September the 5th, 2018, Namur

Timoteo Carletti

Simples rules to explain the complex beauty of fractals

naxys
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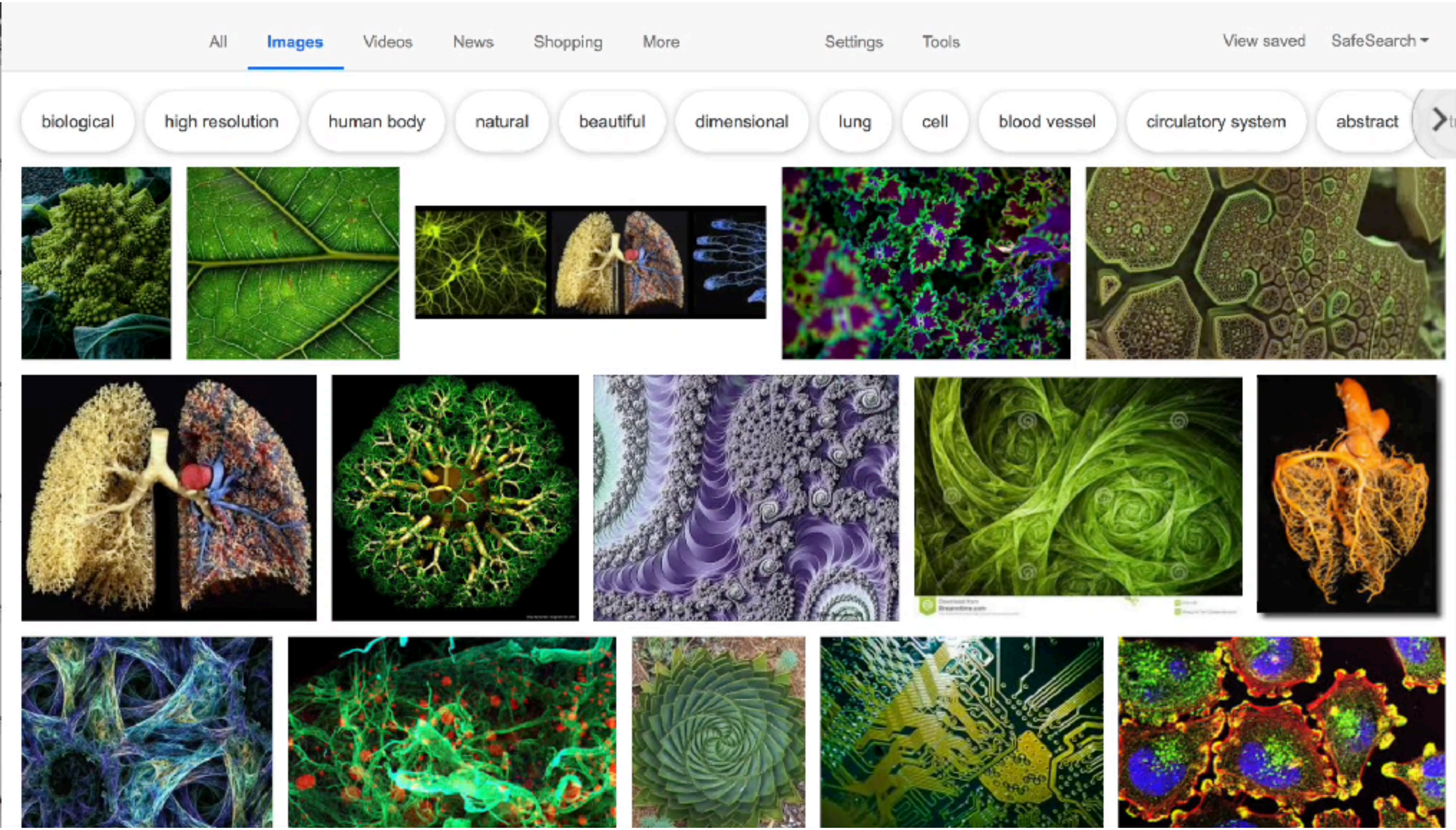
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biological
high resolution
human body
natural
beautiful
dimensional
lung
cell
blood vessel
circulatory system
abstract

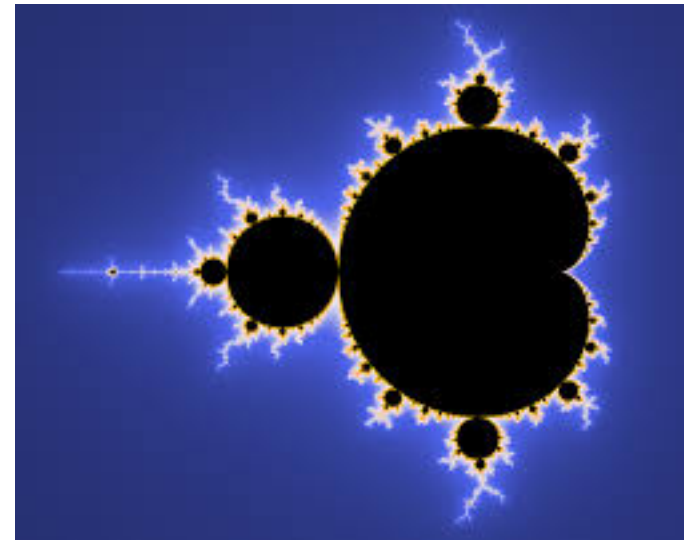
What exactly is a fractal?

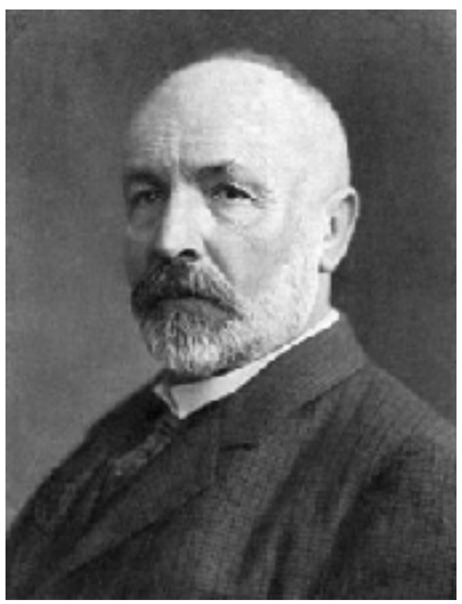
Why are they so widespread?

How can we measure/recognize fractality?

How difficult is to create a (synthetic) fractal object?

Benoit Mandelbrot (1924-2010)



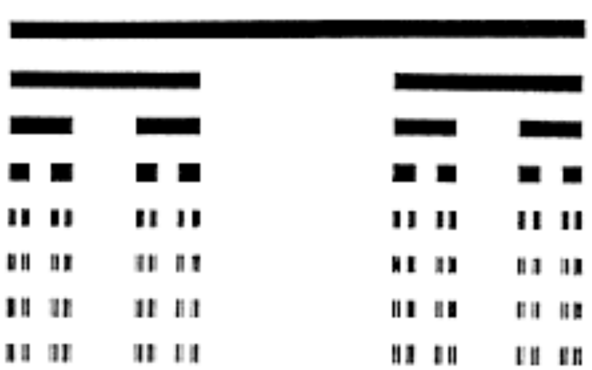


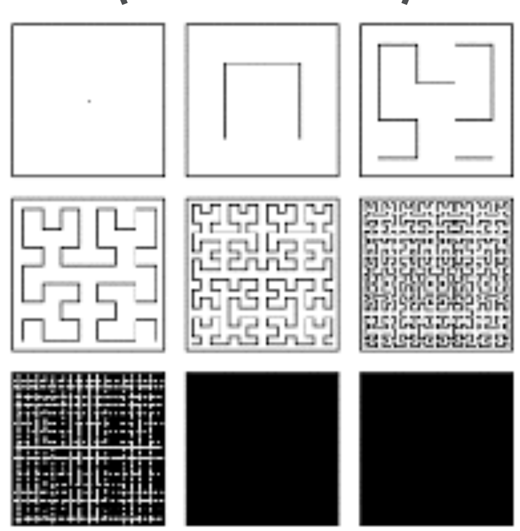
Georg Cantor
(1845-1918)

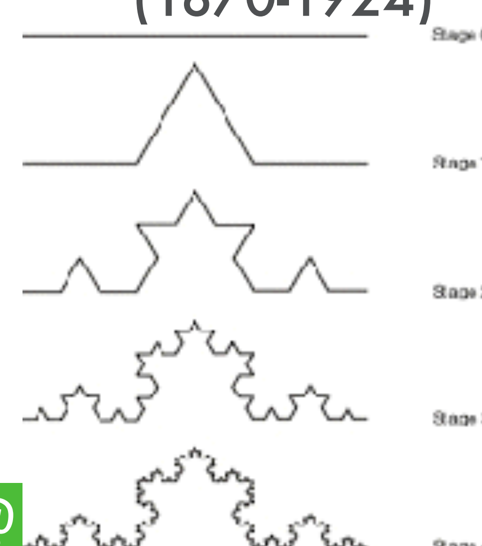
Giuseppe Peano
(1858-1932)

Helge von Koch
(1870-1924)

Wacław Sierpiński
(1882-1969)







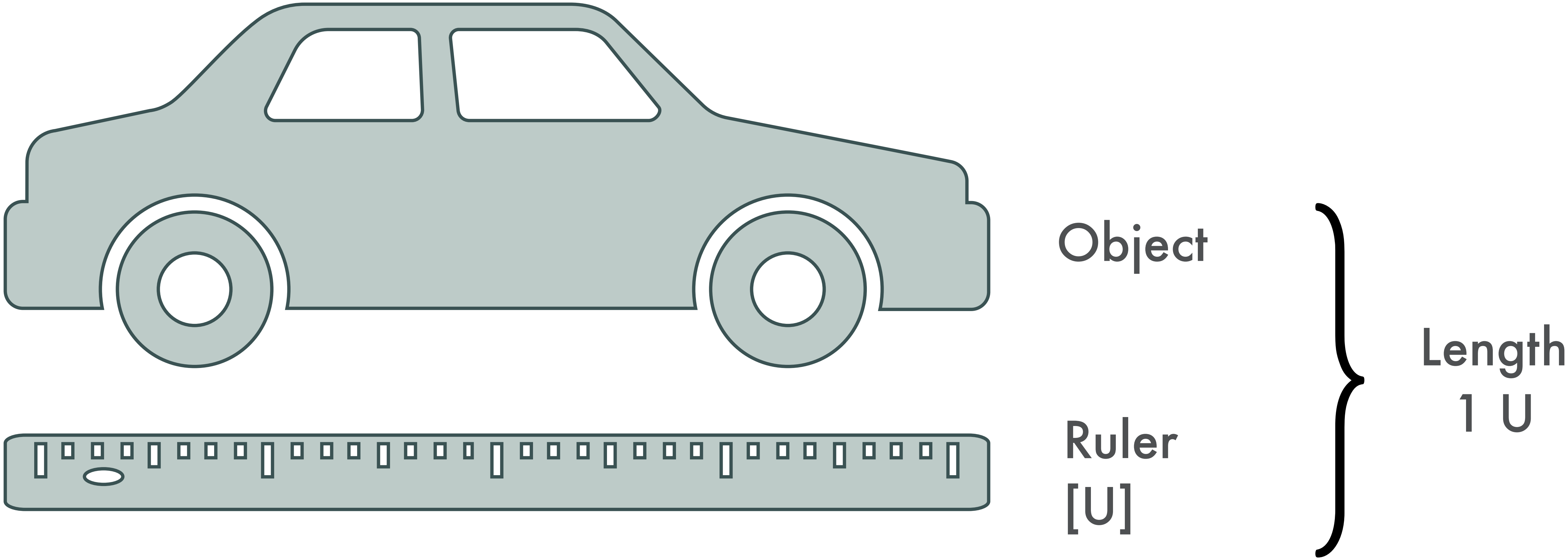
Fractal (etymology)

Latin : fractus, frangere
=
broken / rough not smooth

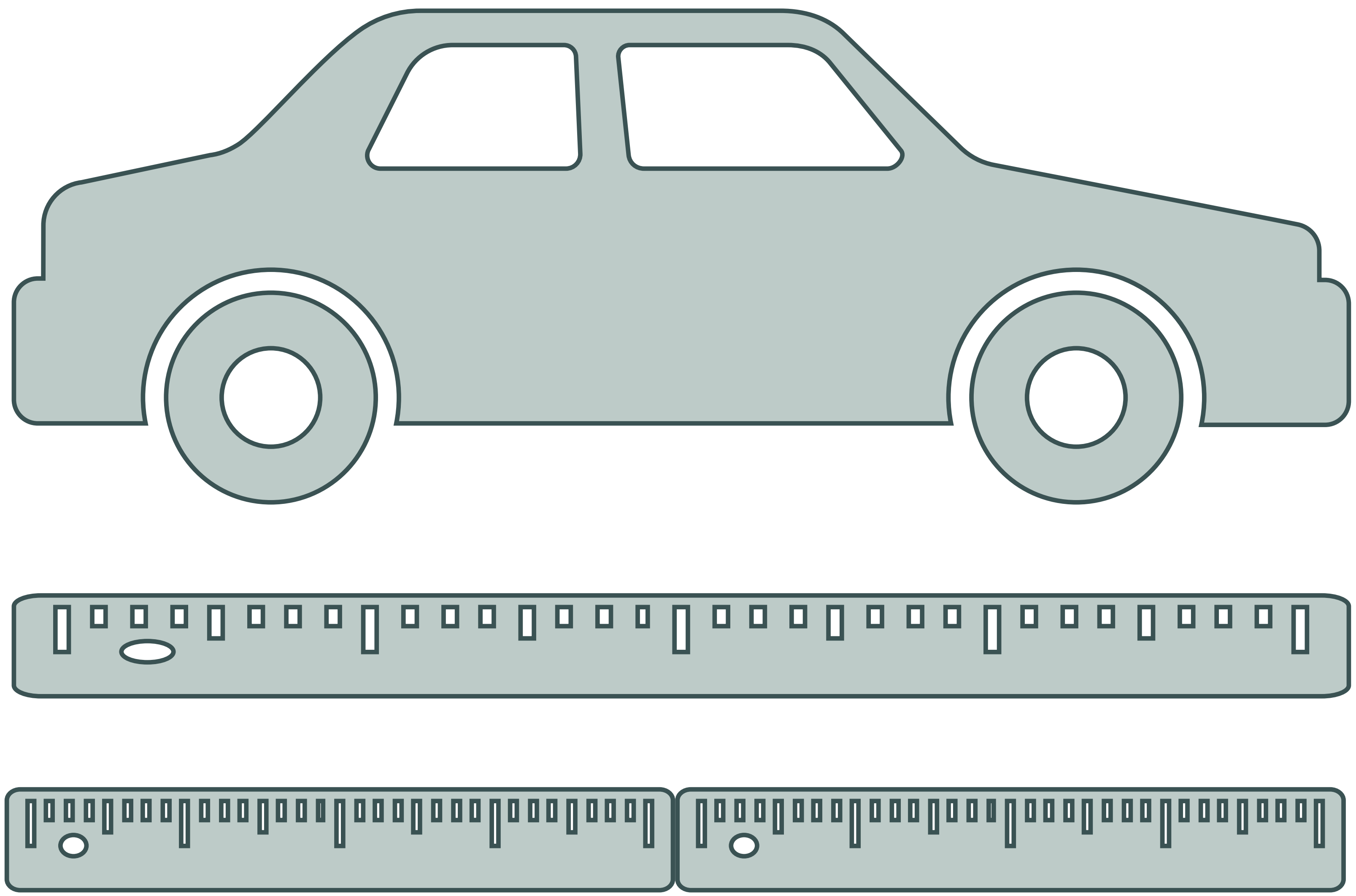
English : fraction,
not an integer number

Linked ideas

Dimension … an operational definition



Dimension … an operational definition



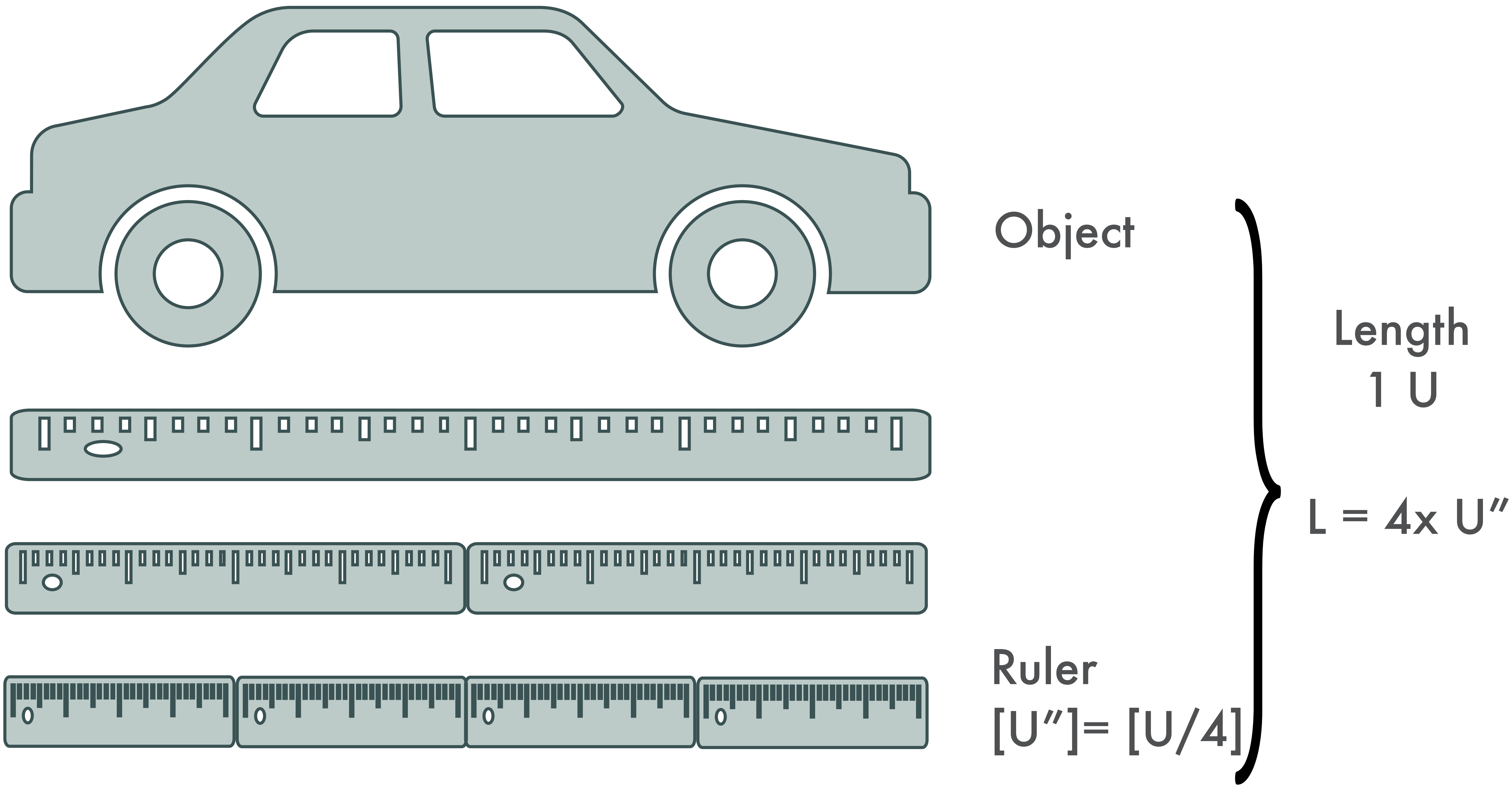
Object

Length
1 U

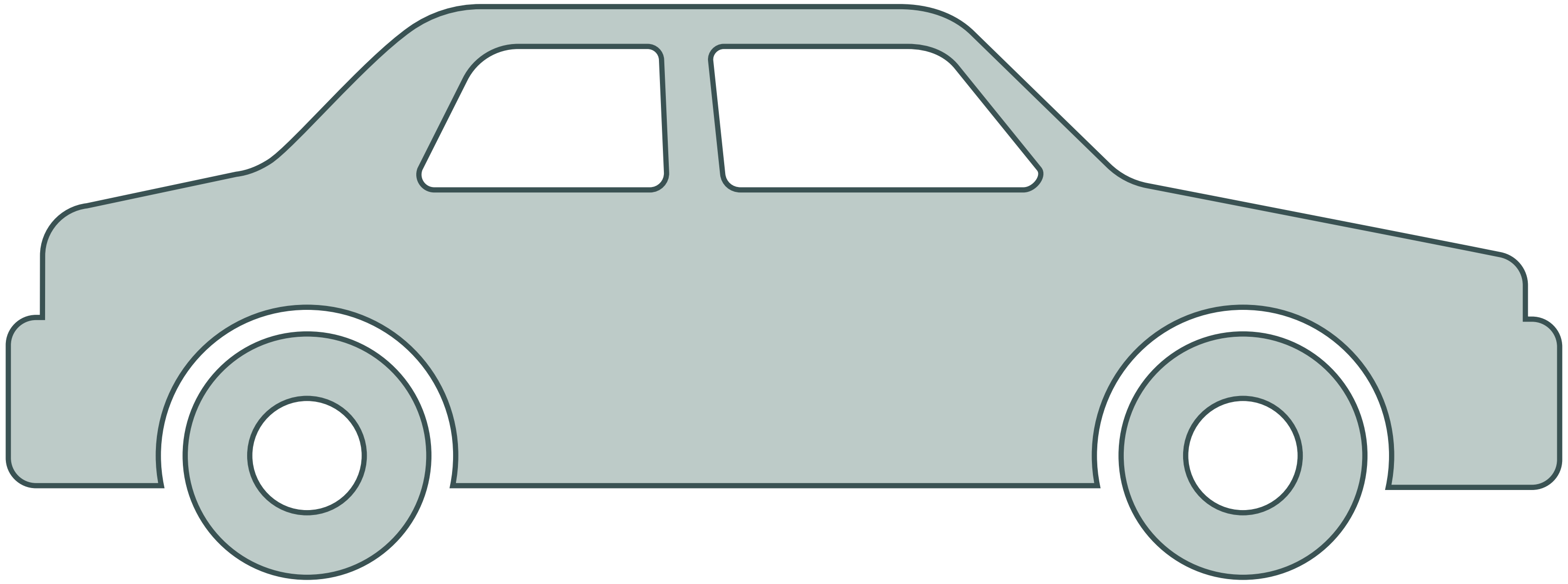
L = 2x U′

Ruler
[U′]= [U/2]

Dimension … an operational definition

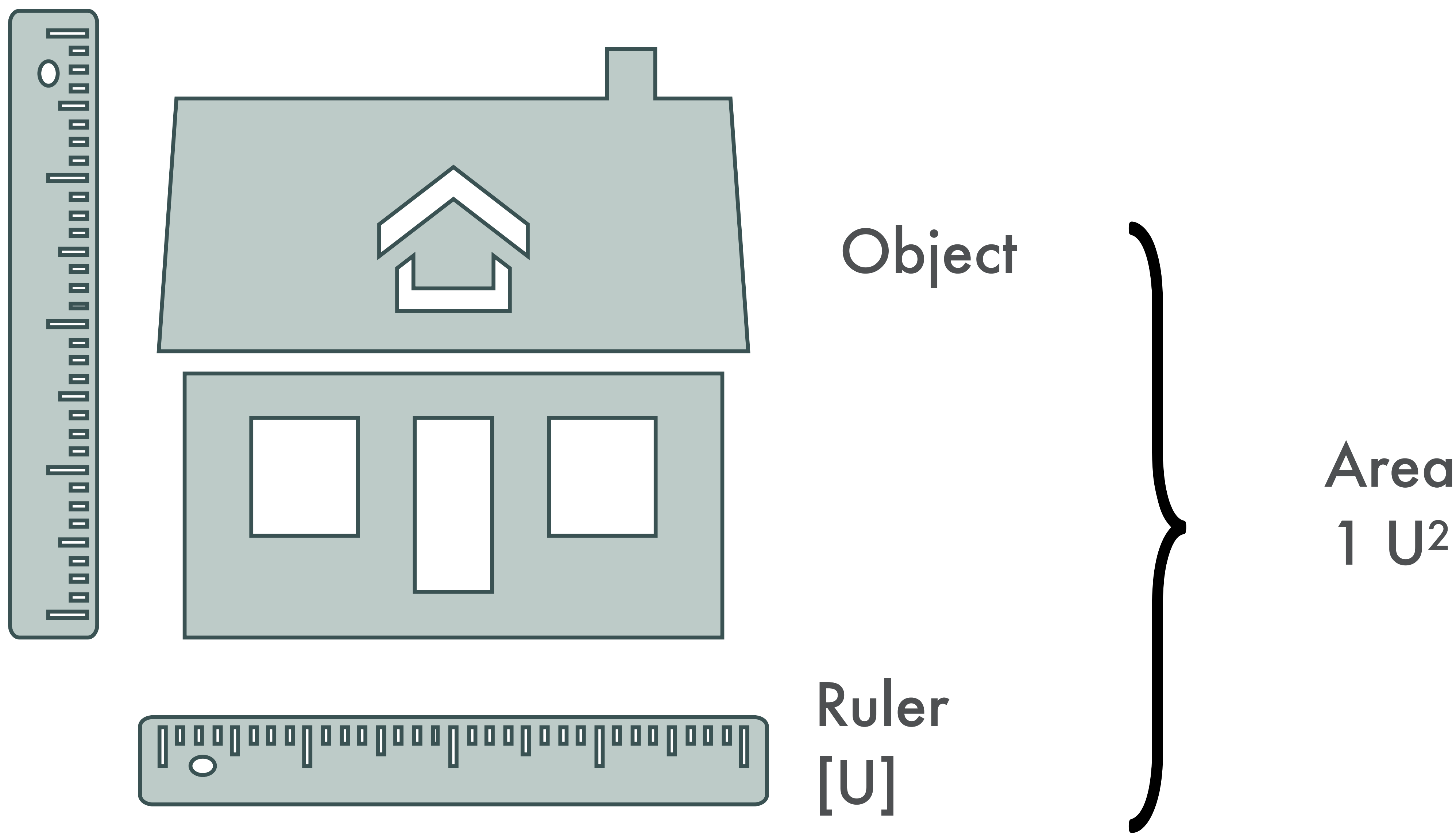


Dimension … an operational definition

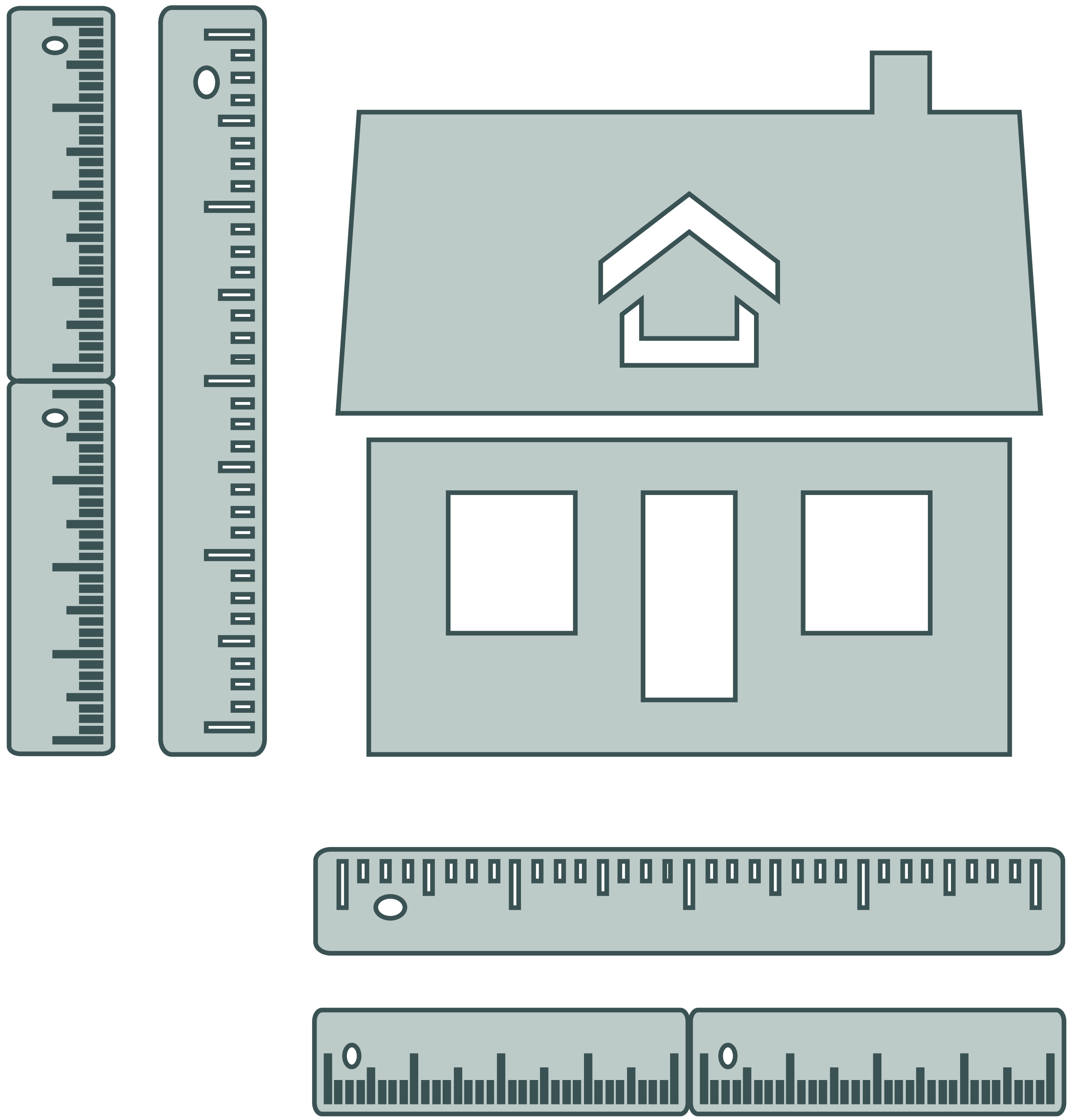


$$\text{number of small rulers} = \left(\frac{1}{\text{reduction factor}}\right)^1$$

Dimension … an operational definition



Dimension … an operational definition



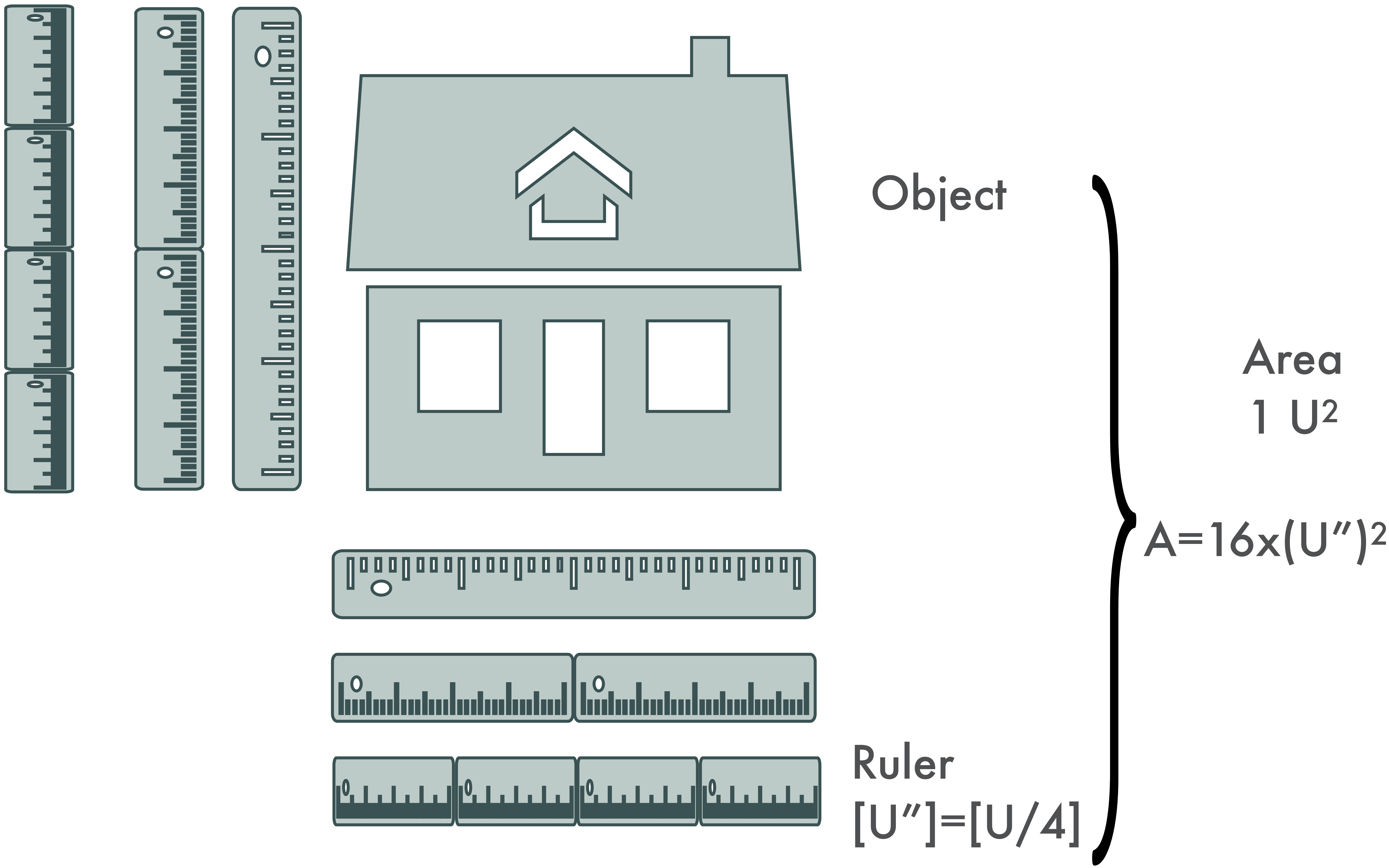
Object

Area
$1\ U^2$

Ruler
$[U'] = [U/2]$

$A = 4 \times (U')^2$

Dimension … an operational definition



Dimension ... an operational definition

$$\text{number of small rulers} = \left(\frac{1}{\text{reduction factor}}\right)^2$$

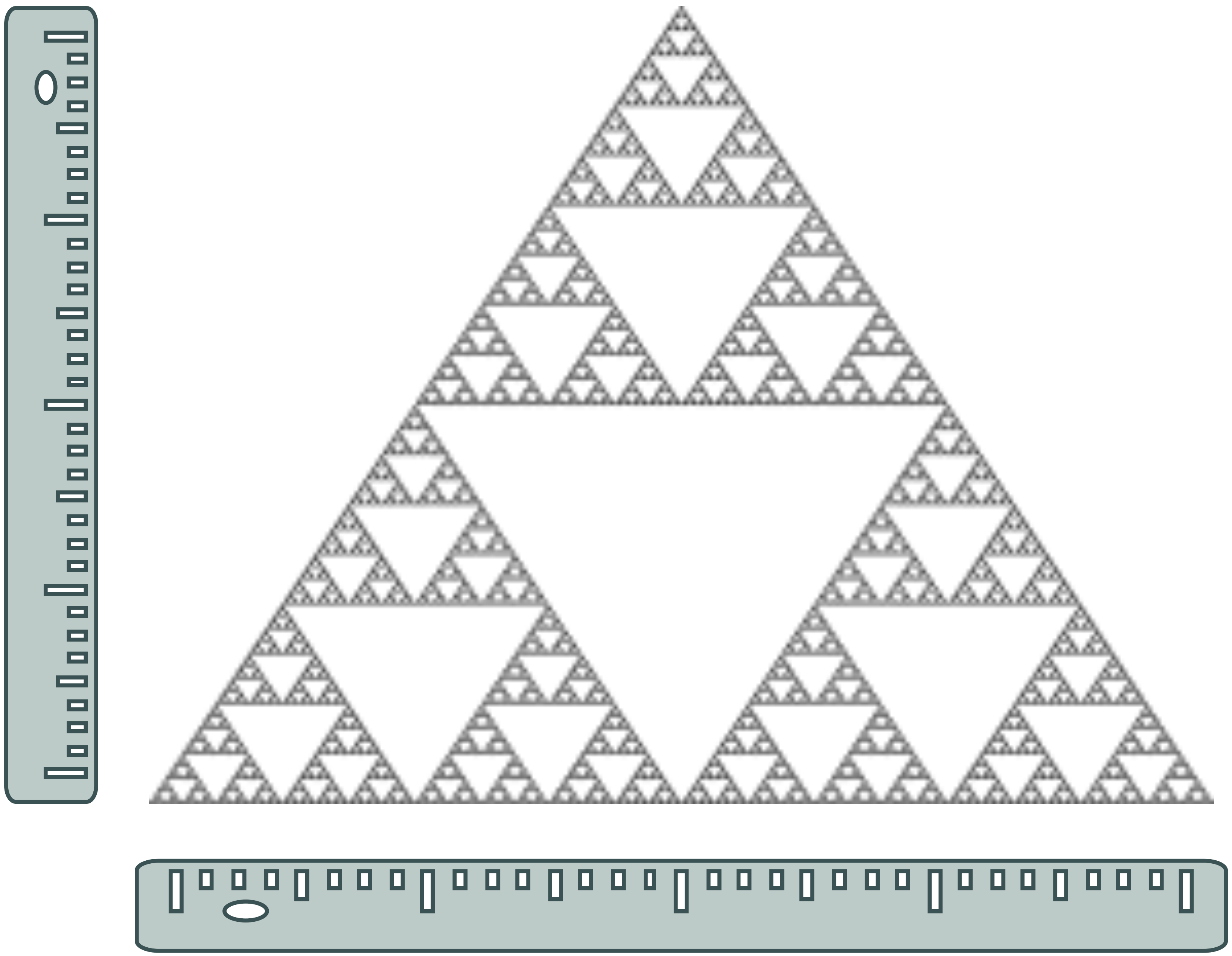
Dimension … an operational definition

Given object (i.e. a set), then define:

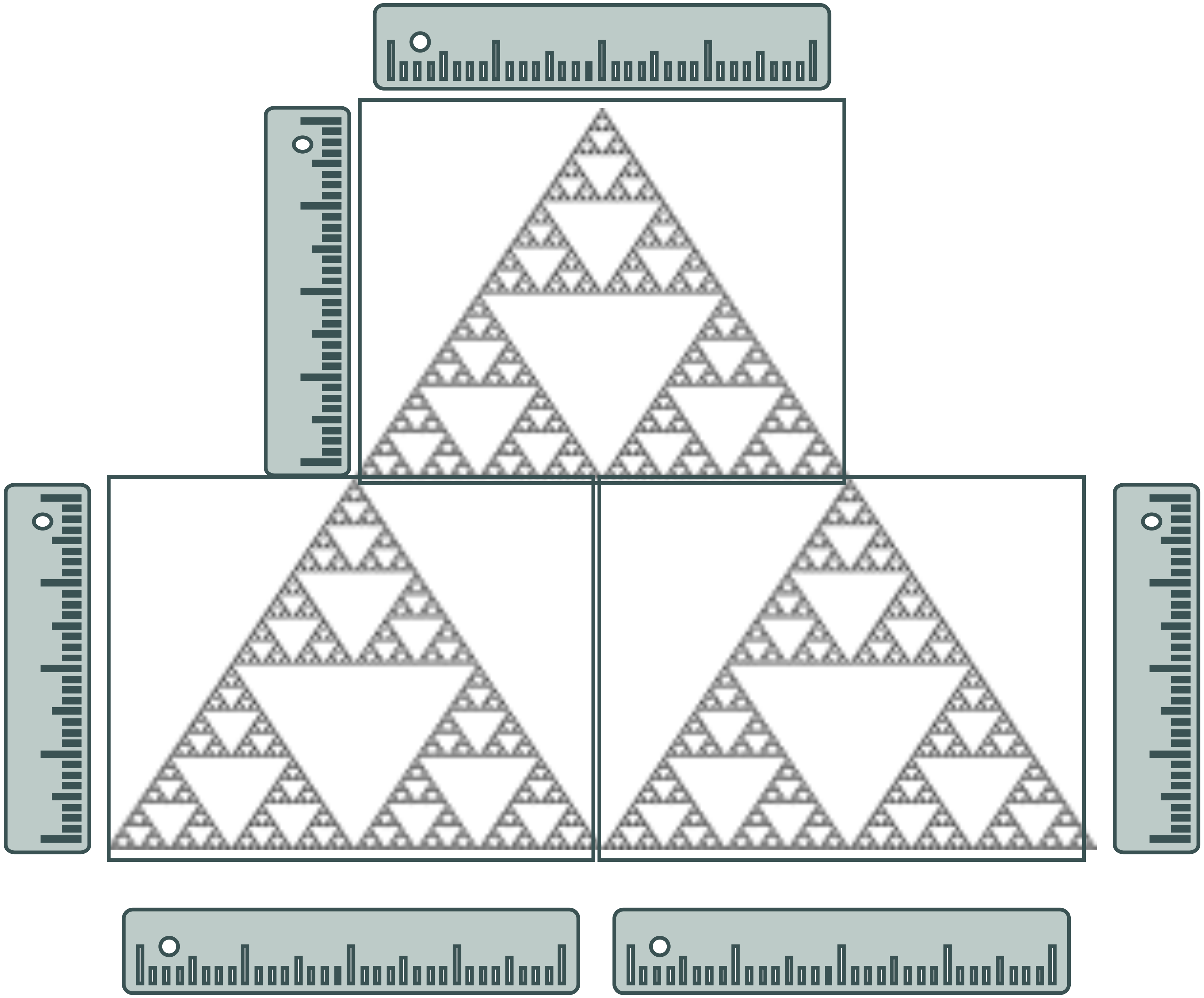
$$\text{number of small rulers} = \left(\frac{1}{\text{reduction factor}}\right)^{D}$$

$$D = \frac{\log(\text{number of small rulers})}{-\log(\text{reduction factor})}$$

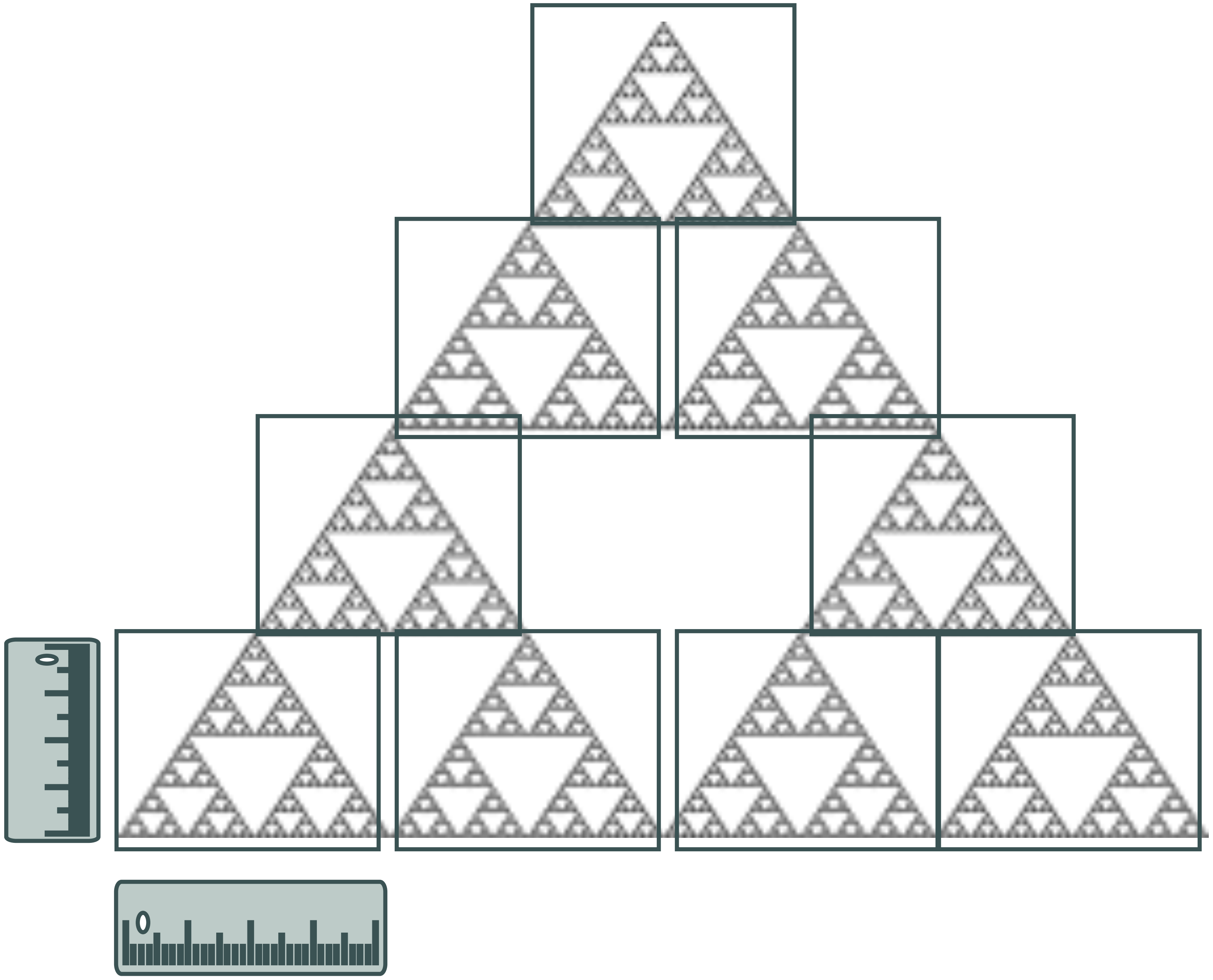
Fractal dimension / self-similarity dimension / counting box dimension



Reduct. fact.	Numb. rulers
1	2



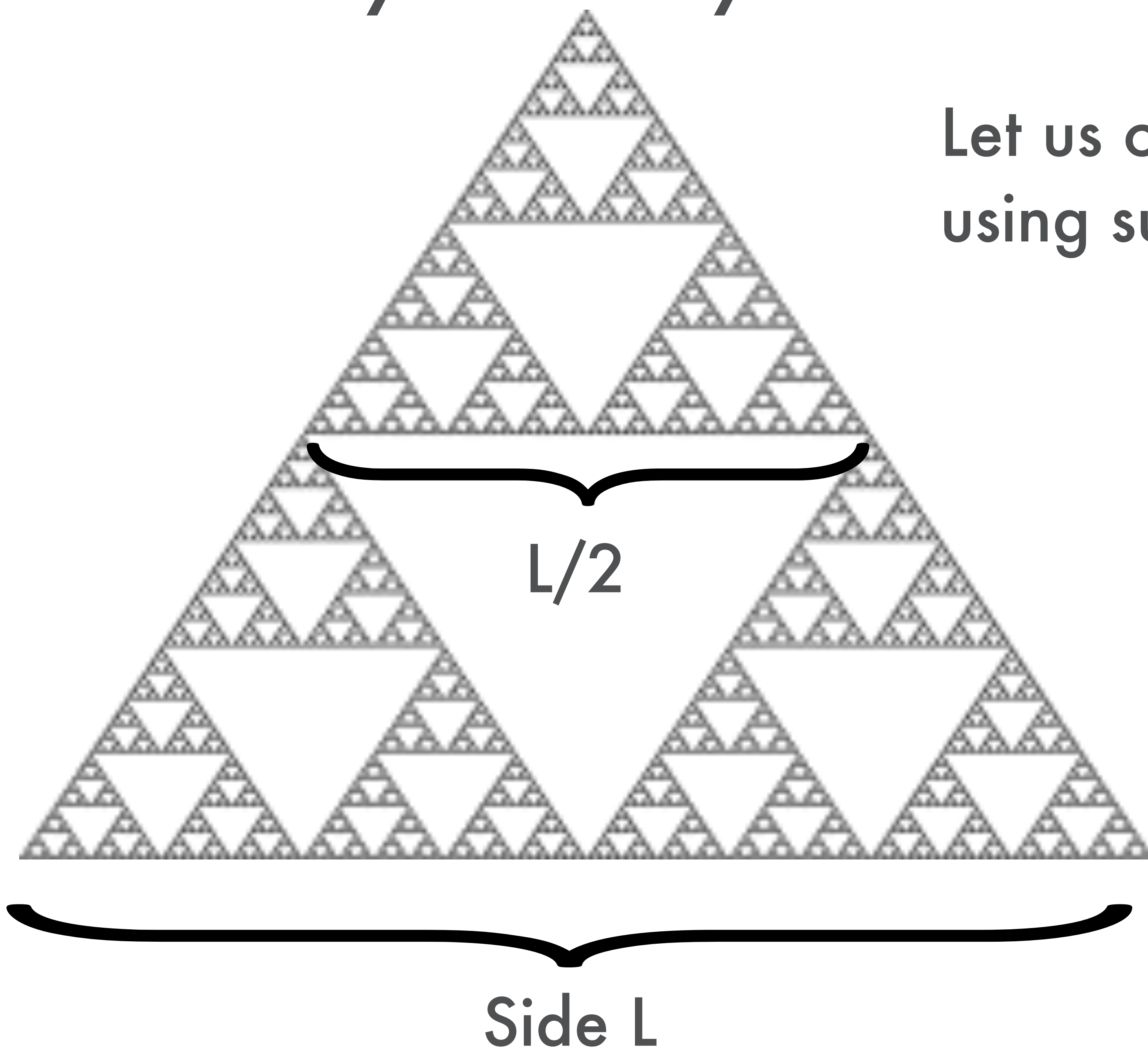
Reduct. fact.	Numb. rulers
1	2
1/2	6



Reduct. fact.	Numb. rulers
1	2
1/2	6=2x3
$1/4=1/2^2$	$18=2x3^2$
⋮	⋮
$1/2^n$	$2x3^n$

$$D = \frac{\log(\text{number of small rulers})}{-\log(\text{reduction factor})} = \frac{\log(2\times 3^n)}{-\log(1/2^n)} = \frac{n\log(3)+\log(2)}{n\log(2)} \to \frac{\log(3)}{\log(2)} \sim 1.585$$

Why are they "weird"?

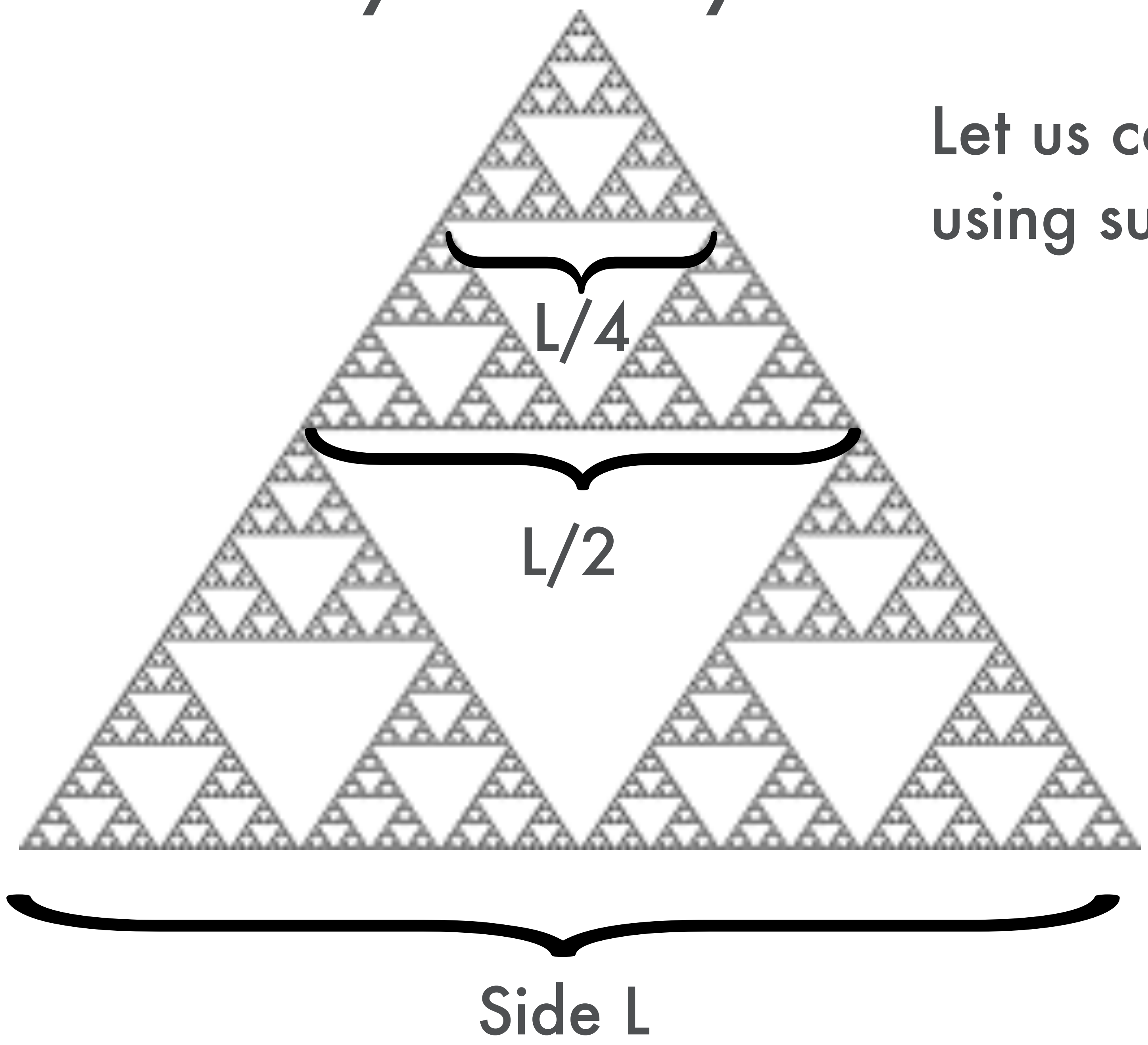


Let us compute its area
using successive approximations:

Step	Area
1	A_1
2	$A_2 = \frac{3}{4} A_1$

$$A_2 = 3 \times (\text{area } A_1/4)$$

Why are they "weird"?

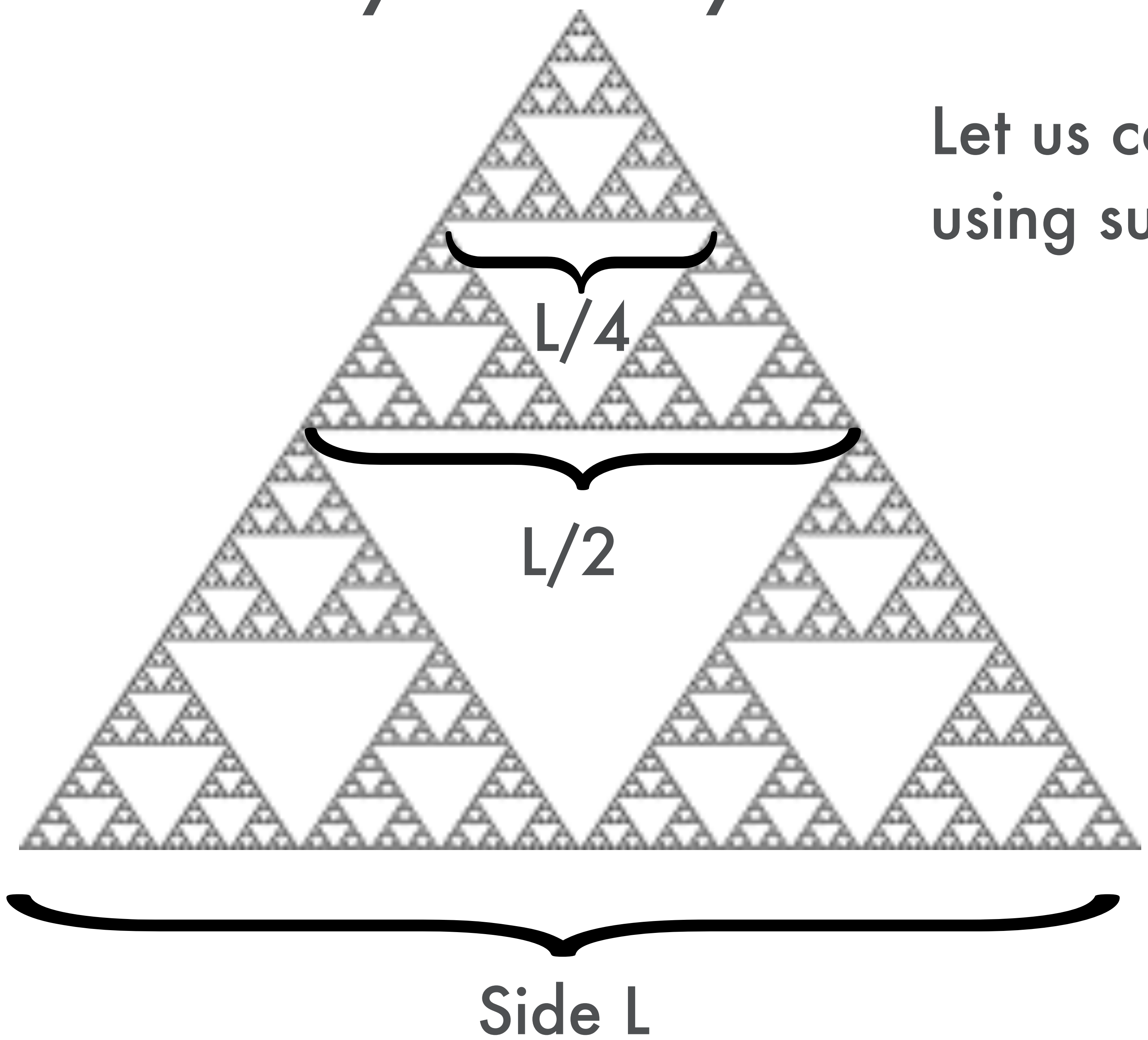


Let us compute its area
using successive approximations:

Step	Area
1	A_1
2	$A_2 = \frac{3}{4} A_1$
3	$A_3 = \left(\frac{3}{4}\right)^2 A_1$

$$A_2 = 9 \times (\text{area } A_1/16)$$

Why are they "weird"?



Let us compute its area using successive approximations:

Step	Area
1	A_1
2	$A_2 = \frac{3}{4} A_1$
3	$A_3 = \left(\frac{3}{4}\right)^2 A_1$
⋮	⋮
n	$A_n = \left(\frac{3}{4}\right)^{n-1} A_1$

$$\text{Area} \to 0 \qquad \text{Perimeter} \to \infty$$

How long is the coast of Britain?
Statistical self-similarity and fractional dimension
Science: 156, 1967, 636-638

B. B. Mandelbrot

Geographical curves are so involved in their detail that their lengths are often infinite or more accurately, undefinable. However, many are statistically ''self-similar,'' meaning that each portion can be considered a reduced-scale image of the whole. In that case, the degree of complication can be described by a quantity **D** that has many properties of a ''dimension,'' though it is fractional. In particular, it exceeds the value unity associated with ordinary curves.

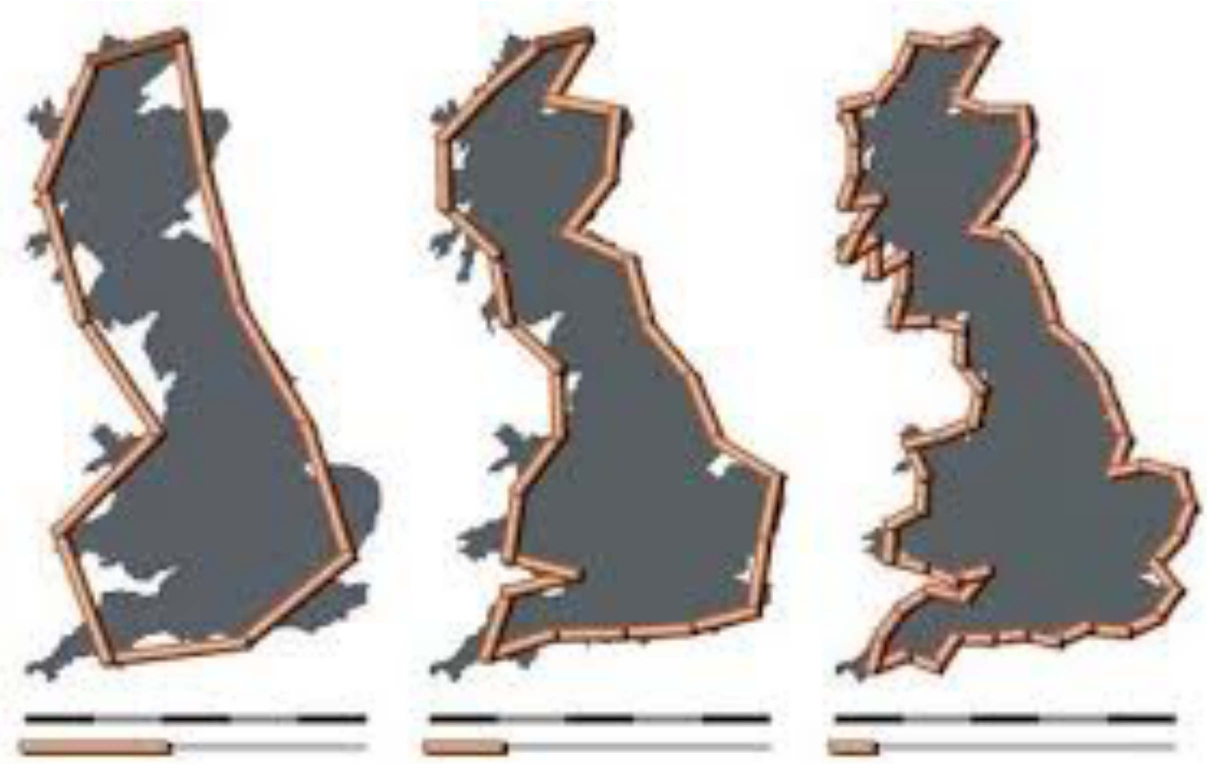
1. Introduction

Seacoast shapes are examples of highly involved curves with the property that — in a statistical sense — each portion can be considered a reduced-scale image of the whole. This property will be referred to as ''statistical self-similarity.'' The concept of ''length'' is usually meaningless for geographical curves. They can be considered superpositions of features of widely scattered characteristic sizes; as even finer features are taken into account, the total measured length increases, and there is usually no clear-cut gap or crossover, between the realm of geography and details with which geography need not be concerned.

Quantities other than length are therefore needed to discriminate between various degrees of complication for a geographical curve. When a curve is self-similar, it is characterized by an exponent of similarity, *D*, which possesses many properties of a dimension, though it is usually a fraction greater that the dimension 1 commonly attributed to curves. I propose to reexamine in this light, some empirical observations in Richardson 1961 and interpret them as implying, for example, that the dimension of the west coast of Great Britain is $D = 1.25$. Thus, the so far esoteric concept of a ''random figure of fractional dimension'' is shown to have simple and concrete applications of great usefulness.

Figure 1 :fig width=page frame=none depth='4.25i' place=bottom. :figcap. :figdesc. Data from Richardson 1961, Fig. 17, reporting on measurements of lengths of geographical curves by way of polygons which have equal sides and have their corners on the curve. For the circle, the total length tends to a limit as the side goes to zero. In all other cases, it increases as the side becomes shorter, the slope of the doubly logarithmic graph having an absolute value equal to $D - 1$. (Reproduced by permission.) :efig.

Self-similarity methods are a potent tool in the study of chance phenomena, wherever they appear, from geostatistics to economics (M 1963b{E }), and physics (M 1967i{N9}). Very similar considerations apply in the study of turbulence, where the characteristic sizes of the ''features'' (which are the eddies) are also very widely scattered, a fact first pointed out by Richardson himself in the 1920's. In fact, many noises have dimensions *D*between 0 and 1; therefore, scientists ought to consider dimension as a continuous quantity ranging from 0 to infinity.



2. Review of the methods used when attempting to measure the length of a seacoast

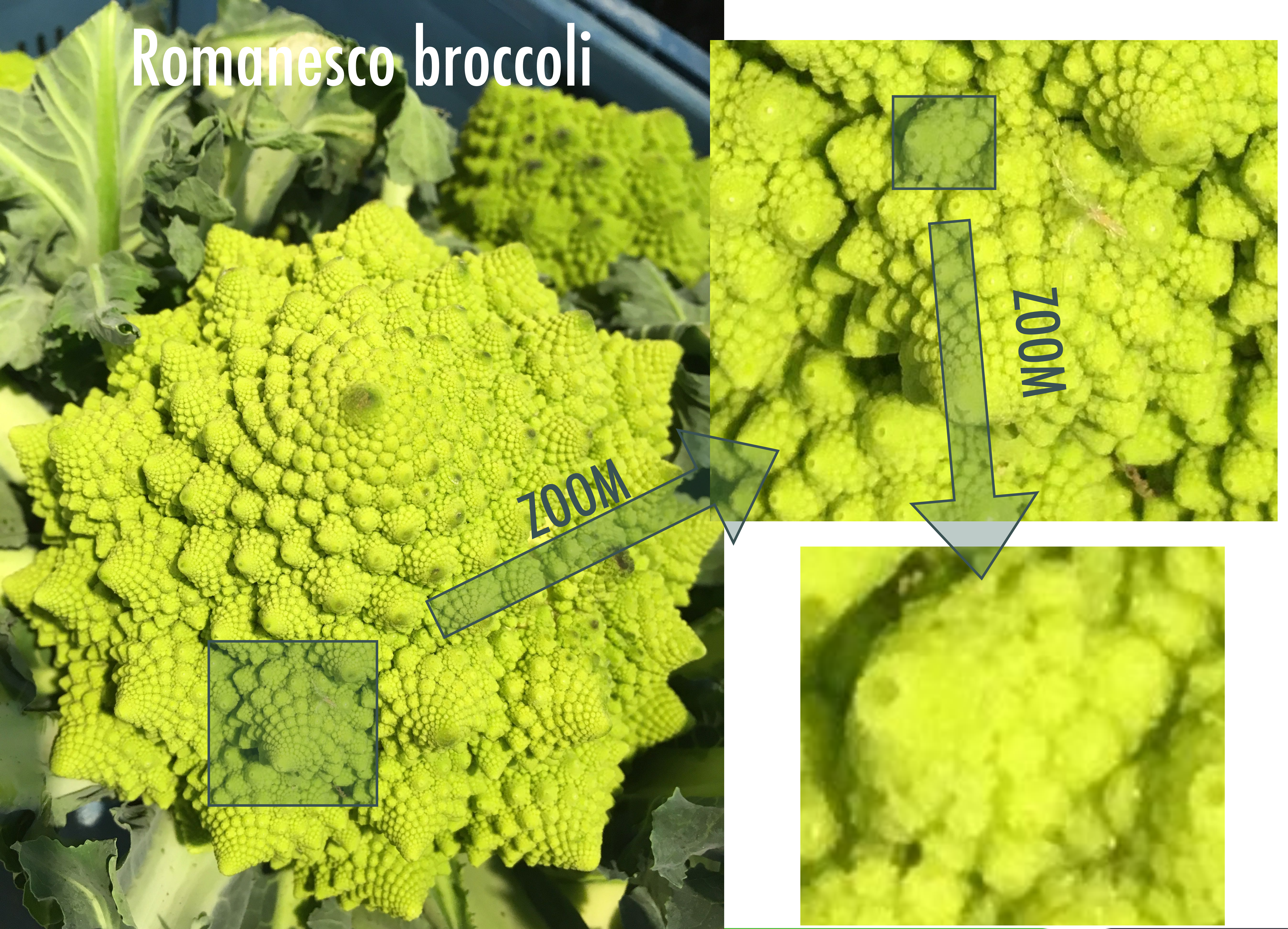
Since geography is unconcerned with minute details, one may choose a positive scale *G* as a lower limit to the length of geographically meaningful features. Then, to evaluate the length of a coast between two of its points *A* and *B* , one may draw the shortest inland curve joining *A* and *B* while staying within a distance *G* of the sea. Alternatively, one may draw the shortest line made of straight segments of length at most *G*, whose vertices are points of the coast which include *A* and *B* . There are many other possible definitions. In practice, of course, the shortest paths must be approximated. We shall suppose that measurements are made by walking a pair of dividers along a map so as to count the number of equal sides of length *G* of an open polygon whose corners lie on the curve. If *G* is small enough, it does not matter whether one starts from *A* or *B*. Thus, one obtains an estimate of the length to be called *L(G)*.

Unfortunately, geographers will disagree about the value of *G*, and *L(G)* depends greatly upon *G*. Consequently, it is necessary to know *L(G)* for several values of *G*. It would be better still to have an analytic formula linking *L(G)* with *G*. Such a formula was proposed in Richardson 1961, but unfortunately it has

A fractal is an object that (among other things):

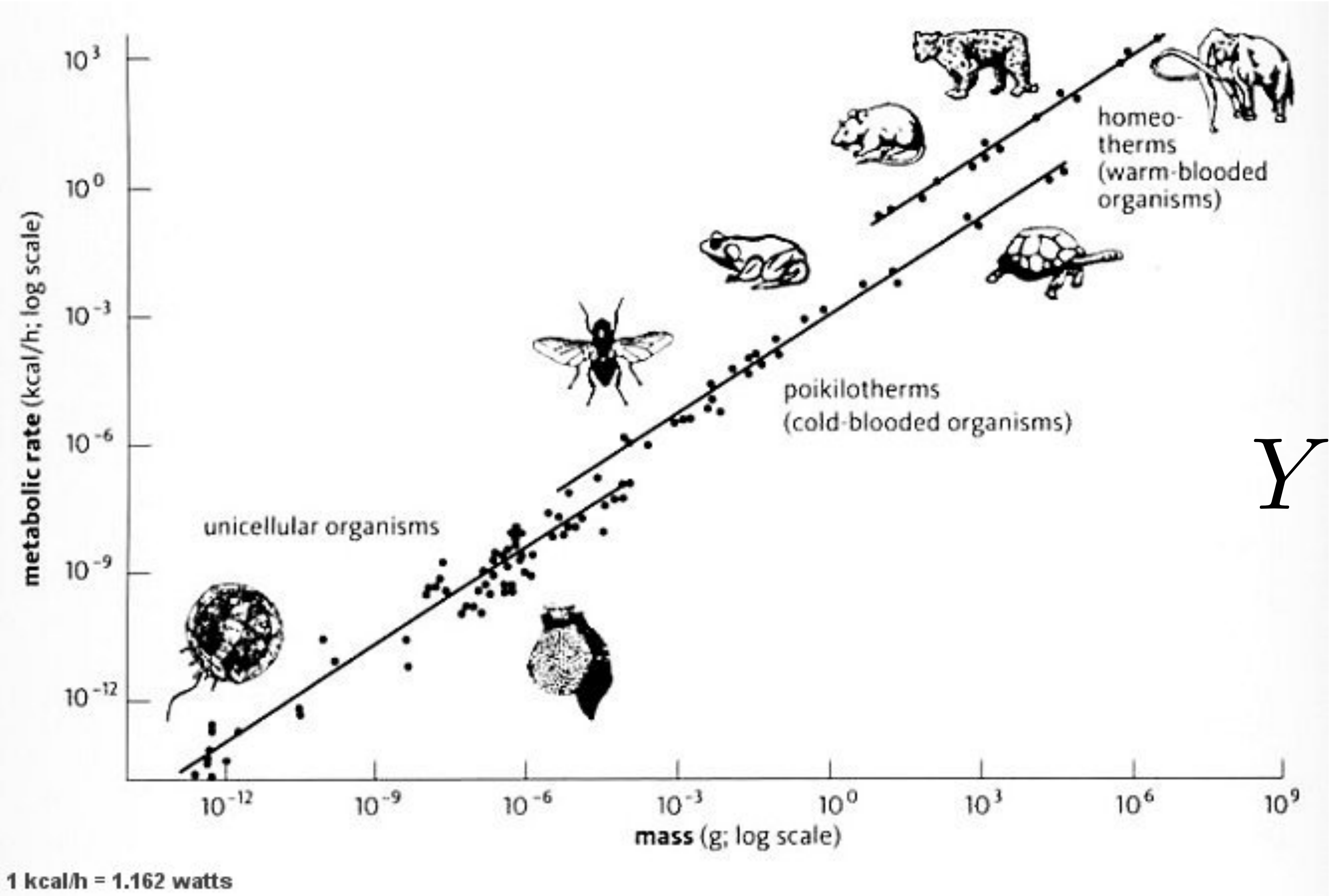
- It has a non trivial fine structure at arbitrarily small scales (self-similarity).
- It is too irregular ("rough") to be easily described by Euclidean geometry.
- It can have a fractional dimension.
- It has a "simple" and recursive definition.

Romanesco broccoli



Fern

Kleiber’s law



$$Y = cM^{3/4}$$

The metabolic rate (energy consumption per hour) vs body mass

Explanation: "The flask cow assumption"

1) The animal is a "bag of blood":

$$\text{body mass} \sim \text{animal volume} \sim L^3$$

2) Metabolism to regulate the body temperature

$$\text{heat exchange} \sim \text{animal surface} \sim L^2$$

$$\text{metabolism} \sim \text{heat exchange} \sim L^2 \sim (\text{body mass})^{2/3}$$

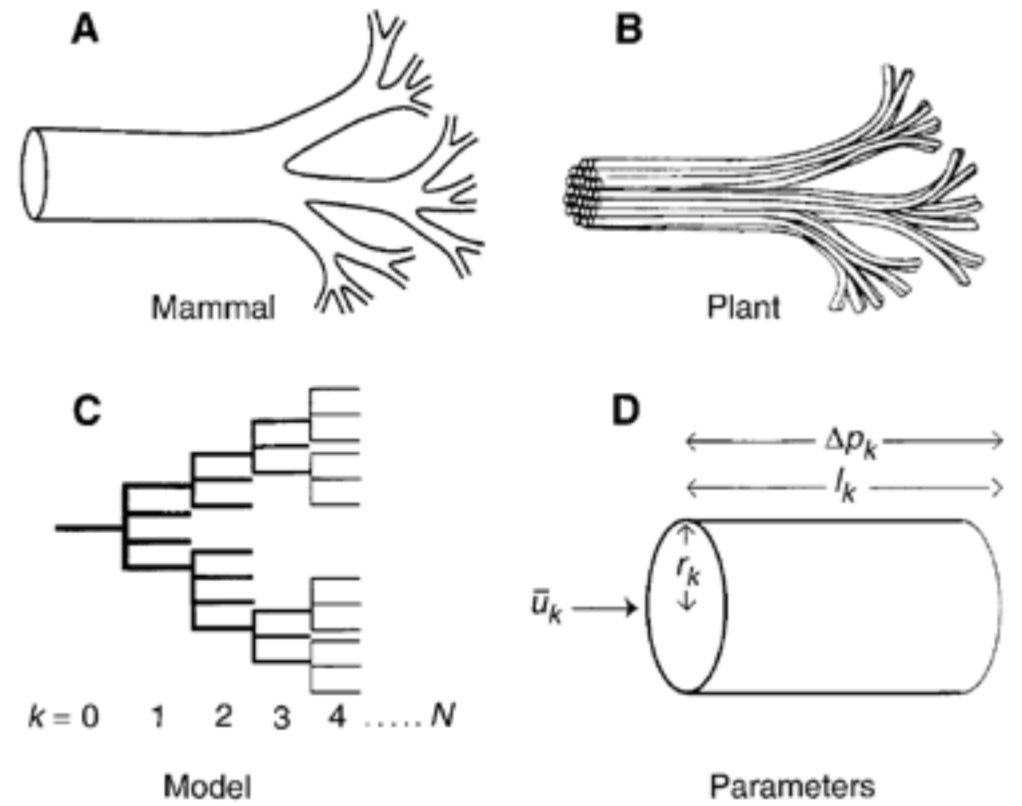
Wrong scaling !

A General Model for the Origin of Allometric Scaling Laws in Biology

Geoffrey B. West, James H. Brown,* Brian J. Enquist

Allometric scaling relations, including the 3/4 power law for metabolic rates, are characteristic of all organisms and are here derived from a general model that describes how essential materials are transported through space-filling fractal networks of branching tubes. The model assumes that the energy dissipated is minimized and that the terminal tubes do not vary with body size. It provides a complete analysis of scaling relations for mammalian circulatory systems that are in agreement with data. More generally, the model predicts structural and functional properties of vertebrate cardiovascular and respiratory systems, plant vascular systems, insect tracheal tubes, and other distribution networks.

SCIENCE • VOL. 276 • 4 APRIL 1997 • http://www.sciencemag.org



Cardiovascular

Variable	Exponent	
	Predicted	Observed
Aorta radius r_0	3/8 = 0.375	0.36
Aorta pressure Δp_0	0 = 0.00	0.032
Aorta blood velocity u_0	0 = 0.00	0.07
Blood volume V_b	1 = 1.00	1.00
Circulation time	1/4 = 0.25	0.25
Circulation distance l	1/4 = 0.25	ND
Cardiac stroke volume	1 = 1.00	1.03
Cardiac frequency ω	−1/4 = −0.25	−0.25
Cardiac output $\dot{E}$	3/4 = 0.75	0.74
Number of capillaries N_c	3/4 = 0.75	ND
Service volume radius	1/12 = 0.083	ND
Womersley number α	1/4 = 0.25	0.25
Density of capillaries	−1/12 = −0.083	−0.095
O_2 affinity of blood P_{50}	−1/12 = −0.083	−0.089
Total resistance Z	−3/4 = −0.75	−0.76
Metabolic rate B	3/4 = 0.75	0.75

Sizing Up Allometric Scaling Theory

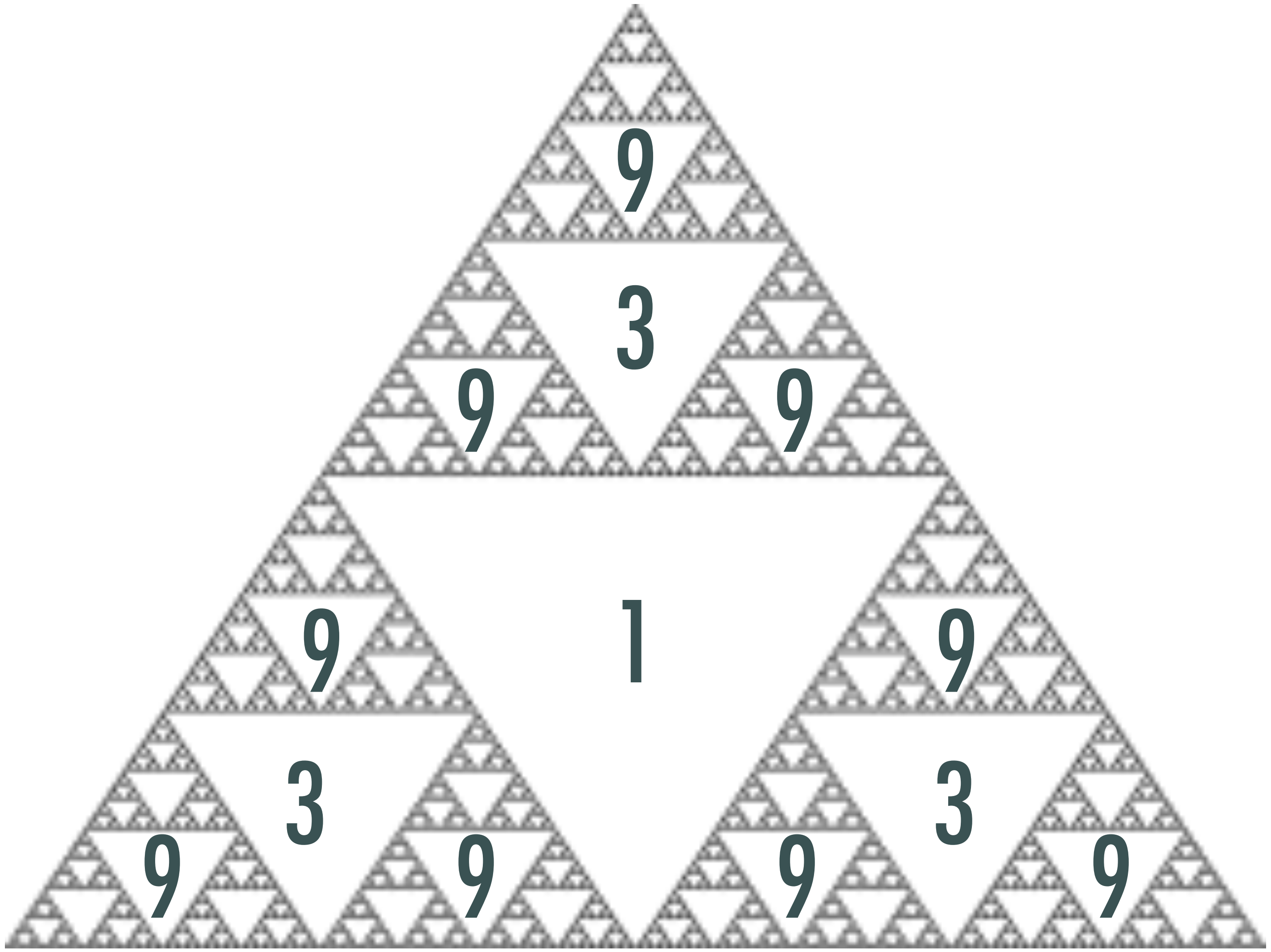
Van M. Savage[¤], Eric J. Deeds[¤], Walter Fontana*

Department of Systems Biology, Harvard Medical School, Boston, Massachusetts, United States of America

Abstract

Metabolic rate, heart rate, lifespan, and many other physiological properties vary with body mass in systematic and interrelated ways. Present empirical data suggest that these scaling relationships take the form of power laws with exponents that are simple multiples of one quarter. A compelling explanation of this observation was put forward a decade ago by West, Brown, and Enquist (WBE). Their framework elucidates the link between metabolic rate and body mass by focusing on the dynamics and structure of resource distribution networks—the cardiovascular system in the case of mammals. Within this framework the WBE model is based on eight assumptions from which it derives the well-known observed scaling exponent of 3/4. In this paper we clarify that this result only holds in the limit of infinite network size (body mass) and that the actual exponent predicted by the model depends on the sizes of the organisms being studied. Failure to clarify and to explore the nature of this approximation has led to debates about the WBE model that were at cross purposes. We compute analytical expressions for the finite-size corrections to the 3/4 exponent, resulting in a spectrum of scaling exponents as a function of absolute network size. When accounting for these finite-size corrections over a size range spanning the eight orders of magnitude observed in mammals, the WBE model predicts a scaling exponent of 0.81, seemingly at odds with data. We then proceed to study the sensitivity of the scaling exponent with respect to variations in several assumptions that underlie the WBE model, always in the context of finite-size corrections. Here too, the trends we derive from the model seem at odds with trends detectable in empirical data. Our work illustrates the utility of the WBE framework in reasoning about allometric scaling, while at the same time suggesting that the current canonical model may need amendments to bring its predictions fully in line with available datasets.

Scale-free is a signature of fractality



Numb. Triang.	Area
1	W_1
3	$W_2 = \frac{W_1}{2}$
$9=3^2$	$W_3 = \frac{W_2}{2} = \frac{W_1}{2^2}$
$\vdots$	$\vdots$
3^n	$W_n = \frac{W_{n-1}}{2} = \frac{W_1}{2^{n-1}}$

$$\log\left(\text{Numb. Triang.}(W)\right) = -\frac{\log(3)}{\log(2)}\log(W) + C$$

Growth, innovation, scaling, and the pace of life in cities

Luís M. A. Bettencourt*[†], José Lobo[‡], Dirk Helbing[§], Christian Kühnert[§], and Geoffrey B. West*[¶]

*Theoretical Division, MS B284, Los Alamos National Laboratory, Los Alamos, NM 87545; [‡]Global Institute of Sustainability, Arizona State University, P.O. Box 873211, Tempe, AZ 85287-3211; [§]Institute for Transport and Economics, Dresden University of Technology, Andreas-Schubert-Strasse 23, D-01062 Dresden, Germany; and [¶]Santa Fe Institute, 1399 Hyde Park Road, Santa Fe, NM 87501

Edited by Elinor Ostrom, Indiana University, Bloomington, IN, and approved March 6, 2007 (received for review November 19, 2006)

Humanity has just crossed a major landmark in its history with the majority of people now living in cities. Cities have long been known to be society's predominant engine of innovation and wealth creation, yet they are also its main source of crime, pollution, and disease. The inexorable trend toward urbanization worldwide presents an urgent challenge for developing a predictive, quantitative theory of urban organization and sustainable development. Here we present empirical evidence indicating that the processes relating urbanization to economic development and knowledge creation are very general, being shared by all cities belonging to the same urban system and sustained across different nations and times. Many diverse properties of cities from patent production and personal income to electrical cable length are shown to be power law functions of population size with scaling exponents, *β*, that fall into distinct universality classes. Quantities

PNAS | April 24, 2007 | vol. 104 | no. 17 | 7301–7306

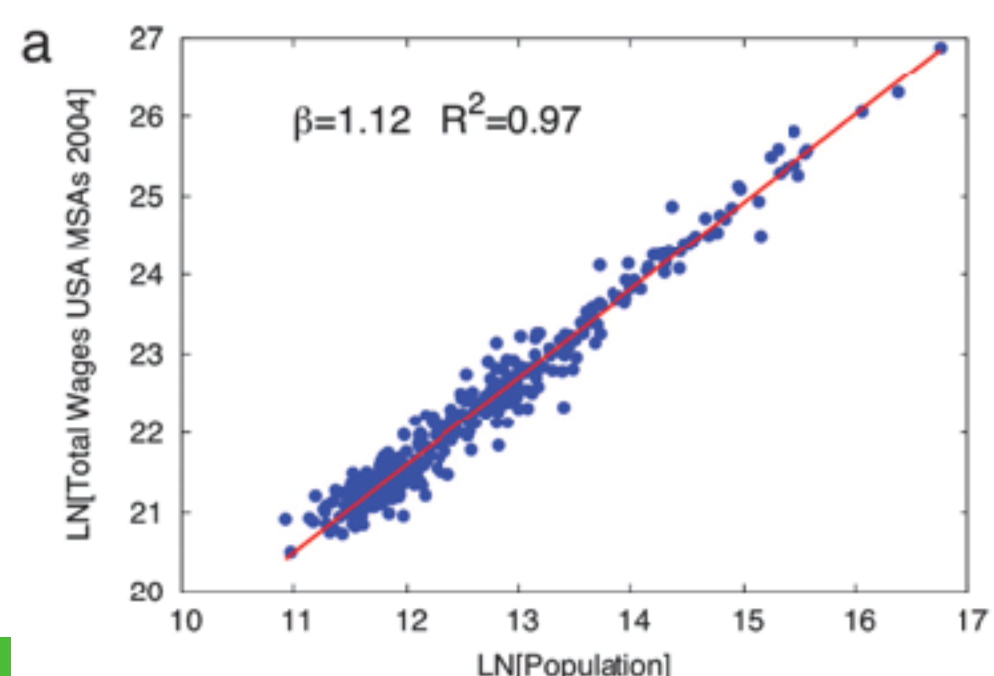
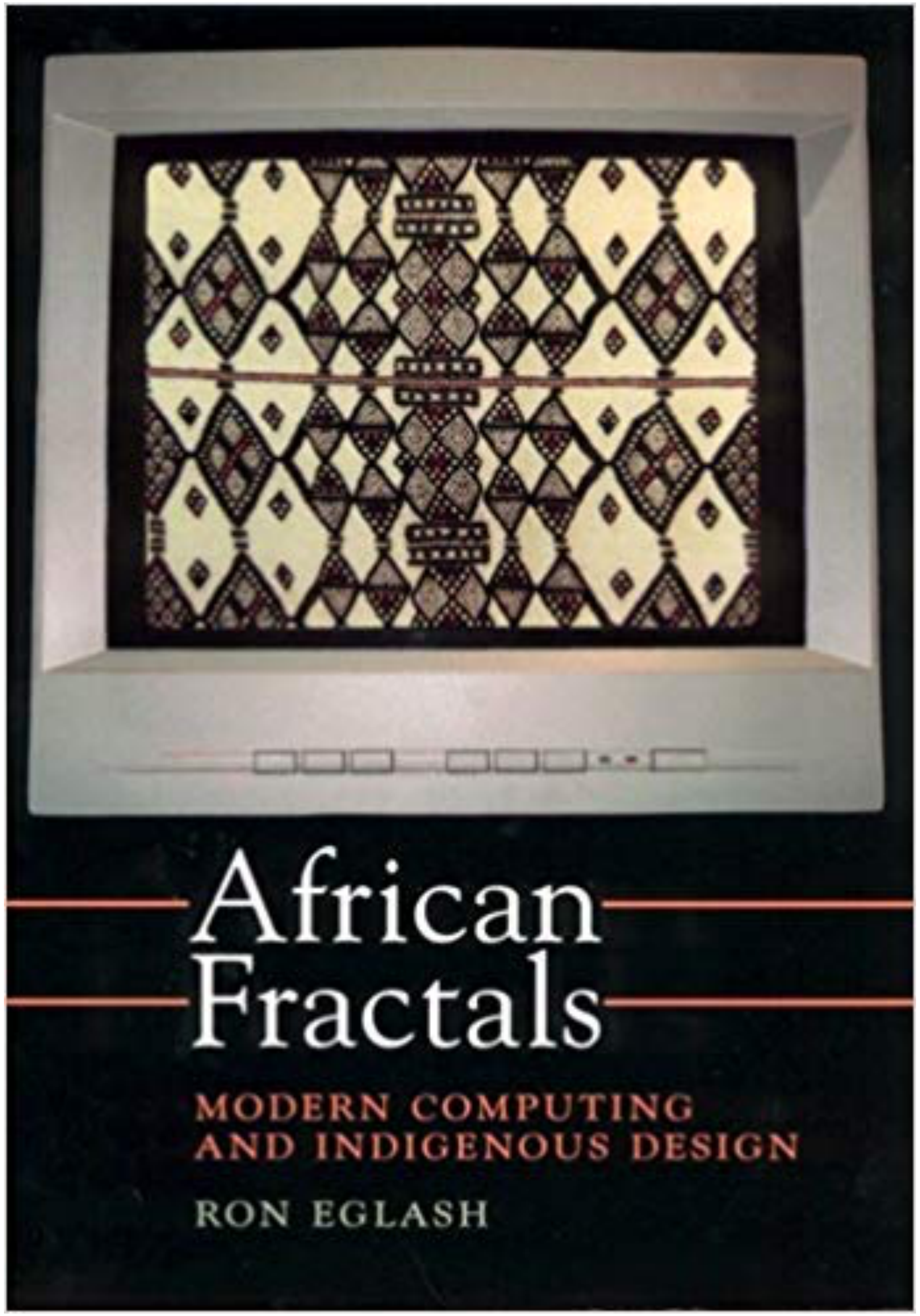
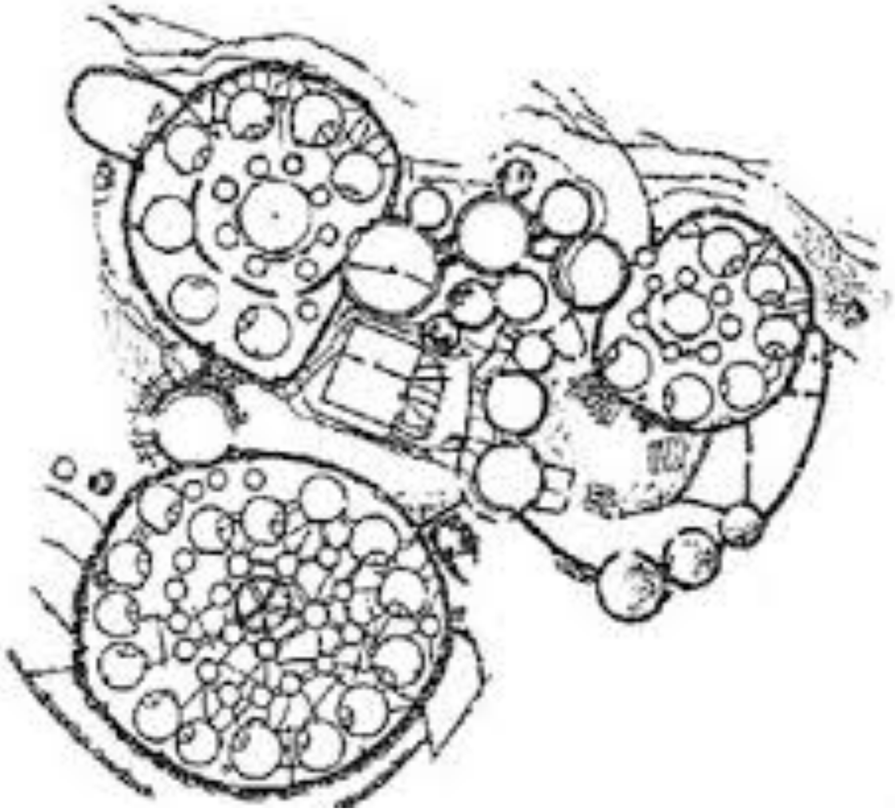
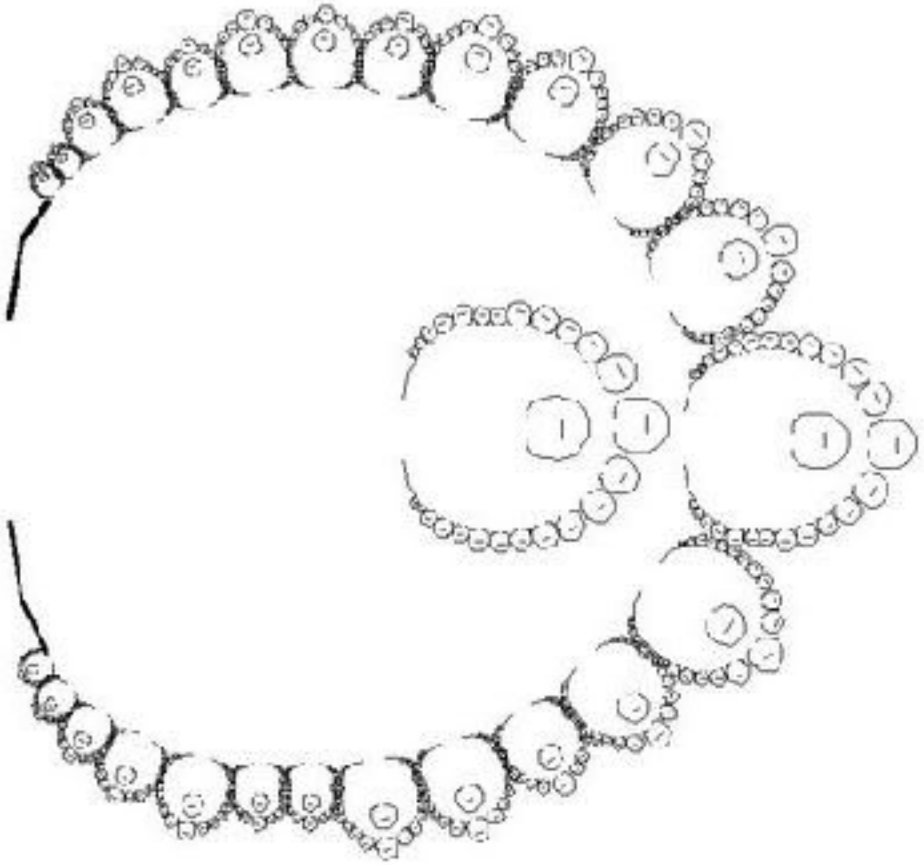


Table 1. Scaling exponents for urban indicators vs. city size

Y	*β*	95% CI	Adj-R^2	Observations	Country–year
New patents	1.27	[1.25,1.29]	0.72	331	U.S. 2001
Inventors	1.25	[1.22,1.27]	0.76	331	U.S. 2001
Private R&D employment	1.34	[1.29,1.39]	0.92	266	U.S. 2002
"Supercreative" employment	1.15	[1.11,1.18]	0.89	287	U.S. 2003
R&D establishments	1.19	[1.14,1.22]	0.77	287	U.S. 1997
R&D employment	1.26	[1.18,1.43]	0.93	295	China 2002
Total wages	1.12	[1.09,1.13]	0.96	361	U.S. 2002
Total bank deposits	1.08	[1.03,1.11]	0.91	267	U.S. 1996
GDP	1.15	[1.06,1.23]	0.96	295	China 2002
GDP	1.26	[1.09,1.46]	0.64	196	EU 1999–2003
GDP	1.13	[1.03,1.23]	0.94	37	Germany 2003
Total electrical consumption	1.07	[1.03,1.11]	0.88	392	Germany 2002
New AIDS cases	1.23	[1.18,1.29]	0.76	93	U.S. 2002–2003
Serious crimes	1.16	[1.11, 1.18]	0.89	287	U.S. 2003
Total housing	1.00	[0.99,1.01]	0.99	316	U.S. 1990
Total employment	1.01	[0.99,1.02]	0.98	331	U.S. 2001
Household electrical consumption	1.00	[0.94,1.06]	0.88	377	Germany 2002
Household electrical consumption	1.05	[0.89,1.22]	0.91	295	China 2002
Household water consumption	1.01	[0.89,1.11]	0.96	295	China 2002
Gasoline stations	0.77	[0.74,0.81]	0.93	318	U.S. 2001
Gasoline sales	0.79	[0.73,0.80]	0.94	318	U.S. 2001
Length of electrical cables	0.87	[0.82,0.92]	0.75	380	Germany 2002
Road surface	0.83	[0.74,0.92]	0.87	29	Germany 2002

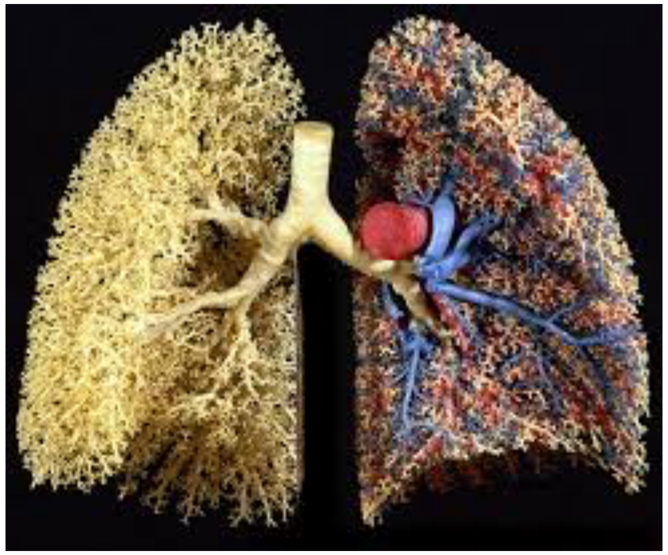

African
Fractals
MODERN COMPUTING
AND INDIGENOUS DESIGN
RON EGLASH





Fractals in ...

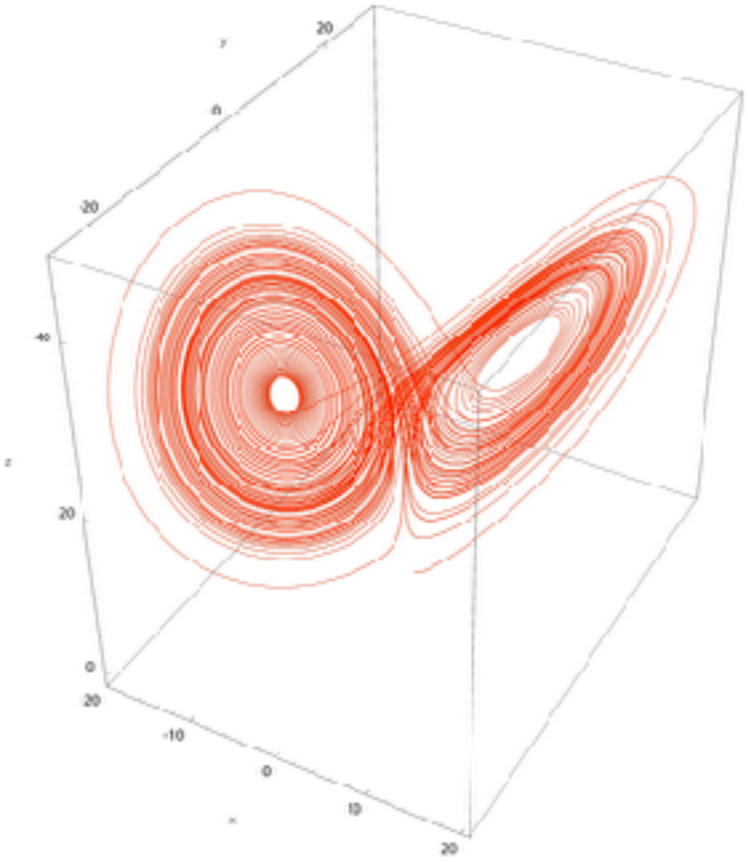
Medicine: vascular system in the kidney, lung, tissues, ...

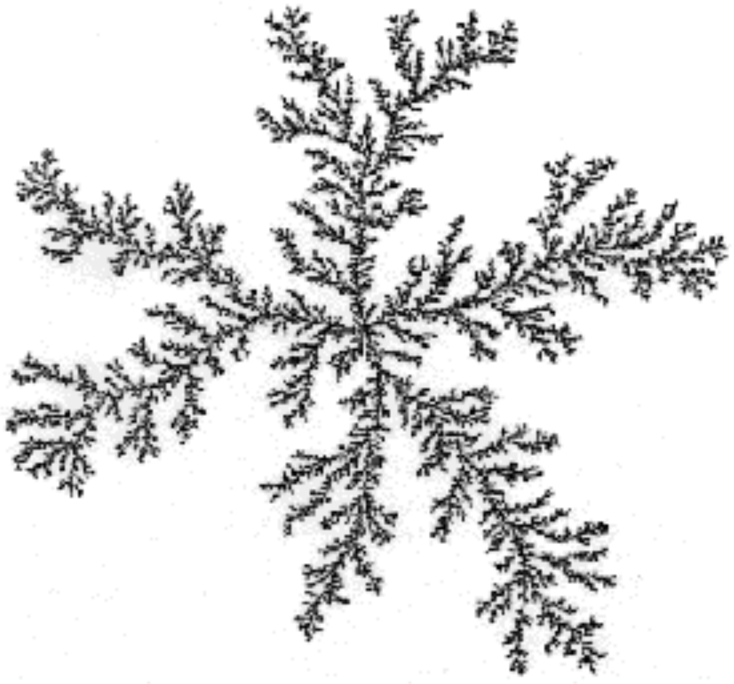


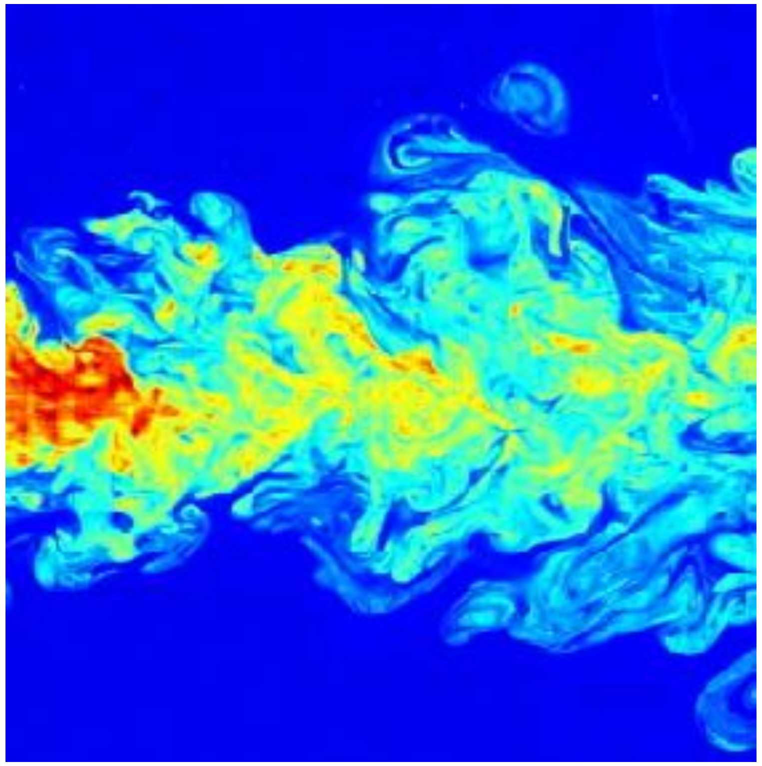
Idea: determine if a disease modifies the natural (fractal) structure

Fractals in ...

Physics/mathematics: strange attractors of dynamical systems, diffusion limited aggregates (DLA), turbulence, ...







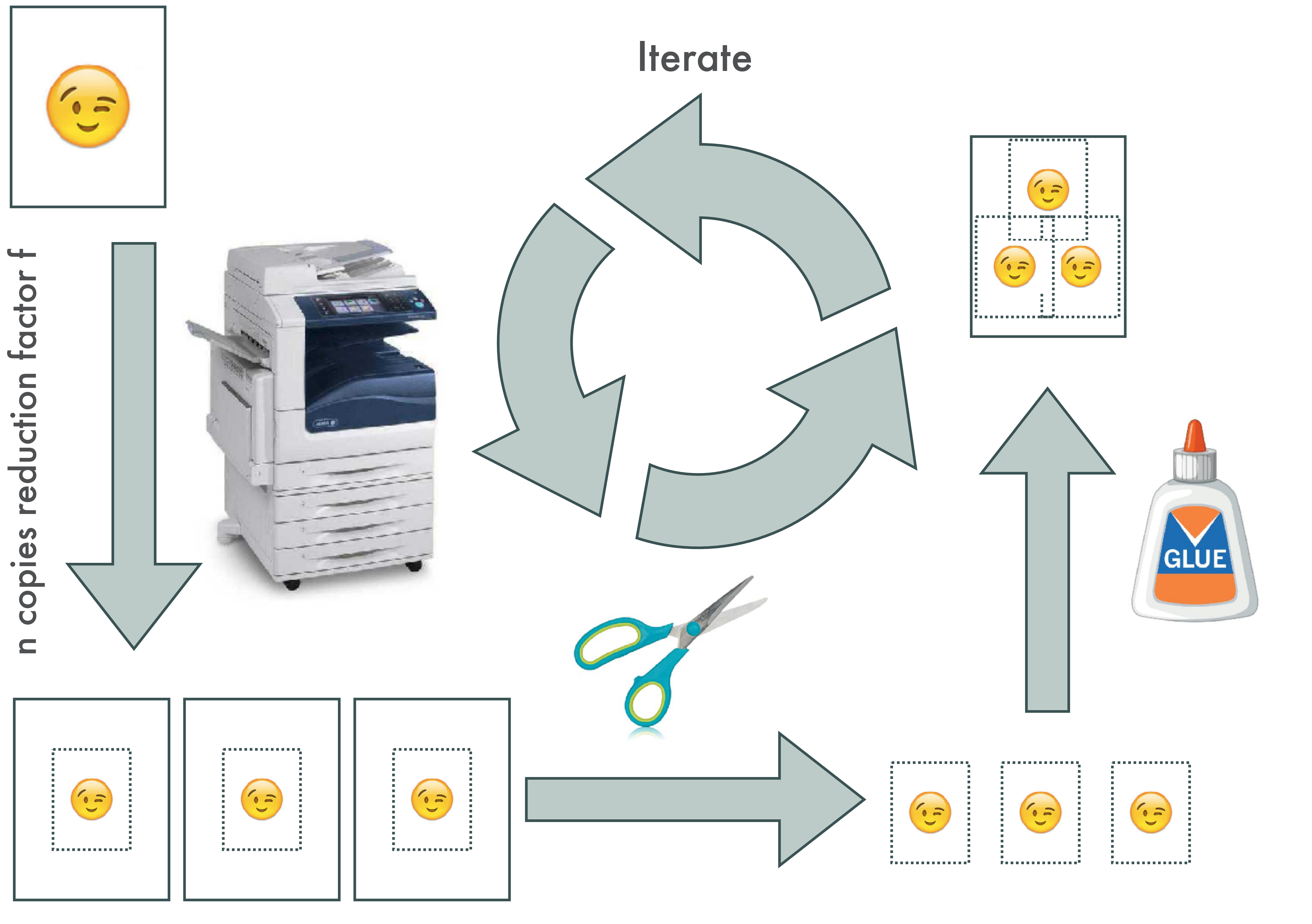
Deterministic chaos

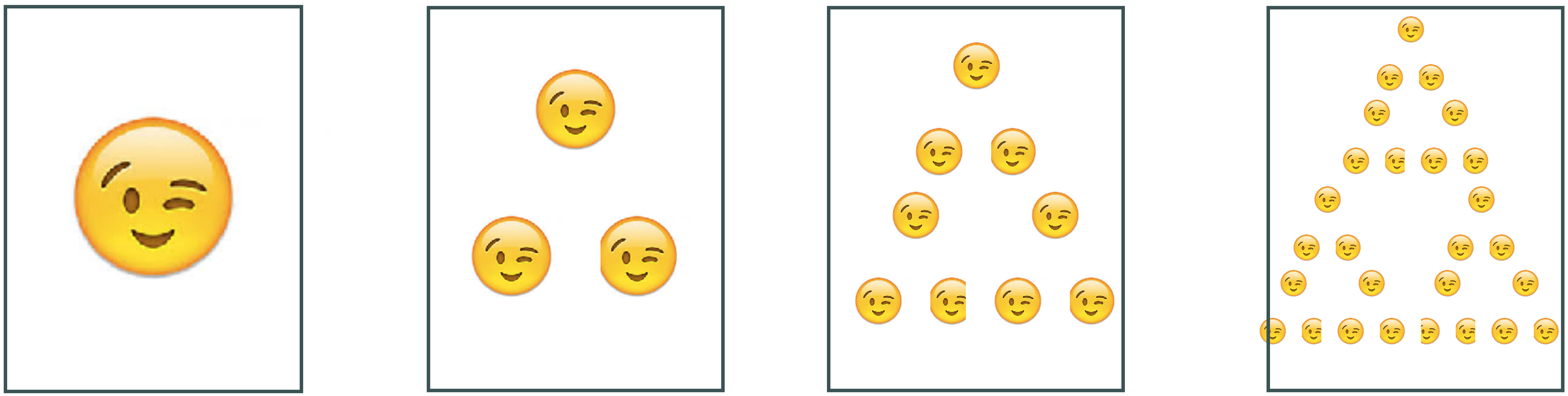
Growing materials

Hand made fractal recipe
(a theorem you can prove with your hands …)

+

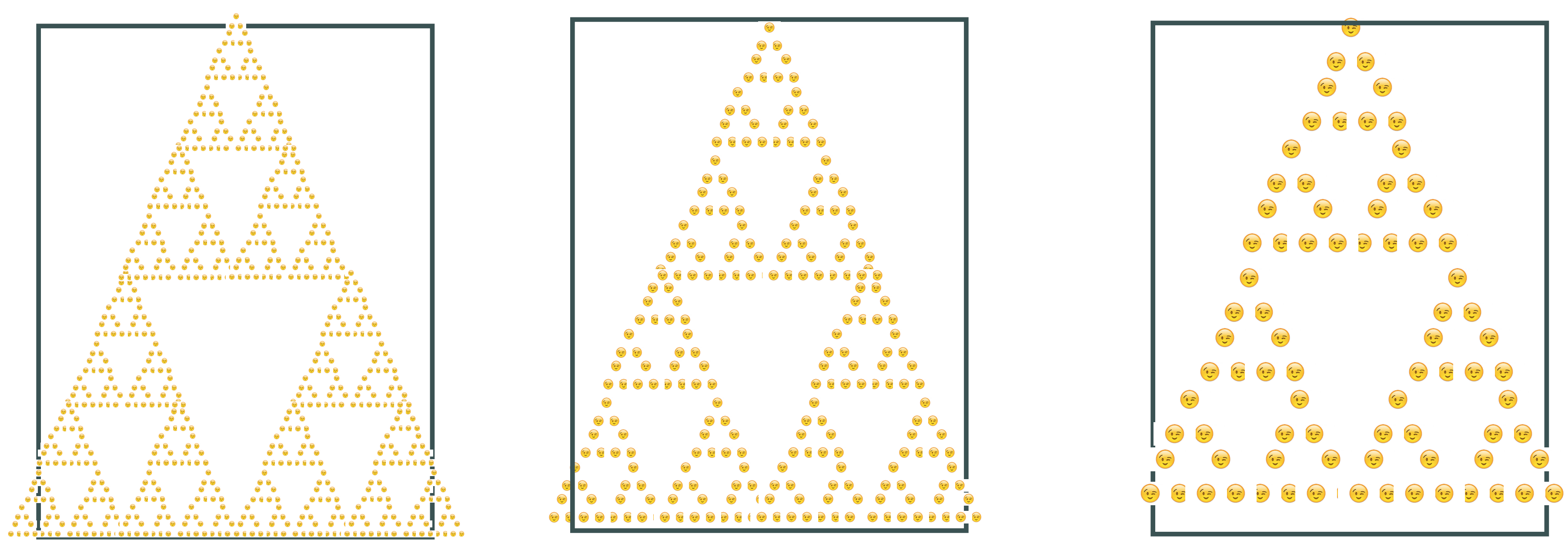
+


Iterate
n copies reduction factor f
GLUE



Sierpinski triangle!

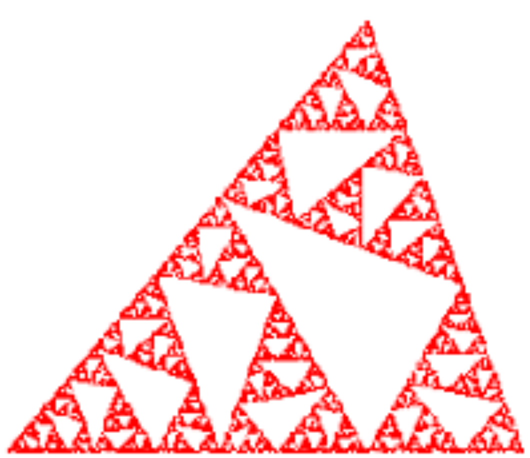
Which dimension?

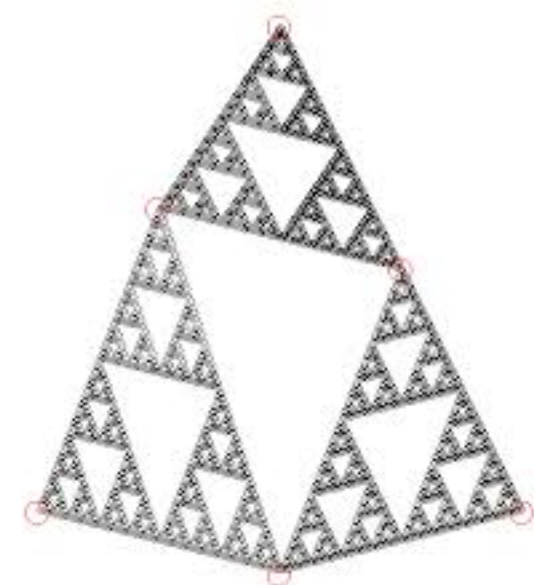


The fractal dimension depends only on the number of copies and on the scaling factor, not on the initial shape.

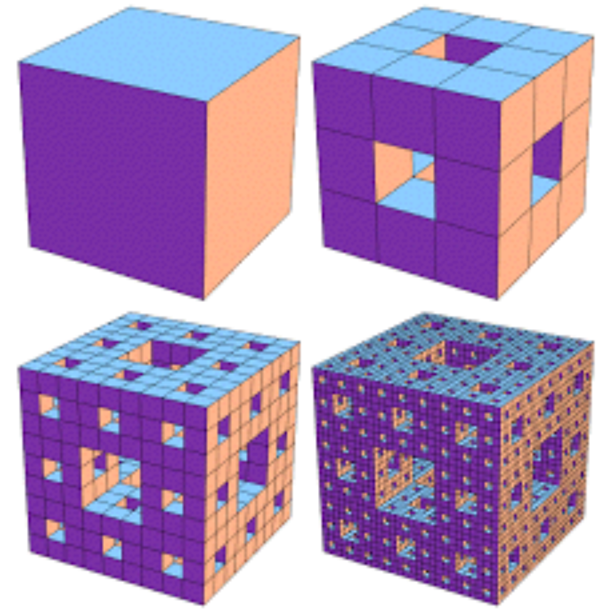
$$D = -\frac{\log(\text{number of copies})}{\log(\text{reduction factor})}$$

The fractal set depends also on the relative position of each copy.





Fractals maximize the volume/surface ratio, so they are relevant once diffusion is a key ingredient.



Fractals can be constructed iterating simple rules: grow, split/branch, scale down and iterate.

September the 5th, 2018, Namur

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Simples rules to explain the complex beauty of fractals

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