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# Nice Port-Hamiltonian systems are Riesz-spectral systems ⋆

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**Abstract:** It is shown that the class of infinite-dimensional nice port-Hamiltonian systems including a large range of distributed parameter systems with boundary control is a subclass of Riesz-spectral systems. This result is illustrated by an example of a vibrating string.

*Keywords:* Distributed-parameter system - Infinite-dimensional system - Exponential stability - port-Hamiltonian system - Riesz basis - Riesz-spectral system

## 1. INTRODUCTION

The concept of Riesz basis is fundamental in system theory and an extensive literature is devoted to it. Results regarding controllability, stabilizability, their dual concepts and stability are easily checkable for a large class of systems whose dynamic generator has a Riesz basis of eigenvectors, see (Curtain and Zwart, 1995). Furthermore, this concept allows to describe the dynamics of a system under the form of eigenfunction expansions of non-harmonic Fourier series.

The class of port-Hamiltonian systems includes numerous physical models, e.g. the wave equation, traveling waves, the heat exchanger, the Timoshenko beam, diffusive tubular reactors. On the other hand, all these particular models are known to be Riesz-spectral systems, see (Curtain and Zwart, 1995), (Xu, 2005) and (Delattre et al., 2003) respectively. This begs the question whether port-Hamiltonian systems are Riesz-spectral. Important hints towards this general result are available in the literature: see (Tretter, 2000) and (Villegas, 2007, Chapter 4). The objective of this note is to prove this result, on the basis of these references, for a specific subclass, viz. nice port-Hamiltonian systems introduced in this extended abstract. This should be seen as an attempt to the study of the Riesz-spectral property of port-Hamiltonian systems. (Villegas, 2007, Chapter 4) bears the marks of a thorough research and much material is presented but, not everything is needed. In Section 3, some results are gathered from (Villegas, 2007, Chapter 4) and developed in a straightforward manner in order to deduce the Riesz basis property from (Tretter, 2000, Theorem 3.11). Further contributions are the authors' willingness of giving a clarification of (Villegas, 2007, Chapter 4), while being self-contained, and the tutorial aspect of this note.

This extended abstract is organized as follows. First, we introduce the class of systems under study. Then, we establish the Riesz basis property for these systems. Next, we prove the main result of this paper: the class of nice port-Hamiltonian systems is a sub-class of Riesz-spectral systems. Eventually, the theory is illustrated through an example of a vibrating string.

## 2. PORT-HAMILTONIAN SYSTEM

We consider the port-Hamiltonian system definition introduced in (Le Gorrec et al., 2005) and (Jacob and Zwart, 2012).

*Definition 1.* A port-Hamiltonian system is governed by a PDE of the form

$$\frac{\partial x}{\partial t}(\zeta,t) = P_1\frac{\partial}{\partial \zeta}(\mathscr{H}(\zeta)x(\zeta,t)) + P_0(\mathscr{H}(\zeta)x(\zeta,t)), \quad (1)$$

where $P_1 \in \mathbb{K}^{n\times n}$ is invertible and self-adjoint ($P_1^* = P_1$), $P_0 \in \mathbb{K}^{n\times n}$ is skew-adjoint ($P_0^* = -P_0$) and $\mathscr{H} \in L^\infty([a,b];\mathbb{K}^{n\times n})$ is self-adjoint and satisfies $mI \leq \mathscr{H}(\zeta) \leq MI$ for a.e. $\zeta \in [a,b]$, for some constants $m$, $M > 0$ ($\mathbb{K}$ denotes the field of real or complex numbers). The associated Hamiltonian $E : L^2([a,b];\mathbb{K}^n) \to \mathbb{K}$ evaluated along the trajectory $x(t)$ is given by

$$E(x(t)) = \frac{1}{2}\int_a^b x(\zeta,t)^*\mathscr{H}(\zeta)x(\zeta,t)d\zeta. \quad (2)$$

These systems rely on the natural state space $X := L^2([a,b];\mathbb{K}^n)$. The inner product

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$$\langle x_1, x_2\rangle = \frac{1}{2}\int_a^b x_2(\zeta)^* \mathscr{H}(\zeta) x_1(\zeta) d\zeta \tag{3}$$

induces the norm $||x||_X = \sqrt{\langle x, x\rangle}$, which makes the state space $L^2([a,b];\mathbb{K}^n)$ complete.

The boundary port-variables that are used to express the boundary conditions related to the PDE of a physical system like (1) are given by the boundary flow variable $f_\partial$ and the boundary effort variable $e_\partial$, defined here as

$$\begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} = R_0 \begin{pmatrix} (\mathscr{H}x)(b) \\ (\mathscr{H}x)(a) \end{pmatrix} \tag{4}$$

with $R_0 := \frac{1}{\sqrt{2}}\begin{pmatrix} P_1 & -P_1 \\ I & I \end{pmatrix}$, where $I \in \mathbb{K}^{n\times n}$ denotes the identity matrix. Additionally, we specify the boundary conditions associated to the PDE (1) as $\tilde{W}_B \begin{bmatrix} (\mathscr{H}x)(b) \\ (\mathscr{H}x)(a) \end{bmatrix} = 0$. As a matter of fact, the best choice is to express them through the boundary variables, i.e.,

$$W_B \begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix} = 0, \tag{5}$$

where $W_B = \tilde{W}_B R_0^{-1}$. Notice that the invertibility of $R_0$ follows from the invertibility of $P_1$. Considering $x(t) \in L^2([a,b];\mathbb{K}^n)$ as system state, the PDE (1) can be rewritten as the differential equation $\dot{x}(t) = Ax(t)$ by defining the differential operator

$$Ax = P_1 \frac{d}{d\zeta}(\mathscr{H}x) + P_0(\mathscr{H}x) \tag{6}$$

with domain

$$D(A) = \Big\{ x \in L^2([a,b];\mathbb{K}^n) : \mathscr{H}x \in H^1([a,b];\mathbb{K}^n), \\ W_B \begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix} = 0 \Big\}. \tag{7}$$

The necessary and sufficient condition stated in the following theorem will be assumed to hold throughout.

*Theorem 2.* (Jacob and Zwart, 2012, Theorem 7.2.4) Consider the operator $A$ with domain $D(A)$ given by (6)-(7), associated to a port-Hamiltonian system (1)-(5). Assume that $W_B$ is a $n \times 2n$ matrix of full rank. Then A is the generator of a contraction $C_0$-semigroup on $L^2([a,b];\mathbb{K}^n)$ if and only if $W_B \Sigma W_B^* \geq 0$ where $\Sigma = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \in \mathbb{K}^{2n\times 2n}$.

## 3. RIESZ BASIS PROPERTY

In this section we shall prove that under some assumptions on the spectrum of the dynamical operator $A$ described by (6)-(7), the eigenfunctions $(\phi_n)_{n\in\mathbb{N}}$ of an auxiliary eigenvalue problem form a Riesz basis.

*Definition 3.* A vector sequence $(\phi_n)_{n\in\mathbb{N}}$ on a Hilbert space $X$ is a Riesz basis if it satisfies the following conditions:

(1) $\overline{span\{\phi_n\}} = X$;
(2) there exist positive constants $M_1$ and $M_2$ such that for any $N \in \mathbb{N}$ and for any $c_n \in \mathbb{K}$, $n = 1, 2, ..., N$,

$$M_1 \sum_{n=1}^{N} |c_n|^2 \leq || \sum_{n=1}^{N} c_n\phi_n||_X^2 \leq M_2 \sum_{n=1}^{N} |c_n|^2. \tag{8}$$

Any vector $x \in X$ is uniquely decomposed in a Riesz basis $(\phi_n)_{n\in\mathbb{N}}$ as

$$x = \sum_{n=1}^{\infty} c_n\phi_n, \tag{9}$$

where the scalars $c_n$ are uniquely determined by $x$.

*Remark 4.* It is well-known that any Riesz basis is an orthonormal basis with respect to an equivalent inner product, which means that $(\phi_n)_{n\in\mathbb{N}}$ is a Riesz basis if and only if there exists an invertible bounded linear operator $U$ that transforms $(\phi_n)_{n\in\mathbb{N}}$ into some orthonormal basis $(e_n)_{n\in\mathbb{N}}$, i.e.,

$$\forall n \in \mathbb{N},\ e_n = U\phi_n. \tag{10}$$

For details of the proof of this result, see (Young, 2001, Theorem 7).

*Definition 5.* Consider a closed linear operator $A$ on a Hilbert space $X$ with a discrete spectrum consisting of simple eigenvalues $\sigma_p(A) := \{\lambda_n : n \in \mathbb{N}\}$ and corresponding eigenvectors $(\phi_n)_{n\in\mathbb{N}}$. If the closure of $\{\lambda_n : n \in \mathbb{N}\}$ is totally disconnected and if $(\phi_n)_{n\in\mathbb{N}}$ is a Riesz basis on $X$, then $A$ is said to be a Riesz-spectral operator.

Recall that the set $D := \overline{\{\lambda_n : n \in \mathbb{N}\}} \subset \mathbb{C}$ is totally disconnected if every point in $D$ cannot be joined with any other point in $D$ by a segment lying entirely in $D$. In order to establish the Riesz-basis property, we shall use (Tretter, 2000, Theorem 3.11). To do so, we shall follow the approach developed in (Villegas, 2007, Section 4.2).

*Assumption 6.* The multiplication operator $P_1^{-1}\mathscr{H}^{-1}$ is assumed to be diagonalizable, i.e.,

$$P_1^{-1}\mathscr{H}^{-1}(\zeta) = S(\zeta)A_1(\zeta)S(\zeta)^{-1}, \qquad \zeta \in [a,b], \tag{11}$$

where $A_1$ is a diagonal matrix-valued function whose diagonal entries are the eigenvalues $(r_\nu)_{\nu=1}^n$ of $P_1^{-1}\mathscr{H}^{-1}$, whereas $S$ is a matrix-valued function whose columns are corresponding eigenvectors. $S$ and $A_1$ are continuously differentiable on $[a,b]$.

Observe that this assumption is not very strong and will almost always be satisfied in practice if $\mathscr{H}$ is continuously differentiable. Moreover, $P_1^{-1}\mathscr{H}^{-1}$ may have eigenvalues that are not simple. Therein, we shall consider that $P_1^{-1}\mathscr{H}^{-1}$ has $l$ different eigenvalues such that $l \leq n$.

*Lemma 7.* The eigenvalue problem

$$P_1\frac{d}{d\zeta}((\mathscr{H}x)(\zeta)) + P_0((\mathscr{H}x)(\zeta)) = \lambda x(\zeta), \tag{12}$$

where $\lambda \in \sigma_p(A)$ and $x \in D(A)$ is a corresponding eigenfunction can be formulated under the form:

$$\frac{df}{d\zeta}(\zeta) = (\lambda A_1(\zeta) + A_0(\zeta))f(\zeta), \qquad \zeta \in [a,b], \\ W_b(Sf)(b) + W_a(Sf)(a) = 0, \tag{13}$$

where $W_b := W_1P_1 + W_2$ and $W_a := -W_1P_1 + W_2$ with $W_B := [\, W_1\ W_2 \,]$ and $f \in W^{1,2}([a,b];\mathbb{K}^{n\times n}) := \Big\{ f \in L^2([a,b];\mathbb{K}^n) : \frac{df}{d\zeta} \in L^2([a,b];\mathbb{K}^n)\Big\}$ with coefficients $A_0$, $A_1 \in L^\infty([a,b];\, \mathbb{K}^{n\times n})$.

**Proof.** Let us consider the eigenvalue problem

$$P_1\frac{d}{d\zeta}((\mathscr{H}x)(\zeta)) + P_0((\mathscr{H}x)(\zeta)) = \lambda x(\zeta),$$

where $\lambda \in \sigma_p(A)$ and $x \in D(A)$ is a corresponding eigenfunction satisfying the boundary conditions:

$$\tilde{W}_B \begin{bmatrix} (\mathscr{H}x)(b) \\ (\mathscr{H}x)(a) \end{bmatrix} = 0. \qquad (14)$$

By using the basis transformation $S(\zeta)$ that diagonalizes $P_1^{-1}\mathscr{H}^{-1}$, this eigenvalue problem becomes

$$\frac{df}{d\zeta}(\zeta) = (\lambda A_1(\zeta) + A_0(\zeta))f(\zeta), \qquad \zeta \in [a,b], \qquad (15)$$

where $A_1 = S^{-1}(\mathscr{H}P_1)^{-1}S$, $A_0 = -S^{-1}(P_1^{-1}P_0S + \frac{d}{d\zeta}S)$ and $f(\zeta) = (S^{-1}\mathscr{H}x)(\zeta)$. Furthermore, the boundary condition $\tilde{W}_B \begin{bmatrix} \mathscr{H}(b)x(b,t) \\ \mathscr{H}(a)x(a,t) \end{bmatrix} = 0$ becomes $\tilde{W}_B \begin{bmatrix} (Sf)(b) \\ (Sf)(a) \end{bmatrix} = 0$, which finally yields the boundary condition:

$$W_b(Sf)(b) + W_a(Sf)(a) = 0, \qquad (16)$$

where $W_b := W_1P_1 + W_2$ and $W_a := -W_1P_1 + W_2$ with $W_B := [\, W_1 \; W_2 \,]$.

Related to the eigenvalue problem (13), we define the operator $A_f$ as

$$\begin{aligned} A_f f &= A_1^{-1}\frac{d}{d\zeta}f - A_1^{-1}A_0 f \\ &= S^{-1}\mathscr{H}P_1\frac{d}{d\zeta}(Sf) + S^{-1}\mathscr{H}P_0(Sf) \end{aligned} \qquad (17)$$

with domain

$$\begin{aligned} D(A_f) = \{f \in L^2([a,b];\mathbb{K}^n) : Sf \in H^1([a,b];\mathbb{K}^n), \\ W_b(Sf)(b) + W_a(Sf)(a) = 0\}. \end{aligned} \qquad (18)$$

*Lemma 8.* Consider the operator $A$ with domain $D(A)$ given by (6)-(7) and the operator $A_f$ with domain $D(A_f)$ defined by (17)-(18). Then,

$$S^{-1}\mathscr{H}Ax = A_f S^{-1}\mathscr{H}x, \qquad x \in D(A), \qquad (19)$$

and $x \in D(A)$ if and only if $f = S^{-1}\mathscr{H}x \in D(A_f)$. Moreover, the eigenvalues of $A$ that are given by (12),(5) are the same as those of the operator $A_f$. If $\lambda$ is an eigenvalue of $A_f$ with a corresponding eigenfunction $f \in D(A_f)$, then $\lambda$ is an eigenvalue of $A$ with eigenfunction $x = \mathscr{H}^{-1}Sf \in D(A)$.

**Proof.** (19) is a direct consequence of (6) and (17). Let $x$ be in $D(A)$. Thus $x$ must satisfy $W_b(\mathscr{H}x)(b) + W_a(\mathscr{H}x)(a) = 0$ and by setting $f = S^{-1}\mathscr{H}x$, we get that $f \in D(A_f)$ from (18). Let us now consider an eigenvalue $\lambda \in \sigma_p(A_f)$ with corresponding eigenfunction $f$. From the definition of $A_f$, it follows that

$$\lambda f = A_f f = S^{-1}\mathscr{H}P_1\frac{d}{d\zeta}(Sf) + S^{-1}\mathscr{H}P_0(Sf)$$

and from the identity $Sf = \mathscr{H}x$ we get $\lambda x = Ax$.

In order to use (Tretter, 2000, Theorem 3.11), we have to make a further assumption.

*Assumption 9.* For $\nu \in \{1,...,l\}$, let us define $R_\nu(z) := \int_a^z r_\nu(\zeta)d\zeta$, where $(r_\nu(\zeta))_{\nu=1}^l$ are the $l$ different eigenvalues of $P_1^{-1}\ \mathscr{H}^{-1}(\zeta)$ and $E_\nu(z,\lambda) := e^{\lambda R_\nu(z)}I_{n_\nu}$, where $n_\nu$ is the multiplicity of $r_\nu(\cdot)$ and $I_{n_\nu}$ denotes the $n_\nu$-dimensional unit matrix such that $\sum_{\nu=1}^{l} n_\nu = n$. We set $E(z,\lambda) = \operatorname{diag}(E_0(z,\lambda),\ldots,E_l(z,\lambda))$, $z \in [a,b]$. We shall assume that the eigenvalue problem (13) is normal, i.e., for sufficiently large $\lambda$, the asymptotic expansion of the characteristic determinant of (13) given by

$$p(\lambda) = \sum_{c\in\mathscr{E}}(b_c + \{o(1)\}_\infty)e^{\lambda c} \qquad (20)$$

has non-zero minimum and maximum coefficients, where

$$\mathscr{E} = \left\{\sum_{\nu=1}^{l}\delta_\nu R_\nu(b) : \delta_\nu \in \{0,1\}\right\} \subset \mathbb{R} \qquad (21)$$

and $\{o(1)\}_\infty$ means that for each $c \in \mathscr{E}$ the remaining part depending on $z \in [a,b]$ divided by $\lambda$ tends to 0 in the uniform norm when $|\lambda| \to \infty$.

From (Villegas, 2007, Theorem 4.10), it follows that the non-zero coefficients are given by

$$\sum_{c\in\mathscr{E}} b_c e^{\lambda c} = \det(W_bS(b)\Phi_0(b)E(b,\lambda) + W_aS(b)), \qquad (22)$$

where $\Phi_0 \in W^{1,\infty}([a,b];\mathbb{K}^{n\times n})$ is determined by

$$\begin{aligned} \Phi_0(\zeta)A_1 = A_1\Phi_0(\zeta), \qquad \Phi_0(a) = I, \\ \frac{d\Phi_{0,\nu\nu}}{d\zeta} - A_{0,\nu\nu}\Phi_{0,\nu\nu} = 0, \qquad \nu = 1,...,l \end{aligned} \qquad (23)$$

where $A_{0,\nu\nu}$ and $\Phi_{0,\nu\nu}$ are the elements of the $\nu$th row and the $\nu$th column of $A_0$ and $\Phi_0$ respectively. For more details, see (Mennicken and Möller, 2003, Section § 2.8).

*Theorem 10.* Assume that the eigenvalue problem (13) is normal with $A_0, A_1 \in W^{2,\infty}([a,b];\mathbb{K}^{n\times n})$ and $A_1$ is a diagonal matrix with the eigenvalues of $P_1^{-1}\mathscr{H}^{-1}$ as diagonal elements with $\mathscr{H} \in W^{2,\infty}([a,b];\mathbb{K}^{n\times n})$. If the eigenvalues have a uniform gap, i.e., $\inf_{m\neq p}|\lambda_m - \lambda_p| > 0$, then there exists a sequence of eigenfunctions of $A_f$ that forms a Riesz basis of $L^2([a,b];\mathbb{K}^n)$.

**Proof.** This is a direct consequence of (Tretter, 2000, Theorem 3.11) applied to the eigenvalue problem (13). Notice that the canonical system of eigenfunctions in (Tretter, 2000, Theorem 3.11) corresponds to the eigenfunctions of the operator $A_f$. In (Tretter, 2000, Theorem 3.11) the eigenvalue problem is also assumed to be non-degenerate, i.e. $\rho(T) \neq \emptyset$, where $T$ is a linear pencil $T(\lambda) = T_0 - \lambda T_1$ of bounded operators $T_0$, $T_1 \in \mathcal{L}(H^1([a,b];\mathbb{K}^n), L^2([a,b];\mathbb{K}^n) \times \mathbb{C}^n)$ given by

$$T_0 f := \begin{bmatrix} \frac{d}{d\zeta}f - A_0 f \\ W_b(Sf)(b) + W_a(Sf)(a) \end{bmatrix}, \; T_1 f := \begin{bmatrix} A_1 f \\ 0 \end{bmatrix}.$$

From (Villegas, 2007, Theorem 4.2), we have that $\mathbb{C}^+ \subset \rho(T)$. Eventually, since the eigenvalue problem (13) is assumed to be normal and the eigenvalues of $A_f$ are assumed to have a uniform gap, we conclude that there exists a sequence of eigenfunctions $(f_n)_{n\in\mathbb{N}}$ of $A_f$ that forms a Riesz basis.

## 4. MAIN RESULT

In this section we prove the Riesz-spectral property of port-Hamiltonian systems satisfying the assumptions of Theorem 10. First we need the following definition.

*Definition 11.* A dynamical system

$$\dot{x}(t) = Ax(t) \qquad (24)$$

where $A : D(A) \subset X \to X$ is a linear operator on a Hilbert space $X$ is said to be a Riesz-spectral system if it satisfies the following conditions:

(1) $A$ is a Riesz-spectral operator;
(2) $A$ is the infinitesimal generator of a $C_0$-semigroup $(T(t))_{t\geq 0}$ on $X$.

*Definition 12.* A nice port-Hamiltonian system is a port-Hamiltonian system (according to Definition 1) which satisfies the condition $W_B \Sigma W_B^* \geq 0$ and Assumptions 6 and 9, and whose generator $A$ given by (6) has a uniform gap of eigenvalues, i.e., $\inf_{m\neq p} |\lambda_m - \lambda_p| > 0$.

*Theorem 13.* Under the regularity assumptions on the matrix-valued functions $A_0$, $A_1$ and $\mathscr{H}$ in Theorem 10, any nice port-Hamiltonian system (1)-(5) is a Riesz-spectral system.

**Proof.** First observe that, by Theorem 2, $A$ generates a $C_0$-semigroup since it is assumed that $W_B \Sigma W_B^* \geq 0$. It follows that $A$ is closed.
Besides, the closure of the set of the eigenvalues is totally disconnected. Indeed, from (Villegas, 2007, Theorem 2.28), it is known that the resolvent operator of $A$ is compact and thus, the spectrum $\sigma(A)$ is only made of eigenvalues of finite multiplicity such that $\sigma(A) = \sigma_p(A)$. Since we are counting the eigenvalues with multiplicity, the uniform gap between them entails that they are in fact all simple.
The set of corresponding eigenfunctions is a Riesz basis of $L^2([a,b];\mathbb{K}^n)$, i.e., it is an orthonormal basis with respect to an equivalent inner product, see Remark 4. From Theorem 10, it is known that the operator $A_f$ has a Riesz basis and, in view of Lemma 8 its eigenfunctions $(f_n)_{n\in\mathbb{N}}$ are isomorphic to the eigenfunctions $(x_n)_{n\in\mathbb{N}}$ of $A$. Indeed, $f_n = S^{-1}\mathscr{H}x_n$ for $n \in \mathbb{N}$. Therefore, the operator $A$ has a Riesz basis of eigenfunctions.

*Remark 14.* In order to prove the exponential stability of a $C_0$-semigroup on a finite dimensional space, it suffices to compute the eigenvalues of the operator (matrix) which generates the $C_0$-semigroup. In infinite dimensional spaces, this is not so obvious since there is a distinction between the spectral bound and the growth bound. To understand the issue better, we refer the reader to (Curtain and Zwart, 1995). However from (Curtain and Zwart, 1995, Theorem 2.3.5), it is known that the spectral bound determines the growth bound for Riesz-spectral systems. Thus Theorem 13 implies that the exponential stability of a nice port-Hamiltonian system is checkable through its eigenvalues.

## 5. EXAMPLE: VIBRATING STRING

In this section we investigate the exponential stability of a vibrating string. This example is based on (Jacob and Zwart, 2012, Example 7.2.5). The dynamic is governed by the following PDE:

$$\frac{\partial^2 w}{\partial t^2}(\zeta,t) = \frac{1}{\rho(\zeta)}\frac{\partial}{\partial \zeta}\left(T(\zeta)\frac{\partial w}{\partial \zeta}(\zeta,t)\right), \tag{25}$$

where $w(\zeta,t)$ is the vertical position of the string at place $\zeta$ and time $t$. $T(\zeta)$ and $\rho(\zeta)$ are respectively the Young's modulus and the mass density at place $\zeta$. We define $x_1(\zeta,t) = \rho(\zeta)\frac{\partial w}{\partial t}(\zeta,t)$ (momentum) and $x_2(\zeta,t) = \frac{\partial w}{\partial \zeta}(\zeta,t)$ (strain). Thus, the PDE (25) can be rewritten as:

$$\frac{\partial}{\partial t}\begin{bmatrix} x_1(\zeta,t) \\ x_2(\zeta,t)\end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0\end{bmatrix}\frac{\partial}{\partial \zeta}\left(\begin{bmatrix} \frac{1}{\rho(\zeta)} & 0 \\ 0 & T(\zeta)\end{bmatrix}\begin{bmatrix} x_1(\zeta,t) \\ x_2(\zeta,t)\end{bmatrix}\right)$$
$$= P_1\frac{\partial}{\partial \zeta}\left(\mathscr{H}(\zeta)\begin{bmatrix} x_1(\zeta,t) \\ x_2(\zeta,t)\end{bmatrix}\right), \tag{26}$$

The boundary conditions $T(b)\frac{\partial w}{\partial \zeta}(b,t) + \frac{\partial w}{\partial t}(b,t) = 0$ and $T(a)\frac{\partial w}{\partial \zeta}(a,t) = 0$ are under consideration. For this example the boundary flow and effort are respectively given by

$$f_\partial(t) = \frac{1}{\sqrt{2}}\begin{bmatrix} T(b)\frac{\partial w}{\partial \zeta}(b,t) - T(a)\frac{\partial w}{\partial \zeta}(a,t) \\ \frac{\partial w}{\partial t}(b,t) - \frac{\partial w}{\partial t}(a,t)\end{bmatrix},$$
$$e_\partial(t) = \frac{1}{\sqrt{2}}\begin{bmatrix} \frac{\partial w}{\partial t}(b,t) + \frac{\partial w}{\partial t}(a,t) \\ T(b)\frac{\partial w}{\partial \zeta}(b,t) + T(a)\frac{\partial w}{\partial \zeta}(a,t)\end{bmatrix}.$$

The boundary conditions become in these variables

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} T(a)\frac{\partial w}{\partial \zeta}(a,t) \\ T(b)\frac{\partial w}{\partial \zeta}(b,t) + \frac{\partial w}{\partial t}(b,t)\end{bmatrix} = W_B\begin{bmatrix} f_\partial(t) \\ e_\partial(t)\end{bmatrix}, \tag{27}$$

where $W_B = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1\end{bmatrix}$ has rank 2 and $W_B\Sigma W_B^* = \begin{bmatrix} 0 & 0 \\ 0 & 2\end{bmatrix} \geq 0$. The eigenvalues and eigenfunctions of $P_1^{-1}\mathscr{H}^{-1} := \begin{bmatrix} 0 & \frac{1}{T(\zeta)} \\ \rho(\zeta) & 0\end{bmatrix}$ are given by $\lambda(\zeta) = \pm\sqrt{\frac{\rho(\zeta)}{T(\zeta)}}$ and $\begin{bmatrix} \sqrt{\frac{\rho(\zeta)}{T(\zeta)}} \\ \rho(\zeta)\end{bmatrix}$, $\begin{bmatrix} -\sqrt{\frac{\rho(\zeta)}{T(\zeta)}} \\ \rho(\zeta)\end{bmatrix}$ respectively. In addition,

$$W_a = \sqrt{2}\begin{bmatrix} 0 & 1 \\ 0 & 0\end{bmatrix} \text{ and } W_b = \sqrt{2}\begin{bmatrix} 0 & 0 \\ 1 & 1\end{bmatrix}. \tag{28}$$

The verification that this application satisfies the normality assumption is straightforward but requires some computations left to the reader, see Assumption 9.

$$\mathscr{E} = \left\{\int_a^b \sqrt{\frac{\rho(\zeta)}{T(\zeta)}}d\zeta, -\int_a^b \sqrt{\frac{\rho(\zeta)}{T(\zeta)}}d\zeta, 0\right\}$$

$$\begin{aligned}&\det(W_b S(b)\Phi_0(b)E(b,\lambda) + W_a S(b)) \\ &= 2\sqrt{(T\rho)(b)}(1+\sqrt{(T\rho)(b)})E_1(b,\lambda) \\ &\quad + 2\sqrt{(T\rho)(b)}(1-\sqrt{(T\rho)(b)})E_2(b,\lambda),\end{aligned} \tag{29}$$

where $E(z,\lambda) = diag(\exp\left(\lambda\int_a^z \sqrt{\frac{\rho(\zeta)}{T(\zeta)}}d\zeta\right), \exp\left(-\lambda\int_a^z \sqrt{\frac{\rho(\zeta)}{T(\zeta)}}d\zeta\right))$ with $\lambda \in \sigma(A)$. The system is normal if and only if $1 \neq \sqrt{((T\rho)(b))}$.
It is acknowledged that the roots of the characteristic determinant (20) given by

$$\begin{aligned}p(\lambda) &= \{o(1)\}_\infty + [2\sqrt{(T\rho)(b)}(1+\sqrt{(T\rho)(b)}) + \{o(1)\}_\infty] \\ &E_1(b,\lambda) + [2\sqrt{(T\rho)(b)}(1-\sqrt{(T\rho)(b)}) + \{o(1)\}_\infty]E_2(b,\lambda)\end{aligned}$$

are the eigenvalues of $A$. The roots are approximated by (29) and by Rouché's theorem, the eigenvalues are approximated as $\lambda_n = \lambda_n^* + o(\frac{1}{n})$ for $n \in \mathbb{N}$, where

$$\lambda_n^* = \begin{cases} \dfrac{-1}{2\int_a^b \sqrt{\frac{\rho(\zeta)}{T(\zeta)}}} ln \left| \dfrac{\sqrt{(T\rho)(b)}+1}{\sqrt{(T\rho)(b)}-1} \right| - i \dfrac{\pi n}{\int_a^b \sqrt{\frac{\rho(\zeta)}{T(\zeta)}} d\zeta}, \\ \dfrac{-1}{2\int_a^b \sqrt{\frac{\rho(\zeta)}{T(\zeta)}}} ln \left| \dfrac{\sqrt{(T\rho)(b)}+1}{\sqrt{(T\rho)(b)}-1} \right| - i \dfrac{\pi(2n+1)}{2\int_a^b \sqrt{\frac{\rho(\zeta)}{T(\zeta)}} d\zeta}, \end{cases} \tag{30}$$

if $\sqrt{(T\rho)(b)} > 1$ or $\sqrt{(T\rho)(b)} < 1$ respectively. Observe that the uniform gap property is satisfied. Therefore, (26) and (27) describe a nice port-Hamiltonian system and thus, from Theorem 13, define a Riesz-spectral system. Besides, since $w_0$ is given by

$$w_0 = \sup_{n\in\mathbb{N}} Re\lambda_n = \frac{-1}{2\int_a^b \sqrt{\frac{\rho(\zeta)}{T(\zeta)}}} ln \left| \frac{\sqrt{(T\rho)(b)}+1}{\sqrt{(T\rho)(b)}-1} \right| < 0,$$

the Riesz-spectral property entails the exponential stability of the system.

## 6. CONCLUSION AND REMARKS

In this note, nice port-Hamiltonian systems on infinite-dimensional spaces were considered. It was shown that they are a subclass of Riesz-spectral systems, which constitute a class of systems with useful properties. Moreover, as explained in Remark 14, in the case of nice port-Hamiltonian systems, the spectral bound characterizes the exponential stability utterly.

As far as known, whether nice port-Hamiltonian systems are Riesz-spectral was not clearly stated in the literature. Thus, this note may be considered as an attempt to summarize the available literature and to fill this blank left in the literature.

A natural extension would be to consider dissipative effects in the port-Hamiltonian framework in order to see if the Riesz-spectral property still holds. Even though the answer seems intuitively to be positive, the proof of such extension is not straightforward and is still being investigated.

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