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Luo, Ziyang; Qi, Liqun; Toint, Philippe

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Bernstein Concentration Inequalities for Tensors via Einstein Products

Ziyan Luo*, Liqun Qi[†] and Philippe L. Toint[‡]

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Abstract

A generalization of the Bernstein matrix concentration inequality to random tensors of general order is proposed. This generalization is based on the use of Einstein products between tensors, from which a strong link can be established between matrices and tensors, in turn allowing exploitation of existing results for the former.

AMS subject classifications. 15A52, 15A72, 49J55, 60H25. **Keywords:** random tensors, concentration inequality, Einstein products, subsampling, computational statistics.

1 Introduction

The theory of random matrices has a rich history starting with Hurwitz (see [10]) and Wishart [22] in the first half of the 20th century. While it has developed on its own right within probability theory, it has also found applications in many diverse domains of computational statistics, ranging from matrix approximation [11] to compressed sensing [8], graph theory [2], sparsification [1] or subsampling of data [21]. Important tools in several of these fields are matrix concentration theorems that give results on expectation, norm distribution and probability of deviation from the expectation. We refer the interested reader to the excellent book by Tropp [20] for further elaboration and an extensive bibliography.

The purpose of this short paper is to extend one of the prominent matrix concentration results, the Bernstein inequality, to the case of tensors of general order. This extension was originally motivated by the desire to extend the use of the Bernstein inequality in subsampling estimation of gradients and Hessians of additive multivariate real functions [3, 4, 19, 6, 13, 23, 24] to derivatives of higher degree, thereby providing estimation tools for general Taylor's expansions of such functions. It is however clear that applications of the new tensor result has wider potential, including, for instance, randomized tensor sparsification (such as in video streaming) or randomized tensor products for fast computations.

*State Key Laboratory of Rail Traffic Control and Safety, Beijing Jiaotong University, Beijing, China. E-mail: zyluo@bjtu.edu.cn

[†]Department of Applied Mathematics, The Hong Kong Polytechnic University, Hung Hom, Kowloon, Hong Kong. E-mail: liqun.qi@polyu.edu.hk

[‡]Namur Center for Complex Systems (naXys), University of Namur, 61, rue de Bruxelles, B-5000 Namur, Belgium. Email: philippe.toint@unamur.be

Our approach hinges on Einstein products of tensors and associated “matricization” transformations: these recast tensors in the form of large matrices to which known results of matrix concentration inequalities [20] may then be applied.

The paper is organized as follows. Section 2 introduces the Einstein products and states some of its properties that are central to our development. We then state the Bernstein concentration inequality for Einstein-symmetric tensor of even order in Section 3. The more general inequality for Einstein-symmetric tensors of arbitrary order is derived in Section 4 and an “intrinsic dimension” version of this inequality presented in Section 5. Some conclusions and perspectives are finally presented in Section 6.

2 Tensors and the Einstein Product

We start by defining the Einstein tensor product for high-order tensors, first introduced by Lord Kelvin in 1856 [12] and named after Albert Einstein for his work in [9].

Definition 1 (Einstein Product, [9]) Let \mathcal{A} be a tensor in $\mathbb{R}^{I_1 \times \dots \times I_m \times K_1 \times \dots \times K_m}$ and \mathcal{B} be a tensor in $\mathbb{R}^{K_1 \times \dots \times K_m \times J_1 \times \dots \times J_p}$. The Einstein product of \mathcal{A} and \mathcal{B} , denoted by $\mathcal{A} \square \mathcal{B}$, is defined by

$$(\mathcal{A} \square \mathcal{B})_{i_1 \dots i_m j_1 \dots j_p} = \sum_{k_1 \dots k_m} a_{i_1 \dots i_m k_1 \dots k_m} b_{k_1 \dots k_m j_1 \dots j_p}, \quad \text{for all } i_1, \dots, i_m, j_1, \dots, j_p, \quad (2.1)$$

In this definition, each lowercase index varies from 1 to its uppercase equivalent: for instance i_2 varies from 1 to I_2 , k_3 from 1 to K_3 and j_1 from 1 to J_1 .

The Einstein product can be regarded as a higher order generalization of the standard matrix multiplication in which $m = p = 1$. Such a contraction product has been widely used in the areas of continuum mechanics [15] and relativity theory [9]. Notice that in $\mathbb{T}_{m,d}$, the space of real tensors of order m and dimension d , that is the set of multiarrays $\mathcal{A} = (a_{i_1, \dots, i_m})$ where i_j varies from 1 to d for $j = 1, \dots, m$, the Einstein product satisfies the closure property

$$\mathcal{A}, \mathcal{B} \in \mathbb{T}_{2m,d} \implies \mathcal{A} \square \mathcal{B} \in \mathbb{T}_{2m,d}.$$

This nice property allows us to follow [5] and define several new concepts based on the Einstein product for tensors.

Definition 2 Let $\mathcal{A} = (a_{i_1 \dots i_m j_1 \dots j_m}) \in \mathbb{T}_{2m,n}$.

(i) **Transpose:** The transpose of \mathcal{A} , denoted by \mathcal{A}^\top , is defined by the relations

$$\left(\mathcal{A}^\top\right)_{i_1 \dots i_m j_1 \dots j_m} = (\mathcal{A})_{j_1 \dots j_m i_1 \dots i_m} \quad \text{for all } i_1, \dots, i_m, j_1, \dots, j_m.$$

(ii) **Einstein-Symmetric Tensor:** \mathcal{A} is called Einstein-symmetric, or \square -symmetric, if $\mathcal{A}^\top = \mathcal{A}$. The set of all \square -symmetric tensors in $\mathbb{T}_{2m,d}$ is a subspace and is denoted by $\mathbb{S}_{2m,d}$.

(iii) **Diagonal Tensor:** An \square -symmetric tensor \mathcal{A} is said to be diagonal if $a_{i_1 \dots i_m j_1 \dots j_m} = 0$ whenever $\prod_k \delta_{i_k j_k} = 0$, where δ_{ij} is the Kronecker delta.

(iii) **Identity Tensor:** The Einstein-identity tensor, denoted by \mathcal{I}^\square , is a diagonal \square -symmetric tensor with $a_{i_1 \dots i_m i_1 \dots i_m} = 1$ for all i_1, \dots, i_m .

(iv) **Orthogonal Tensor:** \mathcal{A} is called Einstein-orthogonal, or \square -orthogonal, if $\mathcal{A}^\top \square \mathcal{A} = \mathcal{I}^\square$.

(vi) **EVD:** If $\mathcal{A} \in \mathbb{S}_{2m,d}$, then

$$\mathcal{A} = \mathcal{U} \square \mathcal{D} \square \mathcal{U}^\top \quad (2.2)$$

is called an eigenvalue decomposition (EVD) of \mathcal{A} , where \mathcal{U} is \square -orthogonal and \mathcal{D} is \square -symmetric and diagonal. Each $d_{i_1 \dots i_m i_1 \dots i_m}$ in \mathcal{D} is called an Einstein eigenvalue of \mathcal{A} , or \square -eigenvalue. The \square -eigenvalues of \mathcal{A} are denoted by $\lambda_i^\square(\mathcal{A})$ ($i \in \{1, \dots, d^m\}$).

(vii) **Spectral norm and trace:** The Einstein-spectral norm and trace of $\mathcal{A} \in \mathbb{S}_{2m,d}$ are defined by

$$\|\mathcal{A}\|^\square = \max_{i \in \{1, \dots, d^m\}} |\lambda_i^\square(\mathcal{A})| \quad \text{and} \quad \text{tr}^\square(\mathcal{A}) = \sum_{i \in \{1, \dots, d^m\}} \lambda_i^\square(\mathcal{A}).$$

As in [5], we introduce the important bijective ‘‘matricization’’ transformation f that maps each tensor $\mathcal{A} \in \mathbb{T}_{2m,d}$ to a matrix $A \in \mathbb{R}^{d^m \times d^m}$ with $A_{ij} = a_{i_1 \dots i_m j_1 \dots j_m}$, where

$$i = i_1 + \sum_{k=2}^m \left((i_k - 1) d^{k-1} \right) \quad \text{and} \quad j = j_1 + \sum_{k=2}^m \left((j_k - 1) d^{k-1} \right). \quad (2.3)$$

Note that

$$f(x^{\otimes m}) = f(\underbrace{x \otimes \dots \otimes x}_{m \text{ times}}) = \underbrace{x \bullet \dots \bullet x}_{m \text{ times}} = x^{\bullet m} \quad \text{for } x \in \mathbb{R}^d, \quad (2.4)$$

where \otimes denotes the tensor external product and \bullet the Kronecker product. Importantly for our purposes, it is proved in [5] that

$$f(\mathcal{A} \square \mathcal{B}) = f(\mathcal{A}) \cdot f(\mathcal{B}), \quad (2.5)$$

where \cdot is the standard matrix multiplication. Thus the consistency of the concepts introduced in Definitions 2 results from standard matrix analysis.

The property (2.5) in turn implies the following useful results.

Proposition 2.1 *Let $\mathcal{A} = (a_{i_1 \dots i_m j_1 \dots j_m}) \in \mathbb{T}_{2m,d}$ be an \square -symmetric tensor with EVD given by $\mathcal{A} = \mathcal{U} \square \mathcal{D} \square \mathcal{U}^\top$. We then have that*

(i) $f(\mathcal{A}^\top) = f(\mathcal{A})^\top$, and hence $f(\mathcal{A}) = f(\mathcal{U}) \cdot f(\mathcal{D}) \cdot f(\mathcal{U})^\top$;

(ii) All eigenvalues of $f(\mathcal{A})$ are \square -eigenvalues of \mathcal{A} and vice-versa;

(iii) $\text{tr}^\square(\mathcal{A}) = \sum_{i \in \{1, \dots, d^m\}} \lambda_i(f(\mathcal{A}))$;

(iv) $\mathcal{A} \square \mathcal{A} = \mathcal{U} \square (\mathcal{D} \circ \mathcal{D}) \square \mathcal{U}^\top$, where \circ denotes the Hadamard product;

Moreover, we may also establish a relation between the Einstein- and the standard Z-eigenvalues. Here we simply recall that a real scalar λ is called a Z-eigenvalue of a symmetric real tensor $\mathcal{A} \in \mathbb{T}_{2m,d}$, if there exists real unit vector $x \in \mathbb{R}^d$ such that

$$\mathcal{A}x^{2m-1} = \lambda x, \quad \text{where} \quad \mathcal{A}x^{2m-1} = \left(\sum_{i_2 \dots i_{2m}} a_{ii_2 \dots i_{2m}} x_{i_2} \cdots x_{i_{2m}} \right) \in \mathbb{R}^d$$

(see [14]). As pointed out in [17], Z-eigenvalues of even-order symmetric real tensors always exist.

Lemma 2.2 *For an \square -symmetric real tensor $\mathcal{A} \in \mathbb{T}_{2m,d}$, we have that, whenever Z-eigenvalues of \mathcal{A} exist, $\lambda_{\max}^{\square}(\mathcal{A}) \geq \lambda_{\max}^Z(\mathcal{A})$.*

Proof. By direct calculation, we have that

$$\begin{aligned} \lambda_{\max}^{\square}(\mathcal{A}) &= \max_{y \in \mathbb{R}^{d^m} \setminus \{0\}} \frac{y^{\top} f(\mathcal{A})y}{\|y\|_2^2} \\ &\geq \max_{x \in \mathbb{R}^d \setminus \{0\}} \frac{\langle f(\mathcal{A}), (x^{\otimes m}) \cdot (x^{\otimes m})^{\top} \rangle}{\|x^{\otimes m}\|_2^2} \\ &\geq \max_{x \in \mathbb{R}^d, x^{\top} x = 1} \mathcal{A}x^{2m} \\ &\geq \lambda_{\max}^Z(\mathcal{A}), \end{aligned}$$

where the second inequality results from the observation that $x^{\top} x = 1$ implies that $\|x^{\otimes m}\|_2^2 = 1$. Q.E.D.

3 The Bernstein Inequality for Even-Order Tensors

We now turn to random tensors, which are defined as follows. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A real (m, d) random tensor \mathcal{X} is a measurable map from Ω to $\mathbb{T}_{m,d}$. A finite sequence $\{\mathcal{X}_k\}$ of random tensors is independent whenever

$$\mathbb{P}(\mathcal{X}_k \in \mathcal{F}_k \text{ for all } k) = \prod_k \mathbb{P}(\mathcal{X}_k \in \mathcal{F}_k)$$

for every collection $\{\mathcal{F}_k\}$ of Borel subsets of $\mathbb{T}_{m,d}$. $\mathbb{E}(\mathcal{X})$, the expectation of the random tensor \mathcal{X} , is, as is the case for matrices, taken elementwise.

We are now in position to achieve our first objective: the Bernstein inequality for even order real \square -symmetric tensors based on Einstein products.

Theorem 3.1 *Consider a finite sequence $\{\mathcal{X}_k\}$ of independent random real \square -symmetric tensors of order $2m$ and dimension d . Assume that*

$$\mathbb{E}(\mathcal{X}_k) = \mathcal{O} \quad \text{and} \quad \lambda_{\max}^{\square}(\mathcal{X}_k) \leq L \quad \text{for each } k.$$

Consider the random tensor $\mathcal{Y} = \sum_k \mathcal{X}_k$ and let $\nu(\mathcal{Y})$ be the tensor variance statistic of \mathcal{Y} via Einstein product, that is

$$\nu(\mathcal{Y}) = \|\mathbb{E}(\mathcal{Y}^{\square 2})\|^{\square} = \left\| \sum_k \mathbb{E}(\mathcal{X}_k^{\square 2}) \right\|^{\square}.$$

Then

$$\mathbb{E}(\lambda_{\max}^{\square}(\mathcal{Y})) \leq \sqrt{2\nu(\mathcal{Y})m \log d} + \frac{1}{3}Lm \log d. \quad (3.1)$$

Furthermore, for all $t \geq 0$,

$$\mathbb{P}(\lambda_{\max}^{\square}(\mathcal{Y}) \geq t) \leq d^m \cdot \exp\left(\frac{-t^2/2}{\nu(\mathcal{Y}) + Lt/3}\right). \quad (3.2)$$

Proof. First observe that the following equivalences between tensors and matrices hold:

$$\mathbb{E}(\mathcal{X}_k) = \mathcal{O} \iff \mathbb{E}(f(\mathcal{X}_k)) = 0, \quad \|\mathcal{X}_k\|^{\square} \leq L \iff \|f(\mathcal{X}_k)\| \leq L, \quad \lambda_{\max}^{\square}(\mathcal{Y}) = \lambda_{\max}(f(\mathcal{Y})). \quad (3.3)$$

Using those equivalences and applying the matrix Bernstein inequality [20, Theorem 6.6.1] to $f(\mathcal{Y}) = \sum_k f(\mathcal{X}_k)$, we then deduce the desired result. Q.E.D.

Using Lemma 2.2, we then immediately deduce the following corollary involving Z-eigenvalues.

Corollary 3.2 *Suppose that the assumptions of Theorem 3.1 hold and that Z-eigenvalues of \mathcal{Y} exist. Then,*

$$\mathbb{E}(\lambda_{\max}^Z(\mathcal{Y})) \leq \sqrt{2\nu(\mathcal{Y})m \log d} + \frac{1}{3}Lm \log d. \quad (3.4)$$

Furthermore, for all $t \geq 0$,

$$\mathbb{P}(\lambda_{\max}^Z(\mathcal{Y}) \geq t) \leq d^m \cdot \exp\left(\frac{-t^2/2}{\nu(\mathcal{Y}) + Lt/3}\right). \quad (3.5)$$

This result reduces to the matrix Bernstein inequality for real symmetric matrices [20, Theorem 6.6.1] when $m = 1$, since for any symmetric real matrix A , \mathcal{A} is \square -symmetric,

$$\lambda_{\max}^Z(\mathcal{A}) = \lambda_{\max}^{\square}(\mathcal{A}) = \lambda_{\max}(A), \quad \|\mathbb{E}(\mathcal{A}^{\square 2})\|^{\square} = \|\mathbb{E}(A^2)\|$$

and $d^m = d$.

4 The General Tensor Bernstein Inequality

As is the case for the matrix case, extending the condensation inequality to tensors of odd order requires additional work. The notion of Einstein product itself must first be extended to general tensors in $\mathbb{T}_{N,d}$.

Definition 3 (Generalized Einstein Products) *Let \mathcal{A}, \mathcal{B} be two real tensors in $\mathbb{T}_{N,d}$, and $m = \left\lfloor \frac{N}{2} \right\rfloor$. Two generalized Einstein products of \mathcal{A} and \mathcal{B} , denoted by $\mathcal{A} \bar{\square} \mathcal{B}$ and $\mathcal{A} \square \mathcal{B}$, are defined by*

$$\left(\mathcal{A} \bar{\square} \mathcal{B}\right)_{i_1 \dots i_m j_1 \dots j_m} = \sum_{k_1, \dots, k_{N-m}} a_{i_1 \dots i_m k_1 \dots k_{N-m}} b_{j_1 \dots j_m k_1 \dots k_{N-m}} \in \mathbb{T}_{2m,d}, \quad (4.1)$$

and

$$\left(\mathcal{A} \square \mathcal{B}\right)_{k_1 \dots k_{N-m} k'_1 \dots k'_{N-m}} = \sum_{i_1, \dots, i_m} a_{i_1 \dots i_m k_1 \dots k_{N-m}} b_{i_1 \dots i_m k'_1 \dots k'_{N-m}} \in \mathbb{T}_{2(N-m),d}, \quad (4.2)$$

respectively.

We examine two special cases.

- (i) If $N = 1$, the ranges between 1 and $N - m = 0$ in the above definition are interpreted as empty. In this case, \mathcal{A} and \mathcal{B} are vectors in \mathbb{R}^d , say a and b . Thus, $a \bar{\square} b = ab^\top$ and $a \square b = a^\top b$, which are exactly the outer and inner products of vectors.
- (ii) If $N = 2m$, then \mathcal{A} and \mathcal{B} are in $\mathbb{T}_{2m,d}$, and

$$\mathcal{A} \bar{\square} \mathcal{B} = \mathcal{A} \square \mathcal{B}^\top, \quad \text{and} \quad \mathcal{A} \square \mathcal{B} = \mathcal{A}^\top \square \mathcal{B}, \quad (4.3)$$

where \mathcal{B}^\top is defined in Definition 2 and \square is the Einstein product in Definition 1, both for even-order tensors.

We also need to generalize the bijective transformation f which unfolds an even-order tensor to a square matrix (as introduced in Section 2) to operate on tensors of any order. This is done as follows.

Definition 4 (Matricization) Let $N \geq 1$, $d \geq 1$ and $m = \lfloor \frac{N}{2} \rfloor$. Define a bijective linear transformation \bar{f} from $T_{N,d}$ to $\mathbb{R}^{d^m \times d^{N-m}}$ such that for any tensor $\mathcal{A} \in \mathbb{T}_{N,d}$,

$$(\bar{f}(\mathcal{A}))_{ik} = a_{i_1 \dots i_m k_1 \dots k_{N-m}},$$

where

$$i = i_1 + \sum_{l=2}^m \left((i_l - 1) d^{l-1} \right) \quad \text{and} \quad k = k_1 + \sum_{l=2}^{N-m} \left((k_l - 1) d^{l-1} \right).$$

Note that $\bar{f}(\mathcal{A})$ need not be square or (obviously) symmetric. As above, we consider two special cases.

- (i) If $N = 1$, the range between 1 and $N - m = 0$ is again interpreted as empty. It results that \bar{f} is the identity transformation that maps any vector $\mathbf{x} \in \mathbb{R}^d$ to itself.
- (ii) If $N = 2m$, then \bar{f} coincides with the transformation f .

The all important relation (2.5) may also be generalized as follows.

Lemma 4.1 Let $N \geq 1$, $d \geq 1$ and $m = \lfloor \frac{N}{2} \rfloor$. Then we have that, for all $\mathcal{A} \in \mathbb{T}_{N,d}$,

$$f(\mathcal{A} \bar{\square} \mathcal{A}) = \bar{f}(\mathcal{A}) \cdot \bar{f}(\mathcal{A})^\top \quad \text{and} \quad f(\mathcal{A} \square \mathcal{A}) = \bar{f}(\mathcal{A})^\top \cdot \bar{f}(\mathcal{A}). \quad (4.4)$$

Proof. Notice that

$$\begin{aligned} (\mathcal{A} \bar{\square} \mathcal{A})_{i_1 \dots i_m j_1 \dots j_m} &= \sum_{k_1, \dots, k_{N-m}} a_{i_1 \dots i_m k_1 \dots k_{N-m}} a_{j_1 \dots j_m k_1 \dots k_{N-m}} \\ &= \sum_{k_1, \dots, k_{N-m}} a_{j_1 \dots j_m k_1 \dots k_{N-m}} a_{i_1 \dots i_m k_1 \dots k_{N-m}} \\ &= (\mathcal{A} \bar{\square} \mathcal{A})_{j_1 \dots j_m i_1 \dots i_m}, \end{aligned}$$

for any $i_1, \dots, i_m, j_1, \dots, j_m$. Thus, $\mathcal{A} \bar{\square} \mathcal{A} \in \mathbb{S}_{2m, d}$ and hence $f(\mathcal{A} \bar{\square} \mathcal{A})$ is well-defined. Denote $B = f(\mathcal{A} \bar{\square} \mathcal{A})$ and $C = \bar{f}(\mathcal{A}) \cdot \bar{f}(\mathcal{A})^\top$. From the definitions of f and \bar{f} , we know that the matrices B and C have the same size, which is $d^m \times d^m$. For any i and $j \in \{1, \dots, d^m\}$, there exist two m -tuples of indices (i_1, \dots, i_m) and (j_1, \dots, j_m) that uniquely determine by i and j via (2.3). By direct calculation, we then obtain that

$$\begin{aligned}
C_{ij} &= \sum_{l=1}^{d^{N-m}} [\bar{f}(\mathcal{A})]_{il} [\bar{f}(\mathcal{A})]_{jl} \\
&= \sum_{k_1, \dots, k_{N-m}} a_{i_1 \dots i_m k_1 \dots k_{N-m}} a_{j_1 \dots j_m k_1 \dots k_{N-m}} \\
&= (\mathcal{A} \bar{\square} \mathcal{A})_{i_1 \dots i_m j_1 \dots j_m} \\
&= [f(\mathcal{A} \bar{\square} \mathcal{A})]_{ij} \\
&= B_{ij}.
\end{aligned} \tag{4.5}$$

The proof for the case involving \square is similar.

Q.E.D.

We next need to revisit the definition of the spectral norm.

Definition 5 Let $N, d \geq 1$ and $m = \lfloor \frac{N}{2} \rfloor$. Suppose that $\mathcal{A} = (a_{i_1 \dots i_m k_1 \dots k_{N-m}}) \in \mathbb{T}_{N, d}$. The spectral norm of \mathcal{A} in the sense of generalized Einstein products, denoted as $\|\mathcal{A}\|^\square$, is defined by $\|\mathcal{A}\|^\square = \sqrt{\|\mathcal{A} \bar{\square} \mathcal{A}\|^\square}$.

Using Proposition 2.1 (iii) and (4.3), one verifies that $\|\mathcal{A}\|^\square = \|\mathcal{A}\|^E$ whenever $\mathcal{A} \in \mathbb{T}_{2m, d}$.

As for the matrix case [20], we now use a construct to build a symmetric even-order object from (possibly) odd-order non-square parts. This is achieved by using the Hermitian dilation defined, for any real matrix B , by

$$H(B) = \begin{pmatrix} O & B \\ B^\top & O \end{pmatrix}.$$

It is then possible to establish a link between this construct and the spectral norm just defined: first note that

$$\lambda_{\max}(H(B)) = \|H(B)\| = \|B\|. \tag{4.6}$$

We may then use this identity to establish the following result.

Lemma 4.2 Let $N, d \geq 1$ and $m = \lfloor \frac{N}{2} \rfloor$. Suppose $\mathcal{A} = (a_{i_1 \dots i_m k_1 \dots k_{N-m}}) \in \mathbb{T}_{N, d}$. We then have that

$$\|\mathcal{A}\|^\square = \sqrt{\|\mathcal{A} \bar{\square} \mathcal{A}\|^\square} = \|\bar{f}(\mathcal{A})\| = \|H(\bar{f}(\mathcal{A}))\| = \lambda_{\max}(H(\bar{f}(\mathcal{A}))). \tag{4.7}$$

Proof. By direct calculation, we have

$$\begin{aligned}
\|\mathcal{A}\|_{\bar{\square}} &= \sqrt{\|\mathcal{A} \bar{\square} \mathcal{A}\|_{\square}} = \sqrt{\max_i |\lambda_i^{\square}(\mathcal{A} \bar{\square} \mathcal{A})|} \\
&= \sqrt{\max_i |\lambda_i(f(\mathcal{A} \bar{\square} \mathcal{A}))|} \\
&= \sqrt{\max_i |\lambda_i(\bar{f}(\mathcal{A}) \cdot \bar{f}(\mathcal{A})^{\top})|} \\
&= \|\bar{f}(\mathcal{A})\| \\
&= \sqrt{\|\mathcal{A} \bar{\square} \mathcal{A}\|_{\square}}
\end{aligned} \tag{4.8}$$

where the second equality follows from Definition 2 (vii), the third one by applying Proposition 2.1 (ii), the fourth and the sixth resulting from (4.4). Now, using (4.6),

$$\|\bar{f}(\mathcal{A})\| = \|H(\bar{f}(\mathcal{A}))\| = \lambda_{\max}(H(\bar{f}(\mathcal{A}))),$$

completing the proof. Q.E.D.

We are now in a position to state the general tensor Bernstein inequality for random tensors of any order.

Theorem 4.3 *Consider a finite sequence $\{\mathcal{X}_k\}$ of independent random tensors in $\mathbb{T}_{N,d}$ and let $m = \lfloor \frac{N}{2} \rfloor$. Assume that, for some constant $L \geq 0$,*

$$\mathbb{E}(\mathcal{X}_k) = \mathcal{O} \quad \text{and} \quad \|\mathcal{X}_k\|_{\bar{\square}} \leq L \quad \text{for all } k.$$

Consider now the random tensor $\mathcal{Y} = \sum_k \mathcal{X}_k$ and let $\nu(\mathcal{Y})$ be the generalized tensor variance statistic of the sum given by

$$\begin{aligned}
\nu(\mathcal{Y}) &= \max \left\{ \left\| \mathbb{E}(\mathcal{Y} \bar{\square} \mathcal{Y}) \right\|_{\square}^{\square}, \left\| \mathbb{E}(\mathcal{Y} \square \mathcal{Y}) \right\|_{\square}^{\square} \right\} \\
&= \max \left\{ \left\| \sum_k \mathbb{E}(\mathcal{X}_k \bar{\square} \mathcal{X}_k) \right\|_{\square}^{\square}, \left\| \sum_k \mathbb{E}(\mathcal{X}_k \square \mathcal{X}_k) \right\|_{\square}^{\square} \right\}
\end{aligned}$$

Then

$$\mathbb{E}(\|\mathcal{Y}\|_{\bar{\square}}) \leq \sqrt{2\nu(\mathcal{Y}) \log(d^m + d^{N-m})} + \frac{1}{3}L \log(d^m + d^{N-m}). \tag{4.9}$$

Furthermore, for all $t \geq 0$,

$$\mathbb{P}\left(\|\mathcal{Y}\|_{\bar{\square}} \geq t\right) \leq (d^m + d^{N-m}) \cdot \exp\left(\frac{-t^2/2}{\nu(\mathcal{Y}) + Lt/3}\right). \tag{4.10}$$

Proof. The desired result follows from applying [20, Theorem 6.1.1] to the random matrix $\bar{f}(\mathcal{Y}) = \sum_k \bar{f}(\mathcal{X}_k)$ and using the facts that

$$\|\mathcal{X}_k\|_{\bar{\square}} = \|\bar{f}(\mathcal{X}_k)\|, \quad \|\mathcal{Y}\|_{\bar{\square}} = \|\bar{f}(\mathcal{Y})\|,$$

and that

$$\begin{aligned} \nu(\mathcal{Y}) &= \max \left\{ \left\| \mathbb{E} \left(\bar{f}(\mathcal{Y}) \cdot \bar{f}(\mathcal{Y})^\top \right) \right\|, \left\| \mathbb{E} \left(\bar{f}(\mathcal{Y})^\top \cdot \bar{f}(\mathcal{Y}) \right) \right\| \right\} \\ &= \max \left\{ \left\| \mathbb{E} \left(\sum_k \bar{f}(\mathcal{X}_k) \cdot \bar{f}(\mathcal{X}_k)^\top \right) \right\|, \left\| \mathbb{E} \left(\sum_k \bar{f}(\mathcal{X}_k)^\top \cdot \bar{f}(\mathcal{X}_k) \right) \right\| \right\}. \end{aligned} \quad (4.11)$$

Q.E.D.

Observe that the dimension-dependent factor on the right-hand side of (3.2) is $d^m + d^{N-m}$, which is larger than md , the factor one might naively expect as a generalization of the matrix case, where this factor is $2d$. This larger bound somewhat limits the applicability of the results to moderate values of d and m . It is however worthwhile to note that we have merely assumed the \square -symmetry of the random tensors under consideration, which is weaker than true symmetry.

5 The Tensor Bernstein Inequality in Intrinsic Dimension

The above discussion about the dimension-dependent factor of (3.2) prompts the question of the extension of a version of the Bernstein inequality where this factor can be improved. This is the case of “intrinsic dimension” version of this result, which we now consider.

Our approach first introduces Einstein-positive-(semi)definite tensors. The positive semi-definiteness of real tensors has been discussed in [17] and shown to have applications such as in biomedical imaging [18]. Recall that a real tensor $\mathcal{A} \in \mathbb{T}_{2m,d}$ is called positive semi-definite (PSD) if

$$\mathcal{A}x^{2m} = \sum_{i_1 \dots i_m j_1 \dots j_m} a_{i_1 \dots i_m j_1 \dots j_m} x_{i_1} \cdots x_{i_m} x_{j_1} \cdots x_{j_m} \geq 0, \quad \text{for all } x \in \mathbb{R}^d.$$

(see [14]). Moreover, it has been shown in [17] that an even-order symmetric real tensors is PSD if and only if all Z-eigenvalues are nonnegative. Similarly, we can define such a nonnegativity in the sense of Einstein products as follows.

Definition 6 *An Einstein-symmetric tensor $\mathcal{A} \in \mathbb{T}_{2m,d}$ is called Einstein-positive semi-definite (\square -PSD) (\square -positive-definite (\square -PD), respectively) if and only if all its Einstein-eigenvalues are nonnegative (positive, respectively).*

We adopt the notation $\mathcal{A} \succeq^\square$ (\succ^\square) \mathcal{O} to represent that \mathcal{A} is \square -PSD (\square -PD), and similarly $\mathcal{A} \succeq^\square$ (\succ^\square) \mathcal{B} is $\mathcal{A} - \mathcal{B}$ if \square -PSD (\square -PD). Such an \square -PSD (\square -PD) property is actually stronger than the original PSD (PD) property, as stated in the following lemma.

Lemma 5.1 *Suppose that $\mathcal{A} \in \mathbb{S}_{2m,d}$. If \mathcal{A} is \square -PSD (\square -PD), then \mathcal{A} is PSD (PD).*

Proof. Because of (2.5), \mathcal{A} is \square -PSD if and only if $f(\mathcal{A})$ is a PSD matrix. Then, for any $x \in \mathbb{R}^d$, it follows that

$$\mathcal{A}x^{2m} = \langle f(\mathcal{A}), (x^{\otimes m})(x^{\otimes m})^\top \rangle = (x^{\otimes m})^\top f(\mathcal{A})(x^{\otimes m}) \geq 0.$$

The proof for the \square -PD case is similar.

Q.E.D.

Note that the PSD property does not, in general, imply the \square -PSD property. The following counterexample is taken from [16, Example 4.5].

Example 5.1 Let $\mathcal{A} \in \mathbb{S}_{4,3}$ with $a_{1122} = a_{1212} = a_{1221} = a_{2112} = a_{2121} = a_{2211} = 1$ and other entries 0. It is easy to verify that $\mathcal{A}x^4 = 6x_1^2x_2^2 \geq 0$ for any $x \in \mathbb{R}^3$, whereas $y^\top f(\mathcal{A})y = 2y_1y_5 < 0$ for $y = (1, 0, 0, 0, -1, 0, 0, 0, 0)^\top$.

Proposition 5.2 Let $\mathcal{A} \in \mathbb{T}_{N,d}$ and $m = \left\lceil \frac{N}{2} \right\rceil$. Then $\mathcal{A} \bar{\square} \mathcal{A} \in \mathbb{T}_{2m,d}$ and $\mathcal{A} \square \mathcal{A} \in \mathbb{T}_{2(N-d),d}$ are both \square -PSD and PSD.

Proof.

$$\begin{aligned} (\mathcal{A} \bar{\square} \mathcal{A})x^{2m} &= \langle f(\mathcal{A} \bar{\square} \mathcal{A}), (x^{\otimes m}) (x^{\otimes m})^\top \rangle \\ &= (x^{\otimes m})^\top \bar{f}(\mathcal{A}) \cdot \bar{f}(\mathcal{A})^\top (x^{\otimes m}) \\ &= \|\bar{f}(\mathcal{A})^\top (x^{\otimes m})\|^2 \\ &\geq 0. \end{aligned}$$

The proof is similar for $\mathcal{A} \square \mathcal{A}$.

Q.E.D.

Armed with these extended notions and the fundamental relation (2.5) applied to the Einstein EVD, we finally state an intrinsic-dimension version of the Bernstein concentration inequality for tensors.

Theorem 5.3 Consider a finite sequence $\{\mathcal{X}_k\}$ of independent random tensors in $\mathbb{T}_{N,d}$ and let $m = \left\lceil \frac{N}{2} \right\rceil$. Assume that, for some constant $L \geq 0$,

$$\mathbb{E}(\mathcal{X}_k) = \mathcal{O} \quad \text{and} \quad \|\mathcal{X}_k\| \bar{\square} \leq L \quad \text{for all } k.$$

Consider now the random tensor $\mathcal{Y} = \sum_k \mathcal{X}_k$ and let \mathcal{V}_1 and \mathcal{V}_2 be upper bounds for the tensor-valued variance statistics of \mathcal{Y} introduced in Theorem 4.3, that is

$$\mathcal{V}_1 \succeq \square \mathbb{E}(\mathcal{Y} \bar{\square} \mathcal{Y}) = \sum_k \mathbb{E}(\mathcal{X}_k \bar{\square} \mathcal{X}_k) \quad \text{and} \quad \mathcal{V}_2 \succeq \square \mathbb{E}(\mathcal{Y} \square \mathcal{Y}) = \sum_k \mathbb{E}(\mathcal{X}_k \square \mathcal{X}_k). \quad (5.1)$$

Let

$$\nu(\mathcal{Y}) = \max\{\|\mathcal{V}_1\| \square, \|\mathcal{V}_2\| \square\} \quad \text{and} \quad d_{\mathcal{Y}}(\mathcal{Y}) = \frac{1}{\nu(\mathcal{Y})} \left(\text{tr} \square(\mathcal{V}_1) + \text{tr} \square(\mathcal{V}_2) \right).$$

Then, for $t \geq \sqrt{\nu(\mathcal{Y})} + L/3$,

$$\mathbb{P} \left(\|\mathcal{Y}\| \bar{\square} \geq t \right) \leq 4d_{\mathcal{Y}}(\mathcal{Y}) \cdot \exp \left(\frac{-t^2/2}{\nu(\mathcal{Y}) + Lt/3} \right). \quad (5.2)$$

Proof. We first observe that, because of Proposition 5.2, $\mathbb{E}(\mathcal{Y} \bar{\square} \mathcal{Y})$ and $\mathbb{E}(\mathcal{Y} \square \mathcal{Y})$ are \square -positive-semidefinite, which make the \square -PSD ordering in (5.1) well-defined. We also note that $d_{\mathcal{Y}}(\mathcal{Y})$ is identical to the intrinsic dimension of the matrix

$$V = \begin{pmatrix} \bar{f}(\mathcal{V}_1)^T & 0 \\ 0 & \bar{f}(\mathcal{V}_2) \end{pmatrix}, \quad (5.3)$$

where the standard (matrix) intrinsic dimension of a positive-semidefinite matrix M is the ratio $\text{tr}(M)/\|M\|$. The desired result then again follows from applying an existing result for matrices (here [20, Theorem 7.3.1]) to the random matrix $\bar{f}(\mathcal{Y}) = \sum_k \bar{f}(\mathcal{X}_k)$. Q.E.D.

The main difference between this theorem and Theorem 4.3 is the replacement of (4.10) by (5.2): have to relax the range of t for which the inequality is valid but often gain in the “dimension-dependent” factor, since $d_{\mathcal{Y}}(\mathcal{Y})$ never exceeds $d^m + d^{N-m}$ and can be much smaller if V in (5.3) is close to being of low rank.

6 Conclusion

We have considered the Einstein tensor products and reviewed the strong link this concept establishes between standard matrix theory and tensor analysis. This link has allowed us to restate the powerful Bernstein matrix concentration inequality in the case of general tensors of arbitrary order.

Other concentration inequalities do exist for matrices (see [20] for an overview). Whether they can be extended to tensors using a similar approach, although likely, remains open at this stage.

It is interesting (and challenging) to examine if a better “dimension factor” (closer to md) could be achieved by an approach where one does not merely unfold tensors to matrices and use existing concentration results for these, but where a true analysis of the tensor case is conducted. The main difficulty is to find an eigenvalue decomposition of (random) tensors with a number of “eigenvalues” smaller than d^m (this is for instance not necessarily the case of \mathbb{Z} -eigenvalues [7]).

If one is to judge by the vast diversity of applications where matrix concentration inequalities have been useful, our new result potentially opens several research paths in high-dimensional computational statistics and numerical optimization. In particular, its application to sub-sampling methods for the estimation of derivative tensors beyond the Hessian may now be considered, as it makes algorithms based on high-order Taylor’s expansions and models practical. The complexity of optimization methods of this type has been analyzed in [4], but the necessary probabilistic estimation properties were so far limited to quadratic models. The new tensor concentration inequality thus allows further developments in a framework which is central to computational deep learning.

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