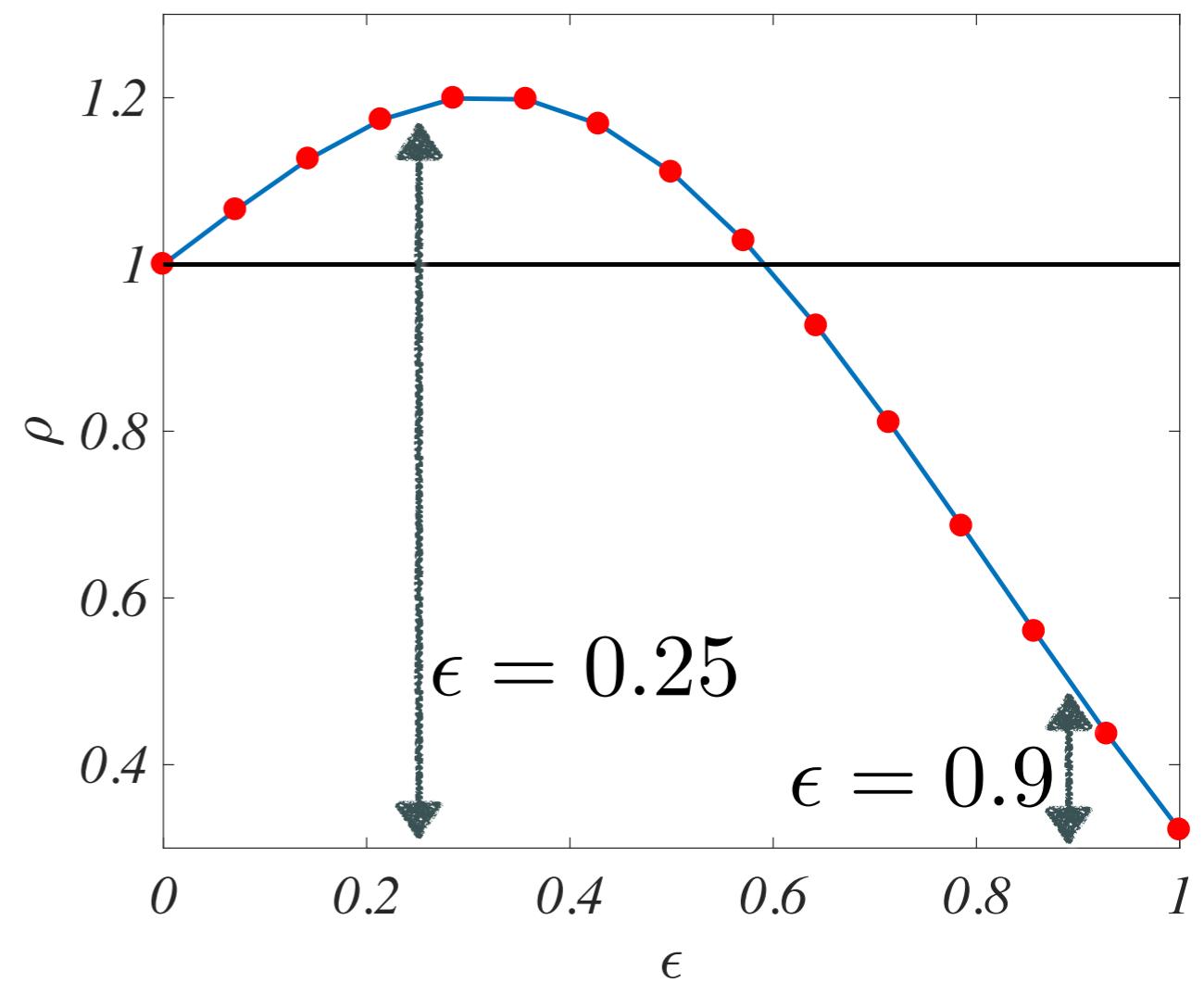
A second example: “Chinese whispers”



A second example: “Chinese whispers”



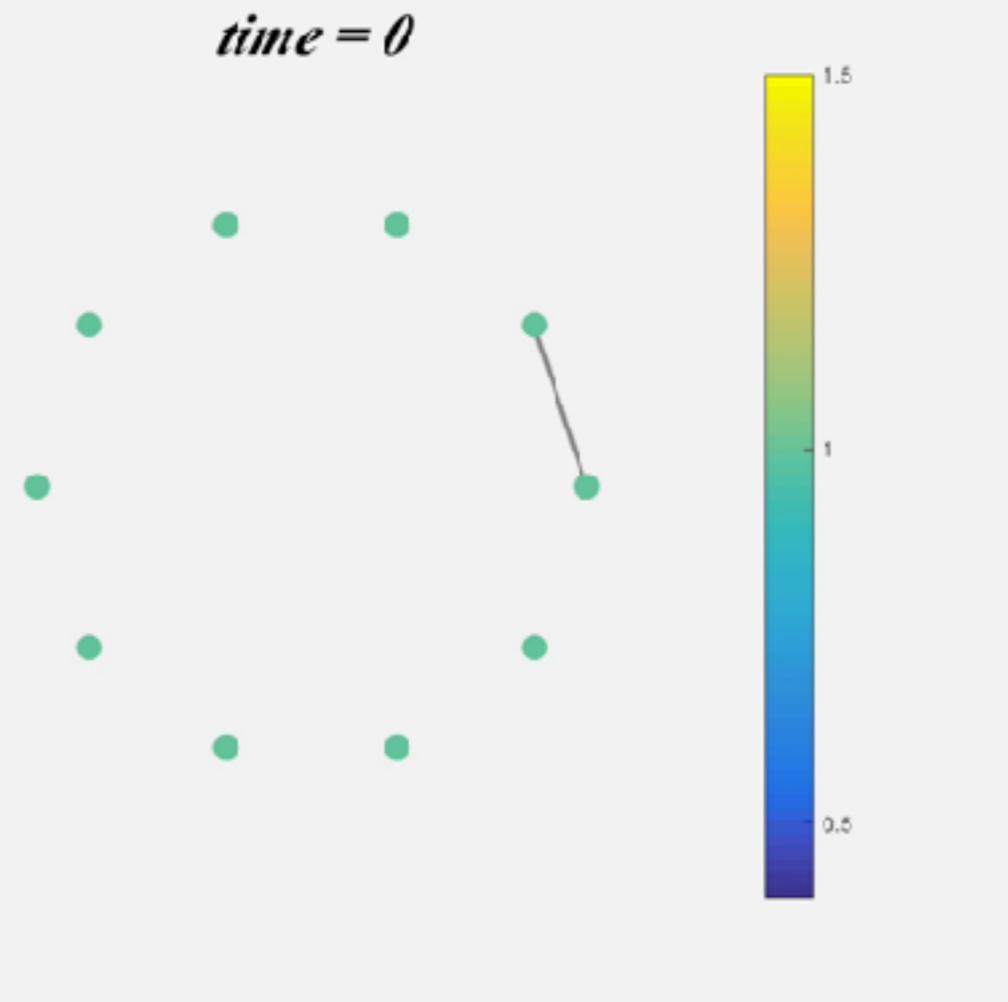
Relation dispersion



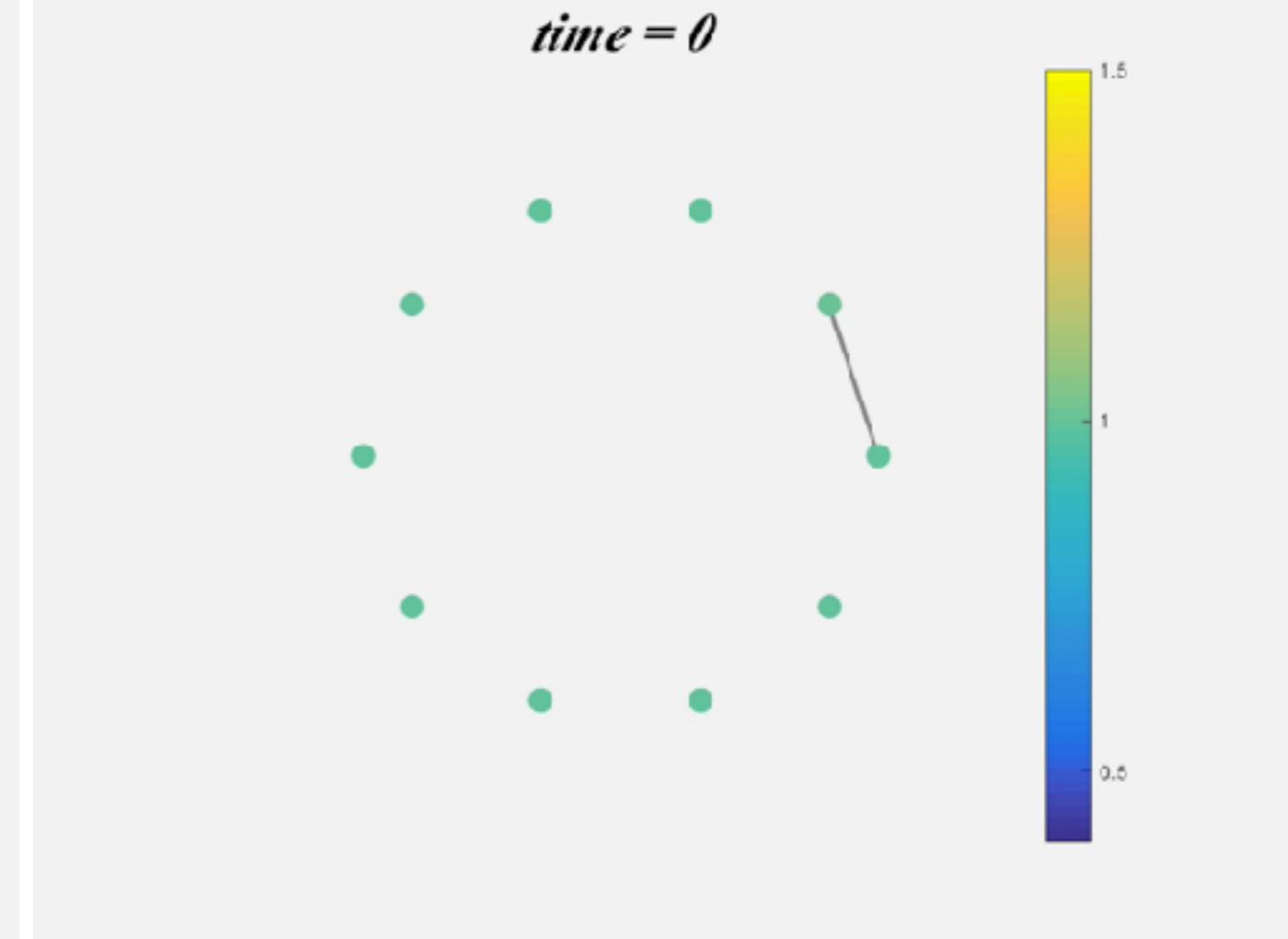
Critical ϵ

$$\epsilon^* = \min\{\epsilon > 0: \quad \forall s \geq \epsilon: \rho(Q_s) \leq 1\}$$

A second example: “Chinese whispers”



$\epsilon = 0.25$



$\epsilon = 0.9$

Generalizations

i) Continuous non-periodic case (use Floquet-Magnus expansion)

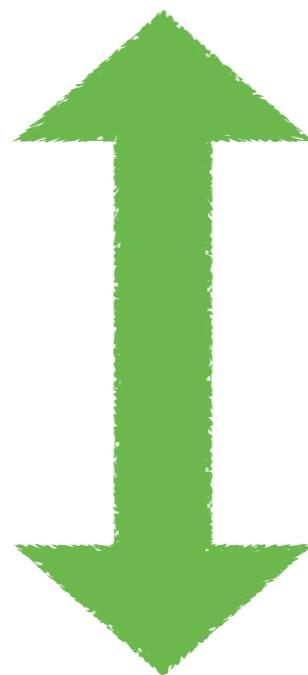
$$\langle \mathbf{L} \rangle = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \mathbf{L}(t) dt$$

ii) Discrete time random switching:

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \mathbf{L}(t) dt = \mathbb{E} [\mathbf{L}(t)] \quad (\text{almost surely})$$

Part II. Oscillation death & time varying network

Can the intrinsic network dynamics suppress the existing synchrony?



Namely from (global) limit cycle behavior to stationary homogeneous equilibrium.

Oscillation death & time varying networks

$$\dot{x}_i = f(x_i, y_i) + D_x \sum_{j=1}^N L_{ij}(t/\epsilon) x_j,$$

$$\dot{y}_i = g(x_i, y_i) + D_y \sum_{j=1}^N L_{ij}(t/\epsilon) x_j,$$

Admits a T-periodic solution $(x_i, y_i) = (\bar{x}(t), \bar{y}(t))$ once $D_x = D_y = 0$

Let $D_x > 0$ and $D_y > 0$

The system synchronizes, if such solution is stable.

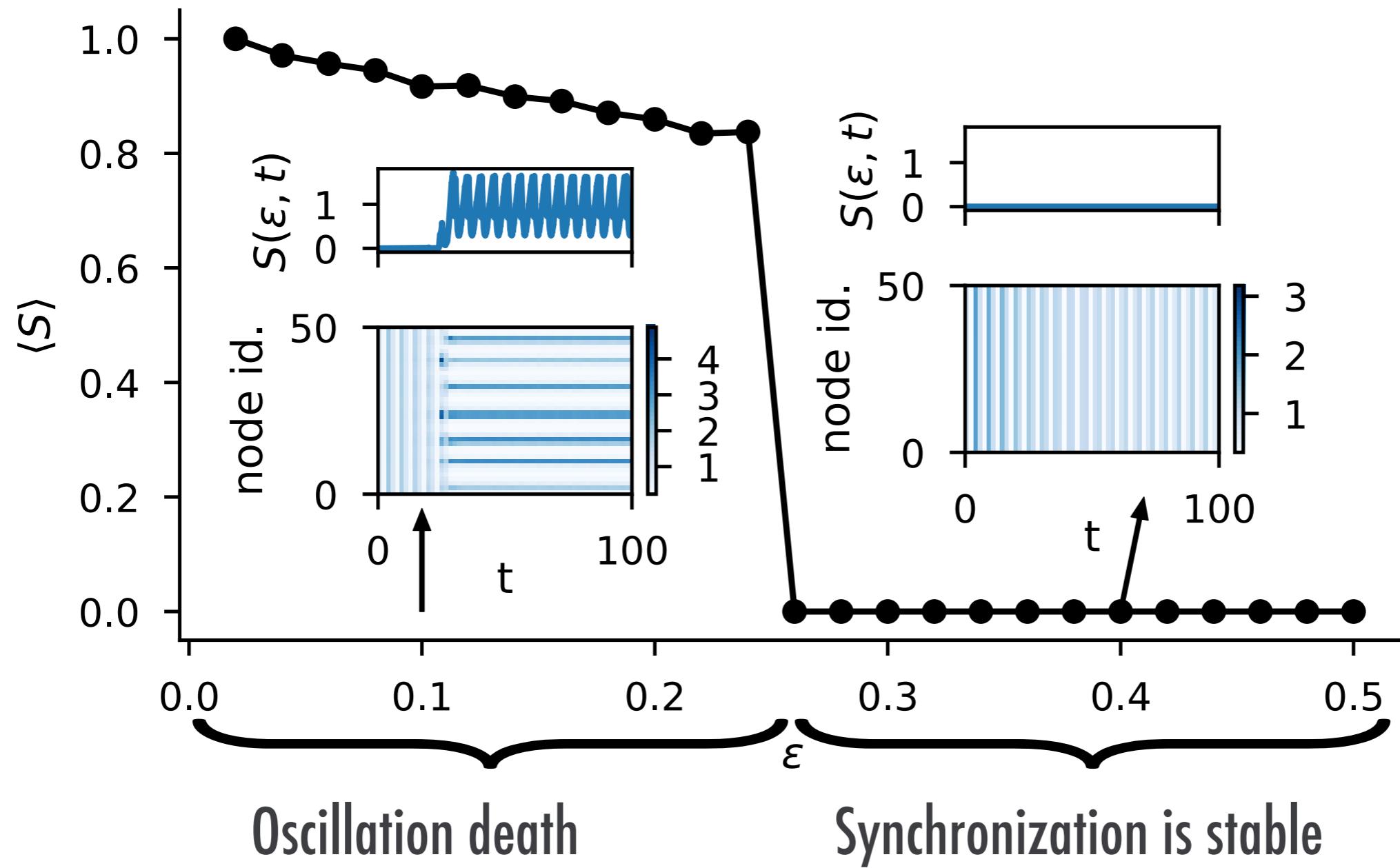
The oscillations can die out, if stability is lost.

Oscillation death induced by fast switching

(Twins network)

$$S(\epsilon, t) = \frac{1}{N} \|\mathbf{x}(t) - \bar{\mathbf{x}}(t)\|^2.$$

$$\langle S \rangle = \frac{1}{T_s} \int_t^{t+T_s} S(\epsilon, u) du$$



Sketch of the proof

Use again the theorem of averaging but in two steps

$$\dot{\mathbf{x}} = \mathcal{F}(\mathbf{x}) + \mathcal{L}(t/\epsilon)\mathbf{x},$$

linearize close to T-periodic solution and
change time scale

$$\delta\mathbf{x}' = \epsilon[(\mathcal{J}(\epsilon\tau) + \mathcal{L}(\tau))\delta\mathbf{x}]$$

1

partial averaging

$$\langle \mathbf{L} \rangle = \frac{1}{T_s} \int_0^{T_s} \mathbf{L}(t) dt$$

$$\dot{\mathbf{y}} = \mathcal{F}(\mathbf{y}) + \langle \mathcal{L} \rangle \mathbf{y}$$

2

Linearization
and change time scale

$$\delta\mathbf{y}' = \epsilon [\mathcal{J}(\epsilon\tau) + \langle \mathcal{L} \rangle] \delta\mathbf{y}$$

Use Floquet and the eigenbasis of $\langle \mathcal{L} \rangle$

Papers for this work

Theory of Turing Patterns on Time Varying Networks, J. Petit, B. Lauwens, D. Fanelli, T. Carletti, *Physical Review Letters*, **119**, pp. 148301-1–5, (2017)

Desynchronization induced by time-varying network, M. Lucas, D. Fanelli, T. Carletti, J. Petit, *Europhysics Letters*, **121**, pp. 50008, (2018)

Conclusions

The theory of Turing Patterns has been extended as to consider “fast” time varying networks;

Main (possible) applications: social contact network & epidemics spreadings;

Other ideas/suggestions for a collaboration ?

Some other related papers

Homogeneous-per-layer patterns in multiplex networks, D.M. Busiello, D. Fanelli, T. Carletti, *Europhysics Letters*, **121**, pp. 48006, (2018)

Tune the topology to create or destroy patterns, M. Asllani, T. Carletti, D. Fanelli, *European Physical Journal B*, **89**, pp. 260 (2016)

Pattern formation in a two-component reaction-diffusion system with delayed processes on a network, J. Petit, M. Asllani, D. Fanelli, B. Lauwens, T. Carletti, *Physica A*, **462**, pp.230, (2016)

Delay induced Turing-like waves for one species reaction–diffusion model on a network, J. Petit, T. Carletti, M. Asllani, D. Fanelli, *Europhysics Letters*. **111**, 5, pp. 58002, (2015)

Turing instabilities on Cartesian product networks, M. Asllani, D.M. Busiello, T. Carletti, D. Fanelli, G. Planchon, *Scientific Reports*. **5**, pp. 12927, (2015)

Turing patterns in multiplex networks, M. Asllani, D.M. Busiello, T. Carletti, D. Fanelli, G. Planchon, *Physical Review E*, **90**, 4, pp. 042814, (2014)

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Turing patterns on time varying networks



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