THESIS / THÈSE

DOCTOR OF SCIENCES

Analysis and LQG Control of Infinite-dimensional Stochastic Port-Hamiltonian Systems

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Award date:
2019

Awarding institution:
University of Namur

UNIVERSITÉ DE NAMUR

FACULTÉ DES SCIENCES
DÉPARTEMENT DE MATHÉMATIQUE

Analysis and LQG Control of Infinite-dimensional Stochastic Port-Hamiltonian Systems

Thèse présentée par
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pour l'obtention du grade
de Docteur en Sciences

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Août 2019

Rempart de la Vierge, 13
B-5000 Namur (Belgique)

Imprimé en Belgique

ISBN : 978-2-39029-073-5
Dépôt légal : D/2019/1881/24

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Analyse et Commande LQG de Systèmes Stochastiques Hamiltoniens à Ports en Dimension Infinie
par François Lamoline

Résumé : Nous considérons les systèmes stochastiques et déterministes Hamiltoniens à ports. Les actions de commande et d'observation sur ces systèmes sont effectuées de manière distribuée sur le domaine spatial ou à la frontière de celui-ci. Le concept de système bien posé au sens de Weiss et Salamon est généralisé aux systèmes stochastiques de dimension infinie. Cette définition étendue permet de démontrer que, sous certaines hypothèses, les systèmes stochastiques Hamiltoniens à ports sont bien posés. Nous traitons ensuite le problème de commande LQG pour ces systèmes stochastiques Hamiltoniens à ports dans le cadre d'opérateurs de commande, d'observation et de bruit bornés. En outre, nous dérivons des conditions sous lesquelles la structure Hamiltonienne est préservée pour la description dynamique du compensateur LQG. Tout au long de cette thèse, la théorie est illustrée sur un modèle de corde vibrante inhomogène sujette à du bruit blanc Gaussien, à dépendance spatiale et temporelle, représentant les perturbations liées à l'environnement dans lequel le système évolue. Finalement, nous proposons un nouveau modèle basé sur une approche Hamiltonienne à ports ainsi qu'une loi de commande pour un endoscope bio-médical actionné au moyen de polymères électroactifs.

Analysis and LQG Control of Infinite-dimensional Stochastic Port-Hamiltonian Systems
by François Lamoline

Abstract: Stochastic and deterministic port-Hamiltonian systems with both distributed and boundary controls along with distributed and boundary observations are considered in this work. The concept of well-posedness in the sense of Weiss-Salamon is generalized to infinite-dimensional stochastic systems. Under this extended definition, stochastic port-Hamiltonians systems are proved to be well-posed under some assumptions. We then address the LQG control problem for stochastic port-Hamiltonian systems with bounded control, observation and noise operators. We further derive conditions under which the Hamiltonian framework is preserved in the LQG controller dynamics. Throughout this thesis, the theory is illustrated on an example of an inhomogeneous vibrating string subject to some space and time Gaussian white noise process representing environment disturbances. Finally, we propose a new model based on the port-Hamiltonian approach and a control law for a compliant bio-medical endoscope actuated by electro-active polymers.

Thèse de doctorat en Sciences Mathématiques (Ph.D. thesis in Mathematics)
Date: 20/08/2019
Département de Mathématique
Promoteur (Advisor): Joseph WINKIN

Remerciements

Bien que l'on rédige une thèse seul, de nombreuses personnes ont contribué à sa réalisation durant ces quatre années.

Je souhaite remercier en premier lieu mon promoteur, Joseph Winkin, de m'avoir encadré et soutenu durant toutes ces années. Je le remercie pour son optimisme à toute épreuve, pour m'avoir transmis sa passion pour l'étude des systèmes commandés et m'avoir aidé à m'épanouir scientifiquement durant ma thèse. J'ai apprécié la grande marge de liberté qu'il m'a laissée pour mener ma recherche. Mes remerciements vont également à mon comité d'accompagnement, Yann Le Gorrec (Femto-ST, Besançon) et Birgit Jacob (University of Wuppertal), ainsi qu'à Anne Lemaître et Alexandre Mauroy (UNamur) pour avoir accepté de faire partie de mon jury de thèse.

Je remercie plus particulièrement Yann pour avoir rendu possible mon séjour à l'institut de recherche Femto-Sciences et Technologies au sein du département AS2M à Besançon en France. Une partie du travail présenté dans cette thèse provient d'une collaboration avec Yann Le Gorrec et Yongxin Wu (ASEM, Femto-ST). Merci à eux pour leur accueil et pour avoir permis à ce travail d'avoir un cadre appliqué et expérimental. Merci à Yann de m'avoir aidé à mieux cerner les concepts physiques de la structure Hamiltonienne à ports et pour sa bienveillance par rapport à toutes mes questions. Merci à Yongxin pour son aide précieuse. Je tiens aussi à remercier Alexandre pour ses discussions enrichissantes, sa disponibilité et pour avoir contribué à l'amélioration de ce manuscrit.

Les résultats de recherche développés dans ce manuscrit ont pu être réalisés grâce à ma bourse FRIA. Je remercie le FNRS-FRS de m'avoir financé durant ces quatre années.

Des remerciements particuliers vont à mon collègue de bureau, Julien Blanchard, pour avoir fait de notre bureau un agréable lieu de travail. Cette thèse doit également beaucoup aux nombreuses relectures d'Anthony Hastir. Je le remercie pour ses remarques avisées, pour les moments de réflexion et de détente, et pour avoir fait exploser ma note café. Ce travail doit également beaucoup à la formidable ambiance qui règne au sein du département de mathématique de Namur. Merci à tous mes collègues et amis du département.

Enfin, un mot particulier pour remercier ma famille et belle-famille: leurs encouragements et leur soutien furent sans failles. Merci à Margaux pour m'avoir soutenu et accompagné dans les bons moments comme dans les mauvais. Je n'y serais pas arrivé sans toi.

Merci à tous pour avoir contribué à cette formidable expérience humaine qu'a été la réalisation de cette thèse de doctorat.

François Lamoline

Contents

List of Symbols and Acronyms

Symbols

t	time variable
ζ	spatial variable
$\mathbb{R}$	set of real numbers
$\mathbb{R}^+$	set of nonnegative real numbers
$\mathbb{R}_0^+$	set of positive real numbers
$\mathbb{C}$	set of complex numbers
$\mathbb{R}^n$	set of real vectors of dimension n
x^T	transpose of the vector x
$\frac{d}{dt}x(t)$ (or $\dot{x}(t)$)	time derivative of $x(t)$
$\frac{\partial}{\partial \zeta}w$ (or $\partial_\zeta w$)	partial derivative of w with respect to ζ
$a \wedge b$	minimum between the real values a and b
$D(A)$	domain of the operator A
$\rho(A)$	resolvent set of the linear operator A, i.e., the set of all complex values λ such that $(\lambda I - A)^{-1}$ is a bounded linear operator on $\mathcal{X}$
Ker A	kernel (or null space) of the operator A
Ran A	range of the operator A
$W^{p,m}$	Sobolev space
$\mathcal{L}(Z, \mathcal{X})$	vector space of bounded linear operators from Z to $\mathcal{X}$
$\mathcal{L}(Z)$	vector space of bounded linear operators from Z to Z
$L_1(Z, \mathcal{X})$	vector space of all bounded linear operators from Z to $\mathcal{X}$ that are of trace class

$L_2(Z,\mathcal{X})$	vector space of all bounded linear operators from Z to $\mathcal{X}$ that are Hilbert-Schmidt operators
$\mathrm{Tr}\,Q$	trace of an operator $Q \in L_1(Z)$
$\Sigma(A,B,C,D)$	state space representation of a system with a dynamical operator A, a control operator B, an observation operator C and a feedthrough operator D
Ω	sample space of events ω
$\mathcal{F}$	σ-algebra of subsets of the set Ω
$\mathbb{P}$	probability measure on the σ-algebra $\mathcal{F}$
$\mathbf{B}(\mathcal{X})$	Borel σ-algebra of space $\mathcal{X}$
$L^2(\Omega;Z)$	vector space of random variables, taking values in Hilbert space Z, with finite variance
$(\mathcal{F}_t)_{t\geq 0}$	filtration
$\mathbb{E}$	expectation with respect to the measure $\mathbb{P}$
$\mathcal{F}_t^W$	filtration generated by a Wiener process $w(s)$ with $0 \leq s \leq t$
$\mathcal{N}(\mu,\sigma^2)$	normal (or Gaussian) distribution with mean μ and variance σ^2

Acronyms

a.e.	almost everywhere
a.s.	almost surely
BCO	boundary controlled and observed
CGT	canonical generalized transformation
DPS	distributed parameter system
EAP	electro-active polymer
IPMC	ionic polymer metal composite
PDE	partial differential equation
PHS	port-Hamiltonian system
SGCT	stochastic generalized canonical transformation
SPDE	stochastic partial differential equation
SPHS	stochastic port-Hamiltonian system

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Introduction

A quick tour

In every form of our modern life, we are surrounded by control systems. It helps us in our daily life and routines, sometimes, without even noticing it. This goes to the simple thermostat to regulate the ambient temperature to much complex devices such as automotive control systems. Even our human body is full of control systems. Cells, organs or tissues are all governed and controlled by chemical and biological rules. From a denominational point a view, the system being controlled is commonly referred as the plant, while the system applying the control is referred as the controller.

Most of dynamical systems considered in applications are governed by partial differential equations (PDEs). When the physical quantity of interest depends on both the position and the time, PDEs describe the distribution in space of this physical quantity. These dynamical systems are often also called distributed parameter systems (DPSs). DPSs as opposed to lumped parameter systems are dynamical systems for which the state space is infinite-dimensional, which means that the solution is taking values in an infinite-dimensional space. Another class of infinite-dimensional systems are delayed systems, which are governed by delay differential equations. The solutions of such equations also take values in infinite-dimensional spaces. However, delayed systems will not be considered in this thesis.

DPSs can be controlled at sections of their physical domain, at punctual locations or at their physical boundaries. This leads to distributed, point or boundary control problems. More particularly, boundary control or point control lead to substantial mathematical difficulties due to the unboundedness of the operators. In a similar manner, measurements may be realised on sections of the physical domains or at punctual positions. When boundary control and observation come into play, things become more complicated. Natural questions including admissibility and well-posednes arise. Pioneered developments were undertaken by Fattorini in [Fat68], where he proposed an abstract general theory to deal with PDEs having boundary control, that were considered also in [CZ95] afterwards. More recently, the class of boundary control systems with boundary observation was studied in [JZ12] and in [RDW16] with Yosida-Type approximate boundary observation.

Basically, Hamiltonian systems have been introduced to reproduce the inner physical properties of physical systems encountered in practice. Over the last decades, linear port-controlled Hamiltonian systems (PHSs) have been proved to be an efficient framework for the modeling, the analysis and the control of dynamical systems governed either by ordinary differential equations (ODEs) or partial differential equations (PDEs). This class of dynamical systems covers a wide range of applications including flexible beams, tubular reactors, electrical networks, irrigation channels, among many others. A good overview of the theory of finite and infinite-dimensional port-Hamiltonian systems can be found in [vdSJ14] and [JZ12]. The control of port-Hamiltonian systems can be performed inside the spatial domain or at the boundary. The port-Hamiltonian framework is naturally well-adapted for control design and has been used advantageously with passivity to control complex physical systems. As in the modeling step, the energy plays a central role in the controller implementation, which most of the time has an energy interpretation. Lyapunov techniques based on the intrinsic link between energy and dynamics of the system have been quite popular for stability analysis or control design for PHSs. It consists in choosing a proper Lyapunov function related to the Hamiltonian and including the boundary conditions. Energy shaping methods were developed in [MM04] and [Mac12]. Furthermore, the dissipation operator of the closed-loop system can be modified with interconnection and damping assignment passivity-based control. This was first proposed for finite-dimensional systems in [OvdSME02] and extended for boundary controlled systems in [RLGMZ14]. More recently, the LQG controller implementation was studied in [WHGM18] for finite-dimensional port-Hamiltonian systems in the perspective of designing reduced-order controllers.

From a practical point of view, various disturbances such as modeling inaccuracies or environment disturbances can occur when real plants are to be controlled. This motivates the stochastic extension of the class of port-Hamiltonian systems. The philosophy of the port-Hamiltonian framework and some hints for a stochastic extension with respect to a port-based approach will be presented in Section 0.2. In a control context, stochastic port-Hamiltonian systems were first introduced in [SF13] on euclidean spaces in the nonlinear time varying case, as the stochastic extension of [MvdS92]. More recently, the author proposed a stochastic generalization in [LW17b] of infinite-dimensional linear port-Hamiltonian systems with boundary control and observation introduced in [LZM05]. Here, the uncertainties in the dynamical systems coming from the environment will be assumed to be white noise processes. The main reason lies in the fact that white noise processes allow one to represent many different kind of noises encountered in experimental applications, in particular in engineering. This leads us to the study of stochastic partial differential equations (SPDEs). Many mathematicians have left their footprints on the analysis and the study of SPDEs. Some of them are Lions, Bensoussan, Pardoux, Curtain, Da Prato, Zabczyk and Hairer, among many others. Most of the SPDEs are treated as stochastic differential equations on infinite-dimensional spaces. The theory of stochastic differential equations was initiated by the early work of Itô in the mid-1940s, see [Ito44]. One of the main difficulties lies in the mathematical interpretation of stochastic disturbances. This requires a proper theory of integration to define the so-called stochastic Itô integrals. One of the most well-known references devoted to the study of infinite-dimensional SDEs is [DPZ14] by Da Prato and Zabzcyk. The latter usually requires some understanding and prior knowledge of infinite-dimensional stochastic analysis and, from the author's perspective, would not be recommended as an introductory book on this subject. In chapter 2, prerequisites on probability measure theory and stochastic processes are presented. In addition, a construction of the stochastic integral in the Itô sense is given and some of its properties are reviewed. The author has tried to make it accessible for readers who are not familiar with stochastic analysis in general.

Linear well-posed systems in the sense of Salamon [Sal89] were introduced to deal with systems with boundary control and observation operators. This class of systems is also known to enjoy many useful properties (see e.g. [Sta05]) involving feedback control, dynamic stabilization, and tracking/disturbance rejection. This is one of the main motivations for generalizing the well-posedness concept to stochastic systems. As far as known, there are not as many references devoted to stochastic well-posed systems as for the deterministic case, see [Sta05], [TW14], [WST01] and [ZGMV10]. Although the study of SPDEs has attracted a lot of attention, the well-posedness of stochastic systems still seems to be an uncultivated field in systems and control. As a matter of fact, few works on this topic are available in the literature. See [Lü15], where a generalization of well-posed linear systems to the stochastic context is undertaken by providing a formulation of stochastic well-posed linear systems. The well-posedness study of boundary controlled and observed SPHSs and more generally of stochastic systems developed in this work falls in line within [Lü15] as an attempt to fill this blank left in the literature.

Moreover, the control theory of stochastic partial differential equations is still at its beginning. The main difficulty comes from the lack of tools when compared to the deterministic setting or even to finite-dimensional SDEs. Nevertheless, paramount efforts have been made to overcome this difficulty. The Linear Quadratic Gaussian (LQG) control problem is an efficient way of considering uncertainties in the control process. It mainly concerns linear dynamical systems subject to some additive noises on the state and output processes. The system and measurement noises are assumed to be Gaussian white noise processes. This specific control method usually requires strong assumptions on the plant such as bounded control and observation operators, and stabilizability and detectability conditions, see [CP78]. The LQG control problem is solved by using a separation principle [CI77a], which states that the LQG control problem can be divided into two separate problems, namely the mean-square estimation of the state process and the optimal control problem with complete observation. In this work we shall focus more particularly on the LQG control of SPHSs. Since the LQG controller is somewhat mimicking the dynamics of the controlled system with a correction, the question of conserving the port-Hamiltonian framework in the LQG controller dynamics arises naturally.

In this thesis it will be assumed that the reader is familiar with functional analysis and measure theory. One is referred to the books [DS88], [Yos95] and [Bar95] among many others. In Sections 0.1, 0.3 and 0.4, we introduce some background material to ease the reading. An overview of the contributions of this work and the organization of this manuscript ends this chapter.

0.1 Notations and definitions

Throughout this manuscript, we shall use standard notations commonly found in the literature. Some of them are recalled here below.
Let $\mathbb{K}$ denotes the field of real or complex numbers. More particularly, the spaces of complex, real and real nonnegative numbers are denoted by $\mathbb{C}$, $\mathbb{R}$ and $\mathbb{R}^+$, respectively. The functional space $L^2([a,b];\mathbb{K}^n)$ consists of square-integrable $\mathbb{K}^n$-valued functions with the usual L^2 inner product $\langle\cdot,\cdot\rangle_{L^2}$. The Sobolev space $H^1([a,b];\mathbb{K}^n)$ is the space of all $\mathbb{K}^n$-valued functions, which are square integrable, absolutely continuous, and the derivative yields again a continuous functions. Let X, Y be Hilbert spaces with corresponding inner products $\langle\cdot,\cdot\rangle_X$ and $\langle\cdot,\cdot\rangle_Y$. We shall simply denote by $\mathcal{L}(X,Y)$ the space of bounded linear operators from X to Y. When $X = Y$, it will be shorten by $\mathcal{L}(X)$. Notice that throughout this book, the functional state space will be denoted by $\mathcal{X}$ instead of the usual notation X to avoid any confusion with the capital letter notation of random variables.

We conclude this section by introducing some standard definitions for differential calculus in Hilbert spaces. We start with the Fréchet differentiability.

Definition 0.1.1. *Let $\mathcal{X}$ and Y be separable Hilbert spaces. Given $x \in \mathcal{X}$ and Ω a neighbourhood of x, a function $f : \mathcal{X} \to Y$ is said to be Fréchet differentiable at x if there exists $df(x) \in \mathcal{L}(\mathcal{X},Y)$ such that for all $h \in \mathcal{X}$,*

$$\lim_{\|h\|_{\mathcal{X}}\to 0} \frac{\|f(x+h)-f(x)-df(x)h\|_Y}{\|h\|_{\mathcal{X}}} = 0. \tag{0.1.1}$$

In this case $df(x)$ is unique and said to be the Fréchet derivative of f with respect to x.

Any function $f : \mathcal{X} \to Y$ is said to be differentiable if f is Fréchet differentiable. In the sequel, the derivative of a function $f : \mathcal{X} \to Y$ in the sense of Fréchet will be simply denoted by f'_x. It may happen that the bounded linear operator f'_x is also differentiable at x. Following Definition 0.1.1, a function is said to be two times Fréchet differentiable at x if there exists $d^2f(x) \in \mathcal{L}(\mathcal{X}\times\mathcal{X},Y)$ such that for any $h_1,h_2 \in \mathcal{X}$,

$$\lim_{\|h_2\|_{\mathcal{X}}\to 0} \frac{\|df(x+h_2)(h_1)-df(x)(h_1)-d^2f(x)(h_1,h_2)\|_Y}{\|h_2\|_{\mathcal{X}}} = 0. \tag{0.1.2}$$

The second Fréchet derivative of f is denoted by f''_{xx}. The higher order of derivatives are defined recursively in a similar manner. Note that the concept of first and second Fréchet derivatives will turn out to be useful when introducing the Itô's Lemma in Chapter 2. Let $\mathcal{X}$ denotes the Hilbert space $L^2([a,b];\mathbb{R}^n)$. We will now compute the Fréchet derivative of a quadratic functional

$$E(x) = \frac{1}{2}\langle x, \mathcal{H}x\rangle_{L^2} = \frac{1}{2}\int_a^b x^T(\zeta)\mathcal{H}(\zeta)x(\zeta)d\zeta, \tag{0.1.3}$$

for all $x \in \mathcal{X}$, with $\mathcal{H} \in L^\infty([a,b];\mathbb{R}^{n\times n})$ satisfying $mI \leq \mathcal{H}(\zeta) \leq MI$ for almost every $\zeta \in [a,b]$. As a natural candidate, one would consider $E'_x(x)$ as the functional which associates to any $h \in X$, $\langle h, \mathcal{H}x\rangle_{L^2}$. To prove this assertion, we show that this candidate satisfies Definition 0.1.1.
Equation (0.1.1) is equivalent to

$$\begin{aligned}
&\frac{|\frac{1}{2}\langle x+h, \mathcal{H}[x+h]\rangle - \frac{1}{2}\langle x, \mathcal{H}x\rangle - \langle h, \mathcal{H}x\rangle|}{\|h\|_{\mathcal{X}}} \\
&= \frac{|\frac{1}{2}\langle x, \mathcal{H}x\rangle + \frac{1}{2}\langle x, \mathcal{H}h\rangle + \frac{1}{2}\langle h, \mathcal{H}x\rangle + \frac{1}{2}\langle h, \mathcal{H}h\rangle - \frac{1}{2}\langle x, \mathcal{H}x\rangle - \langle h, \mathcal{H}x\rangle|}{\|h\|_{\mathcal{X}}} \\
&= \frac{1}{\|h\|_{\mathcal{X}}}|\frac{1}{2}\langle h, \mathcal{H}x\rangle + \frac{1}{2}\langle h, \mathcal{H}x\rangle + \frac{1}{2}\langle h, \mathcal{H}h\rangle - \langle h, \mathcal{H}x\rangle| \\
&= \frac{1}{2\|h\|_{\mathcal{X}}}|\langle h, \mathcal{H}h\rangle| \\
&\leq \frac{1}{2\|h\|_{\mathcal{X}}}\|h\|_{\mathcal{X}}\|\mathcal{H}h\|_{\mathcal{X}} = \frac{1}{2}\|\mathcal{H}h\|_{\mathcal{X}} \leq \frac{1}{2}\|\mathcal{H}\|_\infty\|h\|_{\mathcal{X}},
\end{aligned}$$

which goes to 0 as $\|h\|_{\mathcal{X}}$ tends to 0. By a similar argument, the second Fréchet derivative E''_{xx} is the functional which associates $\langle h_1, \mathcal{H}h_2\rangle_{L^2}$ to any $h_1, h_2 \in \mathcal{X}$.

0.2 Port-Hamiltonian formalism with a stochastic approach

In this section the port-Hamiltonian framework and its connexion with the classical Hamiltonian equations is developed. In addition, the stochastic component is considered and expressed while following the port-based approach. To ease the introduction of the port-Hamiltonian formalism, some technical details are left aside and will be explained in the next chapters.

Historically, Hamiltonian systems originate from mechanics and are governed by Hamiltonian equations of motion (or Hamilton's equations) given by

$$\dot{q} = \frac{\partial H}{\partial p}(q,p), \tag{0.2.1}$$

$$\dot{p} = -\frac{\partial H}{\partial q}(q,p) + u, \tag{0.2.2}$$

with the generalized coordinates $(q,p) \in \mathbb{R}^n$, where q and p are the position and the momentum, respectively. The input u is an external force applied to the Hamiltonian system. The equations (0.2.1) and (0.2.2) are obtained by applying the Legendre transformation to the Euler-Lagrange equations. On the other hand, there is the port-based approach initiated by Paynter and Breedveld in the sixties and which takes its roots in the modeling of multi-domain systems (mechanical, electrical, thermal, ...). This approach consists in representing any physical systems as the interconnection of subcomponents and by identifying the main physical elements: energy storage, energy dissipation and energy transportation. The port-Hamiltonian philosophy consists in mixing both approaches by combining the Hamiltonian equations of motion with the port-based modeling. This motivates the denomination "port-Hamiltonian systems".

A generalization of equations (0.2.1) and (0.2.2) on the functional state space $L^2([a,b];\mathbb{R}^n)$ is given by

$$\dot{x}(t) = \mathcal{J}\mathcal{H}x(t) + Bu(t), \tag{0.2.3}$$

$$y(t) = B^*\mathcal{H}x(t), \tag{0.2.4}$$

where $\mathcal{J}$ is a skew-adjoint unbounded linear operator with domain $D(\mathcal{J})$, $\mathcal{H} = \mathcal{H}^* \in L^\infty([a,b];\mathbb{R}^{n\times n})$, $B \in \mathcal{L}(\mathbb{R}^m, L^2([a,b];\mathbb{R}^n))$ and $y(t) \in \mathbb{R}^m$ denotes the output of the system. The Hamiltonian (i.e. the total energy of the system) is given by

$$E(x(t)) = \frac{1}{2}\int_a^b x^T(\zeta,t)\mathcal{H}(\zeta)x(\zeta,t)d\zeta. \tag{0.2.5}$$

Notice that (0.2.1) and (0.2.2) are a particular case of (0.2.3) with $x = (q,p)$ and J being the skew-symmetric operator described by

$$J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} 0 \\ I \end{bmatrix},$$

where I denotes the identity matrix.

The Dirac structure plays a central role for port-Hamiltonian systems and represents their underlying structure. Let us consider two Hilbert spaces $\mathcal{F}$ and $\mathcal{E}$ with inner products $\langle\cdot,\cdot\rangle_{\mathcal{F}}$ and $\langle\cdot,\cdot\rangle_{\mathcal{E}}$. The spaces $\mathcal{F}$ and $\mathcal{E}$ denotes the flow and the effort spaces. The flow space represents the space of rate energy variables, while the effort space is the space of co-energy variables. In addition, let us define the bond space as $\mathcal{B} = \mathcal{F} \times \mathcal{E}$ equipped with the inner product

$$\left\langle \begin{pmatrix} f_1 \\ e_1 \end{pmatrix}, \begin{pmatrix} f_2 \\ e_2 \end{pmatrix} \right\rangle_{\mathcal{B}} = \langle f_1, f_2\rangle_{\mathcal{F}} + \langle e_1, e_2\rangle_{\mathcal{E}} \tag{0.2.6}$$

for all $(f_1,e_1),(f_2,e_2) \in \mathcal{B}$.

The effort space $\mathcal{E}$ is defined as the dual Hilbert space of $\mathcal{F}$. Therefore, there exists an isometric isomorphism $j : \mathcal{F} \to \mathcal{E}$ such that

$$\langle jf_1, jf_2 \rangle_{\mathcal{E}} = \langle f_1, f_2 \rangle_{\mathcal{F}}, \tag{0.2.7}$$

for all $f_1, f_2 \in \mathcal{F}$. In order to build a Dirac structure, the bond space is endowed with the bilinear symmetric pairing given by

$$\left\langle \begin{pmatrix} f_1 \\ e_1 \end{pmatrix}, \begin{pmatrix} f_2 \\ e_2 \end{pmatrix} \right\rangle_+ = \langle f_1, j^{-1} e_2 \rangle_{\mathcal{F}} + \langle e_1, jf_2 \rangle_{\mathcal{E}}, \tag{0.2.8}$$

where $j^{-1} : \mathcal{F} \to \mathcal{E}$. The bilinear pairing $\langle \cdot, \cdot \rangle_+$ represents the power.

Let $\mathcal{V}$ be a linear subspace of $\mathcal{B}$. The orthogonal subspace with respect to the bilinear pairing $\langle \cdot, \cdot \rangle_+$ is defined as

$$\mathcal{V}^T := \{ b \in \mathcal{B} : \langle b, v \rangle_+ = 0, \text{ for all } v \in \mathcal{V} \}. \tag{0.2.9}$$

Let us now define a Dirac structure.

Definition 0.2.1. *[vdSM02] A linear subspace $\mathcal{D}$ of the bond space $\mathcal{B}$ is said to be a Dirac structure if*

$$\mathcal{D}^T = \mathcal{D}. \tag{0.2.10}$$

Note that the condition (0.2.10) implies that the power of any element of the Dirac structure is equal to zero, i.e.,

$$\left\langle \begin{pmatrix} f \\ e \end{pmatrix}, \begin{pmatrix} f \\ e \end{pmatrix} \right\rangle_+ = 2 \langle f, j^{-1} e \rangle_{\mathcal{F}} = 0,$$

for any $(f, e) \in \mathcal{D}$. The underlying structure of port-Hamiltonian systems forms a Dirac structure, which links the port-variables in a way that the total power is equal to zero.

The port-variables are split in two parts: internal ports and external ports. Within the internal ports, there are resistive ports (or elements) corresponding to the internal energy dissipation and the energy storing ports corresponding to energy storage. On the other hand, the external ports represent the interaction of a port-Hamiltonian system with its environment or even with other systems. These interactions can occur either along the domain or at the boundary. The Dirac structure of port-variables is represented in Figure 0.1.

In this work we shall restrict ourselves to first-order linear port-Hamiltonian systems. Note that a Dirac structure can be obtained for this class of port-Hamiltonian systems, see Section 1.1 and [LZM05].

Considering randomness in the modeling entails that further noise ports have to be added to the port-based structure depending on the nature of the uncertainties.

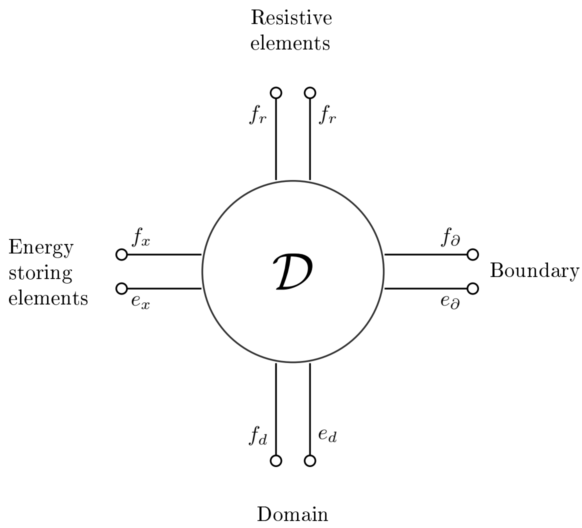

Figure 0.1 – Dirac structure with port-variables

Throughout this thesis, we shall mainly consider some state and measurement noises. Let us consider the following stochastic port-Hamiltonian system with distributed inputs and outputs described by

$$\begin{aligned} \dot{x}(t) &= \mathcal{J}\mathcal{H}x(t) + B_d u_d(t) + H\eta(t), \\ y_d(t) &= B^*\mathcal{H}x(t), \\ y_\eta(t) &= H^*\mathcal{H}x(t), \end{aligned} \tag{0.2.11}$$

where $B_d \in \mathcal{L}(\mathbb{R}^m, \mathcal{X})$ is the control port and $H \in \mathcal{L}(Z, \mathcal{X})$ represents the noise port. The process $\eta(t)$ describes an Hilbert space Z-valued white noise process. The pairs of inputs and outputs $[(u_d(t), y_d(t)), (\eta(t), y_\eta(t))]$ correspond to the external ports (e_d, f_d). The flow and effort variables are given by $f_x = \frac{\partial}{\partial t}x$ and $e_x = \mathcal{H}x$. One could also consider some input noise along the domain or at the boundary. Nevertheless, such generalization will not be studied in this manuscript.

0.3 Boundary controlled and observed systems

In this section we introduce the general setting of boundary controlled and observed systems. Let $\mathcal{X}, U, U_b, Y$ be Hilbert spaces and let us consider the following abstract control system with boundary observation

$$\begin{aligned} \dot{x}(t) &= \mathcal{A}x(t) + B_d u_d(t), \\ u(t) &= \mathcal{B}x(t), \\ y(t) &= \mathcal{C}x(t). \end{aligned} \tag{0.3.1}$$

where $\mathcal{A} : D(\mathcal{A}) \to \mathcal{X}$ and $\mathcal{B} : D(\mathcal{B}) \to U$ are unbounded linear operators such that $D(\mathcal{A}) \subset D(\mathcal{B}) \subset \mathcal{X}$. The distributed control operator B_d belongs to $\mathcal{L}(U_d, \mathcal{X})$.

Definition 0.3.1. *[CZ95, Definition 3.3.2] An abstract control system with boundary observation described by (0.3.1) is said to be a boundary controlled and observed system if the following conditions are satisfied:*

1. *the operator $A : D(A) \to \mathcal{X}$ defined for every $x \in D(A) = D(\mathcal{A}) \cap Ker(\mathcal{B})$ by $Ax = \mathcal{A}x$, is the infinitesimal generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on $\mathcal{X}$,*
2. *there exists an operator $B \in \mathcal{L}(U, \mathcal{X})$ such that, for every $u \in U$, $Bu \in D(\mathcal{A})$, $\mathcal{A}B \in \mathcal{L}(U, \mathcal{X})$ and $\mathcal{B}Bu = u$.*
3. *The observation operator $\mathcal{C} \in \mathcal{L}(D(A), Y)$, where $D(A)$ is endowed with the graph norm of A.*

Condition 1 ensures the generation of a C_0-semigroup for the homogeneous system, i.e. for $u(t) = 0$. Besides, observe that a direct consequence of Condition 2 is the surjectivity of $\mathcal{B}$ and that the operator $B \in \mathcal{L}(U, D(A))$ is the right inverse of $\mathcal{B}$. Moreover, note that the last condition on the boundary observation operator $\mathcal{C}$ is equivalent to the existence of $a, b \in \mathbb{R}$ such that

$$\|\mathcal{C}x\|_Y \leq a\|x\|_{\mathcal{X}} + b\|Ax\|_{\mathcal{X}}, \qquad x \in D(A). \tag{0.3.2}$$

The class of boundary controlled and observed port-Hamiltonian systems is known to satisfy all the conditions of Definition 0.3.1, see [JZ12, Theorem 11.2.2].

0.4 Dissipative systems

Passivity originates from circuit theory and is a cornerstone property in system and control theory leading to some useful results for control design. This property relates to conservation, dissipation, transport and storage of energy. In this preparatory section, some useful concepts such as passivity, dissipativity, storage functions and supply rate are introduced and discussed. This section is mainly based on [HM76], [Wil72] and [vdS16].

Let $\mathcal{X}$, U and Y denote the state, the input and the output Hilbert spaces, respectively. The concepts of dissipativity and passivity are introduced in a deterministic setting for the following large enough (but not restricted) class of control systems described by

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t)), \qquad && x(t) \in \mathcal{X} \\ y(t) &= g(x(t), u(t)), \qquad && u(t) \in U, y(t) \in Y \end{aligned} \tag{0.4.1}$$

The system (0.4.1) includes in particular nonlinear systems affine in the input, static nonlinearity and linear systems. The notion of passivity is intimately related to the way that a dynamical system is interacting with its environment with respect to its inputs and corresponding outputs. This is expressed by the function of supply rate $s : U \times Y \to \mathbb{R}$.

Definition 0.4.1. *A dynamical system (0.4.1) is said to be dissipative with respect to a supply rate $s(u,y)$ if there exists a storage function $S : \mathcal{X} \to \mathbb{R}^+$, such that, for all $t_0 \leq t_f$,*

$$S(x(t_f)) - S(x(t_0)) \leq \int_{t_0}^{t_f} s(u(r),y(r))dr. \tag{0.4.2}$$

Remark 0.4.1.

1. If (0.4.2) holds with an equality, the system is said to be lossless.

2. If the storage function is differentiable, (0.4.2) can be rewritten as

$$\frac{d}{dr} S(x(r)) \leq s(u(r),y(r)), \qquad r \in [t_0,t_f]. \tag{0.4.3}$$

The supply rate represents the energy supplied to the system from some external sources. Thus, dissipativity in the sense of Definition 0.4.1 means that the system cannot store more energy than supplied during the time interval $[t_0,t_f]$. In other words, the system cannot generate energy on its own. In the lossless case, the system stores exactly the supplied energy.

A specific choice of bilinear supply rate leads to the concept of passivity.

Definition 0.4.2. *A dynamical system is said to be passive if the system is dissipative with respect to the supply rate $s(u,y) = u^T y$.*

Passivity leads to several stability results through passivity based control. For instance, the negative output feedback enables to stabilize asymptotically a passive system around zero. For further results, the reader is referred to [HM76], [Wil72], and [vdS16]. Interested readers may also be referred to [KMX$^+$14], where a good survey including the existing connections between passive and positive real systems is presented and a list of classical results for stability of passive, positive real and dissipative systems is given.

Since deterministic port-Hamiltonian systems are known to be passive if the Hamiltonian is bounded from below, this begs naturally the question whether their stochastic extension preserves this property. In this thesis, on the basis of Definition 0.4.2, we shall extend this notion to stochastic systems as given in [Flo99]. More particularly, this new notion of passivity will be studied for distributed and boundary controlled port-Hamiltonian systems.

Contributions and outline

The novelty of this work is the exploration of a stochastic extension of boundary controlled and observed port-Hamiltonian systems as presented in [LZM05] and [JZ12]. This aims at describing the plant as a stochastic system and taking advantages of the port-Hamiltonian framework to express its dynamics and interactions with its environment. From a mathematical perspective, the uncertainties coming from the external disturbances and measurement noises are modeled by Gaussian white noise processes. The notions and concepts of the theory of stochastic integration and the port-Hamiltonian approach to physical systems modeling are merged with a semigroup approach for the study of infinite-dimensional stochastic systems.

We believe that this thesis brings several contributions to the field of control of distributed parameter systems represented under the port-Hamiltonian formalism. In this thesis we consider both deterministic and stochastic port-Hamiltonian systems on infinite-dimensional spaces. In Chapter 1, a clarification of [Vil07, Chapter 4] is given together with a first attempt to summarize the available literature and to investigate the question on whether eigenvectors may form a Riesz basis in the case of port-Hamiltonian systems. The first main contribution of this thesis is to provide the stochastic counterpart of the port-controlled Hamiltonian systems defined in [JZ12] with additive system's noise and to describe them as boundary controlled and observed stochastic systems. As a central part of this work, a study of well-posed linear stochastic systems in the context of boundary control and observation is developed and an extension of the results of [ZGMV10] to the stochastic context is proposed. In addition, the passivity property proposed by Florchinger in [Flo99] for finite-dimensional stochastic systems is extended to infinite-dimensional ones and studied for SPHSs. Furthermore, the LQG control problem is addressed and solved for this new class of stochastic systems. As an infinite-dimensional generalization of [WHGM18], conditions are derived to preserve the stochastic port-Hamiltonian framework in the LQG controller dynamics. In addition, results regarding the exponential stability and detectability of first order linear port-Hamiltonian systems with specific choices of structure matrices are developed. In the last part of this thesis, we focus on the LQG control problem of a compliant endoscope actuated by means of electro-active polymers. An interconnected port-Hamiltonian model is proposed and validated on an experimental setup. A control law is implemented on the model with real-physical parameters to improve the time response of the system and to damp out the oscillations of the endoscope.

The central part of this thesis is divided in 5 chapters. The content of each chapter is briefly summarized as follows.

Chapter 1. This first chapter begins with an introduction of linear first order port-Hamiltonian systems with boundary control and observation. Besides, the Riesz-basis property is investigated for this class of distributed parameter systems . It is proved that nice port-Hamiltonian systems are Riesz-spectral systems. Finally, this result is applied on an illustrative example of a vibrating string which is let free at the origin and damped at the other end.

Chapter 2. This chapter introduces the reader with the stochastic calculus in Hilbert spaces, to the concepts and notions needed for the study of stochastic port-Hamiltonian systems on infinite-dimensional spaces. Some preliminary results of probability theory and concepts of the theory of operators on Hilbert spaces are recalled. The Hilbert space-valued Wiener process and the corresponding Itô stochastic integral are presented. In addition, some tools are presented for the manipulation of the Itô integral.

Chapter 3. The class of boundary controlled and observed stochastic port-Hamiltonian systems is introduced and we proceed to the study of this new class of stochastic systems. The concept of well-posedness in the sense of Weiss and Salamon is generalized to the stochastic context. Under this extended definition, stochastic port-Hamiltonian systems are shown to be well-posed. This chapter ends with the case study of a vibrating string subject to some random forcing. This stochastic system is proved to be well-posed in the sense introduced in this chapter.

Chapter 4. The LQG control problem is addressed for stochastic port-Hamiltonian systems with bounded control, observation and noise ports. This control problem is solved by using the separation principle. Furthermore, conditions are derived to keep the stochastic port-Hamiltonian structure of the LQG controller and thus the closed-loop dynamic can be interpreted as the interconnection of infinite-dimensional stochastic port-Hamiltonian systems. We also study briefly the LQG control problem under weaker assumptions, namely strong stabilizability of the plant. We end this chapter by proving that dissipative port-Hamiltonian systems within a specific framework are exponentially stable. This helps us in establishing the stabilizability conditions for the study of a concrete application in Chapter 5.

Chapter 5. In this chapter we mainly focus on a specific application of a compliant endoscope actuated by means of electro-active polymers. A model for this interconnected system is proposed in the port-Hamiltonian framework. The validity of the proposed model is verified on an experimental setup. We also implement and design a control law consisting of a positive damping injection to improve the time response of the system and a LQG controller to damp out the induced vibrations. The analysis and the results presented in this chapter were essentially obtained within the framework of a collaboration with Professors Yann Le Gorrec and Yongxin Wu from AS2M Department of FEMTO-ST Institute in Besançon, France.

At the end of each chapter, conclusions regarding the results discussed in that chapter are drawn. Finally, this work is closed with a general conclusion and some recommendations on possible future research directions are also provided. For ease of reading, a table of notations and a list of abbreviations have been included.

Most of the research presented in this manuscript has been published in international conference proceedings (with peer review process) or submitted for publication in scientific journals or has been the object of a communication as a talk or a poster presentation, at scientific and engineering conferences, and workshops. The most important publications and communications are listed below:

Scientific journals

- F. Lamoline and J.J. Winkin. Well-posedness of boundary controlled and observed stochastic port-Hamiltonian systems on infinite-dimensional spaces. IEEE Transactions on Automatic Control, 2019, conditionally accepted;
- A. Hastir, F. Lamoline, J.J. Winkin and D. Dochain. Analysis of the existence of equilibrium profiles in nonisothermal axial dispersion tubular reactors. IEEE Transactions on Automatic Control, 2019, accepted, DOI: 10.1109/TAC.2019.2921675;

International conference proceedings and Preprints

- Y. Wu, F. Lamoline, Y. Le Gorrec and J.J. Winkin. Modelling and control of an IPMC actuated flexible beam under the port-Hamiltonian framework. In Proceedings of the 3rd IFAC Workshop on Control of Systems governed by Partial Differential Equations, 2019, (May 22-24, Oaxaca, Mexico);
- F. Lamoline and J.J. Winkin. On LQG control of stochastic port-Hamiltonian systems on infinite-dimensional spaces. In Proceedings of the 23rd Symposium on Mathematical Theory of Networks and Systems, pages 197-203, 2018 (July 16-20, Hong-Kong), with invited oral communication;
- F. Lamoline and J.J. Winkin. On stochastic port-Hamiltonian systems with boundary control and observation. In Proceedings of the 56th IEEE Conference on Decision and Control, pages 2492-2497, 2017 (December 12-15, Melbourne, Australia), with oral communication;
- F. Lamoline and J.J. Winkin. Nice port-Hamiltonian systems are Riesz-spectral systems. Preprints of the 20th IFAC World Congress. Ed. IFAC, p. 695-699, 2017 (July 9-14, Toulouse, France) with oral communication;

Communications

- Poster presented at the 2nd Workshop on Stability and Control of Infinite-Dimensional Systems in Würzburg, Germany:" *On LQG control of stochastic port-Hamiltonian systems*, 10 to 12 October 2018;

- Poster presented at the 10th Workshop on Control of Distributed Parameter Systems (CDPS) in Bordeaux, France: "*Well-posedness of stochastic port-Hamiltonian systems on infinite-dimensional spaces*, 3 to 5 July 2017;
- Participation to an international group project under the supervision of Professors Markus Kunze and Manfred Sauter from Universität Ulm and presentation at the ISEM 2016 Workshop at Santa Chiara institute, Casalmaggiore, Italy: "*The strong Feller property for Ornstein-Uhlenbeck semigroups and its control theoretic background*", 30 May to 4 June 2016.

Chapter 1

Deterministic port-Hamiltonian systems on infinite-dimensional spaces

Linear port-Hamiltonian systems have been the object of much attention over the last two decades. This class of systems was first introduced in [MvdS92] in the language of differential forms. The infinite-dimensional extension was proposed in [vdSM02]. So far, the port-Hamiltonian framework has been proved to be powerful for the modeling and the control of distributed parameter systems. This class of systems encompasses mechanical, electronical and electromechanical systems and can be employed for a wide range of control applications: reactors [HLDWon], beam equations [MM04], heat and mass transfer equations [BCE$^+$09], and irrigation channels [HLM06]. Many others applications can be found in [DMSB09] and references therein.

In addition to introducing linear port-controlled Hamiltonian systems, the Riesz-basis property will be one of the main concerns of this chapter. This property has a paramount importance in system and control theory and an extended literature is devoted to it. This property leads to some efficient results for establishing controllability, stabilizability, their dual concepts, and stability. So far, numerous applications including the wave equation, traveling waves, heat exchangers, the Timoshenko beam, diffusive tubular reactors falling within the port-Hamiltonian formalism have been proved to be Riesz-spectral systems. This begs naturally the question of generalizing this result to the unifying framework of port-Hamiltonian systems.

This chapter is articulated around a main result, namely that nice port-Hamiltonian systems are Riesz-spectral systems. This is mainly based on [LW17a] and results gathered from [Vil07] and developed in a straightforward manner to deduce the Riesz-basis property from [Tre00].

This Chapter is organized as follows. Section 1.1 specifies the considered class of linear first-order port-Hamiltonian systems. In Sections 1.2 and 1.3, the notions of Riesz basis and the class of Riesz-spectral systems are presented. An interesting subclass of port-Hamiltonian systems, namely nice port-Hamiltonian systems, are proved to be Riesz-spectral systems. Finally, an example of a inhomogeneous vibrating string is proved to be a nice port-Hamiltonian system and this feature is used to establish its exponential stability.

1.1 Deterministic port-Hamiltonian systems

In this thesis we consider the class of distributed port-Hamiltonian systems (PHSs) introduced in [JZ12] and [LZM05].

Definition 1.1.1. *A first order linear port-Hamiltonian system is governed by a PDE of the form*

$$\frac{\partial x}{\partial t}(\zeta,t) = P_1 \frac{\partial}{\partial \zeta}(\mathcal{H}(\zeta)x(\zeta,t)) + P_0\mathcal{H}(\zeta)x(\zeta,t), \tag{1.1.1}$$

where $P_1 \in \mathbb{R}^{n\times n}$ is invertible and symmetric ($P_1^T = P_1$), $P_0 \in \mathbb{R}^{n\times n}$ is skew-symmetric ($P_0^T = -P_0$) and $\mathcal{H} \in L^\infty([a,b];\mathbb{R}^{n\times n})$ is symmetric and satisfies $mI \leq \mathcal{H}(\zeta) \leq MI$ for a.e. $\zeta \in [a,b]$, for some constants m, $M > 0$. As state space, we consider $\mathcal{X} := L^2([a,b];\mathbb{R}^n)$ endowed with inner product $\langle x_1,x_2\rangle_{\mathcal{X}} = \langle x_1,\mathcal{H}x_2\rangle_{L^2}$ for any $x_1,x_2 \in \mathcal{X}$.

Note that, since $mI \leq \mathcal{H}(\zeta) \leq MI$, the induced norm $\|\cdot\|_{\mathcal{X}}$ is equivalent to the standard L^2-norm. This choice of norm is made in order to match the Hamiltonian/energy $E : \mathcal{X} \to \mathbb{R}$ of the system given by

$$E(x) = \frac{1}{2}\langle x,\mathcal{H}x\rangle_{L^2} = \frac{1}{2}\int_a^b x^T(\zeta)\mathcal{H}(\zeta)x(\zeta)d\zeta. \tag{1.1.2}$$

To the PDE (1.1.1), we associate some controlled and homogeneous boundary conditions given by

$$u(t) = W_{B,1}\begin{bmatrix} f_\partial(t) \\ e_\partial(t) \end{bmatrix}, \qquad 0 = W_{B,2}\begin{bmatrix} f_\partial(t) \\ e_\partial(t) \end{bmatrix}, \tag{1.1.3}$$

where the boundary port-variables, namely the flow and the effort, are expressed in the following way:

$$\begin{pmatrix} f_\partial(t) \\ e_\partial(t) \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} P_1 & -P_1 \\ I & I \end{pmatrix}\begin{pmatrix} (\mathcal{H}x(t))(b) \\ (\mathcal{H}x(t))(a) \end{pmatrix} =: R_0\begin{pmatrix} (\mathcal{H}x(t))(b) \\ (\mathcal{H}x(t))(a) \end{pmatrix}, \tag{1.1.4}$$

and $W_B := \begin{bmatrix} W_{B,1} \\ W_{B,2} \end{bmatrix} \in \mathbb{R}^{n\times 2n}$. Notice that the number of rows of $W_{B,1}$ is given by the number of boundary controls applied to the PDE (1.1.1).

In order to establish the existence of a unique mild solution for any port-Hamiltonian system described by the PDE (1.1.1) with the boundary conditions (1.1.3) we rewrite the PDE (1.1.1) with its boundary conditions (1.1.3) as an abstract differential equation given by

$$\dot{x}(t) = Ax(t), \tag{1.1.5}$$

where we define the unbounded linear operator

$$Ax := P_1 \frac{d}{d\zeta}(\mathcal{H}x) + P_0 \mathcal{H}x = \mathcal{J}\mathcal{H}x \tag{1.1.6}$$

on the domain

$$D(A) = \left\{ x \in L^2([a,b];\mathbb{R}^n) : \mathcal{H}x \in H^1([a,b];\mathbb{R}^n), W_B \begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix} = 0 \right\}, \tag{1.1.7}$$

The following result establishes the generation of a C_0-semigroup for (1.1.5), see [Vil07, Theorem 2.13].

Theorem 1.1.1. *Consider the operator A with domain $D(A)$ given by (1.1.6)-(1.1.7), associated to a port-Hamiltonian system (1.1.1) and (1.1.3). Assume that W_B is a $n \times 2n$ matrix of full rank. Then the following statements are equivalent.*

1. *A is the generator of a contraction C_0-semigroup on $L^2([a,b];\mathbb{R}^n)$.*
2. *$W_B \Sigma W_B^T \geq 0$, where $\Sigma = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \in \mathbb{R}^{2n \times 2n}$.*
3. *$Re\langle Ax, x\rangle_{\mathcal{X}} \leq 0$.*

Furthermore, A is the infinitesimal generator of a unitary group on $L^2([a,b];\mathbb{R}^n)$ if and only if $W_B \Sigma W_B^T = 0$.

A first order linear port-Hamiltonian system can be formulated on a Dirac structure as follows. The flow and the effort spaces are given by

$$\mathcal{F} = \mathcal{E} = L^2([a,b];\mathbb{R}^n) \times \mathbb{R}^n. \tag{1.1.8}$$

In addition, the flow and the effort variables are taken as

$$f_x = \frac{\partial x}{\partial t} \qquad \text{and} \qquad e_x = \mathcal{H}x, \tag{1.1.9}$$

respectively. A first order linear port-Hamiltonian system is then described by

$$\left\{ x(\cdot,t) | \begin{pmatrix} f_x \\ f_\partial \\ e_x \\ e_\partial \end{pmatrix} \in \mathcal{D} \right\}, \tag{1.1.10}$$

where $\mathcal{D}$ is given by

$$\mathcal{D} = \left\{ \begin{pmatrix} f_x \\ f_\partial \\ e_x \\ e_\partial \end{pmatrix} \in \mathcal{F} \times \mathcal{E} | e_x \in H^1([a,b];\mathbb{R}^n), f_x = \mathcal{J} e_x, \right. \\ \left. \begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} = R_0 \begin{pmatrix} (e_x)(b) \\ (e_x)(a) \end{pmatrix} \right\}. \tag{1.1.11}$$

In the following proposition, we derive the Hilbert space adjoint of A, see [Vil07, Proposition 2.24] and [Aug16, Proposition 3.4.3].

Proposition 1.1.2. *Let W_B be a $n \times 2n$ matrix written as $W_B = \begin{bmatrix} W_1 & W_2 \end{bmatrix}$. Let us consider the operator A with its associated domain $D(A)$ given by (1.1.6) and (1.1.7), respectively. Assume that W_B has rank n and satisfies $W_B \Sigma W_B^T \geq 0$. Then its adjoint A^* is given by*

$$A^* x = -P_1 \frac{d}{d\zeta}(\mathcal{H}x) - P_0(\mathcal{H}x) = -\mathcal{J}\mathcal{H}x \tag{1.1.12}$$

for all x in

$$D(A^*) = \left\{ x \in L^2([a,b];\mathbb{R}^n) : \mathcal{H}x \in H^1([a,b];\mathbb{R}^n), \right. \\ \left. \begin{bmatrix} -(I+M^T) & (I-M^T) \end{bmatrix} \begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} = 0 \right\}, \tag{1.1.13}$$

where I denotes the identity matrix and $M = (W_1 + W_2)^{-1}(W_1 - W_2)$.

Proof. The adjoint of an unbounded operator is given by

$$A^* y = z \Leftrightarrow \forall x \in D(A), \langle Ax, y \rangle_{\mathcal{X}} = \langle x, z \rangle_{\mathcal{X}} \tag{1.1.14}$$

with domain defined by:

$$y \in D(A^*) \Leftrightarrow \exists z \in \mathcal{X} \text{ s.t. } \forall x \in D(A),\ \langle Ax, y \rangle_{\mathcal{X}} = \langle x, z \rangle_{\mathcal{X}}.$$

On one hand, by integrating by parts, we have that

$$\langle Ax, y \rangle_{\mathcal{X}} = [y^T(\zeta)\mathcal{H}(\zeta)P_1\mathcal{H}(\zeta)x(\zeta)]_a^b - \int_a^b \frac{d}{d\zeta}(y^T(\zeta)\mathcal{H}(\zeta))P_1(\mathcal{H}x)(\zeta)d\zeta \\ + \int_a^b y^T(\zeta)\mathcal{H}(\zeta)P_0(\mathcal{H}x)(\zeta)d\zeta, \tag{1.1.15}$$

and, on the other hand,

$$\langle x, A^* y \rangle_{\mathcal{X}} = \int_a^b (A^* y(\zeta))^T \mathcal{H}(\zeta)x(\zeta)d\zeta. \tag{1.1.16}$$

Since the equality between (1.1.15) and (1.1.16) must hold, we deduce that

$$A^* y = -P_0 \mathcal{H} y - P_1 \frac{d}{d\zeta}(\mathcal{H} y) \tag{1.1.17}$$

and

$$[y^T(\zeta)\mathcal{H}(\zeta)P_1\mathcal{H}(\zeta)x(\zeta)]_a^b = 0. \tag{1.1.18}$$

The relation (1.1.18) can be rewritten as

$$\left(\begin{bmatrix} 0 & P_1 \\ -P_1 & 0 \end{bmatrix} \begin{bmatrix} (\mathcal{H}y)(a) \\ (\mathcal{H}y)(b) \end{bmatrix} \right)^T \begin{bmatrix} (\mathcal{H}x)(b) \\ (\mathcal{H}x)(a) \end{bmatrix} = 0. \tag{1.1.19}$$

The boundary term (1.1.19) can be rewritten as

$$\begin{aligned} &\left(\begin{bmatrix} 0 & P_1 \\ -P_1 & 0 \end{bmatrix} \begin{bmatrix} (\mathcal{H}y)(a) \\ (\mathcal{H}y)(b) \end{bmatrix} \right)^T \begin{bmatrix} (\mathcal{H}x)(b) \\ (\mathcal{H}x)(a) \end{bmatrix} \\ &= \begin{pmatrix} (\mathcal{H}y)(b) \\ (\mathcal{H}y)(a) \end{pmatrix}^T R_0^T \Sigma R_0 \begin{pmatrix} (\mathcal{H}x)(b) \\ (\mathcal{H}x)(a) \end{pmatrix}. \end{aligned} \tag{1.1.20}$$

Since A generates a contraction C_0-semigroup, from [JZ12, Lemma 7.3.2], there exists a $n \times n$-matrix M such that

$$\ker W_B = \operatorname{Ran} \begin{bmatrix} I - M \\ -(I+M) \end{bmatrix}, \tag{1.1.21}$$

where $M = (W_1 + W_2)^{-1}(W_1 - W_2)$ with $W_B = \begin{bmatrix} W_1 & W_2 \end{bmatrix}$. Furthermore, since $\begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix}$ lies in the kernel of W_B, we deduce that

$$\begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix} = \begin{bmatrix} I - M \\ -(I+M) \end{bmatrix} l \tag{1.1.22}$$

for some $l \in \mathbb{R}^n$. Using (1.1.22) in (1.1.20) and defining $\begin{pmatrix} f_{\partial,y} \\ e_{\partial,y} \end{pmatrix} = R_0 \begin{bmatrix} (\mathcal{H}y)(b) \\ (\mathcal{H}y)(a) \end{bmatrix}$, we get that

$$\begin{pmatrix} f_{\partial,y} \\ e_{\partial,y} \end{pmatrix}^T \Sigma \begin{pmatrix} I - M \\ -(I+M) \end{pmatrix} l = 0, \tag{1.1.23}$$

which is equivalent to

$$\begin{pmatrix} f_{\partial,y} \\ e_{\partial,y} \end{pmatrix} \in \operatorname{Ker} \begin{pmatrix} -(I+M^T) & I - M^T \end{pmatrix}. \tag{1.1.24}$$

□

Let us define the observation taken at the boundary as

$$y(t) = W_C \begin{bmatrix} f_\partial(t) \\ e_\partial(t) \end{bmatrix} := \mathcal{C}x(t), \qquad W_C \in \mathbb{R}^{p \times 2n}. \tag{1.1.25}$$

Throughout this work, we shall consider two distinct control actions and observations, on a section of the spatial domain $[a,b]$ and at the boundary of it. Thus, let us consider

$$\begin{aligned} \dot{x}(t) &= \mathcal{J}\mathcal{H}x(t) + B_d u_d(t), \\ u(t) &= W_{B,1}\begin{bmatrix} f_\partial(t) \\ e_\partial(t) \end{bmatrix}, \qquad 0 = W_{B,2}\begin{bmatrix} f_\partial(t) \\ e_\partial(t) \end{bmatrix}, \\ y(t) &= W_C\begin{bmatrix} f_\partial(t) \\ e_\partial(t) \end{bmatrix}, \\ y_d(t) &= B_d^*\mathcal{H}x(t), \end{aligned} \tag{1.1.26}$$

where $x(t) \in \mathcal{X}$, $u_d(t) \in \mathbb{R}^k$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^p, y_d(t) \in \mathbb{R}^k$. Here, the distributed control operator $B_d \in \mathcal{L}(\mathbb{R}^k, \mathcal{X})$ represents the action of the inputs $u_d(t)$ on a spatial domain and $y_d(t)$ is the corresponding power-conjugated output (with respect to inner product $\langle \cdot,\cdot \rangle_{\mathcal{X}}$). Observe that the distributed inputs and outputs are collocated. From a practical perspective, this arises when actuators and sensors are implemented at the same location. As stated in Proposition 1.1.3, the choice of a power-conjugated output is made to recover the balance equation (1.1.27) with the power supplied $u_d^T y_d$. The boundary control and observation are made through the port-variables $f_\partial(t)$ and $e_\partial(t)$. The Hamiltonian of the system represents the total amount of energy stored into the system. Exchanges of energy with the environment occur through the port-variables of the system inside the domain and at the boundary. Furthermore, the choice of inputs and outputs in (1.1.26) entails that the variation of energy of the system is lower or equal to the power fed through the boundary and the domain. The following proposition establishes the balance equation for the PHSs described by (1.1.26).

Proposition 1.1.3. *Consider a port-Hamiltonian system with distributed inputs/outputs and boundary inputs/outputs given by (1.1.26). The balance equation for the Hamiltonian /energy is given by*

$$\frac{dE(t)}{dt} = f_\partial^T(t)e_\partial(t) + u_d(t)^T y_d(t). \tag{1.1.27}$$

Proof. First, notice that

$$\frac{dE(t)}{dt} = \frac{1}{2}\langle \mathcal{J}\mathcal{H}x(t) + B_d u_d(t), x(t)\rangle_{\mathcal{X}} + \frac{1}{2}\langle x(t), \mathcal{J}\mathcal{H}x(t) + B_d u_d(t)\rangle_{\mathcal{X}}. \tag{1.1.28}$$

Since

$$\frac{1}{2}(\langle \mathcal{H}x(t), \mathcal{J}\mathcal{H}x(t)\rangle_{L^2} + \langle \mathcal{J}\mathcal{H}x(t), \mathcal{H}x(t)\rangle_{L^2}) = \frac{1}{2}\begin{bmatrix} f_\partial(t) \\ e_\partial(t) \end{bmatrix}^T \Sigma \begin{bmatrix} f_\partial(t) \\ e_\partial(t) \end{bmatrix},$$

and by considering that

$$\langle B_d u_d(t), x(t)\rangle_{\mathcal{X}} = \langle u_d(t), B_d^*\mathcal{H}x(t)\rangle_{L^2} = u_d(t)^T y_d(t),$$

we obtain

$$\frac{dE(t)}{dt} = f_\partial^T(t)e_\partial(t) + u_d(t)^T y_d(t), \tag{1.1.29}$$

which completes the proof. □

By plugging the relations between the port-variables and inputs-outputs given by

$$\begin{bmatrix} u(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} W_{B,1} \\ W_C \end{bmatrix} \begin{bmatrix} f_\partial(t) \\ e_\partial(t) \end{bmatrix},$$

where $\begin{bmatrix} W_{B,1} \\ W_C \end{bmatrix}$ is full rank or equivalently invertible in the ODE (1.1.29), we deduce that

$$\frac{dE(t)}{dt} = \begin{bmatrix} u(t) \\ y(t) \end{bmatrix}^T P_{W_{B,1}W_C} \begin{bmatrix} u(t) \\ y(t) \end{bmatrix} + u_d(t)^T y_d(t), \tag{1.1.30}$$

where

$$P_{W_{B,1},W_C} = \left(\begin{bmatrix} W_{B,1} \\ W_C \end{bmatrix} \Sigma \begin{bmatrix} W_{B,1} \\ W_C \end{bmatrix} \right)^{-1}. \tag{1.1.31}$$

Note that the system (1.1.26) does not contain any internal source.

Example 1.1. Let us consider the example of an inhomogeneous vibrating string on a spatial domain $[a,b]$. This example is based on [JZ12, Example 7.2.5]. The dynamics of this system are governed by the following PDE

$$\frac{\partial^2 z}{\partial t^2}(\zeta,t) = \frac{1}{\rho(\zeta)} \frac{\partial}{\partial \zeta} \left(T(\zeta) \frac{\partial z}{\partial \zeta}(\zeta,t) \right), \tag{1.1.32}$$

where $z(\zeta,t)$ is the vertical position of the string at place ζ and time t. $T(\zeta)$ and $\rho(\zeta)$ are the Young's modulus and the mass density at place ζ, respectively. Let us define the momentum $p(\zeta,t) = \rho(\zeta)\frac{\partial z}{\partial t}(\zeta,t)$ and the strain $q(\zeta,t) = \frac{\partial z}{\partial \zeta}(\zeta,t)$. This yields

$$\begin{aligned} \frac{\partial}{\partial t} \begin{bmatrix} p(\zeta,t) \\ q(\zeta,t) \end{bmatrix} &= \begin{bmatrix} 0 & \partial_\zeta \\ \partial_\zeta & 0 \end{bmatrix} \left(\begin{bmatrix} \frac{1}{\rho(\zeta)} & 0 \\ 0 & T(\zeta) \end{bmatrix} \begin{bmatrix} p(\zeta,t) \\ q(\zeta,t) \end{bmatrix} \right) \\ &= P_1 \frac{\partial}{\partial \zeta} \left(\mathcal{H}(\zeta) \begin{bmatrix} p(\zeta,t) \\ q(\zeta,t) \end{bmatrix} \right). \end{aligned} \tag{1.1.33}$$

Notice that $\frac{1}{\rho}p$ and Tq correspond to the velocity and the stress of the string, respectively. Boundary conditions depend on the physical situation of the string. In this example, we consider a string clamped at the extremity a and let free at the extremity b, i.e.

$$T(b)\frac{\partial z}{\partial \zeta}(b,t) = u(t) \qquad \text{and} \qquad \frac{\partial z}{\partial t}(a,t) = 0. \tag{1.1.34}$$

In addition, the string is assumed to be actuated near the extremity a by distributed forces $b(\zeta)u_d(t)$ on $[a+\varepsilon_1, a+\varepsilon_2]$ with $0 < \varepsilon_1 < \varepsilon_2$, where

$$b(\zeta) = \begin{cases} 1, & \zeta \in [a+\varepsilon_1, a+\varepsilon_2], \\ 0, & \text{elsewhere.} \end{cases}$$

The port variables are given by

$$f_\partial(t) = \frac{1}{\sqrt{2}} \begin{bmatrix} T(b)\frac{\partial z}{\partial \zeta}(b,t) - T(a)\frac{\partial z}{\partial \zeta}(a,t) \\ \frac{\partial z}{\partial t}(b,t) - \frac{\partial z}{\partial t}(a,t) \end{bmatrix},$$

$$e_\partial(t) = \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{\partial z}{\partial t}(b,t) + \frac{\partial z}{\partial t}(a,t) \\ T(b)\frac{\partial z}{\partial \zeta}(b,t) + T(a)\frac{\partial z}{\partial \zeta}(a,t) \end{bmatrix}.$$

Thus, in these variables, the boundary conditions read as follows:

$$\begin{bmatrix} u(t) \\ 0 \end{bmatrix} = \begin{bmatrix} T(b)\frac{\partial z}{\partial \zeta}(b,t) \\ \frac{\partial z}{\partial t}(a,t) \end{bmatrix},$$

hence $W_B = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \end{bmatrix}$. The boundary matrix W_B is full rank and $W_B \Sigma W_B^T = 0$, and thus A generates a unitary group, see Theorem 1.1.1.
The velocity is assumed to be observed at the extremity b, i.e.

$$y(t) = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} f_\partial(t) \\ e_\partial(t) \end{bmatrix} = \mathcal{C}x(t).$$

The distributed control operator $B_d : \mathbb{R} \to \mathcal{X}$ is given by

$$B_d u_d := \begin{pmatrix} b(\zeta) \\ 0 \end{pmatrix} u_d(t). \tag{1.1.35}$$

The distributed output, which corresponds to the mean velocity observed on $[a+\varepsilon_1, a+\varepsilon_2]$, is given by

$$y_d(t) = \int_a^b b(\zeta)\frac{\partial z}{\partial t}(\zeta,t)d\zeta. \tag{1.1.36}$$

The balance equation is then given by

$$\begin{aligned} \frac{dE(t)}{dt} &= \begin{bmatrix} u(t) \\ y(t) \end{bmatrix}^T P_{W_{B,1}W_C} \begin{bmatrix} u(t) \\ y(t) \end{bmatrix} + u_d(t)^T y_d(t), \\ &= u_d(t)^T y_d(t) \end{aligned}$$

where we set $u(t) = 0$.

For readers who feel they need further examples, they are referred to [Vil07] and to Sections 5.2 and 5.3, where an actuated bio-endoscope is modeled under the port-Hamiltonian formalism.

1.2 Riesz basis property

In this section we prove that under some assumptions the dynamical operator described by (1.1.6) and (1.1.7) has a Riesz basis of eigenvectors. These are a generalization of the concept of orthonormal basis. The simplest way of constructing a new basis from an original one is through a bounded invertible transformation, see [You01, Theorem 7].

Definition 1.2.1. *Let $\mathcal{X}$ be a Hilbert space. A vector sequence $(\phi_n)_{n\in\mathbb{N}}$ for $\mathcal{X}$ forms a Riesz basis if it is obtained from an orthonormal basis by means of a bounded invertible operator, i.e., there exists $L \in \mathcal{L}(\mathcal{X})$ such that*

$$\forall n \in \mathbb{N},\ \phi_n = Le_n. \tag{1.2.1}$$

for some orthonormal basis $(e_n)_{n\in\mathbb{N}}$.

The reasons for considering this class of basis are numerous: easily checkable criteria regarding controllability, stabilizability, their respective dual concepts, and stability can be derived. Furthermore, this concept allows one to describe the dynamics of a system under the form of eigenfunction expansions, see [CZ95]. In the following theorem, we state equivalent characterization of a Riesz basis.

Theorem 1.2.1. *Let $\mathcal{X}$ be a Hilbert space and $(\phi_n)_{n\in\mathbb{N}}$ be a vector sequence for $\mathcal{X}$. Then the following assertions are equivalent.*

1. *The vector sequence $(\phi_n)_{n\in\mathbb{N}}$ is a Riesz basis for $\mathcal{X}$;*

2. *The vector sequence $(\phi_n)_{n\in\mathbb{N}}$ is complete in $\mathcal{X}$, i.e. $\overline{span\{\phi_n\}} = \mathcal{X}$, and there exist positive constants M_1 and M_2 such that for any $N \in \mathbb{N}$ and for any $c_n \in \mathbb{K}$, $n = 1,2,...,N$,*

$$M_1 \sum_{n=1}^{N} |c_n|^2 \leq || \sum_{n=1}^{N} c_n\phi_n||^2_{\mathcal{X}} \leq M_2 \sum_{n=1}^{N} |c_n|^2. \tag{1.2.2}$$

Any vector $x \in \mathcal{X}$ is uniquely decomposed in a Riesz basis $(\phi_n)_{n\in\mathbb{N}}$ as

$$x = \sum_{n=1}^{\infty} c_n\phi_n, \tag{1.2.3}$$

where the scalars c_n are uniquely determined by x. The concept of Riesz-spectral operator can now be defined.

Definition 1.2.2. *Consider a closed linear operator A on a Hilbert space $\mathcal{X}$ with a discrete spectrum consisting of simple eigenvalues $\sigma_p(A) := \{\lambda_n : n \in \mathbb{N}\}$ and corresponding eigenvectors $(\phi_n)_{n\in\mathbb{N}}$. If the closure of $\{\lambda_n : n \in \mathbb{N}\}$ is totally disconnected and if $(\phi_n)_{n\in\mathbb{N}}$ is a Riesz basis on $\mathcal{X}$, then A is said to be a Riesz-spectral operator.*

Remark 1.2.1. Assuming that $D := \overline{\{\lambda_n : n \in \mathbb{N}\}} \subset \mathbb{C}$ is totally disconnected means that every point in D cannot be joined with any other point in D by a segment lying entirely in D.

Several hints indicating that the class of first order port-Hamiltonian systems satisfies the Riesz basis property are available in the literature. The reader is referred to [XF02], [MH13], [Vil07] and [CZ95]. The main result of this section consists in proving that a port-Hamiltonian operator of order 1 possesses a Riesz basis consisting of eigenvectors. To do so, we mainly use results from [Tre00] and similar reasoning as in [Vil07, Chapter 4].

Assumption 1.2.1. *The multiplication operator $P_1^{-1}\mathcal{H}^{-1}$ is assumed to be diagonalizable, i.e.,*

$$P_1^{-1}\mathcal{H}^{-1}(\zeta) = S(\zeta)A_1(\zeta)S(\zeta)^{-1}, \qquad \zeta \in [a,b], \tag{1.2.4}$$

where A_1 is a diagonal matrix-valued function whose diagonal entries are the eigenvalues $(r_\nu)_{\nu=1}^n$ of $P_1^{-1}\mathcal{H}^{-1}$, whereas S is a matrix-valued function whose columns are corresponding eigenvectors. In addition, S and A_1 are assumed to be continuously differentiable on $[a,b]$.

Observe that $P_1^{-1}\mathcal{H}^{-1}$ may have eigenvalues that are not simple. Hence, thereafter, we shall consider that $P_1^{-1}\mathcal{H}^{-1}$ has l different eigenvalues such that $l \le n$. Let us now express the eigenvalue problem with boundary conditions as described in [MM03] in the case of a port-Hamiltonian operator.

Lemma 1.2.2. *The eigenvalue problem associated with the operator A given by (1.1.6) and (1.1.7), namely*

$$P_1 \frac{d}{d\zeta}((\mathcal{H}x)(\zeta)) + P_0((\mathcal{H}x)(\zeta)) = \lambda x(\zeta), \tag{1.2.5}$$

where $\lambda \in \sigma_p(A)$ and $x \in D(A)$ is a corresponding eigenfunction can be formulated under the form:

$$\begin{aligned} &\frac{df}{d\zeta}(\zeta) = (\lambda A_1(\zeta) + A_0(\zeta))f(\zeta), \qquad \zeta \in [a,b], \\ &W_b(Sf)(b) + W_a(Sf)(a) = 0, \end{aligned} \tag{1.2.6}$$

where $W_b := W_1P_1 + W_2$ and $W_a := -W_1P_1 + W_2$ with $W_B := \begin{bmatrix} W_1 & W_2 \end{bmatrix}$ and $f \in W^{1,2}([a,b];\mathbb{R}^n) := \left\{ f \in L^2([a,b];\mathbb{R}^n) : \frac{df}{d\zeta} \in L^2([a,b];\mathbb{R}^n) \right\}$ with matrix coefficients A_0, $A_1 \in L^\infty([a,b];\mathbb{R}^{n\times n})$.

Proof. Let us consider the eigenvalue problem

$$P_1 \frac{d}{d\zeta}((\mathcal{H}x)(\zeta)) + P_0((\mathcal{H}x)(\zeta)) = \lambda x(\zeta),$$

where $\lambda \in \sigma_p(A)$ and $x \in D(A)$ is a corresponding eigenfunction satisfying the boundary conditions:

$$\tilde{W}_B \begin{bmatrix} (\mathcal{H}x)(b) \\ (\mathcal{H}x)(a) \end{bmatrix} = 0. \tag{1.2.7}$$

By using the basis transformation $S(\zeta)$ that diagonalizes $P_1^{-1}\mathcal{H}^{-1}$, this eigenvalue problem becomes

$$\frac{df}{d\zeta}(\zeta) = (\lambda A_1(\zeta) + A_0(\zeta))f(\zeta), \qquad \zeta \in [a,b], \tag{1.2.8}$$

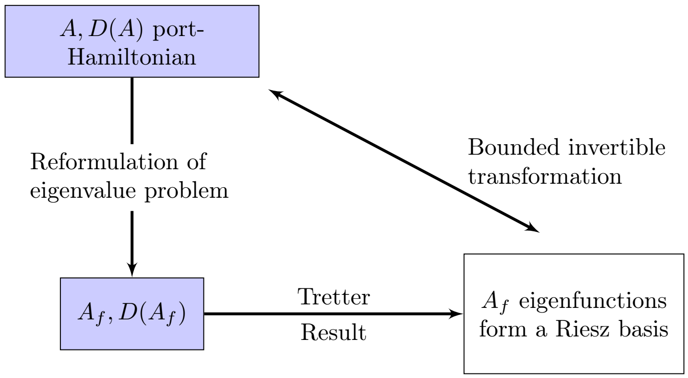

Figure 1.1 – Proof's structure of Theorem 1.2.4

where $A_1 = S^{-1}(\mathcal{H}P_1)^{-1}S$, $A_0 = -S^{-1}(P_1^{-1}P_0 S + \frac{d}{d\zeta}S)$ and $f(\zeta) = (S^{-1}\mathcal{H}x)(\zeta)$. Furthermore, the boundary condition $\tilde{W}_B \begin{bmatrix} \mathcal{H}(b)x(b,t) \\ \mathcal{H}(a)x(a,t) \end{bmatrix} = 0$ becomes $\tilde{W}_B \begin{bmatrix} (Sf)(b) \\ (Sf)(a) \end{bmatrix} = 0$, which finally yields the boundary condition:

$$W_b(Sf)(b) + W_a(Sf)(a) = 0, \tag{1.2.9}$$

where $W_b := W_1 P_1 + W_2$ and $W_a := -W_1 P_1 + W_2$ with $W_B := \begin{bmatrix} W_1 & W_2 \end{bmatrix}$. □

As already mentioned, the rational to prove that the eigenvectors of first order port-Hamiltonian systems form a Riesz basis relies on [Tre00, Theorem 3.11] as depicted in Figure 1.1. In connection with the eigenvalue problem (1.2.6), we define the operator A_f as

$$A_f f = A_1^{-1}\frac{d}{d\zeta}f - A_1^{-1}A_0 f = S^{-1}\mathcal{H}P_1\frac{d}{d\zeta}(Sf) + S^{-1}\mathcal{H}P_0(Sf) \tag{1.2.10}$$

with domain

$$D(A_f) = \{f \in L^2([a,b];\mathbb{R}^n) : Sf \in H^1([a,b];\mathbb{R}^n), W_b(Sf)(b) + W_a(Sf)(a) = 0\}. \tag{1.2.11}$$

The operator A_f is proved to be related to the original operator A through a bounded invertible operator S in Lemma 1.2.3.

Lemma 1.2.3. *Consider the operator A with domain $D(A)$ given by (1.1.6)-(1.1.7) and the operator A_f with domain $D(A_f)$ defined by (1.2.10)-(1.2.11). Then*

$$S^{-1}\mathcal{H}Ax = A_f S^{-1}\mathcal{H}x, \qquad x \in D(A), \tag{1.2.12}$$

and $x \in D(A)$ if and only if $f = S^{-1}\mathcal{H}x \in D(A_f)$. Moreover, the eigenvalues of A that are given by (1.2.6) are the same as those of the operator A_f. If λ is an eigenvalue of A_f with a corresponding eigenfunction $f \in D(A_f)$, then λ is an eigenvalue of A with eigenfunction $x = \mathcal{H}^{-1}Sf \in D(A)$.

Proof. The identity (1.2.12) is directly deduced from (1.1.6) and (1.2.10). Let x be in $D(A)$. Thus x must satisfy $W_b(\mathcal{H}x)(b) + W_a(\mathcal{H}x)(a) = 0$ and by setting $f = S^{-1}\mathcal{H}x$, we get that $f \in D(A_f)$ from (1.2.11). Let us now consider an eigenvalue $\lambda \in \sigma_p(A_f)$ with corresponding eigenfunction f. From the definition of A_f, it follows that

$$\lambda f = A_f f = S^{-1}\mathcal{H}P_1 \frac{d}{d\zeta}(Sf) + S^{-1}\mathcal{H}P_0(Sf)$$

and from the identity $Sf = \mathcal{H}x$ we get $\lambda x = Ax$. □

In order to apply [Tre00, Theorem 3.11], some technical assumptions are needed.

Assumption 1.2.2. *For $\nu \in \{1,...,l\}$, let us define $R_\nu(z) := \int_a^z r_\nu(\zeta)d\zeta$, where $(r_\nu(\zeta))_{\nu=1}^l$ are the l different eigenvalues of $P_1^{-1}\ \mathcal{H}^{-1}(\zeta)$ and $E_\nu(z,\lambda) := e^{\lambda R_\nu(z)} I_{n_\nu}$, where n_ν is the multiplicity of $r_\nu(\cdot)$ and I_{n_ν} denotes the n_ν-dimensional unit matrix such that $\sum_{\nu=1}^{l} n_\nu = n$. We set $E(z,\lambda) = diag(E_0(z,\lambda),\ldots,E_l(z,\lambda))$, $z \in [a,b]$. We shall assume that the eigenvalue problem (1.2.6) is normal, i.e., for sufficiently large λ, the asymptotic expansion of the characteristic determinant of (1.2.6) given by*

$$p(\lambda) = \sum_{c\in\mathcal{E}} (b_c + \{o(1)\}_\infty)e^{\lambda c} \tag{1.2.13}$$

has non-zero minimum and maximum coefficients, where

$$\mathcal{E} = \left\{ \sum_{\nu=1}^{l} \delta_\nu R_\nu(b) : \delta_\nu \in \{0,1\} \right\} \subset \mathbb{R} \tag{1.2.14}$$

and $\{o(1)\}_\infty$ means that for each $c \in \mathcal{E}$ the remaining part depending on $z \in [a,b]$ divided by λ tends to 0 in the uniform norm when $|\lambda| \to \infty$.

From [Vil07, Theorem 4.10], it follows that the non-zero coefficients are given by

$$\sum_{c\in\mathcal{E}} b_c e^{\lambda c} = \det(W_b S(b)\Phi_0(b)E(b,\lambda) + W_a S(b)), \tag{1.2.15}$$

where $\Phi_0 \in W^{1,\infty}([a,b];\mathbb{R}^{n\times n})$ is determined by

$$\begin{aligned} \Phi_0(\zeta)A_1 &= A_1\Phi_0(\zeta), & \Phi_0(a) &= I, \\ \frac{d\Phi_{0,\nu\nu}}{d\zeta} - A_{0,\nu\nu}\Phi_{0,\nu\nu} &= 0, & \nu &= 1,...,l \end{aligned} \tag{1.2.16}$$

where $A_{0,\nu\nu}$ and $\Phi_{0,\nu\nu}$ are the elements of the νth row and the νth column of A_0 and Φ_0 respectively. For more details, see [MM03, Section § 2.8].

Theorem 1.2.4. *Assume that the eigenvalue problem (1.2.6) is normal with $A_0, A_1 \in W^{2,\infty}([a,b];\mathbb{R}^{n\times n})$ and A_1 is a diagonal matrix with the eigenvalues of $P_1^{-1}\mathcal{H}^{-1}$ as diagonal elements with $\mathcal{H} \in W^{2,\infty}([a,b];\mathbb{R}^{n\times n})$. If the eigenvalues have a uniform gap, i.e., $\inf_{m\neq p} |\lambda_m - \lambda_p| > 0$, then there exists a sequence of eigenfunctions of A_f that forms a Riesz basis of $L^2([a,b];\mathbb{R}^n)$.*

Proof. Notice that the canonical system of eigenfunctions in [Tre00, Theorem 3.11] corresponds to the eigenfunctions of the operator A_f. In [Tre00, Theorem 3.11], the eigenvalue problem is also assumed to be non-degenerate, i.e. $\rho(T) \neq \emptyset$, where T is a linear pencil $T(\lambda) = T_0 - \lambda T_1$ of bounded operators T_0, $T_1 \in \mathcal{L}(H^1([a,b];\mathbb{K}^n), L^2([a,b];\mathbb{K}^n) \times \mathbb{C}^n)$ given by

$$T_0 f := \begin{bmatrix} \frac{d}{d\zeta} f - A_0 f \\ W_b(Sf)(b) + W_a(Sf)(a) \end{bmatrix}, \; T_1 f := \begin{bmatrix} A_1 f \\ 0 \end{bmatrix}.$$

From [Vil07, Theorem 4.2], we have that $\mathbb{C}^+ \subset \rho(T)$. Eventually, since the eigenvalue problem (1.2.6) is assumed to be normal and the eigenvalues of A_f are assumed to have a uniform gap, we conclude that there exists a sequence of eigenfunctions $(f_n)_{n\in\mathbb{N}}$ of A_f that forms a Riesz basis. □

Remark 1.2.2. In Theorem 1.2.4, the assumption $\inf_{m\neq p} |\lambda_m - \lambda_p| > 0$ comes from [Tre00, Theorem 3.11]. This assumption enables to deduce that $(f_n)_{n\in\mathbb{N}}$ forms a Riesz basis without parentheses. A vector sequence $(\phi_n)_{n\in\mathbb{N}}$ is said to be a Riesz basis for $\mathcal{X}$ with parentheses if (1.2.3) converges only after putting some of its terms in parentheses (whose arrangement is independent of x), see [Shk86].

1.3 Nice port-Hamiltonian systems are Riesz-spectral systems

As already stated, numerous physical models such as wave equations, traveling waves, the heat exchanger, the Timoshenko beam, and diffusive tubular reactors have been proved to be Riesz-spectral systems (see e.g. [CZ95], [Xu05] and [DDW03]), see Definition 1.3.1. Hence, in order to embed these particular facts into a general framework, the subclass of nice port-Hamiltonian systems is introduced. The latter is proved to satisfy the Riesz-spectral property. We start by defining the notion of Riesz-spectral systems.

Definition 1.3.1. *A dynamical system*

$$\dot{x}(t) = Ax(t), \qquad x(0) = x_0, \tag{1.3.1}$$

where $A : D(A) \subset \mathcal{X} \to \mathcal{X}$ is a linear operator on a Hilbert space $\mathcal{X}$ is said to be a Riesz-spectral system if it satisfies the following conditions:

1. *A is a Riesz-spectral operator;*
2. *A is the infinitesimal generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on $\mathcal{X}$.*

Definition 1.3.2. *A nice port-Hamiltonian system is a port-Hamiltonian system (according to Definition 1.1.1), which satisfies the condition $W_B \Sigma W_B^T \geq 0$ and Assumptions 1.2.1 and 1.2.2, and whose generator A given by (1.1.6) has a uniform gap of eigenvalues, i.e., $\inf_{m\neq p} |\lambda_m - \lambda_p| > 0$.*

Theorem 1.3.1. *Under the regularity assumptions on the matrix-valued functions A_0, A_1 and $\mathcal{H}$ in Theorem 1.2.4, any nice port-Hamiltonian system (1.1.1)-(1.1.3) is a Riesz-spectral system.*

Proof. First, since $W_B \Sigma W_B^T \geq 0$, A generates a contraction C_0-semigroup, see Theorem 1.1.1. It follows that A is closed.
Second, from [Vil07, Theorem 2.28], the resolvent operator of A is known to be compact, which implies that the spectrum $\sigma(A)$ is only made of eigenvalues of finite multiplicity such that $\sigma(A) = \sigma_p(A)$. Since we are counting the eigenvalues with multiplicity, the uniform gap between them entails that they are in fact all simple. It proves that the closure of the set of eigenvalues is totally disconnected.
Finally, from Theorem 1.2.4, it is known that the operator A_f has a Riesz basis of eigenvectors, and in view of Lemma 1.2.3, its eigenfunctions $(f_n)_{n\in\mathbb{N}}$ are isomorphic to the eigenfunctions $(x_n)_{n\in\mathbb{N}}$ of A. Indeed, $f_n = S^{-1}\mathcal{H}x_n$ for $n \in \mathbb{N}$. Therefore, the operator A has a Riesz basis of eigenfunctions. □

Let us recall the growth bound of a C_0-semigroup $(T(t))_{t\geq 0}$

$$\omega_0 = \inf_{t>0}\left(\frac{1}{t}\log\|T(t)\|\right) = \lim_{t\to\infty}\left(\frac{1}{t}\log\|T(t)\|\right). \tag{1.3.2}$$

Definition 1.3.3. *A C_0-semigroup $(T(t))_{t\geq 0}$ on a Hilbert space $\mathcal{X}$ is said to be exponentially stable if there exist positive constants M and α such that*

$$\|T(t)\| \leq Me^{-\alpha t}, \qquad t \geq 0.$$

Equivalently, the growth bound is negative, i.e., $\omega_0 < 0$.

Note that, in infinite dimension, there is a distinction between the spectral bound and the growth bound, i.e.

$$\omega_0 \leq \sup_{n\geq 1} \operatorname{Re}\lambda_n, \qquad \lambda \in \sigma_p(A), \tag{1.3.3}$$

which entails that the eigenvalues does not characterize the exponential stability anymore. To understand this issue better, the reader is referred to [CZ95]. As a straightforward consequence of Theorem 1.3.1 and [CZ95, Theorem 2.3.5], the inequality in (1.3.3) can be replaced by an equality for nice port-Hamiltonian systems. In this case, we say that the spectrum-determined growth condition is satisfied.

Corollary 1.3.2. *Under the regularity assumptions on the matrix-valued functions A_0, A_1 and $\mathcal{H}$ in Theorem 1.2.4, any nice port-Hamiltonian system (1.1.1)-(1.1.3) satisfies the spectrum-determined growth condition. Furthermore, if $\sigma_p(A) \subset \mathbb{C}_0^-$, then the system (1.1.1)-(1.1.3) is exponentially stable.*

Example 3.1. Let us get back to Example 1.1 of a vibrating string. Let us consider now the following boundary conditions:

$$T(b)\frac{\partial w}{\partial \zeta}(b,t) + \frac{\partial w}{\partial t}(b,t) = 0 \qquad \text{and} \qquad T(a)\frac{\partial w}{\partial \zeta}(a,t) = 0, \tag{1.3.4}$$

which means that the string is let free at the extremity a and is damped at the extremity b. This choice of boundary conditions is rewritten as

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} T(a)\frac{\partial w}{\partial \zeta}(a,t) \\ T(b)\frac{\partial w}{\partial \zeta}(b,t) + \frac{\partial w}{\partial t}(b,t) \end{bmatrix} = W_B \begin{bmatrix} f_\partial(t) \\ e_\partial(t) \end{bmatrix}, \tag{1.3.5}$$

where $W_B = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$ has rank 2 and $W_B \Sigma W_B^T = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} \geq 0$. The eigenvalues and eigenfunctions of $P_1^{-1}\mathcal{H}^{-1} := \begin{bmatrix} 0 & \frac{1}{T(\zeta)} \\ \rho(\zeta) & 0 \end{bmatrix}$ are given by

$$r(\zeta) = \pm\sqrt{\frac{\rho(\zeta)}{T(\zeta)}} \qquad \text{and} \qquad \begin{bmatrix} 1 \\ \sqrt{(T\rho)(\zeta)} \end{bmatrix}, \qquad \begin{bmatrix} 1 \\ -\sqrt{(T\rho)(\zeta)} \end{bmatrix}, \tag{1.3.6}$$

respectively. According to Assumption 1.2.2, $P_1^{-1}\mathcal{H}^{-1}$ has 2 different eigenvalues with

$$R_1(z) = \int_a^z \sqrt{\frac{\rho(\zeta)}{T(\zeta)}} d\zeta \qquad \text{and} \qquad R_2(z) = -\int_a^z \sqrt{\frac{\rho(\zeta)}{T(\zeta)}} d\zeta. \tag{1.3.7}$$

This yields

$$\begin{aligned} E(z,\lambda) &= \begin{bmatrix} \exp\left(\lambda \int_a^z \sqrt{\frac{\rho(\zeta)}{T(\zeta)}} d\zeta\right) & 0 \\ 0 & \exp\left(-\lambda \int_a^z \sqrt{\frac{\rho(\zeta)}{T(\zeta)}} d\zeta\right) \end{bmatrix} \\ &:= diag(E_1, E_2)(z,\lambda). \end{aligned} \tag{1.3.8}$$

The matrices W_a and W_b introduced in Lemma 1.2.2 are given by

$$W_a = \sqrt{2} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \text{ and } W_b = \sqrt{2} \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}. \tag{1.3.9}$$

We now verify the normality assumption of the eigenvalue problem for the vibrating string. Towards this end, let us compute the expression of the set $\mathcal{E}$ and express the asymptotic expansion of $p(\lambda)$ given by (1.2.13).
The set $\mathcal{E}$ is given by

$$\mathcal{E} = \{0, R_1(b), R_2(b), R_1(b) + R_2(b)\} = \left\{ \int_a^b \sqrt{\frac{\rho(\zeta)}{T(\zeta)}} d\zeta, -\int_a^b \sqrt{\frac{\rho(\zeta)}{T(\zeta)}} d\zeta, 0 \right\}. \tag{1.3.10}$$

From (1.2.15), the non-zero coefficients are given by

$$\begin{aligned} &\det(W_b S(b) \Phi_0(b) E(b,\lambda) + W_a S(b)) \\ &\quad = 2\sqrt{(T\rho)(b)}(1 + \sqrt{(T\rho)(b)}) E_1(b,\lambda) + 2\sqrt{(T\rho)(b)}(1 - \sqrt{(T\rho)(b)}) E_2(b,\lambda). \end{aligned} \tag{1.3.11}$$

As stated in Assumption 1.2.2, the system is said to be normal if (1.3.11) has non-zero minimum and maximum coefficients, i.e., $(1+\sqrt{(T\rho)(b)})$ and $(1-\sqrt{(T\rho)(b)})$ must be different from zero. This entails that the system is normal if and only if $1 \neq \sqrt{((T\rho)(b))}$.
The roots of the asymptotic expansion of the characteristic determinant (1.2.13) gives the eigenvalues of A. Developing the sum in (1.2.13) for each elements belonging to $\mathcal{E}$ given by (1.3.10) yields

$$\begin{aligned} p(\lambda) = \{o(1)\}_\infty + [2\sqrt{(T\rho)(b)}(1+\sqrt{(T\rho)(b)}) + \{o(1)\}_\infty]E_1(b,\lambda) \\ +[2\sqrt{(T\rho)(b)}(1-\sqrt{(T\rho)(b)}) + \{o(1)\}_\infty]E_2(b,\lambda) \end{aligned}. \tag{1.3.12}$$

The roots of (1.3.12) are approximated by (1.2.15). By Rouché's theorem, the eigenvalues are approximated as $\lambda_n = \lambda_n^* + o(\frac{1}{n})$ for $n \in \mathbb{N}$, where

$$\lambda_n^* = \begin{cases} \frac{-1}{2\int_a^b \sqrt{\frac{\rho(\zeta)}{T(\zeta)}}} \ln \left| \frac{\sqrt{(T\rho)(b)}+1}{\sqrt{(T\rho)(b)}-1} \right| - i \frac{\pi n}{\int_a^b \sqrt{\frac{\rho(\zeta)}{T(\zeta)}} d\zeta}, \\ \frac{-1}{2\int_a^b \sqrt{\frac{\rho(\zeta)}{T(\zeta)}}} \ln \left| \frac{\sqrt{(T\rho)(b)}+1}{\sqrt{(T\rho)(b)}-1} \right| - i \frac{\pi (2n+1)}{2\int_a^b \sqrt{\frac{\rho(\zeta)}{T(\zeta)}} d\zeta}, \end{cases}$$

if $\sqrt{(T\rho)(b)} > 1$ or $\sqrt{(T\rho)(b)} < 1$ respectively. Notice that the condition $1 \neq \sqrt{((T\rho)(b))}$ from the normality assumption of the system ensures the existence of the eigenvalues λ_n^* given by (3.1). When $\sqrt{(T\rho)(b)}$ tends to 1^+, the real part of λ_n^* goes to $-\infty$. Moreover, from Corollary 1.3.2 and since w_0 is given by

$$w_0 = \sup_{n\in\mathbb{N}} \operatorname{Re} \lambda_n = \frac{-1}{2\int_a^b \sqrt{\frac{\rho(\zeta)}{T(\zeta)}}} \ln \left| \frac{\sqrt{(T\rho)(b)}+1}{\sqrt{(T\rho)(b)}-1} \right| < 0, \tag{1.3.13}$$

the vibrating string described by (1.1.32) and (1.3.4) is exponentially stable.

1.4 Conclusion & perspectives

Most of the content presented in this chapter was published in [LW17a]. Since then, research is still ongoing in order to relax the conditions presented in Theorem 1.3.1. Indeed, the Riesz basis property exhibited in Theorem 1.3.1 requires strong assumptions on the eigenvalues of A such as a uniform gap of eigenvalues, which narrows the range of applications. For instance, the coupled vibrating string as described in [JZ12, Example 7.4] has Jordan blocks. A two-dimensional vibrating string does not satisfy the eigenvalue uniform gap assumption. This should be seen as a first attempt to study the Riesz-spectral property of first order port-Hamiltonian systems.

Note that by applying Tretter's result, no use is made of the existence of a contraction C_0-semigroup generated by A. Current researches on Riesz basis of port-Hamiltonian systems tend to go into that direction. See for instance [JK18], where the authors prove that, under some assumptions on matrices $P_1, P_0, \mathcal{H}, W_a$ and W_b, the operator A described by (1.1.6) and (1.1.7) generates a C_0-group if and only if A is a Riesz operator. Nevertheless, a complete characterization for port-Hamiltonian system having a Riesz basis still need to be found.

Moreover, a natural extension of the results devoted to the Riesz-spectral property would be to consider some internal dissipation within the port-Hamiltonian framework as introduced in [Vil07, Chapter 6] and to see whether the Riesz-spectral property still holds. Even though the answer seems intuitively positive, proving that the Riesz-basis property still holds in the presence of dissipative effects is not straightforward and remains a conjecture that would be appealing to prove.

Chapter 2

Stochastic calculus in Hilbert spaces

This chapter serves as an introduction to the basic concepts needed for the study of stochastic partial differential equations. Some notions of the abstract probability theory and some linear spaces are recalled, and Wiener processes are introduced in Section 2.2. A theory of integration is presented and adapted to the range of stochastic disturbances modelized by Wiener processes considered in this work. This chapter ends by stating the stochastic counterpart of Tonelli-Fubini Theorem and the so-called Itô's formula. This chapter will also help to fix the notations used in this thesis.

The different notions of solutions of infinite-dimensional stochastic differential equations are postponed to Chapter 3, where the questions of existence and uniqueness of the SPDE governing the class of stochastic port-Hamiltonian systems will be discussed. Moreover, to keep this chapter within a reasonable length, the proofs are omitted and can be found in standard books such as [DPZ14] and [MPBL14].

This Chapter is organized as follows. Section 2.1 introduces background materials for the study of stochastic processes on functional spaces. In Section 2.2, some of the best known stochastic processes, namely Wiener processes, are studied and some of their properties are presented. The stochastic integral with respect to a Wiener process is defined in Section 2.3. This chapter ends with some results and tools, which will turn out to be useful in the following chapters.

2.1 Preliminaries

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, where Ω is a nonempty set of elements ω, $\mathcal{F}$ denotes a σ-algebra of subsets of Ω and $\mathbb{P}$ is a probability measure. More precisely, the family $\mathcal{F}$ contains the set Ω and is closed when taking the complement and under countable unions of subsets of Ω. Any point $\omega \in \Omega$ is a sample or an experiment, $S \in \mathcal{F}$ is an event and $\mathbb{P}(S)$ represents the probability measure of the event S. As usual, an event $S \in \mathcal{F}$ is said to hold $\mathbb{P}$-almost surely ($\mathbb{P}$-a.s.) if $\mathbb{P}(S) = 1$. A probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is said to be complete if for all $A \subset \Omega$ such that there exists $B, C \in \mathcal{F}$ with $B \subset A \subset C$ and $P(B) = P(C)$, then $A \in \mathcal{F}$.

Let $\mathcal{X}$ and Z be separable Hilbert spaces. U is a $\mathcal{X}$-valued random variable if the map $G : \Omega \to \mathcal{X}$ is strongly measurable with respect to the probability measure $\mathbb{P}$, i.e.

$$G^{-1}(B) := \{\omega \in \Omega : G(\omega) \in B\} \in \mathcal{F} \text{ for all } B \in \mathbf{B}(\mathcal{X}),$$

where $\mathbf{B}(\mathcal{X})$ is a Borel σ-algebra on $\mathcal{X}$, i.e. the smallest σ-algebra containing the open and closed subsets of $\mathcal{X}$. The σ-algebra generated by the random variable G is given by

$$\sigma(G) = \left\{G^{-1}(B) : B \in \mathbf{B}(\mathcal{X})\right\} = \{\{G \in B\} : B \in \mathbf{B}(\mathcal{X})\}$$

and represents the smallest σ-algebra on Ω such that the random variable G is strongly measurable. The σ-algebra $\sigma(G)$ contains all the information about the random variable G. After observing the outcome of G, we can tell whether an event $A \subset \Omega$ has occurred or not. For an integrable (in the Bochner sense) random variable G, we define its expectation as

$$\mathbb{E}[G] = \int_\Omega G(\omega) d\mathbb{P}(\omega). \tag{2.1.1}$$

The most important distribution in probability theory is the Gaussian distribution. We now give the definition of a Gaussian random variable taking values in $\mathcal{X}$.

Definition 2.1.1. *Let G be a $\mathcal{X}$-valued random variable and $(e_i)_{i\in\mathbb{N}}$ be an orthonormal basis in $\mathcal{X}$. G is Gaussian iff for all $i \in \mathbb{N}$, $\langle G, e_i\rangle_{\mathcal{X}}$ is a real Gaussian random variable.*

Now we introduce the space of trace class operators $L_1(Z, \mathcal{X})$ and the space of Hilbert-Schmidt (H-S) operators $L_2(Z, \mathcal{X})$.

Definition 2.1.2. *A bounded linear operator $T : Z \to \mathcal{X}$ is said to be a trace class operator if there exist sequences $(a_i)_{i\in\mathbb{N}} \subset Z$ and $(b_i)_{i\in\mathbb{N}} \subset \mathcal{X}$ such that, for all $z \in Z$,*

$$Tz = \sum_{i=1}^{\infty} \langle z, a_i\rangle_Z b_i \in \mathcal{X}. \tag{2.1.2}$$

with

$$\sum_{i=1}^{\infty} \|a_i\|_Z \|b_i\|_{\mathcal{X}} < \infty. \tag{2.1.3}$$

Definition 2.1.3. *A bounded linear operator $T : Z \to \mathcal{X}$ is said to be a Hilbert-Schmidt operator if*

$$\sum_{i=1}^{\infty} \|Te_i\|_{\mathcal{X}}^2 < \infty, \tag{2.1.4}$$

where $(e_i)_{i\mathbb{N}}$ is an orthonormal basis of Z.

Let us introduce the following subspaces of bounded linear operators:

1. $L_1(Z,\mathcal{X})$ denotes the space of trace class operators endowed with the norm given by
$$\|T\|_{L_1} = \operatorname{Tr}\tilde{T} := \sum_{i=1}^{\infty} \langle \tilde{T} e_i, e_i \rangle_Z, \tag{2.1.5}$$
where $\tilde{T} = (T^*T)^{1/2}$.

2. $L_2(Z,\mathcal{X})$ denotes the space of H-S operators endowed with the norm given by
$$\|T\|_{L_2} := \left(\sum_{i=1}^{\infty} \|Te_i\|_{\mathcal{X}}^2 \right)^{1/2}. \tag{2.1.6}$$

Obviously,

$$L_2(Z,\mathcal{X}) \subset L_1(Z,\mathcal{X}) \subset \mathcal{L}(Z,\mathcal{X})$$

and if T is a self-adjoint operator, $\|T\|_{L_1} = \operatorname{Tr} T$.

2.2 Wiener processes

In what follows, we introduce some notions and definitions related to stochastic processes taking values in Hilbert spaces. To do so, we further assume that the probability space $(\Omega,\mathcal{F},\mathbb{P})$ is equipped with an increasing (in the sense of set inclusion) sequence of σ-algebras $(\mathcal{F}_t)_{t\geq 0}$. Such family of σ-algebras is called a filtration. The usual interpretation of a filtration is that each $\mathcal{F}_t$ contains all the information which is available up to time t.

Definition 2.2.1. *A filtration $(\mathcal{F}_t)_{t\geq 0}$ on a probability space $(\Omega,\mathcal{F},\mathbb{P})$ is said to be normal if the two following conditions hold:*

1. *$\mathcal{F}_0$ contains all the $\mathbb{P}$-null sets of $\mathcal{F}$;*

2. *$(\mathcal{F}_t)_{t\geq 0}$ is right-continuous, i.e. $\mathcal{F}_t = \mathcal{F}_{t^+} := \bigcap_{s\geq t} \mathcal{F}_s$, for all $t \geq 0$.*

Condition 1 means that the filtration contains all the negligible sets (relative to $\mathbb{P}$). Condition 2 implies that taking an infinitesimal step forward in time does not add any information.

A stochastic process $(\varepsilon(t))_{t\geq 0}$ ($\varepsilon(t)$ for short) is a random function of two variables: an experiment $\omega \in \Omega$ and a time argument $t \in [0,\infty)$. For each fixed experiment $\omega \in \Omega$, the map $\varepsilon(\omega,\cdot)$ is called a realisation or sample path of a stochastic process $(\varepsilon(t))_{t\geq 0}$. If we fix now the time argument t, we obtain a random variable $\varepsilon(\cdot,t)$. This corresponds to the naive interpretation of a stochastic process as a collection of random variables parametrized by the time argument.

Definition 2.2.2. *A Z-valued stochastic process $(\varepsilon(t))_{t\geq 0}$ is a measurable map from $\Omega \times [0,\infty)$ to Z with respect to the σ-algebra $\mathcal{F} \times \mathcal{B}([0,\infty))$.*

Observe that the definition of a Hilbert space valued stochastic process is more restrictive since such process has to be measurable in the product space $\mathcal{F} \times \mathcal{B}([0,\infty))$ and not only measurable with respect to $\mathcal{F}$ as for the real case, see [Doo90]. Let us now recall the definition of a Gaussian process.

Definition 2.2.3. *A stochastic process $(\varepsilon(t))_{t\geq 0}$ is said to be Gaussian if for any $n \in \mathbb{N}$ and for any finite partition $0 \leq t_1 \leq \ldots \leq t_n < \infty$, the Z^n-valued random variable $(\varepsilon(t_1),\ldots,\varepsilon(t_n))$ is Gaussian.*

We now define the covariance and the correlation operators.

Definition 2.2.4. *Let Z be a separable Hilbert space with the inner product $\langle\cdot,\cdot\rangle_Z$ and $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space. If the two stochastic processes $(X(t))_{t\geq 0}$, $(Y(t))_{t\geq 0} \in L^2((\Omega,\mathcal{F},\mathbb{P});Z)$, then for any $t \geq 0$ the covariance operator of $X(t)$ and the correlation operator of $X(t)$ and $Y(t)$ are given by the formulae*

$$Cov(X(t)) = \mathbb{E}\left[(X(t)-\mathbb{E}(X(t)))\circ(X(t)-\mathbb{E}(X(t)))\right], \tag{2.2.1}$$

and,

$$Cor(X(t),Y(t)) = \mathbb{E}\left[(X(t)-\mathbb{E}(X(t)))\circ(Y(t)-\mathbb{E}(Y(t)))\right], \tag{2.2.2}$$

respectively, where $X \circ Y \in \mathcal{L}(Z)$ is defined by $(X\circ Y)h = X\langle Y,h\rangle_Z$ for any $h \in Z$.

Definition 2.2.5. *A stochastic process $(\varepsilon(t))_{t\geq 0}$ is stationary if*

1. *for all $t,s \geq 0$, $m(t+r) = m(t)$ and*
2. *for all $t,s,r \geq 0$, $R(t+r,s+r) = R(t,s)$,*

where $m(t) = \mathbb{E}(\varepsilon(t))$ and $R(t,s) = \mathbb{E}((\varepsilon(t)-m(t))\circ(\varepsilon(s)-m(s)))$.

Definition 2.2.6. *A stochastic process $(\varepsilon(t))_{t\geq 0}$ is said to have continuous sample paths almost surely (or continuous) if the mapping $t \mapsto \varepsilon(\omega,t)$ is continuous for almost every $\omega \in \Omega$.*

Definition 2.2.7. *A stochastic process $(\varepsilon(t))_{t\geq 0}$ is said to be $\mathbb{F}$-adapted if $\varepsilon(t)$ is $\mathcal{F}_t$-measurable for all $t \geq 0$.*

This concept is defined similarly on any interval $I \subset \mathbb{R}^+$.

Definition 2.2.8. *Consider a filtration $(\mathcal{F}_t)_{t\geq 0}$ on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a separable Hilbert space Z. A Z-valued stochastic process $(M_t)_{t\geq 0}$ is called a $\mathbb{F}$-martingale on an interval $I \subset \mathbb{R}^+$ if the following conditions hold:*

1. *$(M_t)_{t\in I}$ is $(\mathcal{F}_t)_{t\in I}$-adapted;*
2. *$\mathbb{E}[M_t] < \infty$ for all $t \in I$;*
3. *$\mathbb{E}[M_t \mid \mathcal{F}_s] = M_s$ for all $s,t \in I$ such that $0 \leq s \leq t$,*

where $\mathbb{E}[\cdot \mid \mathcal{F}_s]$ denotes the conditional expectation with respect to the σ-algebra $\mathcal{F}_s$.

A large class of stochastic processes can be assumed to be represented by white noises, which has the nice property to generate uncorrelated shocks. A white noise process $e(t)$ is defined by a stationary stochastic process whose autocorrelation operator is a Dirac distribution, i.e., for any $g, h \in Z$

$$\mathbb{E}\langle e(t), g\rangle = 0 \qquad \text{and} \qquad \mathbb{E}\langle e(t), g\rangle\langle e(s), h\rangle = \delta(t-s)\langle Qg, h\rangle, \tag{2.2.3}$$

where $0 \leq s \leq t$, Q is a covariance operator. One usually models a white noise process as the formal derivative (in the sense of distributions, see [Sch57]) of a Wiener process. Note that the white noise process has mean zero. In this case, we usually say that the stochastic process is centred.

Definition 2.2.9. *Let Z be a separable Hilbert space. A Z-valued stochastic process $(w(t))_{t\geq 0}$ is a Wiener process if it satisfies the following conditions:*

1. *$w(0) = 0$, $\mathbb{P}$-almost surely;*
2. *The trajectories $w(t)$ with $t \geq 0$ are continuous, that is, the mapping $t \mapsto w(t, \omega)$ is continuous for almost every $\omega \in \Omega$;*
3. *$(w(t))_{t\geq 0}$ has independent increments, that is, for any finite partition $0 = t_0 \leq \ldots \leq t_f < \infty$, the random variables $w(t_1)$, $w(t_2) - w(t_1), \ldots, w(t_n) - w(t_{n-1})$ are (jointly) independents ;*
4. *For any $0 \leq s \leq t$, the random variables $w(t) - w(s)$ are normally distributed with mean 0 and variance $(t-s)Q$ (i.e.$w(t) - w(s) \sim \mathcal{N}(0, (t-s)Q)$).*

Moreover, if we further assume that

- $w(t)$ is adapted to $\mathcal{F}_t$ for all $t \geq 0$ and that
- $w(t) - w(s)$ is independent of $\mathcal{F}_s$ for all $0 \leq s \leq t$,

then $w(t)$ is said to be a Wiener process adapted to the filtration $(\mathcal{F}_t)_{t\geq 0}$. The covariance operator Q represents the increments of $w(t)$. It is a self-adjoint nonnegative trace class operator that characterizes the distribution of $w(t)$. For each u in Z with the inner product $\langle\cdot,\cdot\rangle_Z$, $(\langle w(t), u\rangle_Z)_{t\geq 0}$ is a real-valued Wiener process.

Remark 2.2.1. 1. In the literature it is common to call a Hilbert space-valued Wiener process with covariance operator Q, a Q-Wiener process. Nevertheless, for the sake of simplicity and when this is clear from the context, we shall simply use the term Wiener process.

2. Note that in Definition 2.2.9, the assumption of a trace class covariance operator is taken. This choice is required for the definition of well-posed stochastic systems given in Section 3.4. The trace class assumption means that the noise is colored with respect to the space variable. As a matter of fact, in this case, the heterogeneity caused by a finite trace operator induces some correlations in the spatial domain, on which the noise process is operating. For instance, it excludes the choice Q as the identity operator. Throughout this thesis, we restrict ourselves to Q-Wiener processes, but it is also of interest to consider the so-called cyclindrical Wiener process with covariance operator Q such that $\mathrm{Tr}[Q] = \infty$. The definition of a stochastic integral given in Section 2.3 requires the assumption that Q is a trace class operator, but can easily be extended to the case of cylindrical Wiener processes. Further details on cylindrical Wiener processes are available in [DPZ14, Section 4.2.1].

In analogy to the Karhunen-Loève expansion [DPZ14, Proposition 4.3], a Wiener process can be represented as an expansion in the eigenvectors of Q, which is given in the following proposition.

Proposition 2.2.1. *If $(w(t))_{t\geq 0}$ is a Wiener process, then there exists a complete orthonormal basis $(v_i)_{i\in\mathbb{N}}$ of Z, such that*

$$w(t) = \sum_{i=1}^{\infty} \beta_i(t) v_i, \tag{2.2.4}$$

where $(\beta_i(t))_{i\in\mathbb{N}}$ is a sequence of real independent Wiener processes with increments $(q_i)_{i\in\mathbb{N}}$ such that the series $\sum_{i=1}^{\infty} q_i$ is convergent in Z.

An approximation of the sample paths of a Q-Wiener process based on the Karhunen-Loève expansion is represented in Figure 2.1. One can observe that the sample paths of the approximated Wiener process are distributed along the spatial domain $[0,1]$. In addition, notice that the condition 4 of Definition 2.2.9 is recovered since we can observe that the standard deviation of the stochastic process is a function of $\sqrt{t}$. See [LPS14, Chapter 10] for further details.

Remark 2.2.2. Observe that there always exists a normal filtration $\mathbb{F}$ (see Definition 2.2.1) to which a Wiener process is adapted. Indeed, let us consider all $\mathbb{P}$-null sets

$$\mathcal{M} := \{A \in \mathcal{F} : \mathbb{P}(A) = 0\}$$

and let us define

$$\mathcal{F}_t^W = \sigma(w(s) : s \in [0,t]) \subset \mathcal{F}_t, \qquad \forall t \geq 0,$$

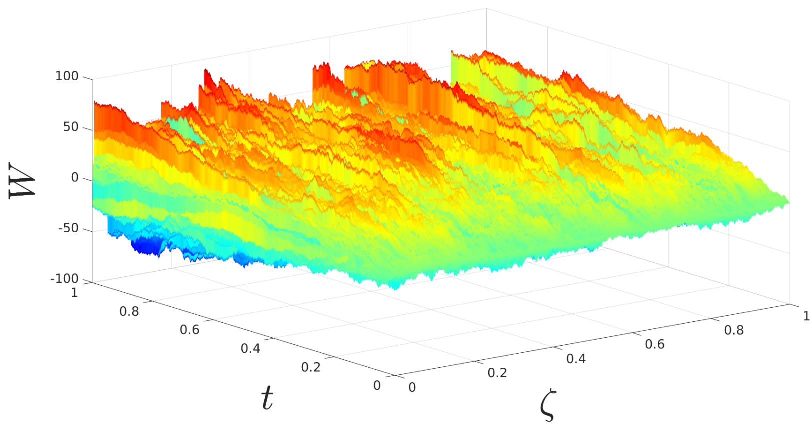

Figure 2.1 – Wiener process: $\Delta t = 0.001$ and $\Delta\zeta = 0.001$

as the filtration generated by $w(t)$, which contains all sets of the form

$$\{A \in \mathcal{F} : w(s)A \in \mathbf{B}(Z), 0 \leq s \leq t\}, \qquad \forall t \geq 0.$$

The augmented filtration given by

$$\hat{\mathcal{F}}_t^W = \bigcap_{r>t} \sigma(\mathcal{M} \cup \mathcal{F}_r^W) \tag{2.2.5}$$

is right-continuous and $w(t)$ is still adapted to $\mathbb{F} := \hat{\mathcal{F}}_t^W$. In the sequel, the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ will be considered, wherein $\mathbb{F}$ defined by (2.2.5) is normal, i.e., it satisfies the assumptions of completeness and right continuity.

Let us conclude this section by introducing some functional spaces, which will turn out to be useful afterwards:

$$L^2_{\mathbb{F}}([0,T]; L^2(\Omega;\mathcal{X})) := \{\varepsilon : \Omega \times [0,T] \to \mathcal{X} : \varepsilon(\cdot) \text{ is } \mathbb{F}-\text{adapted and } \mathbb{E}\int_0^T \|\varepsilon(s)\|^2_{\mathcal{X}} ds < \infty\}$$

endowed with the norm $\|\varepsilon\|^2_{L^2_{\mathbb{F}}([0,T]\times\Omega;\mathcal{X})} := \mathbb{E}\int_0^T \|\varepsilon(s)\|^2_{\mathcal{X}} ds$,

$$C^2_{\mathbb{F}}([0,T]; L^2(\Omega;\mathcal{X})) := \{\varepsilon : \Omega \times [0,T] \to \mathcal{X} : \varepsilon(\cdot) \text{ is } \mathbb{F}-\text{adapted and } \mathbb{E}\|\varepsilon(s)\|^2_{\mathcal{X}}, \mathbb{E}\|\dot{\varepsilon}(s)\|^2_{\mathcal{X}} \text{ are continuous}\}$$

endowed with the norm $\|\varepsilon\|^2_{C^2_{\mathbb{F}}([0,T];L^2(\Omega;\mathcal{X}))} := \sup_{t\in[0,T]} \mathbb{E}\|\varepsilon(t)\|^2_{\mathcal{X}}$ and

$$M^2_{\mathbb{F}}([0,T];Z) := \{M : [0,T] \to Z : M(\cdot) \text{ is a continuous } \mathbb{F}-\text{adapted martingale}, M(0) = 0 \text{ and } \sup_{t\in[0,T]} \mathbb{E}\|M(t)\|^2_Z < \infty\}$$

endowed with norm $\|M\|^2_{M^2_T} := \sup_{t\in[0,T]} \mathbb{E}\|M(t)\|^2_Z$, which is a Banach space. In order to simplify the notations, we denote the first two spaces by $L^2_{\mathbb{F}}([0,T];\mathcal{X})$ and $C^2_{\mathbb{F}}([0,T];\mathcal{X})$, respectively.

Remark 2.2.3. Completness of the probability space $(\Omega,\mathcal{F},\mathbb{F},\mathbb{P})$ and normality of the filtration $\mathbb{F}$ will be assumed to hold throughout this thesis. The theory of stochastic integration is usually developed under these assumptions for convenience. In [DM11], the authors stress the impact on general results of probability theory when these assumptions are suspended.

2.3 Stochastic integration

This section is devoted to a short introduction to the theory of stochastic integration with respect to Wiener processes in Hilbert spaces. The theory developed herein is for stochastic integrals with respect to a Hilbert-space valued Wiener process, which are also called Itô integrals. Notice that we restrict ourselves to the case of non-random integrands. Most of this section is based on [DPZ14, Chapter 4], and [Cho14]. For further reading, we refer to these books.

Let $(\Omega,\mathcal{F},\mathbb{P})$ be a complete filtered probability space with a normal filtration $\mathbb{F} := (\mathcal{F}_t)_{t\geq 0}$. Let $\mathcal{X}$ and Z be separable Hilbert spaces and let us consider a Z-valued Wiener process $(w(t))_{t\geq 0}$ with covariance operator Q as described in Definition 2.2.9. Here, the operator $Q \in \mathcal{L}(Z)$ is assumed to be self-adjoint, nonnegative and to satisfy $\mathrm{Tr}[Q] < \infty$, where Tr denotes the trace operator of Q defined by (2.1.5). Denote by Z_0 the range of the space Z by the square root of the covariance operator: $Z_0 := Q^{1/2}(Z)$, which is a subspace of Z equipped with the norm $\|\cdot\|_0$ associated with the inner product

$$\langle u,v\rangle_0 = \langle Q^{-1/2}u, Q^{-1/2}v\rangle_Z, \qquad u,v \in Z_0,$$

where $Q^{-1/2}$ denotes the pseudo-inverse of $Q^{1/2}$ defined as

$$(Q^{1/2})^{-1}y := \arg\min\left\{\|z\|_Z : z \in Z, Q^{1/2}z = y\right\} \text{ for all } y \in \text{ Ran } Q^{1/2}.$$

The range $Z_0 := Q^{1/2}(Z)$ is called the reproducing kernel or Cameron-Martin space of the process $(w(t))_{t\geq 0}$.

Proposition 2.3.1. *The space of Hilbert-Schmidt operators $L^0_2 := L_2(Z_0,\mathcal{X})$ equipped with the norm*

$$\|T\|^2_{L^0_2} = \|TQ^{1/2}\|^2_{L_2(Z,\mathcal{X})} = Tr[TQ^{1/2}(TQ^{1/2})^*] = Tr[TQT^*] \tag{2.3.1}$$

for any $T \in L^0_2$ is a separable Hilbert space.

Proof. For the proof, we refer to [Kuo75, Theorem 1.3]. □

We can now turn our attention to the stochastic (Itô) integral definition. As proved by Dvoretski, Erdos and Kakutani, real-valued Wiener processes $(\langle w(t), z\rangle)_{t\geq 0}$ for each $z \in Z$ are nowhere differentiable, see [Bre68, Theorem 12.25]. Hence, an integral with respect to a Wiener process cannot be defined by pathwise integration in the Lebesgue-Stieltjes approach by considering ω as a parameter, even with a deterministic integrand.

First let $f(t)$ be a $\mathcal{L}(Z,\mathcal{X})$-valued step function with $t \in [0,T]$. $f(t)$ is a step function if there exist a partition $0 = t_0 < t_1 < ... < t_n = T$ and $\mathcal{L}(Z,\mathcal{X})$-valued functions $\{f_i\}_{i=0}^{n-1}$ strongly measurable such that

$$f(t) = \sum_{i=0}^{n-1} f_i \mathbb{1}_{[t_i,t_{i+1}]}(t), \qquad t \in [0,T]. \tag{2.3.2}$$

Then the stochastic integral of the step function $f(t)$ with respect to $(w(t))_{t\geq 0}$ is given by

$$I_T(f) = \int_0^T f(t)dw(t) = \sum_{i=0}^{n-1} f_i[w(t_{i+1}) - w(t_i)] \in \mathcal{X}, \tag{2.3.3}$$

which also leads to

$$I_t(f) = \int_0^t f(s)dw(s) = \sum_{i=0}^{n-1} f_i[w(t_{i+1}\wedge t) - w(t_i \wedge t)] = \sum_{i=0}^{n-1} f_i[\Delta w_i(t)], \qquad t \in [0,T], \tag{2.3.4}$$

where $t_i \wedge t := \min(t_i,t)$. We denote the linear space of step functions by E.

Theorem 2.3.2. *Consider f as a $\mathcal{L}(Z,\mathcal{X})$-valued step function such that $\int_0^T \|f(t)\|^2_{L_2^0} dt < \infty$. The stochastic integral $I(f)$ defined by (2.3.4) is a continuous, square integrable $\mathcal{X}$-valued martingale on $[0,T]$ and the following holds:*

1. $\mathbb{E}\int_0^t f(s)dw(s) = 0$;
2. $\mathbb{E}\|\int_0^t f(s)dw(s)\|^2_{\mathcal{X}} = \int_0^t Tr\left[f(s)Q^{1/2}(f(s)Q^{1/2})^*\right] ds, \qquad t \in [0,T]$.

Proof. For the proof, we refer to [DPZ14, Proposition 4.20] or [Cho14, Lemma 3.2]. □

In the literature, the identity 2 is called the Itô's isometry. It shows that the stochastic integral is an isometry transformation from the space of $\mathcal{L}(Z,\mathcal{X})$-valued step functions to the space of $\mathcal{X}$-valued square integrable martingales denoted by $\mathcal{M}_T^2$.

Corollary 2.3.3. *Let us consider $\mathcal{L}(Z,\mathcal{X})$-valued step functions f_1 and f_2 such that for any $t_1,t_2 \in [0,T]$, $\int_0^{t_1\wedge t_2} \langle f_1(s), f_2(s)\rangle_{L_2^0} ds < \infty$. Then*

$$\mathbb{E}\left[\langle \int_0^{t_1} f_1(s)dw(s), \int_0^{t_2} f_2(s)dw(s)\rangle_{\mathcal{X}}\right] = \left[\int_0^{t_1\wedge t_2} Tr[f_1(s)Qf_2(s)^*]\, ds\right]. \tag{2.3.5}$$

Proof. It is a direct consequence of the Itô's isometry. □

Since $\mathcal{M}_T^2$ is a Banach space, we may extend the isometry $I(f) : E \to \mathcal{M}_T^2$ to the abstract completion of E. This extension remains isometric and is unique. This will be done in the subsequent step.

The next step is to extend the definition of the stochastic integral to a larger class of integrands taking values in L_2^0 such that $\int_0^T ||f(s)||^2_{L_2^0} ds < \infty$. Since the step functions are dense in

$$\mathcal{N}_w^2([0,T];L_2^0) = \left\{ f : [0,T] \to L_2^0 : \int_0^T ||f(s)||^2_{L_2^0} ds < \infty \right\},$$

for any $f \in L_2([0,T];\mathcal{L}(Z_0,\mathcal{X}))$ there exists a sequence of step functions f_n such that $\lim_{n\to\infty} \int_0^T ||f(s) - f_n(s)||^2_{L_2^0} ds = 0$, see [DPZ14, Proposition 4.22]. As a consequence, the definition of the stochastic integral can be extended to any function $f \in \mathcal{N}_w^2([0,T];L_2^0)$.
The stochastic integral of $f \in \mathcal{N}_w^2([0,T];L_2^0)$ with respect to a Wiener process $w(t)$ is then defined by

$$\int_0^t f(s)dw(s) := \lim_{n\to\infty} \int_0^t f_n(s)dw(s), \qquad t \in [0,T]. \tag{2.3.6}$$

Theorem 2.3.2 and Corollary 2.3.3 still hold for the stochastic integral of $\mathcal{N}_w^2([0,T];L_2^0)$ integrands. This is summarized in the following theorem.

Theorem 2.3.4. *Let $f \in \mathcal{N}_w^2([0,T];L_2^0)$. Then the stochastic integral defined by (2.3.6) is a continuous, square integrable $\mathcal{X}$-valued martingale on $[0,T]$, and*

1. $\mathbb{E}\int_0^T f(s)dw(s) = 0$;
2. $$\begin{aligned}\mathbb{E}\|\int_0^T f(s)dw(s)\|^2_{\mathcal{X}} &= \int_0^T Tr[(f(s)Q^{1/2})(f(s)Q^{1/2})^*]ds \\ &= \int_0^T ||f(s)||^2_{L_2^0} ds \le Tr[Q]\int_0^T \|f(s)\|^2_{L_2} ds.\end{aligned}$$

Proof. This can be shown by applying Theorem 2.3.2 via a sequence of simple functions. For further details, we refer to [DPZ14, P. 98]. □

Several results from the Bochner integration have their natural counterparts in stochastic integration.

Lemma 2.3.5. *[DPZ14, Proposition 4.30]*
Consider a family of bounded linear operators $(f(s))_{s\geq 0}$ on the Hilbert space Z and let $A : D(A) \subset \mathcal{X} \to \mathcal{X}$ be a closed linear operator. If $f(s)Q^{1/2}(Z) \subset D(A)$ for all $s \in [0,t]$ and if the following conditions hold:

$$\int_0^t ||f(s)||^2_{L_2^0} ds < \infty \tag{2.3.7}$$

and

$$\int_0^t ||Af(s)||^2_{L_2^0} ds < \infty, \tag{2.3.8}$$

then $\int_0^t f(s)dw(s) \in D(A)$ *and* $A\int_0^t f(s)dw(s) = \int_0^t Af(s)dw(s)$ $\mathbb{P}$*-a.s.*

For further details, we refer to [DPZ14, Section 4.2].
In order to study port-Hamiltonian systems driven by additive noises, we need to define a stochastic integral of the form

$$\int_0^T S(t,s)f(s)dw(s), \tag{2.3.9}$$

where $S : [0,T] \times [0,T] \to \mathcal{L}(\mathcal{X})$ is bounded and strongly continuous for $s,t \in [0,T]$. The special case $S(t,s) = S(t-s)$ is of great importance and is called the convolutional stochastic integral.

Theorem 2.3.6. *Consider a C_0-semigroup $(S(t))_{t\geq 0}$ whose the operator A is the infinitesimal generator. If*

$$\int_0^T \|S(s)f(s)\|^2_{L_2^0} ds = \int_0^T Tr[S(s)f(s)Q(S(s)f(s))^*]\, ds < \infty,$$

then the process $W_A(t) := \int_0^t S(t-s)f(s)dw(s) \in C([0,T];L^2(\Omega;\mathcal{X}))$ is a Gaussian process with covariance

$$Cov(W_A(T)) = \int_0^T [S(T-s)f(s)Q(S(T-s)f(s))^*]\, ds. \tag{2.3.10}$$

2.4 Some useful tools

In this section we state two important theorems related to stochastic integrals. We first present the stochastic counterpart of Fubini's theorem, taken from [DPZ14, Theorem 4.33]. Let F be a map from $[0,T] \times [0,T] \to \mathcal{L}(Z,\mathcal{X})$. $F(s,t) \in \mathcal{L}(Z,\mathcal{X})$ is said to be strongly measurable if $F(s,t)z$ is measurable for all $z \in Z$ and we denote $\mathcal{B}(\mathcal{L}(Z,\mathcal{X}))$ as the smallest σ-algebra of subsets of $\mathcal{L}(Z,\mathcal{X})$, $F^{-1}(A) = \{F \in \mathcal{L}(Z,\mathcal{X}) : Fz \in A\}$ for all $A \in \mathcal{B}(\mathcal{X})$.

Theorem 2.4.1. *Let Z and $\mathcal{X}$ be separable Hilbert spaces. If the map $F : [0,T] \times [0,T] \to \mathcal{L}(Z,\mathcal{X})$ is strongly measurable and $\int_0^T \int_0^T \|F(s,t)\|^2_{\mathcal{L}(Z,\mathcal{X})} dsdt < \infty$, then*

$$\int_0^T \int_0^T F(s,t)dw(s)dt = \int_0^T \int_0^T F(s,t)dtdw(s). \tag{2.4.1}$$

Another important tool that is worth mentioning is Itô's formula, see [DPZ14, Theorem 4.32]. This formula will play a key role in the seeking of an energy equality for stochastic port-Hamiltonian systems.

Theorem 2.4.2. *Let $\phi(s)$ be a $\mathcal{X}$-valued, Bochner integrable mapping on $[0,T]$, $H \in L_2^0$, and let X_0 be an $\mathcal{F}_0$-measurable, $\mathcal{X}$-valued random variable. Then*

$$X(t) := X_0 + \int_0^t \phi(s)ds + \int_0^t H dw(s), \qquad t \in [0,T]$$

is a well-defined stochastic process. Let $f : [0,T] \times X \to \mathbb{R}$ be a continuous function satisfying:

1. *$f(t,x)$ is differentiable in t and $f_t'(t,x)$ is continuous on $[0,T] \times \mathcal{X}$;*
2. *$f(t,x)$ is twice Fréchet differentiable in x, $f_x'(t,x) \in \mathcal{X}$ and $f_{xx}''(t,x) \in \mathcal{L}(\mathcal{X})$ are continuous on $[0,T] \times \mathcal{X}$.*

Then $\mathbb{P}$-almost surely, for all $t \in [0,T]$

$$\begin{aligned} f(t,X(t)) = f(0,X(0)) + \int_0^t &\langle f_x'(s,X(s)), H dw(s)\rangle_{\mathcal{X}} \\ + \int_0^t &\langle f_t'(s,X(s)) + f_x'(s,X(s)), \phi(s)\rangle_{\mathcal{X}} \\ + \frac{1}{2} Tr &\left[f_{xx}''(s,X(s))(HQ^{1/2})(HQ^{1/2})^* \right] ds. \end{aligned} \tag{2.4.2}$$

Chapter 3

Stochastic port-Hamiltonian systems on infinite-dimensional spaces

Over the last decade, the stochastic counterpart of port-Hamiltonian systems has begun to attract more consideration. This is motivated by the presence of uncertainties and external random fluctuations for dynamical systems operating in random environments. Examples of disturbances are wind gusts, environment turbulences, unpredictable fluctuations in the line voltage, fluctuations of the environment temperature or reaction parameters uncertainty. In order to capture the stochastic nature of these neglected effects, the class of stochastic port-Hamiltonian systems is introduced.

The concept of stochastic Hamiltonian systems was first introduced in [LCO08] on Poisson manifolds. In that paper, conserved quantities and underlying geometric features are characterized. On finite-dimensional spaces, the class of nonlinear time-varying stochastic port-Hamiltonian systems was introduced in [SF13] as the stochastic extension of [MvdS92]. In [SF13], Satoh and Fujimoto depicted the performance degradation and the possible nonstabilization of control systems resulting from stochastic disturbances by considering the problem of controlling a rolling coin on a horizontal plane. Besides, a passivity-based stabilization method via a stochastic generalized canonical transformation is proposed. Afterwards, this method was improved to encompass a wider range of stochastic disturbances, notably input noise, and was applied to stabilize continuous stirred tank reactor processes with stochastic phenomena in [FG17]. Further stabilization methods for finite-dimensional stochastic port-Hamiltonian systems can be found in [SS14] and [Sat17]. More recently, in [HRJ18], a stochastic extension of the passivity-based control framework as proposed in [OvdSME02] and [OvdSME99] was developed for nonlinear stochastic port-

Hamiltonian systems.
So far, most of the literature devoted to stochastic port-Hamiltonian systems (SPHSs) sticks to euclidean state spaces. Recently, a stochastic extension of linear infinite-dimensional first order port-Hamiltonian systems [LZM05] was proposed in [LW17b].

The main purposes of this chapter are to introduce the new class of boundary controlled and observed stochastic port-Hamiltonian systems governed by Itô stochastic differential equations (SDEs) on functional spaces and to study some properties of this new class of stochastic systems. The Sections 3.1, 3.2 and 3.3 are based on the results presented in [LW17b].

This chapter is organized as follows. In Section 3.1, stochastic port-Hamiltonian systems with boundary control and observation are introduced. In Section 3.2, the existence of mild and strong solutions is established with a similar approach as in [CP78]. Next, the passivity concept is extended to infinite-dimensional stochastic systems and is investigated for SPHSs. Section 3.4 is devoted to the extension of the well-posedness concept to boundary controlled and observed stochastic systems. To be more specific, we shall focus on the well-posedness of SPHSs and the verification of this property for this specific class of stochastic systems that is considered here. To conclude this chapter, the theory will be illustrated on an example of a vibrating string subject to some external random force modelized by a space and time Gaussian white noise process.

3.1 Stochastic port-Hamiltonian systems

Let $(\Omega,\mathcal{F},\mathbb{F},\mathbb{P})$ be a complete filtered probability space, wherein $\mathbb{F}$ is a normal filtration. To emphasise the distinction between deterministic and stochastic port-Hamiltonian systems, we shall denote the state as $x(t)$ and $\varepsilon(t)$ for deterministic and stochastic versions, respectively. The state is denoted as a space and time dependent stochastic process $\varepsilon(\zeta,t)$ on the state space $\mathcal{X} := L^2([a,b];\mathbb{R}^n)$. The class of first order linear stochastic port-Hamiltonian systems is governed by the following form of stochastic partial differential equation (SPDE for short) of the form

$$\frac{\partial \varepsilon}{\partial t}(\zeta,t) = P_1 \frac{\partial}{\partial \zeta}(\mathcal{H}(\zeta)\varepsilon(\zeta,t)) + P_0\mathcal{H}(\zeta)\varepsilon(\zeta,t) + (H\eta(t))(\zeta), \tag{3.1.1}$$

where $P_1 \in \mathbb{R}^{n\times n}$ is invertible and symmetric ($P_1^T = P_1$), $P_0 \in \mathbb{R}^{n\times n}$ is skew-symmetric ($P_0^T = -P_0$) and $\mathcal{H} \in L^\infty([a,b];\mathbb{R}^{n\times n})$ is self-adjoint and satisfies $mI \le \mathcal{H}(\zeta) \le MI$ for all $\zeta \in [a,b]$, for some constants m, $M > 0$. The system's noise $\eta : \Omega \times [0,T] \to Z$ is a Gaussian white noise process taking values in a Hilbert space Z with intensity $H \in \mathcal{L}(Z,\mathcal{X})$ and covariance $Q \in \mathcal{L}(Z)$.

The class of stochastic port-Hamiltonian systems (SPHSs) is interacting with its environment by means of ports located at the boundary of the spatial domain. The class of system under consideration is assumed to be controlled at the boundary (boundary control) and on small intervals inside the spatial domain (distributed control). This entails that energy exchanges occur either at the boundary and/or inside the domain.

Based on Definition 0.3.1, let us explicit the class of boundary controlled and observed (BCO for short) stochastic systems. This class of stochastic systems is described by equations of the form:

$$\begin{aligned} d\varepsilon(t) &= (\mathcal{A}\varepsilon(t) + B_d u_d(t))dt + H dw(t), \qquad \varepsilon(0) = \varepsilon_0, \\ u(t) &= \mathcal{B}\varepsilon(t), \\ y(t) &= \mathcal{C}\varepsilon(t), \\ y_d(t) &= B_d^* \mathcal{H}\varepsilon(t), \end{aligned} \tag{3.1.2}$$

where $\varepsilon(t) \in \mathcal{X}$, $u_d(t) \in \mathbb{R}^k$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^p, y_d(t) \in \mathbb{R}^k$. Let us recall that the distributed control operator $B_d \in \mathcal{L}(\mathbb{R}^k, \mathcal{X})$ represents the action of the inputs $u_d(t)$ on a spatial domain and $y_d(t)$ is the corresponding power-conjugated output (with respect to inner product $\langle \cdot, \cdot \rangle_{\mathcal{X}}$). The boundary control operator $\mathcal{B}$ represents the action of the inputs $u(t)$ pointwisely only at the boundary of the spatial domain. As already stated, the operators $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are assumed to be unbounded linear operators on the state space $\mathcal{X}$.

Definition 3.1.1. *A BCO stochastic system is a stochastic system described by (3.1.2) which satisfies the conditions of Definition 0.3.1.*

To give a complete characterization of the considered class of SPHSs, some boundary controlled and homogeneous conditions are added to the SPDE (3.1.1) as follows:

$$u(t) = W_{B,1} \begin{bmatrix} f_\partial(t) \\ e_\partial(t) \end{bmatrix}, \qquad 0 = W_{B,2} \begin{bmatrix} f_\partial(t) \\ e_\partial(t) \end{bmatrix},$$

where the boundary port-variables $f_\partial(t)$ and $e_\partial(t)$ are given by (1.1.4).
This yields the following definition of BCO stochastic port-Hamiltonian systems.

Definition 3.1.2. *A boundary controlled and observed stochastic port-Hamiltonian system is a BCO stochastic system, which is described by*

$$d\varepsilon(t) = (\mathcal{A}\varepsilon(t) + B_d u_d(t))dt + H dw(t), \qquad \varepsilon(0) = \varepsilon_0, \tag{3.1.3}$$

$$u(t) = W_{B,1} \begin{bmatrix} f_\partial(t) \\ e_\partial(t) \end{bmatrix} =: \mathcal{B}[\varepsilon(t)], \tag{3.1.4}$$

$$0 = W_{B,2} \begin{bmatrix} f_\partial(t) \\ e_\partial(t) \end{bmatrix}, \tag{3.1.5}$$

$$y(t) = W_C \begin{bmatrix} f_\partial(t) \\ e_\partial(t) \end{bmatrix} =: \mathcal{C}[\varepsilon(t)], \tag{3.1.6}$$

$$y_d(t) = B_d^* \mathcal{H}\varepsilon(t), \tag{3.1.7}$$

where $W_B := \begin{bmatrix} W_{B,1} \\ W_{B,2} \end{bmatrix} \in \mathbb{R}^{n\times 2n}$ *and* $W_C \in \mathbb{R}^{p\times 2n}$, $\mathcal{A}$ *is a linear operator given by*

$$\mathcal{A}\varepsilon := P_1 \frac{d}{d\zeta}(\mathcal{H}\varepsilon) + P_0(\mathcal{H}\varepsilon) \tag{3.1.8}$$

and $\mathcal{B} : D(\mathcal{B}) \to \mathbb{R}^m$ is a linear operator, with the same domain

$$\begin{aligned} D(\mathcal{A}) &:= \left\{ \varepsilon(t) \in \mathcal{X} : \mathcal{H}\varepsilon(t) \in H^1([a,b];\mathbb{R}^n) \text{ and } W_{B,2} \begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix} = 0 \right\} \\ &= D(\mathcal{B}). \end{aligned} \tag{3.1.9}$$

The class of SPHSs introduced in Definition 3.1.2 is the stochastic extension (with distributed disturbance) of deterministic port-Hamiltonian systems described by (1.1.26). Depending upon the control problem considered, pure boundary or distributed control can be regarded as particular cases of Definition 3.1.2.

To study the state equation (3.1.3) with the boundary conditions (3.1.4) and (3.1.5), we make the following assumptions.

Assumption 3.1.1. *The matrices W_B and W_C are full rank, W_B satisfies $W_B \Sigma W_B^T \geq 0$ and rank $\begin{bmatrix} W_{B,1} \\ W_C \end{bmatrix} = m + p$.*

Assumption 3.1.2. *$H \in L_2^0$, i.e. $\|H\|^2_{L_2^0} := Tr[HQH^*] < \infty$, which ensures that the Itô integrals $\int_0^t H dw(s)$ and $\int_0^t T(t-s) H dw(s)$ are well-defined.*

Note that the boundedness of the control operator B_d yields a well-defined and continuous input-output mapping with respect to $(u_d(t), y_d(t))$. Hence, unless stated otherwise, we shall set $B_d = 0$ (pure boundary control and observation) throughout the rest of this chapter, without affecting the validity of the results herefater.

From [JZ12, Theorem 11.3.2], it is known that the SPHS (3.1.3)-(3.1.5) is a boundary controlled stochastic system as specified in Definition 3.1.1, and thus the change of state variables: $X(\zeta,t) = \varepsilon(\zeta,t) - Bu(t)$ applied to (3.1.3) leads to an associated SDE given by

$$\begin{aligned} dX(t) &= (AX(t) - B\dot{u}(t) + \mathcal{A}Bu(t))dt + Hdw(t), \\ X(0) &= X_0. \end{aligned} \tag{3.1.10}$$

Definition 3.1.3. *A Hilbert space-valued process $(X(t))_{t\in[0,T]}$ is said to be a mild solution of (3.1.10) with respect to $(w(t))_{t\in[0,T]}$ if*

1. *$X(t)$ is $\mathbb{F}$-adapted;*
2. *$X(t) \in C([0,T]; L^2(\Omega;\mathcal{X}))$;*
3. *For all $t \in [0,T]$, $\mathbb{P}(\omega \in \Omega : \int_0^T \|X(\omega,t)\|^2_{\mathcal{X}} ds < \infty) = 1$ and*

$$X(t) = T(t)X_0 + \int_0^t T(t-s)(\mathcal{A}Bu(s) - B\dot{u}(s))ds + \int_0^t T(t-s)Hdw(s). \tag{3.1.11}$$

Theorem 3.1.1. *Consider a stochastic port-Hamiltonian system (3.1.3)-(3.1.5) as in Definition 3.1.2, satisfying Assumptions 3.1.1 and 3.1.2. In this setting, the mild solution of (3.1.10) is represented as state trajectories of a stochastic process given by (3.1.11) and satisfies the following estimate: for any $t > 0$, there is a constant $K(t) > 0$ such that*

$$\mathbb{E}\,\|X(t)\|_{\mathcal{X}}^2 \leq K(t)\left[\mathbb{E}\,\|X_0\|_{\mathcal{X}}^2 + \mathbb{E}\,\|u\|_{H^1([0,t];\mathbb{R}^m)}^2 + Tr[Q]\right], \tag{3.1.12}$$

where $\|\cdot\|_{H^1([0,t];\mathbb{R}^m)} = \int_0^t \|\cdot\|_{\mathbb{R}^m} ds + \int_0^t \|\frac{d(\cdot)}{ds}\|_{\mathbb{R}^m} ds$. *Moreover, if u is deterministic such that $u(t) = \mathcal{B}\mathbb{E}[X(t)]$, for every $t > 0$:*

1. *The mean of $X(t)$ is governed by the abstract differential equation (ADE)*

$$\dot{m}_X(t) = Am_X(t) + \mathcal{A}Bu(t) - B\dot{u}(t), \tag{3.1.13}$$

   whose mild solution is $m_X(t) = T(t)m_{X_0} + \int_0^t T(t-s)(\mathcal{A}Bu(s) - B\dot{u}(s))ds$.

2. *The variance of $X(t)$ is governed by the Lyapunov type ADE*

$$\dot{Cov}(X(t)) = ACov(X(t)) + Cov(X(t))A^* + HQH^*, \tag{3.1.14}$$

   whose mild solution is $Cov(X(t)) = T(t)Cov(X_0)T(t)^* + \int_0^t T(t-s)HQH^*\,T(t-s)^*ds$.

Proof. The existence and uniqueness of a mild solution can be directly deduced by using a probabilistic fixed point argument and its expression is obtained from the variational constant formula (3.1.11). The estimate is obtained by using Itô's isometry and the boundedness of the operators $\mathcal{A}B$, B and $H \in L_2^0$:

$$\begin{aligned}\mathbb{E}\,\|X(t)\|_{\mathcal{X}}^2 &= \mathbb{E}\,\|T(t)X_0 + \int_0^t T(t-s)(\mathcal{A}Bu(s) - B\dot{u}(s))ds + \int_0^t T(t-s)Hdw(s)\|_{\mathcal{X}}^2 \\ &\overset{(1)}{\leq} 3\,\mathbb{E}\,\|X_0\|_{\mathcal{X}}^2 + 3\,\mathbb{E}\int_0^t \|\mathcal{A}Bu(s) - B\dot{u}(s)\|_{\mathcal{X}}^2 ds + 3t\|H\|_{L_2^0}^2 \\ &\leq K(t)\left[\mathbb{E}\,\|X_0\|_{\mathcal{X}}^2 + \mathbb{E}\,\|u\|_{H^1([0,t];\mathbb{R}^m)}^2 + \mathrm{Tr}[Q]\right].\end{aligned}$$

$X(t)$ given by (3.1.11) is $\mathbb{F}$-adapted since $W_A(t)$ and $u(t)$ are $\mathbb{F}$-adapted, and X_0 is $\mathcal{F}_0$-measurable. The mean-square continuity of $X(t)$ is a straightforward consequence of the mean-square continuity of $W_A(t)$, see Theorem 2.3.6.
Using the vanishing property of the stochastic integral and the fact that X_0 has mean m_{X_0}, (3.1.13) is obtained.
Using the independence of X_0 and $w(t)$, (3.1.14) is deduced by Leibniz' differentiation rule. □

Remark 3.1.1. 1. Theorem 3.1.1 also holds for general BCO stochastic systems as defined in Definition 3.1.1.

(1) $\|a+b+c\|_{\mathcal{X}}^2 \leq 3\|a\|_{\mathcal{X}}^2 + 3\|b\|_{\mathcal{X}}^2 + 3\|c\|_{\mathcal{X}}^2$, with $a,b,c \in \mathcal{X}$

2. Since the mean and the covariance operators determine the Gaussian distribution of the process $X(t)$ given by (3.1.11), one can compute the distribution of $X(t)$ by solving the SDEs (3.1.13) and (3.1.14).

Theorem 3.1.2. *Assume that the stochastic port-Hamiltonian system (3.1.3)-(3.1.5) satisfying Assumptions 3.1.1 and 3.1.2 admits a mild solution $X(t)$ given by (3.1.11). Then $X(t)$ has continuous sample paths.*

Proof. Since we already know that the Bochner integral $\int_0^t T(t-s)(\mathcal{A}Bu(s)-B\dot{u}(s))ds$ and $T(t)X_0$ are continuous, it remains to prove that $\int_0^t T(t-s)Hdw(s)$ is continuous, $\mathbb{P}$-a.s. By making use of the Sz-Nagy-Foias theory of dilations [SN53] as done in [HS01], there exists a larger Hilbert space $\mathcal{X}_1$ on which the contraction C_0-semigroup $(T(t))_{t\geq 0}$ has a unitary dilation $(\bar{T}(t))_{t\geq 0}$. Besides, the state space $\mathcal{X}$ is embedded as a closed subspace of $\mathcal{X}_1$ and $(\bar{T}(t))_{t\geq 0}$ is a strongly continuous unitary group on $\mathcal{X}_1$ with $T(t) = P\bar{T}(t)$ for all $t \geq 0$, where P is the orthogonal projection of $\mathcal{X}_1$ onto $\mathcal{X}$. We denote the infinitesimal generator of $(\bar{T}(t))_{t\geq 0}$ by $\bar{A}$. Hence the stochastic convolution of the operator $\bar{A}$ can be decomposed as

$$\int_0^t \bar{T}(t-s)Hdw(s) = \bar{T}(t)\int_0^t \bar{T}(-s)Hdw(s). \tag{3.1.15}$$

Notice that the sample path continuity of $\int_0^t \bar{T}(t-s)Hdw(s)$ is directly deduced from the continuity of $\int_0^t \bar{T}(-s)Hdw(s)$. It is known that if $\left[\int_0^t \|\bar{T}(-s)H\|^2_{L_2^0} ds\right] < \infty$, then $\int_0^t \bar{T}(-s)Hdw(s)$ is continuous. The continuity of the orthogonal projection P implies that the stochastic convolution term $\int_0^t T(t-s)Hdw(s)$ has continuous sample paths, which concludes the proof. □

Remark 3.1.2. In proof of Theorem 3.1.2, observe that $\int_0^t \bar{T}(-s)Hdw(s)$ is well-defined since

$$\int_0^t \mathrm{Tr}[\bar{T}^*(s)HQH^*\bar{T}(s)]ds = t\,\mathrm{Tr}[HQH^*] < \infty.$$

Unfortunately, the stochastic convolution $W_A(t)$ is no longer a martingale, and thus the Itô's formula cannot be applied directly to mild solutions. This implies that applying Itô's formula to determine the Hamiltonian SDE requires stronger concepts of solution. In the next section we establish existence and uniqueness theorems for weak and strong solutions.

3.2 Existence and uniqueness theorems of weak and strong solutions

To trace the first studies of solutions of stochastic partial differential equations, we need to go back to [Bak63]. In this paper, the author proved existence theorems for both stochastic parabolic and hyperbolic equations by rewriting them as integral equations. This originates the semigroup approach that was developed afterwards for existence results of weak and strong solutions for SPDEs, notably by Bensoussan, Pardoux, Rozovskii, Curtain and Pritchard, among many others.

The notions of weak and strong solutions considered in this work have the same spirit as those defined for deterministic PDEs. Note that different meanings of weak and strong solutions with a probabilistic point of view are also available in the literature.

First, we investigate the concept of weak solution, which is obtained by applying $z \in D(A^*)$ to both parts of the SDE (3.1.10).

Definition 3.2.1. *A $\mathcal{X}$-valued process $(X(t))_{t\in[0,T]}$ with $T \geq 0$ is said to be a weak solution of (3.1.10) with respect to the Wiener process $(w(t))_{t\in[0,T]}$ if the trajectories $X(t)$ are $\mathbb{P}$-a.s Bochner integrable and if for all $z \in D(A^*)$ and $t \in [0,T]$*

$$\begin{aligned}\langle X(t),z\rangle_{\mathcal{X}} = \langle X_0,z\rangle_{\mathcal{X}} + \int_0^t [\langle X(s),A^*z\rangle_{\mathcal{X}} + \langle \mathcal{A}Bu(t) - B\dot{u}(t),z\rangle_{\mathcal{X}}]\,ds \\ + \langle Hw(t),z\rangle_{\mathcal{X}},\ \mathbb{P}-a.s.\end{aligned} \tag{3.2.1}$$

Theorem 3.2.1. *Consider a stochastic port-Hamiltonian system (3.1.3)-(3.1.5) and let Assumptions 3.1.1 and 3.1.2 hold. Then for every input $u \in C^2_{\mathbb{F}}([0,T];\mathbb{R}^m)$, $\mathcal{H}\varepsilon_0 \in H^1([a,b];\mathbb{R}^n)$ and $u(0) = W_B \begin{bmatrix} f_\partial(0) \\ e_\partial(0) \end{bmatrix}$, the stochastic differential equation (3.1.10) admits a unique weak solution given by (3.1.11). Since $X(t)$ defined by (3.1.11) is almost surely integrable, the mild and weak solutions coincide.*

Proof. From [JZ12, Theorem 10.1.8], it is already known that $x(t)$ given by

$$x(t) = T(t)x_0 + \int_0^t T(t-s)(\mathcal{A}Bu(s) - B\dot{u}(s))ds, \qquad t \geq 0$$

is the unique weak solution of

$$\dot{x}(t) = Ax(t) + \mathcal{A}Bu(t) - B\dot{u}(t), \qquad x(0) = x_0. \tag{3.2.2}$$

Therefore, it is enough to prove that the process $\int_0^t T(t-s)Hdw(s)$ is a unique weak solution of

$$dX(t) = AX(t)dt + Hdw(t), \qquad X(0) = 0. \tag{3.2.3}$$

with $t > 0$. For this we refer to the proof of [DPZ14, Theorem 5.4] □

Generally speaking, the strong solution requires more restrictive conditions such as taking values in $D(A)$. Therefore, the usual way of defining the solution by integrating both parts of the stochastic equation (3.1.10) can be applied.

Definition 3.2.2. *A $\mathcal{X}$-valued process $(X(t))_{t\in[0,T]}$ with $T \geq 0$ is said to be a strong solution of (3.1.10) with respect to the Wiener process $(w(t))_{t\in[0,T]}$ with the covariance operator Q satisfying $Tr\,Q < \infty$ if*

1. *$X(t)$ belongs to $D(A)$ a.s. and is adapted to $\mathbb{F}$;*

2. *$X(t)$ is continuous in $t \in [0,T]$ a.s.;*

3. *$\int_0^\infty \|AX(s)\| ds < \infty$ $\mathbb{P}$-a.s and the process $(X(t))_{t\in[0,T]}$ is given by*

$$X(t) = X_0 + \int_0^t (AX(s) + \mathcal{A}Bu(s) - B\dot{u}(s))\, ds + \int_0^t H dw(s), \qquad \mathbb{P}-a.s. \tag{3.2.4}$$

Observe that $\mathcal{A}Bu(s) - B\dot{u}(s) \in D(A) = D(\mathcal{A}) \cap \mathrm{Ker}\mathcal{B}$ would be too restrictive on $u(t)$. See for instance the example in Section 3.4.2, where $u = 0$ would have to be taken. As a consequence, the SDE (3.1.10) does not admit a strong solution. To encounter this issue, we shall extend the state space $\mathcal{X}$ and introduce a family of approximate systems by using the Yosida approximate. The rationale is then based on a limiting argument. The extended state space is defined as $\mathcal{X}^e := \mathbb{R}^m \oplus \mathcal{X}$, where the (extended) state is defined as $X^e(t) := \begin{pmatrix} u(t) & X(t) \end{pmatrix}^T$ and $\tilde{u}(t) = \dot{u}(t)$. The approximating system of (3.1.10) is given as follows:

$$dX^e(t) = \begin{pmatrix} 0 & 0 \\ \mathcal{A}B & A \end{pmatrix} X^e(t)dt + \begin{pmatrix} I \\ -B \end{pmatrix} \tilde{u}(t)dt + \begin{pmatrix} 0 \\ H \end{pmatrix} dw(t). \tag{3.2.5}$$

Let us define $A^e := \begin{pmatrix} 0 & 0 \\ \mathcal{A}B & A \end{pmatrix}$ and $B^e := \begin{pmatrix} I \\ -B \end{pmatrix}$ with domains $D(A^e) = \mathbb{R}^m \oplus D(A)$ and $D(B^e) = \mathbb{R}^m$ and $H^e = \begin{bmatrix} 0 \\ H \end{bmatrix}$. The operator A^e is the infinitesimal generator of a C_0-semigroup $T^e(t) = \begin{pmatrix} I & 0 \\ S(t) & T(t) \end{pmatrix}$, where $S(t)u := \int_0^t T(t-s)\mathcal{A}Bu(s)ds$ for all $u(t) \in \mathbb{R}^m$. To build a family of approximating systems having strong solutions, let us introduce the following Yosida approximate control operators for all $\lambda \in \rho(A^e)$:

$$B^e_\lambda : \mathbb{R}^m \to \mathcal{X} : \tilde{u} \mapsto B^e_\lambda \tilde{u} := \lambda R(\lambda, A^e) B^e \tilde{u}, \tag{3.2.6}$$

where the resolvent operator $R(\lambda, A^e) = (\lambda I - A^e)^{-1}$.

Theorem 3.2.2. *Consider the stochastic port-Hamiltonian system (3.1.3)-(3.1.5) satisfying Assumptions 3.1.1 and 3.1.2. In addition, we assume that $HQ^{1/2}(Z) \subset D(A)$ and that $X_0 \in D(A)$. If the following condition holds for all $t \geq 0$:*

$$\int_0^t \|AT(t-s)H\|^2_{L_2^0} ds < \infty; \tag{3.2.7}$$

then for all $\lambda \in \rho(A^e)$,

$$dX^e_\lambda(t) = A^e X^e_\lambda(t)dt + B^e_\lambda \tilde{u}(t)dt + H^e dw(t); \qquad X^e(0) = \begin{pmatrix} u(0) & X_0 \end{pmatrix}^T \in D(A^e), \tag{3.2.8}$$

has a unique strong solution $X^e_\lambda(t)$ with respect to $w(t)$, where $B^e_\lambda \tilde{u} = \lambda R(\lambda, A^e) B^e \tilde{u}$ for all $\tilde{u} \in \mathbb{R}^m$, such that

$$\sup_{0 \leq s \leq t} \mathbb{E} \|X^e_\lambda(s) - X^e(s)\|^2_{\mathcal{X}^e} \to 0 \qquad as\ \lambda \to \infty, \tag{3.2.9}$$

where $X^e(t)$ is the mild solution of (3.2.5).

Proof. First, notice that the uniqueness is a direct outcome of the uniqueness of the mild solution of (3.2.5). Thus, only the existence of a strong solution needs to be showed for every $t \in [0,T]$. $X^e_\lambda(t) \in D(A^e)$ and $X^e_\lambda(t)$ given by

$$X^e_\lambda(t) = T^e(t)X^e_0 + \int_0^t T^e(t-s)B^e_\lambda \tilde{u}(s)ds + \int_0^t T^e(t-s)H^e dw(s) \tag{3.2.10}$$

satisfies the integral equation

$$X^e_\lambda(t) = X^e_0 + \int_0^t (A^e X^e_\lambda(s) + B^e_\lambda \tilde{u}(s))ds + \int_0^t H^e dw(s). \tag{3.2.11}$$

Since $X^e_0 \in D(A^e)$, we have that $T^e(t)X^e_0 \in D(A)$ by the first part of [JZ12, Theorem 5.2.2]. Moreover, for any $\tilde{u} \in L^2([0,T];\mathbb{R}^m)$, once more from [JZ12, Theorem 5.2.2], we obtain that $\int_0^t T^e(t-s)B^e\tilde{u}(s)\ ds \in D(A^e)$. Eventually, Lemma 2.3.5 entails that $\int_0^t T(t-s)Hdw(s) \in D(A)$ and that $A\int_0^t T(t-s)Hdw(s) = \int_0^t AT(t-s)Hdw(s)$. This implies that $\int_0^t T^e(t-s)H^e dw(s) \in D(A^e)$ and thus proves that $X^e_\lambda(t)$ belongs to $D(A^e)$. In order to prove that $X^e_\lambda(t)$ satisfies (3.2.11), we mainly base the following rational on [Liu05]. From the stochastic Fubini Theorem 2.4.1, we have that

$$\begin{aligned}\int_0^t\int_0^s A^e T^e(s-v)H^e dw(v)ds &= \int_0^t\int_v^t A^e T^e(s-v)H^e ds dw(v)\\ &= \int_0^t T^e(t-v)H^e dw(v) - \int_0^t H^e dw(v).\end{aligned} \tag{3.2.12}$$

Moreover, by stochastic Fubini's Theorem once again,

$$\begin{aligned}\int_0^t\int_0^s A^e T^e(s-v)B^e_\lambda \tilde{u}(v)dvds &= \int_0^t\int_v^t A^e T^e(s-v)B^e_\lambda \tilde{u}(v)dsdv\\ &= \int_0^t T^e(t-v)B^e_\lambda \tilde{u}(v)dv\\ &\quad - \int_0^t B^e_\lambda \tilde{u}(v)dv.\end{aligned} \tag{3.2.13}$$

By applying A^e to both sides of (3.2.10) and by integrating on $[0,t]$, we get that

$$\begin{aligned}\int_0^t A^e X^e_\lambda(s)ds = \int_0^t A^e T^e(s)X^e_0 ds &+ \int_0^t\int_0^s A^e T^e(s-v)H^e dw(v)ds\\ &+ \int_0^t\int_0^s A^e T^e(s-v)B^e_\lambda \tilde{u}(v)dvds.\end{aligned} \tag{3.2.14}$$

Using the relations (3.2.12) and (3.2.13), it follows that

$$X^e_0 + \int_0^t A^e X^e_\lambda(s)ds = X^e_\lambda(t) - \int_0^t H^e dw(s) - \int_0^t B^e_\lambda \tilde{u}(s)ds,$$

which means that $X^e_\lambda(t)$ satisfies the integral equation (3.2.11).
Moreover, the continuity of $X^e_\lambda(t)$ can be deduced from the continuity of $\int_0^t H^e dw(s)$,

and since $A^eX^e_\lambda(t)$ is assumed to be integrable, $\int_0^t A^eX^e_\lambda(s)ds$ has continuous sample paths.
We know that $\lim_{\lambda\to\infty} \lambda R(\lambda,A^e)z = z$, $z \in \mathcal{X}^e$. Therefore, since $(T(t))_{t\geq 0}$ is a contraction C_0-semigroup and by using $\|\lambda R(\lambda,A^e)\| \leq 2$ for λ large enough, we have that

$$\sup_{0\leq s\leq t} \mathbb{E}\,\|X^e_\lambda(s) - X^e(s)\|^2_{\mathcal{X}^e} \leq \int_0^t \mathbb{E}\,\|(I-\lambda R(\lambda,A^e))B^e\tilde{u}(r)\|^2_{\mathcal{X}^e}dr.$$

So, $\mathbb{E}\,\|X^e_\lambda(s) - X^e(s)\|^2_{\mathcal{X}^e} \to 0$ uniformly on $[0,t]$ as $\lambda \to \infty$. □

Remark 3.2.1. The condition $\int_0^t \|AT(t-s)H\|^2_{L_2^0}ds < \infty$ ensures that the stochastic Fubini Theorem 2.4.1 can be used. This condition can be replaced by the stronger assumption that $AHQ^{1/2}$ is a Hilbert-Schmidt operator, i.e. $AHQ^{1/2} \in L_2(Z,\mathcal{X})$. Indeed, note that

$$\begin{aligned}\int_0^t \|AT(t-s)H\|^2_{L_2^0} &= \int_0^t \|T(t-s)AHQ^{1/2}\|^2_{L_2}ds \\ &= \int_0^t \|T(s)AHQ^{1/2}\|^2_{L_2}ds \\ &\leq \|AHQ^{1/2}\|^2_{L_2}\int_0^t \|T(s)\|^2 ds < \infty.\end{aligned}$$

We now determine the SDE governing the evolution of the energy of SPHSs. By applying the Itô's formula to the SDE (3.1.3) with $u(t) = 0$ and $u_d(t) = 0$, we get the following balance equation

$$dE(t) = \langle E'_x(\varepsilon(t)), A\varepsilon(t)\rangle_{L^2}dt + \langle E'_x(\varepsilon(t)), Hdw(t)\rangle_{L^2} + \frac{1}{2}\operatorname{Tr}\left[E''_{xx}(\varepsilon(t))HQH^*\right]dt. \tag{3.2.15}$$

The diffusion term $Hdw(t)$ can be interpreted as the incremental work of an external force representing the uncertainty about the environment in which the system is operating. The power supplied to a stochastic port-Hamiltonian system by a random component can be represented as

$$\langle \mathcal{H}x, Hdw(t)\rangle_{L^2} = \langle H^*\mathcal{H}, dw(t)\rangle_Z = \langle z_2(t), dw(t)\rangle_Z, \tag{3.2.16}$$

where $z_2(t)$ consists of a further power-conjugated output disturbance. In the next result, the energy increments due to the Wiener process $w(t)$ are explicited.

Proposition 3.2.3. *The expected energy increment with respect to the Hamiltonian $E(t)$ due to the noise effect is given by*

$$\mathbb{E}[dE(\varepsilon(t))|\varepsilon_0 = x] - dE(\mathbb{E}[\varepsilon(t)|\varepsilon_0 = x]) = \frac{1}{2}Tr[\mathcal{H}HQH^*]dt, \tag{3.2.17}$$

where $\varepsilon(t)$ is the stochastic port-Hamiltonian process defined by (3.1.3)-(3.1.6) with $u = 0$, $u_d = 0$ and starting at $x \in \mathcal{X}$.

Proof. First, we compute the expected value of the energy of the process $\varepsilon(t)$ starting at x. Applying Itô's formula, we have

$$\begin{aligned}\mathbb{E}^x\left[dE(\varepsilon(t))\right] &= \mathbb{E}^x[\langle E'_x(\varepsilon(t)), A\varepsilon(t)\rangle_{L^2} dt \\ &+ \langle E'_x(\varepsilon(t)), Hdw(t)\rangle_{L^2} + \frac{1}{2}\operatorname{Tr}\left[E''_{xx}(\varepsilon(t))HQH^*\right]dt].\end{aligned} \tag{3.2.18}$$

Noticing that the first and second Fréchet derivatives are given by $E'_x(\varepsilon(t)) = \mathcal{H}\varepsilon(t)$ and $E''_{xx}(\varepsilon(t)) = \mathcal{H}$. Moreover, since the expectation of the Wiener process increments vanishes, one gets

$$\begin{aligned}\mathbb{E}^x\left[dE(\varepsilon(t))\right] &= \mathbb{E}^x\langle \mathcal{H}\varepsilon(t), A\varepsilon(t)\rangle_{L^2} dt + \frac{1}{2}\operatorname{Tr}\left[\mathcal{H}HQH^*\right]dt \\ &= \mathbb{E}^x[f_\partial(t)^T e_\partial(t)]dt + \frac{1}{2}\operatorname{Tr}\left[\mathcal{H}HQH^*\right]dt,\end{aligned} \tag{3.2.19}$$

where e_∂ and f_∂ are given by (1.1.4).
Second, we compute the expected value of the energy E at time t without noise, which gives

$$dE(\mathbb{E}^x[\varepsilon(t)]) = f_\partial(t)^T e_\partial(t)dt. \tag{3.2.20}$$

Finally, by subtracting (3.2.20) from (3.2.19) we get (3.2.17), i.e., $\frac{1}{2}\operatorname{Tr}\left[\mathcal{H}HQH^*\right]$ represents the expected energy increment due to the noise effect. □

The energy increment $\frac{1}{2}\operatorname{Tr}\left[\mathcal{H}HQH^*\right]$ highlighted in Proposition 3.2.3 is a direct consequence of the diffusion part generated by $w(t)$ and is specific to the Itô calculus chosen here. Notice that this energy increment is somehow hidden in the Stratonovich theory of integration. For the definition of the Itô integral, we have considered the left extremity t_i value of the integrand on each interval $[t_i, t_{i+1}]$, see (2.3.3). Whereas for the Stratonovich integral, the middle point $\frac{t_i+t_{i+1}}{2}$ is considered. In the case of additive noise (considered here) the two coincide. For further details on the Stratonovich integral in Hilbert spaces, we refer to [DW14, Chapter 4].

3.3 Passivity of stochastic systems

The notions of dissipativity and passivity have been the object of a lot of attention over the years. In this section the concept of passivity is generalized for stochastic systems. When one wants to extent the passivity property of deterministic systems to stochastic ones, the following questions beg naturally:

- Which regularity conditions must the storage function satisfy?
- With what the time derivative should be replaced?

Let us consider the dynamical system (0.4.1) driven by some Wiener process $w(t)$, which yields

$$\begin{aligned} dX(t) &= f(X(t), u(t))dt + Hdw(t), \qquad X(t) \in \mathcal{X}, \\ y(t) &= g(X(t), u(t)), \qquad u(t) \in U, y(t) \in Y. \end{aligned} \tag{3.3.1}$$

In order to define the notion of passivity for stochastic systems, we define the infinitesimal generator of stochastic processes.

Definition 3.3.1. *The infinitesimal generator $\mathcal{L}$ of a stochastic process $(X(t))_{t\geq 0}$ with an initial condition $X(0) = x$ is the linear operator given by*

$$(\mathcal{L}f)(x) = \lim_{t\to 0^+} \frac{\mathbb{E}^x[f(X(t))] - f(x)}{t}, \tag{3.3.2}$$

and acts on the set of Itô functionals

$$C_b^2(\mathcal{X}) := \left\{ f \in C(\mathcal{X}) : f'_x \in \mathcal{L}(\mathcal{X}), f''_{xx} \in \mathcal{L}(\mathcal{X}, \mathcal{L}(\mathcal{X})) \right\},$$

that lies in

$$D(\mathcal{L}) := \left\{ f \in C_b^2(\mathcal{X}) : \lim_{t\to 0^+} \frac{\mathbb{E}^x[f(X(t))] - f(x)}{t} \text{ exists} \right\}.$$

Notice that the operator $\mathcal{L}$ given by (3.3.2) is the generator of the stochastic Koopman family

$$U_s^t(f(x)) = \mathbb{E}^x[f(X(t))]. \tag{3.3.3}$$

The stochastic Koopman family is also known as the transition semigroup for Markov processes, see [DPZ14, Chapter 9].

According to Itô's formula, the SDE of a storage function $S \in C_b^2(\mathcal{X})$ is given by

$$dS(X(t)) = \mathcal{L}(S(X(t)))dt + \langle S'_x(x(t)), Hdw(t)\rangle_{L^2}, \tag{3.3.4}$$

where

$$\mathcal{L}(\cdot) := \langle \frac{d(\cdot)}{dx}, f(X(t), u(t))\rangle + \frac{1}{2}\operatorname{Tr}\left\{ \frac{d^2(\cdot)}{dx^2} HQH^* \right\}. \tag{3.3.5}$$

The action of the generator $\mathcal{L}$ on $f \in C_b^2(\mathcal{X})$ can be deduced by using (3.3.2), and applying Itô's formula and the property that the expectation of a stochastic integral vanishes (see Theorem 2.3.4). A detailed proof for the real case $C_b^2(\mathbb{R})$ can be found in [Hol08, Section 3.3].

The diffusion operator $\mathcal{L}$ plays a similar role to the differentiation operator for deterministic systems. Roughly speaking, the time derivative is replaced by the expectation of the time derivative for stochastic dynamical systems.
The passivity for infinite-dimensional stochastic systems (3.3.1) can now be defined.

Definition 3.3.2. *The stochastic system (3.3.1) is said to be passive (with respect to the supply rate $s(u, y) = u^T y$) if there exists a function $S : \mathcal{X} \to \mathbb{R}^+$, called the storage function, such that $S \in C_b^2(\mathcal{X})$ satisfies for all $t \geq 0$,*

$$\mathcal{L}S(X(t)) \leq u^T(t)y(t). \tag{3.3.6}$$

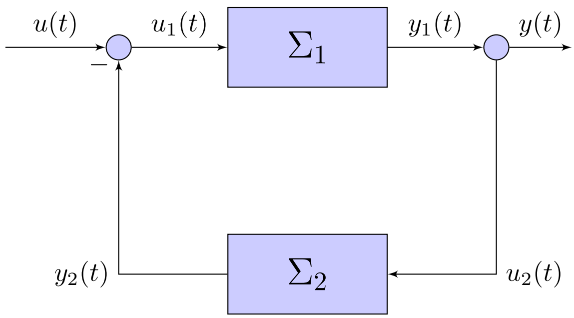

Figure 3.1 – Interconnection of passive systems: $u = u_1 + y_2$ and $y = y_1 = u_2$

Remark 3.3.1. 1. Taking the expectation, integrating both sides of (3.3.6) and using Itô's formula, passivity can be expressed in terms of expectation:

$$\begin{aligned}\mathbb{E}\int_0^t u^T(s)y(s)ds &\geq \int_0^t \mathbb{E}\mathcal{L}(E(X(s)))ds \\ &= \mathbb{E}\left[E(X(t)) - E(X(0))\right].\end{aligned}$$

The integral $\mathbb{E}\int_0^t u^T(s)y(s)ds$ represents the expected supplied energy to the system. Hence, passivity means that a stochastic system cannot store in mean more energy than supplied.

2. If (3.3.6) holds with equality, then the stochastic system (3.3.1) is said to be lossless with respect to the supply rate $u^T(t)y(t)$. The energy is a conserved quantity.

3. Deterministic passive systems can be seen as a special case of stochastic passive systems in the sense that when $H = 0$, the diffusion operator reduces to the time derivative, and then the deterministic definition of passivity is recovered.

The passivity fundamental result, namely that a negative feedback loop of two passive systems is again passive, has its stochastic counterpart. This is illustrated in Figure 3.1.

Proposition 3.3.1. *Let us consider two passive stochastic systems Σ_1 and Σ_2 with corresponding inputs u_1, u_2 and outputs y_1, y_2. The negative feedback interconnection given by*

$$u = u_1 + y_2 \qquad \text{and} \qquad u_2 = y_1 = y \tag{3.3.7}$$

of Σ_1 and Σ_2 yields a new interconnected passive stochastic system Σ with a new input u and output y.

Proof. Since the stochastic systems Σ_1 and Σ_2 are assumed to be passive in the sense of Definition 3.3.2, there exist storage functions S_1 and S_2 such that

$$\mathcal{L}S_1(X_1(t)) \le u_1^T(t)y_1(t) \qquad \text{and} \qquad \mathcal{L}S_2(X_2(t)) \le u_2^T(t)y_2(t).$$

By defining the storage function of Σ as $S = S_1 + S_2$, we get the following relation

$$\mathcal{L}S(X(t)) = \mathcal{L}S_1(X_1(t)) + \mathcal{L}S_2(X_2(t)) \le (u(t) - y_2(t))^T y(t) + y(t)^T y_2(t) = u(t)^T y(t), \tag{3.3.8}$$

which proves the passivity property for the interconnected system Σ. □

In the following result, we state necessary and sufficient conditions for SPHSs to satisfy the passivity property.

Theorem 3.3.2. *Consider the stochastic port-Hamiltonian system (3.1.3)-(3.1.6) with the maximal number of controls and the same number of inputs and outputs at the boundary (i.e. $m = p = n$). Then*

1. *the associated infinitesimal generator is described by*

$$\mathcal{L}(E(\varepsilon(t))) = \frac{1}{2}\begin{bmatrix} u(t) \\ y(t) \end{bmatrix}^T P_{W_B,W_C} \begin{bmatrix} u(t) \\ y(t) \end{bmatrix} + y_d^T u_d(t) + \frac{1}{2} Tr[\mathcal{H}HQH^*], \tag{3.3.9}$$

where P_{W_B,W_C} is given by

$$P_{W_B,W_C} = \left(\begin{bmatrix} W_B \\ W_C \end{bmatrix} \Sigma \begin{bmatrix} W_B \\ W_C \end{bmatrix}^T \right)^{-1} = \begin{bmatrix} W_B\Sigma W_B^T & W_B\Sigma W_C^T \\ W_C\Sigma W_B^T & W_C\Sigma W_C^T \end{bmatrix}^{-1}.$$

2. *Moreover, the system is lossless if and only if the following relations are verified*

$$I = 2S_C(I - L_C L^T)S^T, \tag{3.3.10}$$

$$Tr[\mathcal{H}HQH^*] = 0, \tag{3.3.11}$$

where the matrices W_B and W_C are of the form

$$\begin{aligned} W_B &= S\begin{bmatrix} I+L & I-L \end{bmatrix} \\ W_C &= S_C\begin{bmatrix} I+L_C & I-L_C \end{bmatrix} \end{aligned} \tag{3.3.12}$$

such that S and S_C are nonsingular and L and L_C are unitary.

Proof. 1. Relation (3.3.9) is deduced from the following calculation:

$$\begin{aligned} \mathcal{L}(E(\varepsilon(t))) =& \langle \mathcal{H}\varepsilon(t), \mathcal{J}\mathcal{H}\varepsilon(t)\rangle_{L^2} + \langle \mathcal{H}\varepsilon(t), B_d u_d(t)\rangle_{L^2} + \frac{1}{2}\mathrm{Tr}[\mathcal{H}HQH^*] \\ =& \frac{1}{2}\begin{bmatrix} u(t) \\ y(t) \end{bmatrix}^T P_{W_B,W_C} \begin{bmatrix} u(t) \\ y(t) \end{bmatrix} + \langle B_d^* \mathcal{H}\varepsilon(t), u_d(t)\rangle_{L^2} \\ &+ \frac{1}{2}\mathrm{Tr}[\mathcal{H}HQH^*]. \end{aligned}$$

2. For the sufficiency, assume that (3.3.10) and (3.3.11) hold. As a consequence of (3.3.10), W_B and W_C satisfy

$$W_B^T W_C + W_C^T W_B = \Sigma, \tag{3.3.13}$$

where $\Sigma = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \in \mathbb{R}^{2n\times 2n}$. The specific choice (3.3.12) of matrices W_B and W_C allows to have $P_{W_B,W_C} = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$. Hence, from [LZM05, Theorem 4.2], we get that

$$\mathcal{L}(E(X(t))) = u(t)^T y(t) + y_d(t)^T u_d(t) + \frac{1}{2}\operatorname{Tr}[HQH^*]. \tag{3.3.14}$$

By injecting (3.3.11) in (3.3.14), the lossless property is proved. For the necessity part, since $P_{W_B,W_C} = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$ we have that

$$\Sigma = \begin{bmatrix} W_B \\ W_C \end{bmatrix}^T \Sigma \begin{bmatrix} W_B \\ W_C \end{bmatrix} = \begin{bmatrix} W_B \\ W_C \end{bmatrix}^T \begin{bmatrix} W_C \\ W_B \end{bmatrix}.$$

From (3.3.6) and (3.3.14), we deduce that $\operatorname{Tr}[\mathcal{H}HQH^*] = 0$.

□

Observe that Theorem 3.3.2 entails that the passivity property cannot be preserved for SPHSs under the Itô calculus. Indeed, condition (3.3.11) would imply that $\lambda_i = 0$ for all $i \in \mathbb{N}$ such that $Qv_i = \lambda_i v_i$, which means that there is no noise on the system. As already mentioned, the incremental work $Hdw(t)$ induces some energy increments $\frac{1}{2}\operatorname{Tr}[\mathcal{H}HQH^*]$, which breaks the passivity property. In order to recover the passivity property, this energy surplus has to be compensated by internal energy dissipation (mechanical friction, electrical resistance, etc.) or feedback control. In addition, note that since the covariance operator is nonnegative, assuming that $\operatorname{Tr}[\mathcal{H}HQH^*] \leq 0$ would imply that $\operatorname{Tr}[\mathcal{H}HQH^*] = 0$.

The main interest for the passivity property comes from the passivity-based stabilization methods. One can take advantage of this property for output-feedback stabilizations, see [Wil72] and [DMSB09].

A method to recover the passivity property via a stochastic canonical generalized transformation (SCGT) was proposed in [SF13] for nonlinear and finite-dimensional SPHSs. To be more specific, the SCGT enables to design both the coordinate transformation and the feedback controller while preserving the stochastic port-Hamiltonian structure. The difficulty consists in the identification of a set of transformations that allows mapping any SPHS to a new target one belonging to the same class of stochastic systems. To recover the passivity property, the energy increments are compensated by changing the Hamiltonian (i.e. the internal structure) and the input-output mapping. A generalization of the method proposed in [SF13] to infinite-dimensional SPHSs is not as straightforward as it seems to be and faces numerous difficulties, mainly the spatial coordinate dependence.

In a deterministic setting, a canonical generalized transformation (CGT) was studied in [MGR17] for boundary controlled port-Hamiltonian systems. This is an extension to the infinite-dimensional setting case of the CGT proposed in [FS01]. Nevertheless, a similar approach as proposed in [MGR17] and based on exponential transformations seems to be too limited when the energy function depends on the spatial coordinate. A systematic identification of a set of transformations is not possible anymore due to the boundary conditions to be satisfied.

3.4 Well-posedness

Now, we come to the question of well-posedness of the class of BCO SPHSs and how well-posedness has to be defined. So far, we have only treated the question of existence and uniqueness of the stochastic state equation (3.1.3) with boundary conditions (3.1.4) and (3.1.5).

On finite-dimensional spaces, the question of well-posedness of a deterministic system does not really come into play. The reason is that the existence of a unique solution for sufficiently regular inputs and initial conditions such that the output is square integrable is usually not even mentioned. This is quite different for infinite-dimensional systems. The well-posedness becomes of paramount importance, and even if this is not really a goal in itself from an engineering point of view, it paves the way for dealing with control/estimation, transfer function, etc.

Since distributed control and observation operators yield well-defined state trajectories and outputs, they do not require an analysis. Hence, throughout this section, we shall set $u_d(t) = y_d(t) = 0$. Denote by $\mathcal{X}_{-1}$ the completion of the state space $\mathcal{X}$ with respect to the norm $\|\cdot\|_{-1} := \|(\alpha I - A)\cdot\|_{\mathcal{X}}$, where $\alpha \in \rho(A)$ is fixed. Notice that $D(A) \subset \mathcal{X} \subset \mathcal{X}_{-1}$.
Let us consider the following control system:

$$\begin{aligned} \dot{x}(t) &= A_{-1}x(t) + \mathbb{B}u(t), \qquad x(0) = x_0, \\ y(t) &= \mathbb{C}x(t), \end{aligned} \tag{3.4.1}$$

where $A_{-1} \in \mathcal{L}(\mathcal{X}, \mathcal{X}_{-1})$, $\mathbb{B} \in \mathcal{L}(\mathbb{R}^m, \mathcal{X}_{-1})$ and $\mathbb{C} \in \mathcal{L}(D(A), \mathbb{R}^p)$. The C_0-semigroup $(T(t))_{t\geq 0}$ is uniquely extended on X_{-1} to $(T_{-1}(t))_{t\geq 0}$, whose generator A_{-1} is the unique extension of A. Moreover, $(T_{-1}(t))_{t\geq 0}$ and A_{-1} are unitarily similar to $(T(t))_{t\geq 0}$ and A, respectively, i.e. $T_{-1}(t) = (\lambda I - A_{-1})T(t)(\lambda I - A_{-1})^{-1}$ and $A_{-1} = (\lambda I - A_{-1})A(\lambda I - A_{-1})^{-1}$. To recover the boundedness of the boundary control and observation operators $\mathbb{B}$ and $\mathbb{C}$, the spaces $D(A)$ and $\mathcal{X}_{-1}$ are considered. Nevertheless, this leads to some mathematical inconveniences. One is that state trajectories take values in a larger space, namely $\mathcal{X}_{-1}$. A second is that the pointwise output equation of control system (3.4.1) does not make any sense unless the observation operator $\mathbb{C}$

is bounded on the state space $\mathcal{X}$. Notice that in the case of a boundary control system as in Definition 0.3.1, the operator $\mathbb{B}$ is described by

$$\mathbb{B}u = \mathcal{A}Bu - A_{-1}Bu,$$

and belongs to $\mathcal{L}(\mathbb{R}^m, \mathcal{X}_{-1})$, see [Vil07, Lemma 2.32 and Lemma 2.35].
In order to close the gap between the spaces, the notions of admissible control and observation operators were introduced, see e.g. [TW09]. Let us recall the concepts of admissible control and observation operators.

Definition 3.4.1. *1. Let $\mathbb{B}$ be in $\mathcal{L}(\mathbb{R}^m, \mathcal{X}_{-1})$ and let us define the family of operators $(\Phi_t)_{t\geq 0} \subset \mathcal{L}(L^2([0,t];\mathbb{R}^m), \mathcal{X}_{-1})$ as*

$$\Phi_t u = \int_0^t T(t-s)\mathbb{B}u(s)ds. \tag{3.4.2}$$

The control operator $\mathbb{B}$ is said to be admissible for $(T(t))_{t\geq 0}$ if for some $t_0 \geq 0$, Range $(\Phi_{t_0}) \subset \mathcal{X}$.

2. Let us define the family of operators $(\Psi_t)_{t\geq 0} \subset \mathcal{L}(D(A), L^2([0,\infty);\mathcal{X}))$ as

$$(\Psi_t z)(s) = \begin{cases} \mathbb{C}T(s)z, & s \in [0,t], \\ 0, & s \in (t,\infty). \end{cases} \tag{3.4.3}$$

The observation operator $\mathbb{C} \in \mathcal{L}(D(A), \mathbb{R}^p)$ is said to be admissible for $(T(t))_{t\geq 0}$ if for some $t_0 \geq 0$, Ψ_{t_0} has a continuous extension to $\mathcal{X}$.

Furthermore, linear well-posed systems as described in Definition 3.4.2 were introduced by Salamon [Sal89] to deal with systems with boundary control and observation operators. This class of systems is also known to enjoy many useful properties (see e.g. [Sta05]) involving feedback control, dynamic stabilization, and tracking/disturbance rejection.

For all the above reasons, the well-posedness of a control system needs to be checked in first place.

Definition 3.4.2. *The BCO system described by*

$$\dot{\varepsilon}(t) = \mathcal{A}\varepsilon(t), \qquad \varepsilon(0) = \varepsilon_0 \in \mathcal{X}, \tag{3.4.4}$$

$$u(t) = \mathcal{B}\varepsilon(t), \tag{3.4.5}$$

$$y(t) = \mathcal{C}\varepsilon(t), \tag{3.4.6}$$

where $\mathcal{A} : D(\mathcal{A}) \to \mathcal{X}$, $\mathcal{B} : D(\mathcal{B}) \to \mathbb{R}^m$ and $\mathcal{C} : D(\mathcal{C}) \to \mathbb{R}^p$ are unbounded linear operators as defined in Definition 0.3.1, is said to be well-posed if:

- *The operator $A : D(A) \to \mathcal{X}$ with $D(A) = D(\mathcal{A}) \cap ker(\mathcal{B})$ and*

$$A\varepsilon = \mathcal{A}\varepsilon \qquad \text{for } \varepsilon \in D(A)$$

is the infinitesimal generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on $\mathcal{X}$;

- *There exist* $t_f > 0$ *and* $m_f \geq 0$ *such that the following inequality holds for all* $\varepsilon_0 \in D(\mathcal{A})$ *and* $u \in C^2([0,t_f);\mathbb{R}^m)$ *with* $u(0) = \mathcal{B}\varepsilon(0)$ *(compatibility conditions):*

$$\|\varepsilon(t_f)\|^2_{\mathcal{X}} + \int_0^{t_f} \|y(t)\|^2_{\mathbb{R}^p} dt \leq m_{t_f} \left(\|\varepsilon_0\|^2_{\mathcal{X}} + \int_0^{t_f} \|u(t)\|^2_{\mathbb{R}^m} dt \right). \tag{3.4.7}$$

Remark 3.4.1. Observe that the inequality (3.4.7) implies that the boundary observation and control operators are admissible for $(T(t))_{t\geq 0}$. We refer the reader to [TW09] for further details on admissible observation and control operators.

As already pointed out in [Lü15], admissibility is also a suitable concept for the study of stochastic well-posed systems.

Definition 3.4.3. *1. Let* $\mathbb{B}$ *be in* $\mathcal{L}(\mathbb{R}^m, \mathcal{X}_{-1})$ *and let us consider the family of operators* $(\Phi_t)_{t\geq 0} \subset \mathcal{L}(L^2_{\mathbb{F}}([0,t];\mathbb{R}^m), L^2_{\mathcal{F}_t}(\Omega;\mathcal{X}_{-1}))$ *described by (3.4.2). The control operator* $\mathbb{B}$ *is said to be stochastically admissible for* $(T(t))_{t\geq 0}$ *if for some* $t_0 \geq 0$*, Range* $(\Phi_{t_0}) \subset L^2_{\mathcal{F}_{t_0}}(\Omega;\mathcal{X})$*.*

2. *Let us consider the family of operators* $(\Psi_t)_{t\geq 0} \subset \mathcal{L}(L^2(\Omega;D(A)), L^2_{\mathbb{F}}([0,\infty);\mathcal{X}))$ *described by (3.4.3). The observation operator* $\mathbb{C} \in \mathcal{L}(D(A), \mathbb{R}^p)$ *is said to be stochastically admissible for* $(T(t))_{t\geq 0}$ *if for some* $t_0 \geq 0$*,* Ψ_{t_0} *has a continuous extension to* $L^2(\Omega;\mathcal{X})$*.*

For the stochastic extension, the same spirit as in the deterministic case is followed, which is that the inputs (x_0,u,w) - outputs (ε, y) mapping has to be bounded. The extended notion of well-posedness used here for boundary controlled and observed (BCO) stochastic systems is based on the deterministic definition introduced by Salomon and Weiss, see [Sal89], [Sta05] and [TW09]. The notion of well-posed BCO stochastic systems is introduced as follows.

Definition 3.4.4. *A BCO stochastic system (3.1.2) is said to be well-posed if:*

- *The operator* $A : D(A) \to \mathcal{X}$ *with domain* $D(A) = D(\mathcal{A}) \cap ker(\mathcal{B})$ *and given by*

$$A\varepsilon = \mathcal{A}\varepsilon \qquad \text{for } \varepsilon \in D(A)$$

is the infinitesimal generator of a C_0*-semigroup* $(T(t))_{t\geq 0}$ *on* $\mathcal{X}$*;*

- *There exist* $t_f > 0$ *and* $m_f \geq 0$ *such that the following inequality holds for all* $\varepsilon_0 \in D(\mathcal{A})$ *and* $u \in C^2_{\mathbb{F}}([0,t_f);\mathbb{R}^m)$ *with* $u(0) = \mathcal{B}\varepsilon(0)$*:*

$$\begin{aligned} &\|\varepsilon(t_f)\|^2_{L^2_{\mathcal{F}_{t_f}}(\Omega;\mathcal{X})} + \|\mathcal{C}\varepsilon\|^2_{L^2_{\mathbb{F}}([0,t_f];\mathbb{R}^p)} \\ &\leq m_{t_f} \left(\|\varepsilon_0\|^2_{L^2_{\mathcal{F}_0}(\Omega;\mathcal{X})} + \|u(t)\|^2_{L^2_{\mathbb{F}}([0,t_f];\mathbb{R}^m)} + t_f\, Tr[Q] \right). \end{aligned} \tag{3.4.8}$$

Remark 3.4.2. 1. The inequality (3.4.8) is equivalent to

$$\begin{aligned} &\mathbb{E}\|\varepsilon(t_f)\|^2_{\mathcal{X}} + \mathbb{E}\int_0^{t_f} \|\mathcal{C}\varepsilon(t)\|^2_{\mathbb{R}^p} dt \\ &\le m_{t_f}\left(\mathbb{E}\,\|\varepsilon_0\|^2_{\mathcal{X}} + \mathbb{E}\int_0^{t_f} \|u(t)\|^2_{\mathbb{R}^m} dt + t_f\,\mathrm{Tr}[Q]\right). \end{aligned} \tag{3.4.9}$$

2. From Theorem 3.1.1, the process

$$\begin{aligned} \varepsilon(t) = T(t)(\varepsilon_0 - Bu(0)) + \int_0^t T(t-s)(\mathcal{A}Bu(s) - B\dot{u}(s))ds \\ + \int_0^t T(t-s)Hdw(s) + Bu(t) \end{aligned} \tag{3.4.10}$$

is the mild solution of the boundary controlled and observed stochastic system (3.1.2) for every $\varepsilon_0 \in \mathcal{X}$ and $u \in H^1_{\mathbb{F}}([0,t_f];\mathbb{R}^m)$. The well-posedness of (3.1.2) entails that the mild solution (3.4.10) can be extended to any $u \in L^2_{\mathbb{F}}([0,t_f];\mathbb{R}^m)$ such that the output process is mean-square integrable.

For the following study, two cases will be distinguished: either the input $u(t)$ is stochastic or deterministic. In a first step, it will be showed that if well-posedness is satisfied at least at one time, it holds for any time (see Theorem 3.4.1). In a second step, we shall consider the well-posedness of a SPHS (3.1.3)-(3.1.6) with a deterministic input acting on the mean of the process through the boundaries such that $u(t) = \mathcal{B}\mathbb{E}[\varepsilon(t)]$, where the leitmotiv will be to consider separately the deterministic and the stochastic dynamics.

3.4.1 Stochastic input $u(t) \in L^2_{\mathbb{F}}([0,t];\mathbb{R}^m)$

Theorem 3.4.1. *If the BCO stochastic system (3.1.2) as in Definition 3.1.1 is well-posed, then for all $t_f > 0$ there exists a constant $m_{t_f} > 0$ such that (3.4.9) holds.*

Proof. In order to prove the time invariance of the well-posedness definition, we rely on the well-posedness at t_0. The main argumentation consists in establishing the inequality (3.4.8) for any $t \in [0,t_0]$ by means of the system nodes formalism [Sta05] expressed in the stochastic context; next one can prove it for any $t \in [t_0, 2t_0]$; finally the general case $t > 2^n t_0$ for every $n \in \mathbb{N}$ is deduced by induction.
Step 1.
Let t be in $[0,t_0]$. The inequality (3.4.8) is given through the system nodes operator $S^b(t)\begin{bmatrix}\varepsilon_0\\ u\\ w\end{bmatrix} = (\varepsilon(t), y(t))$ by

$$\begin{aligned} \|S^b(t)\begin{bmatrix}\varepsilon_0\\ u\\ w\end{bmatrix}\|^2 &\le m_t\|\begin{bmatrix}\varepsilon_0\\ u\\ w\end{bmatrix}\|^2_{L^2_{\mathcal{F}_0}(\Omega;\mathcal{X})\oplus L^2_{\mathbb{F}}([0,t];\mathbb{R}^m)\oplus M^2_{\mathbb{F}}([0,t];Z)} \\ &= m_t(\mathbb{E}\,\|\varepsilon_0\|^2_{\mathcal{X}} + \mathbb{E}\int_0^t \|u(s)\|^2_{\mathbb{R}^m} ds + t\,\mathrm{Tr}[Q]) \end{aligned}$$

for all $\begin{bmatrix} \varepsilon_0 \\ u \\ w \end{bmatrix}$ in the domain

$$D(S^b(t)) = \left\{ \begin{bmatrix} \varepsilon_0 \\ u \\ w \end{bmatrix} \in L^2_{\mathcal{F}_0}(\Omega;\mathcal{X}) \oplus L^2_{\mathbb{F}}([0,t];\mathbb{R}^m) \oplus M^2_{\mathbb{F}}([0,t];Z) : \varepsilon_0 \in D(\mathcal{A}), u \in C^2_{\mathbb{F}}([0,t];\mathbb{R}^m), \mathcal{B}\varepsilon_0 = u(0) \right\}.$$

The case where $w(t) = 0$ is a straightforward adaptation of the argumentation of the deterministic proof with a random variable ε_0 and an $\mathbb{F}$-adapted input $u(t)$. We may take $\varepsilon_0 = 0$ and $u = 0$ hereinafter. Using the concatenation operator $\diamond$, which is defined for any L^2-functions f,g as

$$(f \underset{\tau}{\diamond} g)(t) = \begin{cases} f(t), & t < \tau, \\ g(t-\tau), & t > \tau, \end{cases} \tag{3.4.11}$$

one observes that $S_1^b(t)[w]$ is bounded for $t \in [0,t_0]$. Indeed,

$$\begin{aligned} \|S_1^b(t)[w]\|^2_{L^2_{\mathbb{F}}(\Omega;\mathcal{X})} &= \|S_1^b(t_0)[0 \underset{t_0-t}{\diamond} w]\|^2_{L^2_{\mathbb{F}}(\Omega;\mathcal{X})} \\ &\leq m(t_0)\|0 \underset{t_0-t}{\diamond} w\|^2_{M^2_{\mathbb{F}}([0,t_0];Z)} \\ &= m(t_0)\|w(\cdot - t_0 + t)\|^2_{M^2_{\mathbb{F}}([t_0-t,t_0];Z)} \\ &= m(t_0)\|w(\cdot)\|^2_{M^2_{\mathbb{F}}([0,t];Z)}, \end{aligned}$$

thanks to the well-posedness at t_0 and since a Wiener process is invariant under time translation.
Consider the continuous extension w_{ext} on $[0,t_0]$ of $w(t)$, such that $\mathbb{P}(w_{\text{ext}} = w,\ \forall s \in [0,t]) = 1$.
Since $S_2^b(t)\,[w]$ and $S_2^b(t_0)\,[w]$ take values in $L^2_{\mathbb{F}}([0,t];\mathbb{R}^p)$, we have that

$$(S_2^b(t)w)(s) = (S_2^b(t_0)w)(s) \tag{3.4.12}$$

for any $s \in [0,t]$. Now consider the particular extension $w_{\text{ext}} = w \underset{t}{\diamond} 0$. Observe that

$$\begin{aligned} \mathbb{E}\int_0^t \|(S_2^b(t)w)(s)\|^2_{\mathbb{R}^p} ds &= \mathbb{E}\int_0^t \|(S_2^b(t_0)(w \underset{t}{\diamond} 0)(s)\|^2_{\mathbb{R}^p} ds \\ &\leq \mathbb{E}\int_0^{t_0} \|(S_2^w(t_0)(w \underset{t}{\diamond} 0)(s)\|^2_{\mathbb{R}^p} ds \\ &\leq m_{t_0}\|w \underset{t}{\diamond} 0\|^2_{M^2_{\mathbb{F}}([0,t_0];Z)} \\ &= m_{t_0}\|w\|^2_{M^2_{\mathbb{F}}([0,t];Z)} = m_{t_0} t \operatorname{Tr}[Q] \end{aligned}$$

from the well-posedness at t_0.
Step 2.
In this second step we prove that the inequality holds for any $t \in [t_0, 2t_0]$. Let us

consider $t \in [t_0, 2t_0]$ which can be formulated as $t = t_0 + t_1$ with $t_1 \in [0, t_0]$. Then we obtain that

$$\begin{aligned} S_1^b(t)w &= \int_{t_0}^{t} T(t-s)Hdw(s) + \int_0^{t_0} T(t-s)Hdw(s) \\ &= \int_0^{t_1} T(t_1-r)Hd[w(r+t_0)-w(t_0)] + T(t_1)S_1^w(t_0)w \\ &= S_1^b(t_1)w(t_0+\cdot) + T(t_1)S_1^b(t_0)w \end{aligned}$$

and that

$$\begin{aligned} S_2^b(t)w(s) &= \mathcal{C}\int_0^s T(s-r)Hdw(r) \\ &= \begin{cases} (S_2^b(t_0)w)(s), & s \le t_0, \\ (S_2^b(t_1)w(t_0+\cdot))(s), & s \in (t_0, t]. \end{cases} \end{aligned}$$

From Step 1 and Step 2, we deduce that $S_1^b(t_1)$ and $S_2^b(t_1)$ have bounded extensions and so do $S_1^b(t)$ and $S_2^b(t)$. Hence, by induction, we can state that the general case $t > 2^n t_0$ holds, which completes the proof. □

Having presented a general concept of well-posedness for BCO stochastic systems, we shall now focus on the specific class of SPHSs. To establish the well-posedness property in the next section, the input $u(t)$ will be assumed to be deterministic.

3.4.2 Deterministic input

From [JZ12, Corollary 10.1.4], the sample paths $\varepsilon(t)$ given by (3.4.10) satisfy the following relation:

$$\begin{aligned} &\mathbb{E}\|\varepsilon(s)\|_{\mathcal{X}}^2 \\ =&\mathbb{E}\,\|T(s)\varepsilon_0 + \int_0^s T(s-r)\mathcal{A}Bu(r)dr - A\int_0^s T(s-r)Bu(r)dr \\ &+\int_0^s T(s-r)Hdw(r)\|_{\mathcal{X}}^2 \\ \le& 3\,\mathbb{E}\,\|T(s)\varepsilon_0\|_{\mathcal{X}}^2 + 3\|\int_0^s T(s-r)\mathcal{A}Bu(r)dr - A\int_0^s T(s-r)Bu(r)dr\|_{\mathcal{X}}^2 \\ &+3\,\mathbb{E}\,\|\int_0^s T(s-r)Hdw(r)\|_{\mathcal{X}}^2. \end{aligned} \tag{3.4.13}$$

For the corresponding output $\mathcal{C}\varepsilon(t)$, we have

$$\begin{aligned} &\mathbb{E}\int_0^t \|\mathcal{C}\varepsilon(s)\|_{\mathbb{R}^p}^2 ds \\ =&\mathbb{E}\int_0^t \|\mathcal{C}T(s)\varepsilon_0 ds + \mathcal{C}(\int_0^s T(s-r)\mathcal{A}Bu(r)dr - A\int_0^s T(s-r)Bu(r)dr) \end{aligned}$$

$$
\begin{aligned}
&+\mathcal{C}\int_0^s T(s-r)Hdw(r)\|^2_{\mathbb{R}^p}ds \\
\leq & 3\,\mathbb{E}\int_0^t \|\mathcal{C}T(s)\varepsilon_0\|^2_{\mathbb{R}^p}ds + 3\,\mathbb{E}\int_0^t \|\mathcal{C}\int_0^s T(s-r)Hdw(r)\|^2_{\mathbb{R}^p}ds \\
&+3\int_0^t \|\mathcal{C}(\int_0^s T(s-r)\mathcal{A}Bu(r)dr - A\int_0^s T(s-r)Bu(r)dr)\|^2_{\mathbb{R}^p}ds. \qquad (3.4.14)
\end{aligned}
$$

Theorem 3.4.2 (Well-posedness of SPHSs)**.** *Consider a stochastic port-Hamiltonian system (3.1.3)-(3.1.6) satisfying Assumptions 3.1.1 and 3.1.2. In addition, assume that:*

1. *the multiplication operator $P_1\mathcal{H}$ can be written as*

$$P_1\mathcal{H}(\zeta) = S^{-1}(\zeta)\Delta(\zeta)S(\zeta), \qquad \zeta\in[a,b], \tag{3.4.15}$$

where Δ is a diagonal matrix-valued function, S is a matrix-valued function and both Δ and S are continuously differentiable on $[a,b]$;

2. *$HQ^{1/2}Z \subset D(A)$;*
3. *$\int_0^t \|AT(s)H\|^2_{L_2^0}ds < \infty$ for all $t\geq 0$.*

Then the SPHS (3.1.3)-(3.1.6) is well-posed and furthermore, for all $t_f>0$ there exists a constant $m_{t_f}>0$ such that

$$
\begin{aligned}
&\mathbb{E}\,\|\varepsilon(t_f)\|^2_{\mathcal{X}} + \mathbb{E}\int_0^{t_f} \|\mathcal{C}\varepsilon(t)\|^2_{\mathbb{R}^p}dt \\
&\qquad \leq m_{t_f}\left(\mathbb{E}\,\|\varepsilon_0\|^2_{\mathcal{X}} + \int_0^{t_f}\|u(t)\|^2_{\mathbb{R}^m}dt + t_f\,Tr[Q]\right).
\end{aligned} \tag{3.4.16}
$$

Proof. The deterministic and the stochastic dynamics are split to obtain

$$
\begin{aligned}
&\mathbb{E}\|\varepsilon(t_f)\|^2_{\mathcal{X}} + \mathbb{E}\int_0^{t_f}\|\mathcal{C}\varepsilon(s)\|^2_{\mathbb{R}^p}ds \\
\leq & 3\,\mathbb{E}\,\|T(t_f)\varepsilon_0\|^2_{\mathcal{X}} + 3\|\int_0^{t_f} T(t_f-s)\mathcal{A}Bu(s)ds - A\int_0^{t_f}T(t_f-s)Bu(s)ds\|^2_{\mathcal{X}} \\
&+3\int_0^{t_f}\mathbb{E}\,\|\mathcal{C}T(s)\varepsilon_0\|^2_{\mathbb{R}^p}ds \\
&+3\int_0^{t_f}\|\mathcal{C}\int_0^s T(s-r)\mathcal{A}Bu(r)dr - \mathcal{C}A\int_0^s T(s-r)Bu(r)dr\|^2_{\mathbb{R}^p}ds \\
&+3\,\mathbb{E}\,\|\int_0^{t_f}T(t_f-s)Hdw(s)\|^2_{\mathcal{X}} + 3\,\mathbb{E}\int_0^{t_f}\|\mathcal{C}\int_0^s T(s-r)Hdw(r)\|^2_{\mathbb{R}^p}ds.
\end{aligned} \tag{3.4.17}
$$

The deterministic part has already been set out in [ZGMV10, Theorem 2.4] with $\varepsilon_0 = 0$. The stochastic part is set out by the stochastic admissibility of $\mathcal{C}$, the fact that $H\in L_2^0$ and the following calculation:

$$\mathbb{E}\int_0^{t_f}\|\mathcal{C}\int_0^s T(s-r)Hdw(r)\|^2_{\mathbb{R}^p}ds = \int_0^{t_f}\int_0^s \|\mathcal{C}T(s-r)HQ^{1/2}\|^2_{L_2}dr\,ds$$

$$\begin{aligned}
&\int_0^{t_f}\int_0^s \|\mathcal{C}T(s-r)HQ^{1/2}\|_{L_2}^2 dr\, ds \\
&\leq \int_0^{t_f}\int_0^s c\, s\|HQ^{1/2}\|_{L_2}^2 dr\, ds + \int_0^{t_f}\int_0^s \|AT(s-r)HQ^{1/2}\|_{L_2}^2 dr\, ds \\
&\leq c(t_f)\|H\|_{L_2}^2 t_f \operatorname{Tr}[Q]
\end{aligned}$$

where we used Assumptions 2., 3. and the boundedness of $\mathcal{C}$ with respect to the graph norm. This concludes the proof of well-posedness. Moreover, Theorem 3.4.1 entails that the well-posedness holds for any $t_f > 0$. □

Theorem 3.4.2 will now be used to establish the well-posedness of a model of an inhomogeneous vibrating string as in Example 1.1 and subject to some space and time dependent Gaussian white noise process. As proved in Example 3.1, the vibrating string with an appropriate choice of boundary conditions is an example of a nice port-Hamiltonian system, and thus is a Riesz-spectral system. The vertical position $z(\zeta,t)$ of the string at position $\zeta \in [a,b]$ and time $t \in [0,\tau]$ satisfies the following stochastic partial differential equation

$$\frac{\partial^2 z}{\partial t^2}(\zeta,t) = \frac{1}{\rho(\zeta)}\frac{\partial}{\partial \zeta}\left(T(\zeta)\frac{\partial z}{\partial \zeta}(\zeta,t)\right) + \frac{1}{\rho(\zeta)}\eta(\zeta,t), \tag{3.4.18}$$

where $z(\zeta,0) = z_0(\zeta)$ is the initial condition, which may be random. $T(\zeta)$ and $\rho(\zeta)$ are the Young's modulus and the mass density at position ζ, respectively. The stochastic disturbance is assumed to have intensity $\frac{1}{\rho(\zeta)}$, which means that making the string heavier decreases the impact of the random force.
As boundary conditions, we consider

$$T(a)\frac{\partial z}{\partial \zeta}(a,t) = u(t), \qquad T(b)\frac{\partial z}{\partial \zeta}(b,t) + \frac{\partial z}{\partial t}(b,t) = 0, \tag{3.4.19}$$

where $u(t) \in L^2([0,t];\mathbb{R}^m)$. As it is known, the deterministic dynamics fit in the port-Hamiltonian framework. Then the SPDE (3.4.18) can be rewritten as (3.1.1), where

$$P_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \text{and} \qquad \mathcal{H}(\zeta) = \begin{bmatrix} \frac{1}{\rho(\zeta)} & 0 \\ 0 & T(\zeta) \end{bmatrix}.$$

As the output, we measure the velocity at extremity a, i.e.

$$y(t) = \frac{\partial z}{\partial t}(a,t). \tag{3.4.20}$$

We are now in position to verify the assumptions of Theorem 3.4.2.

$$\begin{aligned}
\int_0^\tau \|AT(s)H\|_{L_2^0}^2 ds &= \int_0^\tau \operatorname{Tr}\left[AT(s)HQ^{1/2}(AT(s)HQ^{1/2})^*\right] ds \\
&= \sum_{i=1}^\infty \int_0^\tau \langle AT(s)HQ^{1/2}f_i, AT(s)HQ^{1/2}f_i\rangle_{\mathcal{X}} ds
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{\infty} \int_0^{\tau} \|AT(s)HQ^{1/2} f_i\|_{\mathcal{X}}^2 ds \\
&= \sum_{i=1}^{\infty} \int_0^{\tau} \| \sum_{k=1}^{\infty} \lambda_k \langle T(s)HQ^{1/2} f_i, \psi_k \rangle_{\mathcal{X}} \phi_k \|_{\mathcal{X}}^2 ds \\
&= \sum_{i=1}^{\infty} \int_0^{\tau} \| \sum_{k=1}^{\infty} \lambda_k e^{\lambda_k s} \langle HQ^{1/2} f_i, \psi_k \rangle_{\mathcal{X}} \phi_k \|_{\mathcal{X}}^2 ds,
\end{aligned}
$$

where we used the eigenfunction expansion of A and $(T(t))_{t \geq 0}$. This leads to

$$
\begin{aligned}
\int_0^{\tau} \|AT(s)H\|_{L_2^0}^2 ds &\leq M \sum_{i=1}^{\infty} \int_0^{\tau} \sum_{k=1}^{\infty} |\lambda_k|^2 |e^{\lambda_k s}|^2 |\langle HQ^{1/2} f_i, \psi_k \rangle_{\mathcal{X}}|^2 ds \\
&= M \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} |\lambda_k|^2 |\langle HQ^{1/2} f_i, \psi_k \rangle_{\mathcal{X}}|^2 \int_0^{\tau} e^{2 \operatorname{Re} \lambda_k s} ds \\
&= \frac{M}{2} \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} \frac{|\lambda_k|^2}{\operatorname{Re} \lambda_k} (e^{2 \operatorname{Re} \lambda_k \tau} - 1) q_i |\langle H f_i, \psi_k \rangle_{\mathcal{X}}|^2 \\
&\leq K \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} (\operatorname{Im} \lambda_k)^2 q_i |\langle H f_i, \psi_k \rangle_{\mathcal{X}}|^2 < \infty
\end{aligned}
$$

where the Dominated Convergence Theorem is satisfied under the assumptions that

$$
\begin{cases}
\sum_{i=1}^{\infty} \sum_{k=1}^{\infty} (2k+1)^2 \pi^2 (q_i) |\langle H f_i, \psi_k \rangle_{\mathcal{X}}|^2 < \infty, & \text{if } \sqrt{T\rho}(b) < 1 \\
\sum_{i=1}^{\infty} \sum_{k=1}^{\infty} (2k)^2 \pi^2 (q_i) |\langle H f_i, \psi_k \rangle_{\mathcal{X}}|^2 < \infty, & \text{if } \sqrt{T\rho}(b) > 1
\end{cases}. \tag{3.4.21}
$$

Moreover, the multiplication operator $P_1 \mathcal{H}$ can be rewritten as

$$
P_1 \mathcal{H} = \begin{bmatrix} \gamma & -\gamma \\ \frac{1}{\rho} & \frac{1}{\rho} \end{bmatrix} \begin{bmatrix} \gamma & 0 \\ 0 & -\gamma \end{bmatrix} \begin{bmatrix} \frac{1}{2\gamma} & \frac{\rho}{2} \\ -\frac{1}{2\gamma} & \frac{\rho}{2} \end{bmatrix}, \tag{3.4.22}
$$

where $\gamma = \sqrt{\frac{T}{\rho}}$. By using Theorem 3.4.2, one can conclude that the stochastic vibrating string described by (3.4.18)-(3.4.20) is a well-posed BCO system in the sense of Definition 3.4.4, and thus for all $\tau > 0$ there exists a constant $m_\tau > 0$ such that for any $\varepsilon_0 \in L^2_{\mathcal{F}_0}(\Omega; \mathcal{X})$ and $u \in L^2([0,\tau]; \mathbb{R})$ the inequality (3.4.16) holds.

3.5 Conclusion & perspectives

In this chapter we have extended the class of linear port-controlled Hamiltonian systems to a stochastic setting. Some properties of this class have been investigated such as passivity and well-posedness.

This chapter lays the foundation of a theory for boundary controlled and observed stochastic port-Hamiltonian systems and expresses the author's willingness to yield new directions for future research. Further works would be to consider multiplicative noise and noise in the boundary control and/or observation, which would extend the range of considered disturbances. Moreover, well-posed linear systems are closely related to regular systems, i.e. systems for which there is a well-defined feedthrough term given by the strong limit to infinity of the transfer function [JZ12, Chapter 13]. This would be interesting and maybe challenging to broaden the notion of regular systems in a stochastic context. As a more general matter, well-posed linear systems are known to enjoy many useful properties in the context of feedback control and dynamic stabilization. Hence, investigating which results still hold for well-posed stochastic systems would be a relevant future research topic.

Chapter 4

LQG Control of stochastic port-Hamiltonian systems

In the deterministic setting, the linear quadratic (LQ) optimal control problem has attracted a lot of consideration over the years for finite and infinite-dimensional systems. More particularly, in the context of infinite-dimensional systems with bounded control and observation operators, the LQ optimal control problem was mainly studied by Lions [Lio71], Curtain and Pritchard [CP78], and Balakrishnan [Bal76], among others in the seventies. In the stochastic setting, the linear quadratic control problem for finite-dimensional stochastic systems was first studied by Wonham [Won68a] and [Won68b], and Kushner [Kus62] by means of a dynamic programming approach. Soon afterwards, an infinite-dimensional generalization was developed by Ichikawa for bounded control and noise operators [Ich79]. More precisely, the stochastic optimal control is expressed under the form of a feedback control law, via the resolution of an operator Riccati equation in a similar manner as for the finite-dimensional case.

Furthermore, in addition to environment disturbances, measurement noises can occur, which are typically related to the quality of the sensors used to observe the state process. The linear quadratic gaussian (LQG) control problem is an efficient way to take environment and measurement noises into consideration. A study of the LQG control problem and a generalization of the so-called separation principle can be found in [Cur78] and [CI77a]. As far as known, no attempt to develop an adapted approach to study and solve the LQG control problem for infinite-dimensional SPHSs has been undertaken in the literature. So far, most of the attention has been oriented towards either general infinite-dimensional systems or very specific classes of systems.

In this chapter general results concerning the optimal control of stochastic port-Hamiltonian systems with incomplete observation are provided. The LQG control problem for infinite-dimensional SPHSs is addressed in a stochastic context.

The chapter is organized as follows. In Section 4.1, the class of stochastic port-Hamiltonian systems under study is presented and the LQG control problem is addressed and solved for this specific class of stochastic systems. Section 4.2 is mainly devoted to the port-Hamiltonian structure preserving in the LQG controller dynamics. In Section 4.3, the exponential stability is proved for PHSs under a specific structure. This ensures that the usual assumptions for a well-posed LQG control problem are satisfied. Finally, the exponential stabilizablity and detectability assumptions related to the existence of a unique feedback control solution to the LQG control problem are shown to be verified for this subclass of PHSs. This chapter ends with some conclusions and perspectives.

4.1 Problem setting

In this section the LQG control problem for stochastic port-Hamiltonian systems is addressed, with bounded control, observation and noise operators. The ports of the system are assumed to be closed, which means that no exchange of energy occurs at the boundary. The input is applied on intervals within the spatial domain. Furthermore, some dissipation occurs inside the domain and is captured by a semi-definite positive self-adjoint operator $\mathcal{R} \in \mathcal{L}(\mathcal{X})$. The observation model is taken to be the power-conjugated output of the system subject to some measurement Wiener noise $v(t)$ with incremental covariance matrix $V \in \mathbb{R}^{m\times m}$ and intensity $F \in \mathbb{R}^{m\times m}$. Moreover, since we focus on distributed controls throughout this chapter, the distributed input $u_d(t)$ will be simply denoted by $u(t)$. Hence, we consider stochastic port-Hamiltonian systems under the form:

$$d\varepsilon(t) = ((\mathcal{J}-\mathcal{R})\mathcal{H}\varepsilon(t) + Bu(t))dt + Hdw(t), \tag{4.1.1}$$

$$d\gamma(t) = B^*\mathcal{H}\varepsilon(t)dt + Fdv(t), \tag{4.1.2}$$

where $\varepsilon(0) = \varepsilon_0$ is a $\mathcal{X}$-valued Gaussian random variable. The stochastic system is interacting with its environment through a distributed control port and a noise port. The operators $B \in \mathcal{L}(\mathbb{R}^m, \mathcal{X})$ and $H \in \mathcal{L}(Z, \mathcal{X})$ capture the deterministic and stochastic interactions with the environment, respectively.

The LQG control problem consists in minimizing the following functional:

$$J(u) = \lim_{T\to\infty} \mathbb{E}\int_0^T \|B^*\mathcal{H}\varepsilon(t)\|^2_{\mathbb{R}^m} + \|\tilde{R}^{1/2}u(t)\|^2_{\mathbb{R}^m}dt, \tag{4.1.3}$$

under admissible control laws $u \in L^2_{\mathbb{F}}([0,\infty);\mathbb{R}^m)$ with a weighting matrix $\tilde{R}$ assumed to be symmetric and positive definite. To be more specific, we assume that the control process is mean-square integrable, i.e.

$$\mathbb{E}\int_0^\infty \|u(s)\|^2_{\mathbb{R}^m}ds < \infty, \tag{4.1.4}$$

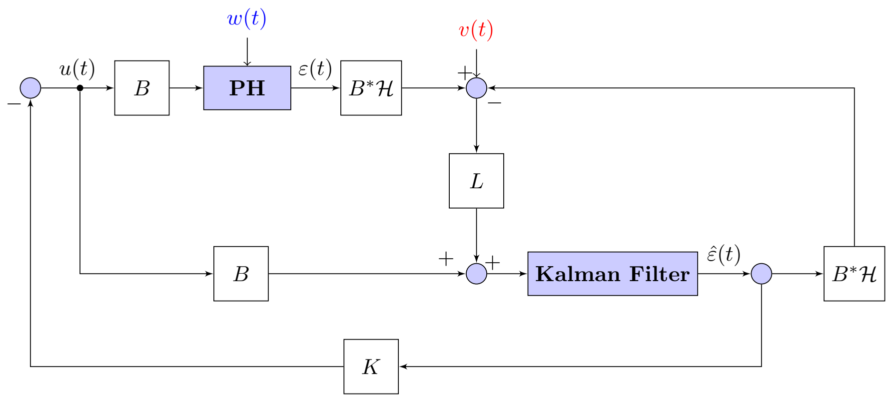

Figure 4.1 – Structure of the LQG control problem

and that for all $0 \le s \le t$, $u(t)$ is independent of the Wiener increments $w(t) - w(s)$. Notice that the independence with respect to the Wiener increments is a usual assumption in stochastic optimal control theory. It ensures that $u(t)$ only relies on the present information of the noise $w(t)$. Besides, notice that $\mathcal{H}BB^*\mathcal{H}$ is positive semi-definite.

We define now the notions of exponentially stabilizable and detectable.

Definition 4.1.1. *1. $((\mathcal{J}-\mathcal{R})\mathcal{H}, B)$ is said to be exponentially (exp.) stabilizable if there exists an operator $K \in \mathcal{L}(\mathcal{X}, \mathbb{R}^m)$ such that $(\mathcal{J}-\mathcal{R})\mathcal{H} - BK$ generates an exp. stable C_0-semigroup $(T_{(\mathcal{J}-\mathcal{R})\mathcal{H}-BK}(t))_{t\ge 0}$.*

2. $(B^*\mathcal{H}, (\mathcal{J}-\mathcal{R})\mathcal{H})$ is said to be exponentially (exp.) detectable if there exists an operator $L \in \mathcal{L}(\mathbb{R}^m, \mathcal{X})$ such that $(\mathcal{J}-\mathcal{R})\mathcal{H} - LB^*\mathcal{H}$ generates an exp. stable C_0-semigroup $(T_{(\mathcal{J}-\mathcal{R})\mathcal{H}-LB^*\mathcal{H}}(t))_{t\ge 0}$.

As it is known from [Cur78], the LQG control problem can be separated in two problems: the best estimation problem based on the observation $(\gamma(t))_{t\ge 0}$ and the LQ optimal control problem with complete observation as depicted in Figure 4.1. Under the assumptions of exp. stabilizability and exp. detectability, we have the following:

- The best estimate $\hat{\varepsilon}(t)$ of the state $\varepsilon(t)$ is governed by the Kalman filter equation, i.e.,

$$\begin{aligned} d\hat{\varepsilon}(t) &= ((\mathcal{J}-\mathcal{R})\mathcal{H}\hat{\varepsilon}(t) + Bu(t))dt + L(d\gamma(t) - B^*\mathcal{H}\hat{\varepsilon}(t)dt), \\ \hat{\varepsilon}(0) &= \mathbb{E}[\varepsilon_0], \end{aligned} \tag{4.1.5}$$

  where $L := P_f\mathcal{H}B(FVF^*)^{-1}$ in which P_f is the stabilizing nonnegative self-adjoint solution of the filter operator Riccati equation (FORE) given by

$$[(\mathcal{J}-\mathcal{R})\mathcal{H}P_f + P_f((\mathcal{J}-\mathcal{R})\mathcal{H})^* - P_f\mathcal{H}B(FVF^*)^{-1}B^*\mathcal{H}P_f + HQH^*]x = 0, \tag{4.1.6}$$

for all $x \in D(((\mathcal{J}-\mathcal{R})\mathcal{H})^*)$, with $P_f(D(((\mathcal{J}-\mathcal{R})\mathcal{H})^*)) \subset D((\mathcal{J}-\mathcal{R})\mathcal{H})$.

- The optimal control problem with complete observation of the state process can be solved using the control operator Riccati equation (CORE)

$$[((\mathcal{J}-\mathcal{R})\mathcal{H})^* P_c + P_c(\mathcal{J}-\mathcal{R})\mathcal{H} + \mathcal{H}BB^*\mathcal{H} - P_c B\tilde{R}^{-1}B^*P_c]x = 0, \quad (4.1.7)$$

  for all $x \in D((\mathcal{J}-\mathcal{R})\mathcal{H})$, with $P_c(D((\mathcal{J}-\mathcal{R})\mathcal{H})) \subset D(((\mathcal{J}-\mathcal{R})\mathcal{H})^*)$. The optimal feedback control is given under the feedback form as

$$u_*(t) = -\tilde{R}^{-1}B^*P_c\hat{\varepsilon}_*(t)$$

  with corresponding optimal estimated state $\hat{\varepsilon}_*(t)$.

Observe that the control Riccati equation (4.1.7) is completely deterministic. As a consequence, the relation between the optimal feedback control and the estimated state $\hat{\varepsilon}(t)$ is also deterministic.

In order to avoid any dependence of the control law with respect to observation process, the cost (4.1.3) is minimized over a certain class of admissible controls denoted by $\mathcal{U}_{ad} \subset L^2_{\mathbb{F}}([0,\infty)\times\Omega;\mathbb{R}^m)$. This admissible set is chosen as

$$\mathcal{U}_{\text{ad}} = \{u : u(t) \text{ is adapted to } \sigma(\gamma(s) : 0 \leq s \leq t) \text{ and to } \sigma(\vartheta(s) : 0 \leq s \leq t)\}, \quad (4.1.8)$$

where $(\vartheta(t))_{t\geq 0}$ is the innovation process given by

$$d\vartheta(t) = d\gamma(t) - B^*\mathcal{H}\hat{\varepsilon}(t)dt \quad (4.1.9)$$

with incremental covariance matrix FVF^*.
In order to solve the LQG control problem (4.1.1)-(4.1.3) the following Lemma 4.1.1 is needed, in which the solution of the LQ optimal control problem assuming the complete observation of the state process is given.

Lemma 4.1.1. *Assume that $((\mathcal{J}-\mathcal{R})\mathcal{H}, B)$ and $((\mathcal{J}-\mathcal{R})\mathcal{H}, HQ^{1/2})$ are exp. stabilizable and that $(B^*\mathcal{H},\ (\mathcal{J}-\mathcal{R})\mathcal{H})$ is exp. detectable. Consider the problem of minimizing $J(u)$ given by (4.1.3) subject to*

$$d\varepsilon(t) = ((\mathcal{J}-\mathcal{R})\mathcal{H} + Bu(t))dt + Hd\kappa(t), \quad (4.1.10)$$

where $H \in \mathcal{L}(\mathbb{R}^m, \mathcal{X})$ and $\kappa(t)$ is a $\mathbb{R}^m$-valued Wiener process with incremental covariance operator F_0. Then there exists a unique optimal control $u_* \in L^2_{\mathbb{F}}([0,\infty);\mathbb{R}^m)$ with corresponding optimal state $\varepsilon_*(t)$ such that u_* is adapted to $\sigma(\kappa(s) : 0 \leq s \leq t)$ and given by

$$u_*(t) = -\tilde{R}^{-1}B^*P_c\varepsilon_*(t) \quad (4.1.11)$$

with

$$\varepsilon_*(t) = \tilde{S}(t)\varepsilon_0(t) + \int_0^t \tilde{S}(t-s)Hd\kappa(s), \quad (4.1.12)$$

where $(\tilde{S}(t))_{t\geq 0}$ is the exponentially stable C_0-semigroup generated by $(\mathcal{J}-\mathcal{R})\mathcal{H} - B\tilde{R}^{-1}B^*P_c$, P_c is the solution of the CORE (4.1.7).

Proof. See the proof of [CP78, Theorem 7.8] for details. □

Remark 4.1.1. Notice that a similar study can be conducted in Lemma 4.1.1 by replacing the assumptions of exp. stabilizability of $((\mathcal{J}-\mathcal{R})\mathcal{H}, B)$ by the assumption of an optimizable cost (4.1.3), i.e., for every $\varepsilon_0 \in \mathcal{X}$, there exists $u \in L^2_{\mathbb{F}}([0,\infty);\mathbb{R}^m)$ such that $J(u)$ is finite. Besides remark that the exp. stabilizability $((\mathcal{J}-\mathcal{R})\mathcal{H}, B)$ implies that the cost (4.1.3) is optimizable. For further details, see [BDPDM06, Part 5] and [WW97b].

By an analysis going along the lines of [CI77b, Theorem 2.3], the separation principle can be stated for SPHSs as follows.

Theorem 4.1.2. *(Separation principle)*
Assume that $((\mathcal{J}-\mathcal{R})\mathcal{H}, B)$ and $((\mathcal{J}-\mathcal{R})\mathcal{H}, HQ^{1/2})$ are exp. stabilizable and that $(B^*\mathcal{H}, (\mathcal{J}-\mathcal{R})\mathcal{H})$ is exp. detectable. Consider the problem of minimizing $J(u)$ given by (4.1.3) subject to (4.1.1) and (4.1.2) over the class of admissible controls $\mathcal{U}_{ad}$. Then there exists a unique optimal control $u_* \in \mathcal{U}_{ad}$ given by

$$u_*(t) = -\tilde{R}^{-1}B^*P_c\hat{\varepsilon}_*(t), \tag{4.1.13}$$

$$\hat{\varepsilon}_*(t) = S(t)\hat{\varepsilon}_0 + \int_0^t S(t-s)P_f\mathcal{H}B(FVF^*)^{-1}d\gamma(s), \tag{4.1.14}$$

where $(S(t))_{t\geq 0}$ is the C_0-semigroup generated by $((\mathcal{J}-\mathcal{R})\mathcal{H} - B\tilde{R}^{-1}B^*P_c - P_f\mathcal{H}B\,(FVF^*)^{-1}\,B^*\mathcal{H})$, P_c and P_f are the solutions of the CORE (4.1.7) and the FORE (4.1.6), respectively. The problem of minimizing $J(u)$ given by (4.1.3) subject to (4.1.1) and (4.1.2) is equivalent to the problem of minimizing

$$\hat{J}(u) = \lim_{T\to\infty}\mathbb{E}\int_0^T \|B^*\mathcal{H}\hat{\varepsilon}(s)\|^2_{\mathbb{R}^m} + \|\tilde{R}^{1/2}u(s)\|^2_{\mathbb{R}^m}ds \tag{4.1.15}$$

subject to (4.1.5).

Proof. Since $((\mathcal{J}-\mathcal{R})\mathcal{H}, B)$ and $((\mathcal{J}-\mathcal{R})\mathcal{H}, HQ^{1/2})$ are exponentially stabilizable and $(B^*\mathcal{H},(\mathcal{J}-\mathcal{R})\mathcal{H})$ is exponentially detectable, (4.1.7) and (4.1.6) have unique exponentially stabilizing nonnegative self-adjoint solutions P_c and P_f, respectively, see [CP78]. The optimal control is characterized among the class of $\sigma(\vartheta(s) : 0 \leq s \leq t)$-adapted controls. The problem of minimizing $J(u)$ is equivalent to minimizing

$$\hat{J}(u) = \lim_{T\to\infty}\mathbb{E}\int_0^T \|B^*\mathcal{H}\hat{\varepsilon}(t)\|^2_{\mathbb{R}^m} + \|\tilde{R}^{1/2}u(t)\|^2_{\mathbb{R}^m}dt, \tag{4.1.16}$$

with

$$\hat{\varepsilon}(t) = T(t)\hat{\varepsilon}_0 + \int_0^t T(t-s)Bu(s)ds + \int_0^t T(t-s)P_f\mathcal{H}B(FVF^*)^{-1}d\vartheta(s). \tag{4.1.17}$$

Since ϑ is a $\mathbb{R}^m$-Wiener process with incremental covariance matrix FVF^*, we can apply Lemma 4.1.1 and deduce the existence of a unique minimizing $u_* \in L^2_{\mathbb{F}}([0,\infty);\mathbb{R}^m)$ adapted to $\sigma(\vartheta(s) : 0 \leq s \leq t)$ and given by

$$u_*(t) = -\tilde{R}^{-1}B^*P_c\hat{\varepsilon}_*(t) \tag{4.1.18}$$

with

$$\hat{\varepsilon}_*(t) = \tilde{S}(t)\hat{\varepsilon}_0 + \int_0^t \tilde{S}(t-s)P_f\mathcal{H}B(FVF^*)^{-1}d\vartheta(s). \tag{4.1.19}$$

Notice that the innovation process is given by

$$d\vartheta(t) = d\gamma(t) - B^*\mathcal{H}\hat{\varepsilon}_*(t). \tag{4.1.20}$$

Substituting (4.1.20) in (4.1.19), we get that

$$\begin{aligned}\hat{\varepsilon}_*(t) &= \tilde{S}(t)\hat{\varepsilon}_0 + \int_0^t \tilde{S}(t-s)P_f\mathcal{H}B(FVF^*)^{-1}d\gamma(s)\\ &\quad - \int_0^t \tilde{S}(t-s)P_f\mathcal{H}B(FVF^*)^{-1}B^*\mathcal{H}\hat{\varepsilon}_*(s)d(s),\end{aligned} \tag{4.1.21}$$

which is equivalent to

$$\hat{\varepsilon}_*(t) = S(t)\hat{\varepsilon}_0 + \int_0^t S(t-s)P_f\mathcal{H}B(FVF^*)^{-1}d\gamma(s), \tag{4.1.22}$$

where $S(t)$ is defined as the perturbation of $\tilde{S}(t)$ by $-P_f\mathcal{H}B(FVF^*)^{-1}B^*\mathcal{H}$. The relation (4.1.22) entails that $\hat{\varepsilon}_*(t)$ is $\sigma(\gamma(s) : 0 \le s \le t)$-adapted, and then $u_*(t) \in \mathcal{U}_{\text{ad}}$. □

4.2 Structure preserving for the LQG controller

As a direct consequence of Theorem 4.1.2, the LQG controller can be expressed as

$$\begin{aligned}u_c(t) &= -K\hat{\varepsilon}(t) := -\tilde{R}^{-1}B^*P_c\hat{\varepsilon}(t),\\ d\hat{\varepsilon}(t) &= (\mathcal{J}-\mathcal{R})\mathcal{H}\hat{\varepsilon}(t)dt + Ld\vartheta(t) + Bu_c(t)dt\end{aligned}$$

where $L := P_f\mathcal{H}B(FVF^*)^{-1}$. Since the output of the controlled system is the input of the controller, the LQG controller is described by

$$u_c(t) = -K\hat{\varepsilon}(t) := -\tilde{R}^{-1}B^*P_c\hat{\varepsilon}(t), \tag{4.2.1}$$

$$d\hat{\varepsilon}(t) = [(\mathcal{J}-\mathcal{R})\mathcal{H} - LB^*\mathcal{H} - BK]\,\hat{\varepsilon}(t)dt + Ld\gamma(t), \tag{4.2.2}$$

$$\hat{\varepsilon}(0) = \mathbb{E}[\varepsilon(0)], \tag{4.2.3}$$

$$\gamma_c(t) = K\hat{\varepsilon}(t). \tag{4.2.4}$$

In (4.2.2) and (4.2.4), the LQG controller is rewritten under the port-Hamiltonian formalism, i.e.,

$$\begin{aligned}d\hat{\varepsilon}(t) &= (\mathcal{J}-\mathcal{R}_c)\mathcal{H}\hat{\varepsilon}(t)dt + Ld\gamma(t),\\ \gamma_c(t) &= K\hat{\varepsilon}(t),\end{aligned} \tag{4.2.5}$$

by considering a new dissipative operator

$$\mathcal{R}_c := \mathcal{R} + P_f\mathcal{H}B(FVF^*)^{-1}B^* + B\tilde{R}^{-1}B^*P_c\mathcal{H}^{-1} \in \mathcal{L}(\mathcal{X}), \tag{4.2.6}$$

but with the same Hamiltonian density $\mathcal{H}$. Observe that the operator $\mathcal{R}_c$ is neither semi-definite positive nor self-adjoint in general.

In this section conditions are derived to preserve the port-Hamiltonian structure of the LQG controller. In this way, the LQG control problem could be interpreted as the power-preserving interconnection of stochastic port-Hamiltonian systems. On finite-dimensional spaces, a modified LQG controller was proposed for port-Hamiltonian systems in [WHGM18] in order to ensure that the port Hamiltonian structure and the passivity are preserved in closed-loop dynamics. This LQG method can be seen as an adaptation of [JS83]. In Proposition 4.2.1, this passive LQG Hamiltonian controller is recalled.

Proposition 4.2.1. *Let us consider a linear dissipative port-Hamiltonian system described by*

$$\begin{aligned} \dot{\tilde{x}}(t) &= (\tilde{J}-\tilde{R})\tilde{H}\tilde{x}(t)+\tilde{B}u(t)+\tilde{w}(t), \\ \tilde{y}(t) &= \tilde{B}^T\tilde{H}\tilde{x}(t)+\tilde{v}(t), \end{aligned} \tag{4.2.7}$$

where the state variable $\tilde{x}(t)\in\mathbb{R}^n$, the input variable $u(t)\in\mathbb{R}^m$, the output variable $\tilde{y}(t)\in\mathbb{R}^m$, and the noise processes $\tilde{w}(t)\in\mathbb{R}^n, \tilde{v}(t)\in\mathbb{R}^m$ with covariance matrices $\tilde{Q}_w$ and $\tilde{Q}_v$, respectively.
Consider the LQG control problem with the functional cost

$$J_c = \lim_{T\to\infty}\mathbb{E}\int_0^T (\tilde{x}^T\tilde{R}_x x + u^T\tilde{R}_u u)dt. \tag{4.2.8}$$

Assume the following relation

$$\tilde{Q}_v = \tilde{R}_u, \tag{4.2.9}$$

between the covariance matrix $\tilde{Q}_v$ and the weighting matrix $\tilde{R}_u$ and the following relation

$$\tilde{Q}_w = \tilde{H}^{-1}(2\tilde{H}\tilde{J}^T\tilde{P}_c + 2\tilde{P}_c\tilde{J}\tilde{H} + \tilde{R}_x)\tilde{H} \tag{4.2.10}$$

between the covariance matrix $\tilde{Q}_w$ and the weighting matrix $\tilde{R}_x$. In this case,

$$\tilde{P}_c\tilde{H}^{-1} = \tilde{H}\tilde{P}_f. \tag{4.2.11}$$

Furthermore, by assuming that the port-Hamiltonian system (4.2.7) is stable, the corresponding control algebraic Riccati equation and the filter algebraic Riccati equation admit unique positive semi-definite solutions, the LQG controller is passive, and the closed-loop system can be written as the feedback interconnection of the port-Hamiltonian system with the port-Hamiltonian realization of the LQG controller.

A generalization of Proposition 4.2.1 is now proposed on infinite-dimensional spaces. Under suitable conditions (see Theorem 4.2.2), the LQG controller is proved to describe a stochastic dissipative port-Hamiltonian system with dissipative operator $\mathcal{R}_c \in \mathcal{L}(\mathcal{X})$ given by (4.2.6).

Theorem 4.2.2. *If we assume that the following link*

$$\tilde{R} = FVF^* \tag{4.2.12}$$

holds and if the Riccati operators satisfy the relation

$$\mathcal{H}P_f x = P_c\mathcal{H}^{-1}x \tag{4.2.13}$$

for all $x \in D((\mathcal{J}-\mathcal{R})\mathcal{H}) \cap D(((\mathcal{J}-\mathcal{R})\mathcal{H})^*)$, then the LQG controller given by (4.2.1)-(4.2.4) describes a dissipative stochastic port-Hamiltonian system. Moreover, the covariance operator Q and the weighting operator $\mathcal{H}BB^*\mathcal{H}$ are related by

$$\begin{aligned}\mathcal{H}BB^*\mathcal{H}x = [\mathcal{H}HQH^*\mathcal{H} + (\mathcal{J}\mathcal{H}+\mathcal{H}\mathcal{J})P_c - P_c(\mathcal{J}\mathcal{H}+\mathcal{H}^{-1}\mathcal{J}\mathcal{H}^2)\\ +(\mathcal{R}\mathcal{H}-\mathcal{H}\mathcal{R})P_c - P_c(\mathcal{H}^{-1}\mathcal{R}\mathcal{H}^2-\mathcal{R}\mathcal{H})]x,\end{aligned} \tag{4.2.14}$$

for all $x \in D((\mathcal{J}-\mathcal{R})\mathcal{H}) \cap D(((\mathcal{J}-\mathcal{R})\mathcal{H})^*)$.

Proof. In order to describe a dissipative stochastic port-Hamiltonian system, the energy dissipation operator

$$\mathcal{R}_c = \mathcal{R} + P_f\mathcal{H}B(FVF^*)^{-1}B^* + B\tilde{R}^{-1}B^*P_c\mathcal{H}^{-1} \tag{4.2.15}$$

must be self-adjoint and positive semi-definite. On one hand, since $\mathcal{H}$ and $\mathcal{R}$ are self-adjoint operators, there holds

$$\mathcal{R}_c^* = \mathcal{R} + B(FVF^*)^{-1}B^*\mathcal{H}P_f + \mathcal{H}^{-1}P_cB(\tilde{R}^*)^{-1}B^*.$$

Hence, $\mathcal{R}_c$ is self-adjoint if the following conditions hold:

$$\begin{aligned}B(FVF^*)^{-1}B^*\mathcal{H}P_f x &= B\tilde{R}^{-1}B^*P_c\mathcal{H}^{-1}x,\\ \mathcal{H}^{-1}P_cB(\tilde{R}^*)^{-1}B^*x &= P_f\mathcal{H}B(FVF^*)^{-1}B^*x,\end{aligned}$$

for all $x \in D((\mathcal{J}-\mathcal{R})\mathcal{H}) \cap D(((\mathcal{J}-\mathcal{R})\mathcal{H})^*)$. These conditions will be satisfied if $\tilde{R} = FVF^*$ and $\mathcal{H}P_f = P_c\mathcal{H}^{-1}$. On the other hand, the LQG controller ensures the exp. stability of the closed-loop system. Therefore, all the eigenvalues of the closed-loop system must be in the left half-plane, and thus the operator $\mathcal{R}_c$ has to be positive semi-definite. Otherwise, there would exist a vector $d \neq 0$ such that $\langle d, \mathcal{R}_c d\rangle_{\mathcal{X}} = \lambda\|d\|^2 < 0$ for which the dynamics of the closed-loop system would not be exponentially stable. Indeed, under the relations (4.2.12) and (4.2.13), the closed-loop dynamics are governed by

$$\begin{aligned}\begin{pmatrix} d\varepsilon(t) \\ d\hat{\varepsilon}(t)\end{pmatrix} = \left[\begin{pmatrix} \mathcal{J} & -BK\mathcal{H}^{-1} \\ LB^* & \mathcal{J}\end{pmatrix} - \begin{pmatrix} \mathcal{R} & 0 \\ 0 & \mathcal{R}_c\end{pmatrix}\right]\begin{pmatrix} \mathcal{H}\varepsilon(t) \\ \mathcal{H}\hat{\varepsilon}(t)\end{pmatrix} dt\\ + \begin{pmatrix} H \\ 0\end{pmatrix} dw(t) + \begin{pmatrix} 0 \\ \mathcal{H}^{-1}P_cB\tilde{R}^{-1}F\end{pmatrix} dv(t).\end{aligned} \tag{4.2.16}$$

Let us set the conservation operator J_{cl} and the dissipation operator R_{cl} of the closed-loop system as follows

$$J_{cl} := \begin{pmatrix} \mathcal{J} & -BK\mathcal{H}^{-1} \\ LB^* & \mathcal{J}\end{pmatrix}, \qquad R_{cl} := \begin{pmatrix} \mathcal{R} & 0 \\ 0 & \mathcal{R}_c\end{pmatrix}. \tag{4.2.17}$$

In addition, we set $A_{cl} := (J_{cl} - R_{cl})\mathcal{H}_{cl}$, where

$$\mathcal{H}_{cl} = \operatorname{diag}(\mathcal{H}, \mathcal{H}). \tag{4.2.18}$$

Notice that for $d_0 = \begin{bmatrix} 0 & \mathcal{H}^{-1}d \end{bmatrix}$, there holds

$$\langle d_0, e^{A_{cl}t} d_0 \rangle = e^{-\lambda t}\|d\|^2 + \int_0^t e^{-\lambda(t-\tau)} \langle d_0, J_{cl} e^{A_{cl}\tau} d_0 \rangle d\tau \tag{4.2.19}$$

$$= e^{-\lambda t}\|d\|^2 + \langle d_0, \int_0^t e^{-\lambda(t-\tau)} J_{cl} e^{A_{cl}\tau} d_0 \, d\tau \rangle \tag{4.2.20}$$

$$= e^{-\lambda t}(\|d\|^2 + \langle d_0, \int_0^t e^{\lambda\tau} J_{cl} e^{A_{cl}\tau} d_0 \, d\tau \rangle), \tag{4.2.21}$$

which goes to ∞ when $t \to \infty$.
By using (4.2.12) and (4.2.13) in the FORE, we get that

$$[(\mathcal{J} - \mathcal{R})P_c\mathcal{H}^{-1} + \mathcal{H}^{-1}P_c\mathcal{H}^{-1}(-\mathcal{J} - \mathcal{R})\mathcal{H} - \mathcal{H}^{-1}P_cB\tilde{R}^{-1}B^*P_c\mathcal{H}^{-1} + HQH^*]x = 0,$$

for $x \in D((\mathcal{J} - \mathcal{R})\mathcal{H}) \cap D(((\mathcal{J} - \mathcal{R})\mathcal{H})^*)$. Factorizing $\mathcal{H}^{-1}$ on both sides and since $\mathcal{H}^{-1}$ is injective, it follows that

$$[\mathcal{H}(\mathcal{J} - \mathcal{R})P_c + P_c\mathcal{H}^{-1}(-\mathcal{J} - \mathcal{R})\mathcal{H}^2 - P_cB\tilde{R}B^*P_c + \mathcal{H}HQH^*\mathcal{H}]x = 0. \tag{4.2.22}$$

Subtracting (4.2.22) from the CORE given by

$$[(-\mathcal{J} - \mathcal{R})\mathcal{H}P_c + P_c(\mathcal{J} - \mathcal{R})\mathcal{H} - P_cB\tilde{R}^{-1}B^*P_c + \mathcal{H}BB^*\mathcal{H}]x = 0, \tag{4.2.23}$$

we deduce (4.2.14), which completes the proof. □

Under the conditions (4.2.12)-(4.2.14) of Theorem 4.2.2, the LQG controller conserves the stochastic port-Hamiltonian structure, and the LQG control problem can then be interpreted as the feedback interconnection of infinite-dimensional stochastic port-Hamiltonian systems.

Remark 4.2.1. The feedback interconnection of a stochastic port-Hamiltonian system with a LQG controller is undertaken as follows

$$u(t) = -\gamma_c(t) \qquad \text{and} \qquad u_c(t) = \gamma(t). \tag{4.2.24}$$

Observe that this interconnection (4.2.24) is power preserving since

$$u^T(t)\gamma(t) + u_c(t)^T\gamma_c(t) = 0.$$

Hence, the port-Hamiltonian structure is even preserved in the closed-loop dynamics.

In Theorem 4.2.2, it may be noticed that condition (4.2.13) somehow breaks the separation methodology of Theorem 4.1.2. The optimal control problem and the mean-square estimation problem cannot be treated separately anymore. By considering the optimal control problem first, the covariance operators are deduced from the specific choice of weighting operators. In this case, the covariance operators do not have any statistical meaning anymore and have to be considered as further control parameters.

Remark 4.2.2. When designing an Hamiltonian LQG controller in Chapter 5 for a bio-medical application, we shall use Proposition 4.2.1 on a finite-dimensional approximation so that we shall not be facing the domain condition of Theorem 4.2.2.

Definition 4.2.1. *A C_0-semigroup $(T(t))_{t\geq 0}$ is said to be strongly stable on a Hilbert space X if for all $x \in \mathcal{X}$,*

$$\lim_{t\to\infty} \|T(t)x\|_{\mathcal{X}} = 0. \tag{4.2.25}$$

Observe that, when no dissipative effect is occurring ($\mathcal{R} = 0$), the SPHSs governed by (4.1.1) would only be strongly stabilizable. Assuming the exponential stabilizability would be equivalent to require the exponential stability. Indeed, according to [CZ95, Theorem 5.2.3], for a bounded control operator B of finite rank, (A,B) cannot be exponentially stabilizable when there is an infinite number of eigenvalues on (or arbitrary closed to) the imaginary axis. This entails that, when no dissipative effects are considered either in the domain or at the boundary, most port-Hamiltonian systems are only strongly stabilizable. Let us recall the definitions of a strongly stable C_0-semigroup, strong stabilizability of $((\mathcal{J}-\mathcal{R})\mathcal{H},B)$ and strong detectability of $((\mathcal{J}-\mathcal{R})\mathcal{H},B^*\mathcal{H})$.

Definition 4.2.2. *Let us consider the system*

$$\begin{aligned} \dot{x}(t) &= (\mathcal{J}-\mathcal{R})\mathcal{H}x(t) + Bu(t), \\ y(t) &= B^*\mathcal{H}x(t), \end{aligned} \tag{4.2.26}$$

denoted by $\Sigma((\mathcal{J}-\mathcal{R})\mathcal{H},B,B^*\mathcal{H})$.

1. *$\Sigma((\mathcal{J}-\mathcal{R})\mathcal{H},B,B^*\mathcal{H})$ is input stable if*

$$\tilde{\Phi}u := \lim_{t\to\infty}\int_0^t T(t-s)Bu(s)ds \in \mathcal{L}(L^2([0,\infty];\mathbb{R}^m),\mathcal{X}). \tag{4.2.27}$$

2. *$\Sigma((\mathcal{J}-\mathcal{R})\mathcal{H},B,B^*\mathcal{H})$ is output stable if*

$$\tilde{\Psi}x := \lim_{t\to\infty} B^*\mathcal{H}T(t)x \in \mathcal{L}(\mathcal{X},L^2([0,\infty];\mathbb{R}^m)). \tag{4.2.28}$$

3. *$\Sigma((\mathcal{J}-\mathcal{R})\mathcal{H},B,B^*\mathcal{H})$ is strongly stabilizable if there exists an operator $K \in \mathcal{L}(\mathcal{X},\mathbb{R}^m)$ such that $(\mathcal{J}-\mathcal{R})\mathcal{H}-BK$ generates a strongly stable C_0-semigroup and $\Sigma((\mathcal{J}-\mathcal{R})\mathcal{H}-BK,B,\begin{bmatrix} K^* & \mathcal{H}B \end{bmatrix}^*)$ is output stable.*
4. *$\Sigma((\mathcal{J}-\mathcal{R})\mathcal{H},B,B^*\mathcal{H})$ is strongly detectable if there exists an operator $L \in \mathcal{L}(\mathbb{R}^m,\mathcal{X})$ such that $(\mathcal{J}-\mathcal{R})\mathcal{H}-LB^*\mathcal{H}$ generates a strongly stable C_0-semigroup and $\Sigma((\mathcal{J}-\mathcal{R})\mathcal{H}-LB^*\mathcal{H},\begin{bmatrix} L & B \end{bmatrix}B^*\mathcal{H})$ is input stable.*

Theorems 4.1.2 and 4.2.2 can be easily extended to strongly stabilizable port-Hamiltonian systems. Nonetheless, due to the nature of the strong convergence, duality relationships, as for exp. stabilizable systems, are not completely satisfied anymore. Indeed, in the case of strongly stabilizable systems, if $(\mathcal{J}-\mathcal{R})\mathcal{H}-BK$ generates a strongly stable C_0-semigroup, then it does not necessarily entail that $((\mathcal{J}-$

$\mathcal{R})\mathcal{H} - BK)^*$ generates a strongly stable C_0-semigroup. To overcome this missing duality property, we take advantage of the compactness of the resolvent operator of $(\mathcal{J} - \mathcal{R})\mathcal{H}$, see [HW97, Remark 3.3]. From [Oos00, Theorem 3.3.2 and 3.3.4] and [Oos00, Corollary 3.3.5], there exist unique strongly stabilizing nonnegative self-adjoint solutions P_c and P_f to equations (4.1.7) and (4.1.6). Moreover, by applying Theorem [Oos00, Theorem 3.3.2] and by duality, the operator $((\mathcal{J} - \mathcal{R})\mathcal{H} - LB^*\mathcal{H})^*$ is strongly stable, and so the compactness of the resolvent of $(\mathcal{J} - \mathcal{R})\mathcal{H}$ implies that $((\mathcal{J} - \mathcal{R})\mathcal{H} - LB^*\mathcal{H})$ is strongly stable.
Therefore, the separation principle still holds under the assumptions of strong stabilizability and detectability. In addition, observe that the arguments of the proof of Theorem 4.2.2 still hold for strongly stablilizable and detectable infinite-dimensional PHSs. In this case, the closed-loop system is strongly stable, which entails that there cannot be eigenvalues on $i\mathbb{R}$. Thus, a similar argument as in the proof of Theorem 4.2.2 can be employed to show that the dissipative operator R_c given by (4.2.6) is self-adjoint and positive semi-definite. This implies that, under the assumptions of Theorem 4.2.2, the port-Hamiltonian structure within the dynamics of the LQG controller is still preserved.

In practice, natural dissipation phenomena such as boundary, environmental or material damping occur, which in most occasions enable to satisfy the exponential stabilizability assumption. Therefore, in the rest of this thesis, we shall consider dissipative effects in the dynamics when studying the exponential stabilizability and detectability in Section 4.3 and when developing an LQG port-Hamiltonian controller for an actuated endoscope in Chapter 5. Further details on strongly stabilizable systems can be found in [Oos00].

4.3 Exponential stability of a class of dissipative port-Hamiltonian systems

This section deals with exponential stability of a class of dissipative port-Hamiltonian systems. We start with a repetition of the class of first order linear port-Hamiltonian systems as introduced in Section 1.1 and subject to some internal dissipation. We consider the following class of PDEs

$$\frac{\partial}{\partial t}x(t,\zeta) = P_1 \frac{\partial}{\partial \zeta}(\mathcal{H}x(t))(\zeta) + (P_0 - G_0)(\mathcal{H}x(t))(\zeta), \tag{4.3.1}$$

where $x(t,\zeta) = (q,p)(t,\zeta) \in \mathbb{R}^{2N}$ with $q(t)$ and $p(t)$ denoting the position and the momentum. Moreover, we assume that $P_1 = P_1^T$ is invertible, $P_0^T = -P_0$, $G_0 = G_0^T \geq 0$ and $\mathcal{H} \in L^\infty([a,b];\mathbb{R}^{2N\times 2N})$ is symmetric and satisfies $mI \leq \mathcal{H}(\zeta) \leq MI$ for a.e. $\zeta \in [a,b]$, for some constants m, $M > 0$. The state space $\mathcal{X} = L^2([a,b];\mathbb{R}^{2N})$ is equipped with inner product $\langle x_1, x_2\rangle_{\mathcal{X}} = \langle x_1, \mathcal{H}x_2\rangle_{L^2}$. As already mentioned, this norm $\|\cdot\|_{\mathcal{X}}$ has been selected to match the total energy of the system $E = \frac{1}{2}\|x\|^2_{\mathcal{X}}$. To the PDE

(4.3.1), we associate the following boundary conditions

$$W_B \begin{bmatrix} f_\partial(t) \\ e_\partial(t) \end{bmatrix} = 0. \tag{4.3.2}$$

We shall now rewrite the class of PDEs (4.3.1) by splitting the position and the momentum $x(t,\zeta) = (q,p)(t,\zeta) \in \mathbb{R}^{2N}$ and by considering specific structures for the matrices P_1, P_0 and G_0 given by

$$P_1 = \begin{bmatrix} 0 & D \\ D^T & 0 \end{bmatrix}, \qquad P_0 = \begin{bmatrix} 0 & D_0 \\ -D_0^T & 0 \end{bmatrix}, \qquad G_0 := \begin{bmatrix} 0 & 0 \\ 0 & R \end{bmatrix}, \tag{4.3.3}$$

where D, D_0 and $R \in \mathbb{R}^N$ with R positive definite. Moreover, observe that $P_1 = P_1^T$ is nonsingular, $P_0 = -P_0^T$ and $G_0 = G_0^T$ is nonnegative. Similarly, we consider

$$\mathcal{H} = \begin{bmatrix} \mathcal{H}_1 & 0 \\ 0 & \mathcal{H}_2 \end{bmatrix}, \tag{4.3.4}$$

where $\mathcal{H}_1, \mathcal{H}_2 \in L^\infty([a,b];\mathbb{R}^{N\times N})$. The PDE (4.3.1) can then be rewritten with respect to (q,p) as follows

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix}(\zeta,t) = \begin{bmatrix} 0 & D\partial_\zeta + D_0 \\ D^T\partial_\zeta - D_0^T & -R \end{bmatrix} \begin{bmatrix} \mathcal{H}_1(\zeta) & 0 \\ 0 & \mathcal{H}_2(\zeta) \end{bmatrix} \begin{bmatrix} q \\ p \end{bmatrix}(\zeta,t). \tag{4.3.5}$$

Besides, we assume that p is set to zero at extremity a and that q is set to zero at extremity b, i.e.

$$p(a,t) = 0 \qquad \text{and} \qquad q(b,t) = 0. \tag{4.3.6}$$

When considering the example of a vibrating string, these conditions are interpreted as a string fixed at the origin and let free at the other end. In addition, let us define the variable $r(t)$ such that

$$\frac{d}{dt} r(\zeta,t) = \mathcal{H}_2(\zeta)p(\zeta,t). \tag{4.3.7}$$

Moreover, observe that, since the momentum at the left extremity is assumed to be set to 0, we deduce that

$$r(a,t) = 0. \tag{4.3.8}$$

Hence, we consider the operator

$$A_d x = P_1 \frac{d}{d\zeta}\mathcal{H}x + (P_0 - G_0)\mathcal{H}x \tag{4.3.9}$$

on a domain including the boundary conditions (4.3.6), i.e.,

$$D(A_d) = \{x \in L^2([a,b];\mathbb{R}^n) : \mathcal{H}x \in H^1([a,b];\mathbb{R}^n), W_B \begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix} = 0\}, \tag{4.3.10}$$

where $W_B = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & D^{-T} & I & 0 \\ -D^{-1} & 0 & 0 & I \end{bmatrix}$. Observe that the operator A with domain $D(A_d)$ generates a unitary group since

$$W_B \Sigma W_B^T = \frac{1}{2} \begin{bmatrix} 0 & D^{-T} & I & 0 \\ -D^{-1} & 0 & 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & I \\ 0 & -D^{-T} \\ D^{-1} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

In what follows, we shall consider the exponential stability of the above specific class of dissipative port-Hamiltonian systems with structure described by (4.3.3). This stability property is of particular interest since it will allow us to deduce the exponential stabilizabity and detectability as defined in Definition 4.1.1.

In order to indicate how the exponential stability is established for PHSs described by (4.3.5) and (4.3.6), the result is first proved for a damped vibrating string in Example 3.1. Note that the exponential stability of boundary damped vibrating string was established in [JZ12, Example 9.2.1]. Here, we consider environment damping, which results from the motion of the string through viscous fluids such as air or water. The energy dissipation resulting from environment damping is proportional to the material velocity. When stability is studied, the difficulty remains in constructing an appropriate Lyapunov functional candidate. The exponential stability is established by means of the Lyapunov's direct method in Appendix B.

Example 3.1. Let us consider a vibrating string under the port-Hamiltonian formalism and governed by

$$\begin{pmatrix} \dot{z}_\zeta \\ \ddot{z} \end{pmatrix} = \begin{pmatrix} 0 & \partial_\zeta \\ \partial_\zeta & -R \end{pmatrix} \begin{pmatrix} z_\zeta \\ \dot{z} \end{pmatrix}, \tag{4.3.11}$$

where $(z_\zeta, \dot{z})(t) \in L^2([a,b];\mathbb{R}^2)$ and R is a positive constant. The rationale of the proof consists in finding a Lyapunov candidate with cross terms between the position and the velocity. After several attempts, the Lyapunov candidate is chosen as

$$V(t) = \int_a^b (\frac{1}{2}(z_\zeta^2 + \dot{z}^2) + \beta R z \dot{z}) d\zeta, \tag{4.3.12}$$

where $\beta \in \mathbb{R}^+$. We first prove that

$$0 \le \lambda_1 E(t) \le V(t) \le \lambda_2 E(t), \tag{4.3.13}$$

where

$$E(t) = \frac{1}{2} \int_a^b (z_\zeta^2 + \dot{z}^2) d\zeta. \tag{4.3.14}$$

Applying the inequalities (B.0.9) and (B.0.10) to the cross term $C(t) = \int_a^b \beta R z \dot{z} d\zeta$ yields

$$|C(t)| \le \beta R \int_a^b |z\dot{z}| d\zeta \le \frac{1}{2} \beta R \int_a^b (z^2 + \dot{z}^2) d\zeta$$

$$
\begin{aligned}
&\leq \frac{1}{2}\beta R \int_a^b (\dot{z}^2 + (b-a)^2 z_\zeta^2) d\zeta \\
&\leq \beta R \max\{1, (b-a)^2\} E(t).
\end{aligned}
$$

This means that

$$
-\beta R \max\{1, (b-a)^2\} E(t) \leq C(t) \leq \beta R \max\{1, (b-a)^2\} E(t), \tag{4.3.15}
$$

which entails that (4.3.13) holds when

$$
\lambda_1 = 1 - \beta R \max\{1, (b-a)^2\} > 0, \tag{4.3.16}
$$

$$
\lambda_2 = 1 + \beta R \max\{1, (b-a)^2\}, \tag{4.3.17}
$$

with β sufficiently small.
Now, by taking the time derivative of V along the state trajectories of (4.3.11), we get

$$
\dot{V}(t) = \int_a^b z_\zeta \dot{z}_\zeta + \dot{z}\ddot{z} + \beta R \dot{z}^2 + \beta R z \ddot{z} d\zeta. \tag{4.3.18}
$$

By plugging (4.3.11) in (4.3.18), we obtain

$$
\begin{aligned}
&\int_a^b z_\zeta \dot{z}_\zeta + \dot{z}(\partial_\zeta z_\zeta - R\dot{z}) + \beta R \dot{z}^2 + \beta R z(\partial_\zeta z_\zeta - R\dot{z}) d\zeta \\
&\qquad = \int_a^b -R\dot{z}^2 + \beta R \dot{z}^2 - \beta R^2 z\dot{z} + z_\zeta \dot{z}_\zeta + (\dot{z} + \beta R z)\partial_\zeta z_\zeta d\zeta.
\end{aligned}
$$

Integrating by parts gives

$$
\int_a^b (\dot{z} + \beta R z)\partial_\zeta z_\zeta d\zeta = [(\dot{z} + \beta R z) z_\zeta]_a^b - \int_a^b (\partial_\zeta \dot{z} + \beta R \partial_\zeta z) z_\zeta d\zeta,
$$

which yields

$$
\begin{aligned}
\dot{V}(t) &= \int_a^b -R\dot{z}^2 + \beta R \dot{z}^2 - \beta R^2 z\dot{z} - \beta R z_\zeta^2 d\zeta \\
&= \int_a^b (\beta - 1) R \dot{z}^2 - \beta R^2 z\dot{z} - \beta R z_\zeta^2 d\zeta \\
&= -\int_a^b \frac{1}{2} R \dot{z}^2 + \frac{1}{2} R^2 z\dot{z} + \frac{1}{2} R z_\zeta^2 d\zeta,
\end{aligned}
$$

where we set $\beta = \frac{1}{2}$. This gives that

$$
\dot{V}(t) = -RV(t),
$$

and thus

$$
V(t) = e^{-Rt} V(0),
$$

which proves the exponential stability of (4.3.11).

In Theorem 4.3.1, the exponential stability of dissipative PHSs given by (4.3.5) and (4.3.6) is established.

Theorem 4.3.1. *Consider the port-Hamiltonian system described by (4.3.5)-(4.3.6) with dynamical operator (4.3.9) and domain (4.3.10). Assume that there exists a strictly positive constant* λ_1 *such that for*

$$L(t) \geq \lambda_1 E(t) \geq 0, \tag{4.3.19}$$

where $E(t) = \frac{1}{2}\int_a^b q^T \mathcal{H}_1 q + p^T \mathcal{H}_2 p d\zeta$ *and where the Lyapunov functional* $L(t)$ *is given by*

$$L(t) = \int_a^b (\frac{1}{2}(q^T \mathcal{H}_1 q + p^T \mathcal{H}_2 p) + \beta r^T R \mathcal{H}_2 p) d\zeta \tag{4.3.20}$$

with $\beta > 0$*. Then the port-Hamiltonian system (4.3.5)-(4.3.6) is exponentially stable.*

Proof. First, notice that the inequality (4.3.19) ensures that the Lyapunov functional (4.3.20) is nonnegative.

By taking the time derivative of (4.3.20) along the state trajectories, we get

$$\begin{aligned}\dot{L}(t) &= \int_a^b (q^T \mathcal{H}_1 \dot{q} + p^T \mathcal{H}_2 \dot{p} + \beta p^T \mathcal{H}_2 R \mathcal{H}_2 p + \beta r^T R \mathcal{H}_2 \dot{p}) d\zeta \\ &= \int_a^b (q^T \mathcal{H}_1 \dot{q} + p^T \mathcal{H}_2 ((D^T \partial_\zeta - D_0^T)\mathcal{H}_1 q - R\mathcal{H}_2 p) + \beta p^T \mathcal{H}_2 R \mathcal{H}_2 p \\ &\quad + \beta r^T R \mathcal{H}_2 ((D^T \partial_\zeta - D_0^T)\mathcal{H}_1 q - R\mathcal{H}_2 p)) d\zeta. \end{aligned} \tag{4.3.21}$$

Rearranging the terms, (4.3.21) becomes

$$\begin{aligned}\dot{L}(t) = \int_a^b (&-p^T \mathcal{H}_2 R \mathcal{H}_2 p + \beta p^T \mathcal{H}_2 R \mathcal{H}_2 p - \beta r^T R \mathcal{H}_2 R \mathcal{H}_2 p + q^T \mathcal{H}_1 \dot{q} \\ &+ (p^T \mathcal{H}_2 + \beta r^T R \mathcal{H}_2)(D^T \partial_\zeta - D_0^T)\mathcal{H}_1 q) d\zeta. \end{aligned} \tag{4.3.22}$$

Integrating by parts the last term of (4.3.22) yields

$$\begin{aligned}&\int_a^b (p^T \mathcal{H}_2 + \beta r^T R \mathcal{H}_2) D^T \partial_\zeta (\mathcal{H}_1 q) d\zeta \\ &\quad = [(p^T + \beta r^T R)\mathcal{H}_2 D^T \mathcal{H}_1 q]_a^b - \int_a^b \partial_\zeta (\mathcal{H}_2 p + \beta \mathcal{H}_2 R r)^T D^T \mathcal{H}_1 q d\zeta.\end{aligned}$$

From the boundary conditions given by (4.3.6) and (4.3.8), we deduce that

$$[(p^T + \beta r^T R)\mathcal{H}_2 D^T \mathcal{H}_1 q]_a^b = 0,$$

which yields

$$\begin{aligned}&\int_a^b (\mathcal{H}_2 p + \beta \mathcal{H}_2 R r)^T (D^T \partial_\zeta - D_0^T)\mathcal{H}_1 q d\zeta \\ &\quad = -\int_a^b (D\partial_\zeta \mathcal{H}_2 p + \beta D \mathcal{H}_2 R \partial_\zeta r)^T \mathcal{H}_1 q d\zeta - \int_a^b [D_0 \mathcal{H}_2 p + \beta D_0 \mathcal{H}_2 R r]^T \mathcal{H}_1 q d\zeta.\end{aligned} \tag{4.3.23}$$

By plugging (4.3.23) in (4.3.22), we get

$$\begin{aligned}
\dot{L}(t) = & \int_a^b (-p^T \mathcal{H}_2 R \mathcal{H}_2 p + \beta p^T \mathcal{H}_2 R \mathcal{H}_2 p - \beta r^T R \mathcal{H}_2 R \mathcal{H}_2 p \\
& - \beta (D \mathcal{H}_2 R \partial_\zeta r + D_0 \mathcal{H}_2 R r)^T \mathcal{H}_1 q) d\zeta \\
= & \int_a^b ((\beta - 1) p^T \mathcal{H}_2 R \mathcal{H}_2 p - \beta r^T R \mathcal{H}_2 R \mathcal{H}_2 p - \beta (D \mathcal{H}_2 R \partial_\zeta r + D_0 \mathcal{H}_2 R r)^T \mathcal{H}_1 q) d\zeta \\
= & \frac{1}{2} \int_a^b (-p^T \mathcal{H}_2 R \mathcal{H}_2 p - r^T R \mathcal{H}_2 R \mathcal{H}_2 p - (D \mathcal{H}_2 R \partial_\zeta r + D_0 \mathcal{H}_2 R r)^T \mathcal{H}_1 q) d\zeta \\
\leq & - m \lambda_{min}(R) \frac{1}{2} \int_a^b (p^T \mathcal{H}_2 p + \beta r^T R \mathcal{H}_2 p + (D \partial_\zeta r + D_0 r)^T \mathcal{H}_1 q) d\zeta \\
\leq & - m \lambda_{min}(R) L(t),
\end{aligned}$$

where m is a positive constant such that $mI \leq \mathcal{H}_2(\zeta)$ for a.e. $\zeta \in [a,b]$, $\lambda_{min}(R)$ denotes the smallest eigenvalue of R and $\beta = \frac{1}{2}$. This implies that

$$L(t) \leq e^{-m\lambda_{min}(R)t} L(0), \tag{4.3.24}$$

which proves the exponential stability. □

Remark 4.3.1. According to Theorem 4.3.1, the operator (4.3.9) with domain (4.3.10) is exp. stable. As a direct consequence, (A_d, B) is exp. stabilizable. Moreover, by duality with respect to $\langle \cdot, \cdot \rangle_{\mathcal{X}}$, $(B^* \mathcal{H}, A_d)$ is exp. detectable if and only if (A_d^*, B) is exp. stabilizable. Noticing that for an exp. stable infinitesimal generator A, there exists a positive constant α such that the generated contraction C_0-semigroup $(T_d(t))_{t \geq 0}$ satisfies

$$\|T_d(t)\| = \|T_d(t)^*\| \leq e^{-\alpha t}.$$

This proves that $(B^* \mathcal{H}, A_d)$ is exp. detectable.

In Theorem 4.3.1, the inequality (4.3.19) is assumed in order to apply Lyapunov's direct method (see Theorem B.0.2). This inequality was proved in the example 3.1 and still has to be proved for port-Hamiltonian systems described by (4.3.5)-(4.3.6).

4.4 Conclusion and perspectives

In this chapter a first investigation of the LQG control problem for infinite-dimensional stochastic port-Hamiltonian systems is exposed. This LQG control problem is solved via a separation principle. Besides, the structure preserving of the port-Hamiltonian framework for the LQG controller is investigated. More specifically, under some conditions, the LQG control problem can be interpreted as the power-preserving interconnection of infinite-dimensional stochastic port-Hamiltonian systems.

The LQG control problem with boundary control and observation is not considered in this thesis. This would be a natural extension of this work. Results regarding the

LQ optimal control problem for infinite-dimensional systems with boundary control and observation are available in [PS87] for a limited class of systems. The LQ control problem for parabolic/hyperbolic equations (but with unbounded control and bounded observation operators) is studied for a more general class via the Riccati equation in [LT14]. Furthermore, a spectral factorization method was developed for regular linear systems in [Sta95] and in [WW97a]. A LQ control problem with boundary control and observation is also solved by using the method of spectral factorization by symmetric extraction in [RDW16]. In this paper a Yosida-Type approximation of the boundary observation based on the resolvent operator of the dynamics operator is undertaken. For further reading, we also refer to [Fra86], [DPI85], [BDPDM06] and references therein.

In [CW92], the method of spectral factorization was considered and investigated. The latter method would probably require to solve challenging numerical problems for the class of SPHSs, due to the lack of knowledge on the eigenvalues and eigenfunctions for port-Hamiltonian systems.

The exponential stability of a subclass of linear port-Hamiltonian systems was investigated. As already pointed out, the condition (4.3.19) still has to be proved for port-Hamiltonian systems described by (4.3.5)-(4.3.6) in order to obtain a complete stability theorem. It is worth noticing that the LQG Hamiltonian controller presented in this paper is itself an infinite-dimensional system. In Chapter 5, an approximation scheme is proposed for solving the LQG control problem and the associated Riccati equations for the case of an actuated endoscope under the port-Hamiltonian framework.

Chapter 5

LQG control of an EAP-actuated port-Hamiltonian system

In the medical field, continuum robots such as endoscope, colonoscopes and arthroscopes have been developed to navigate through the human anatomy and to access narrow spaces in the context of noninvasive surgery. This enables to improve the patient wellness with numerous benefits such as less pain and scaring, and a quicker recovery. A micro-endoscope (45mm) model was proposed in [CRA14]. This model was developed in the context of skull base surgery applications for pituitary gland cancer detection, see [BSR+11]. Further applications are beating heart surgery in [DGV+12], neurosurgery in [BHOC+12] and laser surgery in [RCJW09], for instance. In this chapter we shall more particularly focus on the endoscope application and the design of an optimal control strategy in order to set the endoscope in a desired configuration, while reducing vibrations. The bending of the endoscope is performed by electro-active polymers (EAPs) patched around the tube. Ionic Polymer Metal Composites (IPMCs) are some of the most suitable EAPs for actuation and sensing due to their inherent properties such as light weight, low voltage bending $(1-2V)$ and durability (possibility to bend over 10^6 times) as pointed out in [SK01].

This chapter is mainly intended to focus on the LQG control problem of a compliant endoscope actuated by means of ionic polymer metal composites. We aim at proposing a port-Hamiltonian model for an IPMC actuated endoscope. The bio-medical endoscope and the IPMCs are approximated by a Timoshenko beam model and RLC circuits under the port-Hamiltonian formalism. Thus, the interconnected port-Hamiltonian model of the IPMC actuated beam is presented.

A first contribution of this chapter is to consider the dynamics of the IPMC actuators in the modeling of the control problem. A second contribution is the proof of the exponential stability of the interconnected IPMC-beam model. A third contribution is the implementation of the Hamiltonian LQG controller on the interconnected IPMC-beam model.

This chapter is structured as follows. Section 5.1 is devoted to the description of the experimental setup of an IPMC actuated flexible beam, whereas the two next sections deal with its modeling. In Section 5.4, a coupled PDE-ODE model is proposed for the interconnected system of a flexible beam with an IPMC. Next, the LQG control problem for IPMC-beam model is studied and a finite-dimensional approximation of this system is proposed in Section 5.5. The approximated model is confronted to the experimental setup in Sections 5.6 and 5.7, respectively. This chapter ends with some conclusions and perspectives.

5.1 Motivation: experimental setup

In this section we introduce the experimental setup of an IPMC actuated flexible beam mounted in the AS2M department of FEMTO-ST Institute, see Figure 5.1. The opportunity to study and to review lab testing on this experiment was made possible by Yann Le Gorrec and Yongxin Wu, who introduced both theoretical and experimental questions to me. They also taught me primarily uses of the testing workbench.

The experimental setup consists of a polyethylene plastic beam of 160 *mm* to which an IPMC is patched. The plastic beam is fixed at the origin and let free at the other end. As already mentioned, the endoscope is actuated when a voltage up to $7V$ is applied to the IPMC. To measure the displacements of the beam, a laser sensor from KEYENCE company (LK-G152) is considered. In addition, we also measure the current in the inductor.

A schematic diagram of the experimental setup considered in this study is depicted in Figure 5.2. The inputs and the outputs are handled by means of a dSPACE board connected to a computer and a push and pull amplifier is used to amplify the input signals to the IPMC controller. The dSPACE has 6 DAC out channels that provide the control signals to the actuators and 16 A/D in channels for output signals measurement. It allows the real-time manipulation of the actuated IPMC. From channels 1 and 5, we get the observed displacement and current, respectively. Notice that the laser sensor is placed at $5mm$ from the free end of the beam. The model representation and the programming are done by means of SIMULINK® interface. SIMULINK® blocks are translated into code machine. Once the simulation are undertaken, the data are exported as files with extension .mat and plotted with Matlab®.

5.2 Modeling of a compliant endoscope

As a simplified model, the compliant endoscope is modeled as a Timoshenko type beam equation with frictional dissipative terms, see [CRA14]. The partial differential

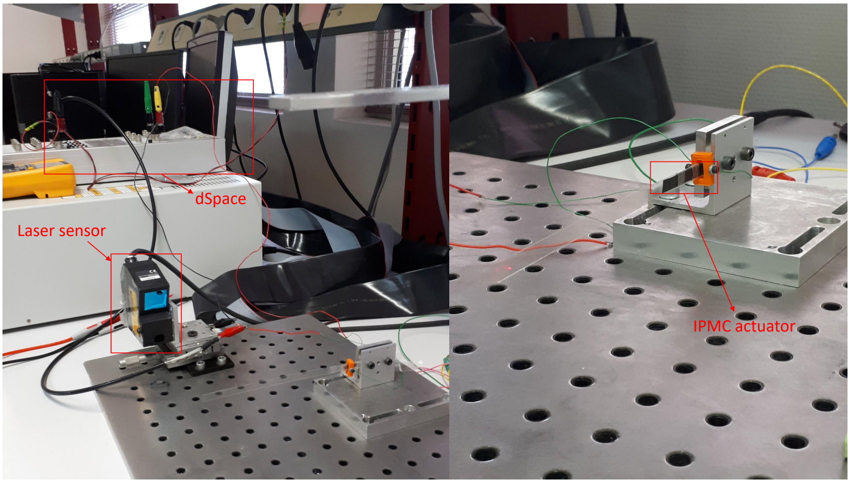

Figure 5.1 – Experimental setup

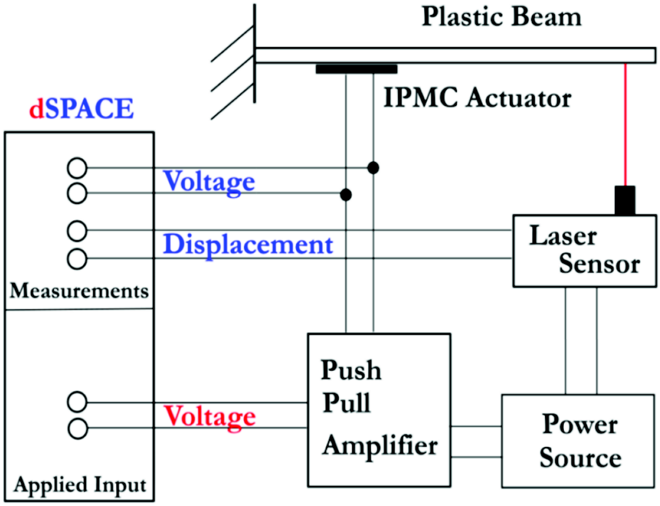

Figure 5.2 – Schematic diagram of the experimental setup [WLGW19]

equations of the Timoshenko beam are given by

$$\rho(\zeta)\frac{\partial^2 z}{\partial t^2}(\zeta,t) = \frac{\partial}{\partial \zeta}\left(K(\zeta)\left(\frac{\partial z}{\partial \zeta}(\zeta,t) - \phi(\zeta,t)\right)\right) - R_t\frac{\partial z}{\partial t}(\zeta,t), \tag{5.2.1}$$

$$I_\rho(\zeta)\frac{\partial^2 \phi}{\partial t^2}(\zeta,t) = \frac{\partial}{\partial \zeta}\left(E_e I(\zeta)\frac{\partial \phi}{\partial \zeta}(\zeta,t)\right) + K(\zeta)\left(\frac{\partial z}{\partial \zeta}(\zeta,t) - \phi(\zeta,t)\right) - R_a\frac{\partial \phi}{\partial t}(\zeta,t), \tag{5.2.2}$$

where $z(\zeta,t)$ and $\phi(\zeta,t)$ are the transverse displacement and the rotational angle at position $\zeta \in [a,b]$ and time t, respectively. The coefficients are ρ (kg/m^3), the mass per unit length, E_e (Pa), the Young modulus of elasticity, I_ρ $(kg\ m)$, the rotary moment of inertia of a cross section, I (m^4), the moment of inertia of a cross section and K (Pa), the shear modulus. The damping coefficients R_t $(kg\ m^3/s)$ and R_a $(kg\ m/s)$ denote the transversal and angular frictions, respectively. Let us consider the state vector $x(\zeta,t) \in \mathbb{R}^4$ whose components are given by

$$\begin{aligned} x_1(\zeta,t) &= \tfrac{\partial z}{\partial \zeta}(\zeta,t) - \phi(\zeta,t) && \text{(shear displacement)} \\ x_2(\zeta,t) &= \rho(\zeta)\tfrac{\partial z}{\partial t}(\zeta,t) && \text{(momentum)} \\ x_3(\zeta,t) &= \tfrac{\partial \phi}{\partial \zeta}(\zeta,t) && \text{(angular displacement)} \\ x_4(\zeta,t) &= I_\rho(\zeta)\tfrac{\partial \phi}{\partial t}(\zeta,t) && \text{(angular momentum).} \end{aligned}$$

The state space is given by $L^2([a,b];\mathbb{R}^4)$. The PDEs (5.2.1) and (5.2.2) can then be rewritten as

$$\frac{\partial x}{\partial t}(\zeta,t) = P_1\frac{\partial}{\partial \zeta}(\mathcal{H}(\zeta)x(\zeta,t)) + (P_0 - G_0)\mathcal{H}(\zeta)x(\zeta,t), \tag{5.2.3}$$

by setting the matrices P_1, P_0, G_0 and $\mathcal{H}$ respectively as

$$P_1 := \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad P_0 := \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix},$$
$$G_0 := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & R_t & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & R_a \end{bmatrix}, \quad \mathcal{H} := \begin{bmatrix} K & 0 & 0 & 0 \\ 0 & \frac{1}{\rho} & 0 & 0 \\ 0 & 0 & E_e I & 0 \\ 0 & 0 & 0 & \frac{1}{I_\rho} \end{bmatrix}. \tag{5.2.4}$$

Furthermore, the mechanical energy is given by:

$$\begin{aligned} E(x(t)) &= \frac{1}{2}\int_a^b (Kx_1^2(t) + \frac{1}{\rho}x_2^2(t) + E_e I x_3^2(t) + \frac{1}{I_\rho}x_4^2(t))d\zeta \\ &= E_k(x(t)) + E_p(x(t)), \end{aligned} \tag{5.2.5}$$

where $E_k(x(t)) = \frac{1}{2}\int_a^b(\frac{1}{\rho}x_2^2(t) + \frac{1}{I_\rho}x_4^2(t))d\zeta$ and $E_p(x(t)) = \frac{1}{2}\int_a^b(Kx_1^2(t) + E_e I x_3^2(t))d\zeta$ represent the kinetic energy and the potential energy, respectively. The medical endoscope is assumed to be clamped at the origin a and let free at its other end b. This corresponds to the following boundary conditions:

$$x_2(a,t) = x_4(a,t) = x_1(b,t) = x_3(b,t) = 0. \tag{5.2.6}$$

The boundary conditions given by (5.2.6) are described through boundary port-variables, namely the efforts e_∂ and the flows f_∂, in the following way:

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = W_B \begin{bmatrix} f_\partial(t) \\ e_\partial(t) \end{bmatrix}, \tag{5.2.7}$$

where

$$W_B = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

By omitting the spatial dependence, (5.2.3) becomes

$$\dot{x}(t) = (\mathcal{J} - \mathcal{R})\mathcal{H}x(t), \tag{5.2.8}$$

where the skew-adjoint operator

$$\mathcal{J}x = P_1 \frac{d}{d\zeta}x + P_0 x \tag{5.2.9}$$

is defined on the domain

$$D(\mathcal{J}) = \{x \in H^1([a,b];\mathbb{R}^4), x_2(a,t) = x_4(a,t) = x_1(b,t) = x_3(b,t) = 0\}, \tag{5.2.10}$$

and the self-adjoint operator $\mathcal{R} \in \mathcal{L}(X)$ is defined as $\mathcal{R} = G_0$.

The endoscope is supposed to be actuated on intervals $[a_i, b_i]$ with $i \in \{1, \dots, m\}$ by distributed forces $u_{d,i}(t)$ through m IPMCs glued to it. The control operator $B_d : \mathbb{R}^m \to \mathcal{X}$ is then given by

$$(B_d u_d(t))(\zeta) = \sum_{i=1}^{m} \begin{bmatrix} 0 \\ 0 \\ 0 \\ b_i(\zeta) \end{bmatrix} u_{d,i}(t), \tag{5.2.11}$$

where $b_i(\zeta) = 1$ for $\zeta \in [a_i, b_i]$ and $b_i(\zeta) = 0$ otherwise for all $i \in \{1, \dots, m\}$. Furthermore, the mean value of the angular velocity is assumed to be measured on each interval $[a_i, b_i]$ with $i \in \{1, \dots, m\}$. The mean value of the angular velocity for each interval $[a_i, b_i]$ is expressed as

$$y_i(t) = \int_a^b b_i(\zeta) \frac{1}{I_\rho(\zeta)} x_4(t,\zeta) d\zeta, \tag{5.2.12}$$

for all $i \in \{1, \dots, m\}$. The power conjugated output is then given by

$$y(t) = [y_1, \dots, y_m] = B_d^* \mathcal{H}x(t). \tag{5.2.13}$$

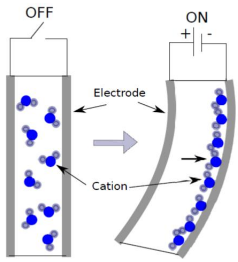

Figure 5.3 – IPMC bending under electric stimulus [WLGW19]

5.3 Modeling of an IPMC actuator as a RLC circuit

An IPMC consists of a polyelectrolyte gel sandwiched between two electrodes as represented in Figure 5.3. IPMCs can be used as sensors or actuators to both monitor and control the time response of the plant. Indeed, these smart materials can generate an electric stimulus when a deformation is applied from a structure, and conversely applying electric stimulus to IPMCs will produce forces and torques to this structure. The working mechanism is the following: under an electric stimulus, which entails a potential difference, cations[(1)] and water molecules are transferred to the negative electrode. The resulting swollen induces a swelling effect, which mechanically bends the IPMC.

In this work the IPMC is approximated by a lumped RLC circuit model as proposed in [GLTY09]. The RLC circuit is represented in Figure 5.4. This RLC circuit model gives a description of the electrical behaviour of the IPMC. To be more specific, the r_1CL circuit models the ionic current produced by the movement of the cations under an electric stimulus. The resistance r_2 represents the internal resistance of the electrolyte between the electrodes. Here, Q is the charge of the capacitor, Φ is the magnetic flux, C is the capacity and L is the inductance. The voltage and the current are given by $V = Q/C$ and $I = \Phi/L$. Let us denote by V_L, V_c, V_R and V_{in} the voltage across the inductor, the capacitor, the resistor and the voltage source, respectively. Besides, I_C, I_R and I_L denote the current across the capacitor, the resistor and the inductor, respectively. In addition, the dynamical relations

$$\begin{aligned} V_L(t) &= \frac{d\Phi}{dt}(t), \\ I_C(t) &= \frac{dQ}{dt}(t), \end{aligned} \tag{5.3.1}$$

(1)Cations are molecules with a positive charge.

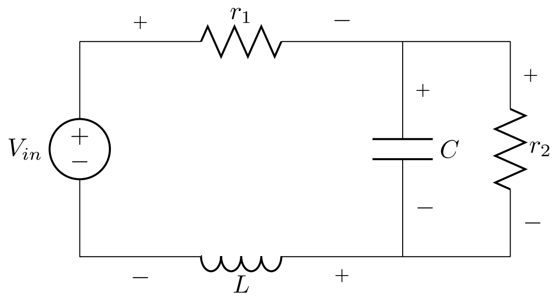

Figure 5.4 – RLC circuit

and the constitutive relations

$$\begin{aligned} V_{r_1}(t) &= r_1 I_{r_1}(t), \\ V_{r_2}(t) &= r_2 I_{r_2}(t), \\ \Phi(t) &= L I_L, \\ Q(t) &= C V_C(t) \end{aligned} \tag{5.3.2}$$

hold for all $t \geq 0$.

The first Kirchkoff's law states that the algebraic sum of currents meeting at any node of an electrical circuit is zero. The second Kirchkoff's law states that the directed sum of potential differences around any closed circuit is zero. By applying the first and second Kirchkoff's laws ($\sum V = 0$ and $\sum I = 0$), we obtain

$$I_C = I_L - I_{r_2} \qquad \text{and} \qquad V_{in} = V_{r_1} + V_C + V_L. \tag{5.3.3}$$

By using relations (5.3.3) together with the relations (5.3.1) and (5.3.2), we deduce the following state space model:

$$\begin{aligned} \frac{dQ}{dt}(t) &= \frac{1}{L}\Phi(t) - \frac{1}{r_2}\frac{1}{C}Q(t), \\ \frac{d\Phi}{dt}(t) &= -\frac{1}{C}Q(t) - r_1\frac{1}{L}\Phi(t) + V_{in}(t). \end{aligned} \tag{5.3.4}$$

Under the port-Hamiltonian formalism, the equations (5.3.4) can be rewritten as

$$\begin{bmatrix} \dot{\Phi} \\ \dot{Q} \end{bmatrix}(t) = \begin{bmatrix} -r_1 & -1 \\ 1 & -\frac{1}{r_2} \end{bmatrix} \begin{bmatrix} \frac{\Phi}{L} \\ \frac{Q}{C} \end{bmatrix}(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} V_{in}(t). \tag{5.3.5}$$

The associated Hamiltonian of the RLC circuit representing the energy of the system is given by

$$E_{rlc}((\Phi, Q)(t)) = \frac{1}{2}\frac{Q^2(t)}{C} + \frac{1}{2}\frac{\Phi^2(t)}{L}. \tag{5.3.6}$$

The port-Hamiltonian system (5.3.5) describes the dynamics of one IPMC with the applied voltage V_{in} as an input $u(t)$. Let us now generalize for m IPMC actuators attached along a flexible beam. This readily extends as follows. The flux and the charge $(\Phi, Q) \in \mathbb{R}^{2m}$ are governed by

$$\begin{bmatrix} \dot{\Phi} \\ \dot{Q} \end{bmatrix}(t) = \begin{bmatrix} -R_1 & -I \\ I & -R_2 \end{bmatrix} \begin{bmatrix} \frac{\Phi}{L} \\ \frac{Q}{C} \end{bmatrix}(t) + \begin{bmatrix} I \\ 0 \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ I \end{bmatrix} u_a(t), \tag{5.3.7}$$

where $R_1 = \text{diag}[r_1, \ldots, r_1] \in \mathbb{R}^{m\times m}$ and $R_2 = \text{diag}[\frac{1}{r_2}, \ldots, \frac{1}{r_2}] \in \mathbb{R}^{m\times m}$ are the resistance matrices. The input $u(t)$ is the applied voltage on the IPMC and the power-conjugated output $y(t)$ given by

$$y(t) = \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} \frac{\Phi}{L} \\ \frac{Q}{C} \end{bmatrix}(t) = \frac{\Phi}{L}(t) \tag{5.3.8}$$

is the current in the inductor. Besides, we denote by $y_a(t)$ the voltage in the capacitor, i.e.,

$$y_a(t) = \begin{bmatrix} 0 & I \end{bmatrix} \begin{bmatrix} \frac{\Phi}{L} \\ \frac{Q}{C} \end{bmatrix}(t) = \frac{Q}{C}(t). \tag{5.3.9}$$

The interconnection between the flexible structure and the RLC circuit is assumed to be perfect, thus $u_a(t)$ and $u_d(t)$ are the current applied on the capacitor due to the movement of the flexible structure and the torque applied on the structure, respectively. On one hand, the torque applied to the beam comes from the voltage V weighted by coefficient $\tilde{k}$ and distributed on the interval $[a_i, b_i]$. On the other hand, a current is generated by the beam motion observed on $[a_i, b_i]$ and acts on the IPMC with coefficient $-\tilde{k}$. To sum it up, the power-preserving interconnection between the flexible structure and the actuator is summarised as follows:

$$\begin{bmatrix} u_d \\ u_a \end{bmatrix}(t) = \begin{bmatrix} 0 & \tilde{k} \\ -\tilde{k}^T & 0 \end{bmatrix} \begin{bmatrix} y_d \\ y_a \end{bmatrix}(t), \tag{5.3.10}$$

with a matrix $\tilde{k} = \text{diag}[k_1, \ldots, k_m] \in \mathbb{R}^{m\times m}$ $[Nm/V]$.

The Hamiltonian associated to (5.3.7) is given by

$$E_{RLC}((\Phi, Q)(t)) = \frac{1}{2}Q^T(t)\frac{1}{C}Q(t) + \frac{1}{2}\Phi^T(t)\frac{1}{L}\Phi(t). \tag{5.3.11}$$

5.4 Interconnection of a Timoshenko beam and a RLC circuit

In this section the interconnected system consisting of a flexible beam with m IPMC actuators is introduced. From the power-preserving interconnecting relations (5.3.10), the IPMC-actuated flexible beam is described as follows:

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \begin{pmatrix} \mathcal{J} - \mathcal{R} & 0 & B_d\tilde{k} \\ 0 & -R_1 & -I \\ -\tilde{k}^T B_d^* & I & -R_2 \end{pmatrix} \mathbf{Q}\mathbf{x}(t) + \begin{pmatrix} 0 \\ I \\ 0 \end{pmatrix} u(t), \\ \mathbf{y}(t) &= \begin{bmatrix} 0 & I & 0 \end{bmatrix} \mathbf{Q}\mathbf{x}(t), \end{aligned} \tag{5.4.1}$$

where the operator B_d is given by (5.2.11), $\mathbf{x} = \begin{bmatrix} x, & \Phi, & Q \end{bmatrix}$ and the energy matrix $\mathbf{Q}$ is given by

$$\mathbf{Q} = \text{diag}[\mathcal{H}, \frac{1}{L}, \frac{1}{C}].$$

The state space is given by $\mathbf{X} := L^2([a,b];\mathbb{R}^4) \times \mathbb{R}^{2m}$ with inner product

$$\langle \mathbf{x}_1, \mathbf{x}_2 \rangle_{\mathbf{X}} := \langle x_1, x_2 \rangle_{\mathcal{X}} + Q^T \frac{1}{C} Q + \Phi^T \frac{1}{L} \Phi.$$

By using matrices D and D_0 introduced in Section 4.3 and given by

$$D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad D_0 = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}, \tag{5.4.2}$$

the abstract differential equation (5.4.1) can be expanded as

$$\begin{bmatrix} \dot{q} \\ \dot{p} \\ \dot{\Phi} \\ \dot{Q} \end{bmatrix} (\zeta, t) = \begin{bmatrix} 0 & D\partial_\zeta + D_0 & 0 & 0 \\ D^T \partial_\zeta - D_0^T & -R & 0 & B\tilde{k} \\ 0 & 0 & -R_1 & -I \\ 0 & -\tilde{k}^T B^* & I & -R_2 \end{bmatrix}
\begin{bmatrix} \mathcal{H}_1(\zeta) & 0 & 0 & 0 \\ 0 & \mathcal{H}_2(\zeta) & 0 & 0 \\ 0 & 0 & \frac{1}{L} & 0 \\ 0 & 0 & 0 & \frac{1}{C} \end{bmatrix} \begin{bmatrix} q \\ p \\ \Phi \\ Q \end{bmatrix} (\zeta, t) + \begin{bmatrix} 0 \\ 0 \\ I_m \\ 0 \end{bmatrix} u(t). \tag{5.4.3}$$

Let us define the unbounded linear operator

$$\begin{aligned} \mathbf{A}\,\mathbf{x} &= \begin{bmatrix} 0 & Dd_\zeta + D_0 & 0 & 0 \\ D^T d_\zeta - D_0^T & -R & 0 & B\tilde{k} \\ 0 & 0 & -R_1 & -I \\ 0 & -\tilde{k}^T B^* & I & -R_2 \end{bmatrix} \begin{bmatrix} \mathcal{H}_1(\zeta) & 0 & 0 & 0 \\ 0 & \mathcal{H}_2(\zeta) & 0 & 0 \\ 0 & 0 & \frac{1}{L} & 0 \\ 0 & 0 & 0 & \frac{1}{C} \end{bmatrix} \mathbf{x} \\ &=: (\mathbf{J} - \mathbf{R})\mathbf{Q}\mathbf{x} \end{aligned} \tag{5.4.4}$$

for $\mathbf{x}$ in the domain

$$D(\mathbf{A}) = \left\{ \begin{bmatrix} x \\ \Phi \\ Q \end{bmatrix} \in \begin{bmatrix} \mathcal{X} \\ \mathbb{R}^m \\ \mathbb{R}^m \end{bmatrix}, \mathcal{H}x \in H^1([a,b];\mathbb{R}^4), \begin{bmatrix} f_\partial(t) \\ e_\partial(t) \end{bmatrix} \in \text{Ker } W_B \right\}, \tag{5.4.5}$$

where $\mathcal{X} = L^2([a,b];\mathbb{R}^4)$.

The total energy of the interconnected system is given by

$$\mathbf{E}(\mathbf{x}(t)) := E(x(t)) + E_{RLC}((\Phi, Q)(t)) \tag{5.4.6}$$

$$= \frac{1}{2}\|x(t)\|_{\mathcal{X}}^2 + \frac{1}{2} Q^T \frac{1}{C} Q + \frac{1}{2} \Phi^T \frac{1}{L} \Phi =: \frac{1}{2} \langle \mathbf{x}(t), \mathbf{Q}x(t) \rangle_{\mathbf{X}}, \tag{5.4.7}$$

where $\mathbf{Q} = \text{diag}[\mathcal{H}_1, \mathcal{H}_2, \frac{1}{L}, \frac{1}{C}]$. Moreover, since the power-preserving interconnection of port-Hamiltonian systems is still a port-Hamiltonian system, we can rely on this structure to establish the C_0-semigroup generation for the interconnected port-Hamiltonian system. From (5.4.3), we define new matrices $\mathbf{P}_1, \mathbf{P}_0$ and $\mathbf{G}_0$ by

$$\mathbf{P}_1 = \begin{bmatrix} 0 & D & 0 & 0 \\ D^T & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{P}_0 = \begin{bmatrix} 0 & D_0 & 0 & 0 \\ -D_0^T & 0 & 0 & B\tilde{k} \\ 0 & 0 & 0 & -I \\ 0 & -\tilde{k}^T B^* & I & 0 \end{bmatrix},$$

$$\mathbf{G}_0 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -R & 0 & 0 \\ 0 & 0 & -R_1 & 0 \\ 0 & 0 & 0 & -R_2 \end{bmatrix}. \tag{5.4.8}$$

The generation of a contraction C_0-semigroup of the operator $\mathbf{A}$ can be directly deduced from Theorem 1.1.1. Since the interconnection is distributed along the spatial domain, it does not alter the boundary conditions, which satisfy $W_B \Sigma W_B^T \geq 0$. The generation of a contraction C_0-semigroup and the compactness property of the related resolvent operator are resumed in the following proposition.

Proposition 5.4.1. *The linear operator* $\mathbf{A}$ *with domain* $D(\mathbf{A})$ *generates a contraction* C_0*-semigroup on the state space* $\mathbf{X}$*. Moreover,* $\mathbf{A}$ *has a compact resolvent.*

Proof. Since $W_B \Sigma W_B^T \geq 0$, $\mathbf{A}$ generates a contraction C_0-semigroup, see Theorem 1.1.1. The compactness of the resolvent operator of $\mathbf{A}$ is a direct consequence of [Vil07, Theorem 2.28]. □

5.5 LQG control problem

In the section we address the LQG control problem for the interconnected IPMC-beam model introduced in Section 5.4. The main objective of the latter control problem consists in setting the endoscope under a specific configuration by means of an admissible control law $u(t)$ minimizing the functional

$$J(u) = \lim_{T \to \infty} \mathbb{E} \int_0^T \langle \mathbf{x}(t), \bar{R}\mathbf{x}(t) \rangle_{\mathbf{X}} + \langle u(t), \tilde{R}u(t) \rangle_{\mathbb{R}^m} dt, \tag{5.5.1}$$

where the operator $\bar{R} = \bar{R}^T$ is positive semi-definite and the matrix $\tilde{R} = \tilde{R}^T$ is positive definite. As already mentioned in Chapter 4, the control system (5.4.1) must satisfy the assumptions of exponential (exp.) stabilizability and detectability as defined in Definition 4.1.1 in order to have a well-defined LQG control problem. These properties ensure the existence of unique nonnegative self-adjoint solutions for the control operator Riccati equation (CORE) and the filter operator Riccati equation (FORE). In Theorem 5.5.1, we shall prove the exponential stability of the system (5.4.1) by considering a Lyapunov functional candidate and showing that its time derivative is bounded by the initial condition of the Lyapunov functional. As a direct consequence, the assumptions of exp. stabilizability and detectability will be satisfied. Notice that, due to domain interconnection, cross terms have to be inserted in the Lyapunov functional candidate.

Theorem 5.5.1. *Let us consider the interconnected system (5.4.1) on the state space* X *with boundary conditions (5.2.6). Assume that there exist* $\beta_0 > 0$ *and* $\lambda_1, \lambda_2 > 0$ *(that may depend on* β_0*) such that for all* $\gamma \in (0, \beta_0)$,

$$0 \leq \lambda_1(E(t) + E_{RLC}(t)) \leq L(t) \leq \lambda_2(E(t) + E_{RLC}(t)), \tag{5.5.2}$$

where the Lyapunov functional $L(t)$ *is given by*

$$\begin{aligned} L(t) = \frac{1}{2}[\int_a^b (q^T \mathcal{H}_1 q + p^T \mathcal{H}_2 p + \gamma r^T p) d\zeta + \Phi^T \frac{1}{L}\Phi + Q^T \frac{1}{C} Q] \\ + \gamma \Phi^T Q + \gamma \Phi^T \tilde{k}^T B^* r \end{aligned} \tag{5.5.3}$$

with r given by (4.3.7). Assume that for all $\gamma \in (0, \beta_0)$*, there exist* K_1 *and* K_2 *such that*

$$|\frac{1}{2}\int_a^b \gamma r^T p d\zeta| \leq K_1 E(t) \qquad |\gamma \Phi^T Q + \gamma \Phi^T \tilde{k}^T B^* r| \leq K_2 E_{RLC}(t). \tag{5.5.4}$$

In addition, assume that the parameter γ *satisfy the conditions:* $\gamma I \leq R_1 \frac{1}{L}$ *and* $\gamma I \leq \mathcal{H}_2 R$*. Then the system (5.4.1) is exponentially stable.*

Proof. Notice that (5.5.2) ensures that the Lyapunov functional (5.5.3) is nonnegative. Taking the time derivative of (5.5.3) along the state trajectories gives

$$\begin{aligned} \dot{L}(t) = \int_a^b \begin{bmatrix} q^T \mathcal{H}_1 & p^T \mathcal{H}_2 + \gamma r^T & \gamma p^T \end{bmatrix} \begin{bmatrix} \dot{q} \\ \dot{p} \\ \mathcal{H}_2 p \end{bmatrix} d\zeta \\ + \begin{bmatrix} \Phi^T L^{-1} & Q^T C^{-1} \end{bmatrix} \begin{bmatrix} \dot{\Phi} \\ \dot{Q} \end{bmatrix} + \gamma \dot{\Phi}^T Q + \gamma \Phi^T \dot{Q} + \gamma \Phi^T \tilde{k}^T B^* \mathcal{H}_2 p + \gamma \dot{\Phi}^T \tilde{k}^T B^* r. \end{aligned} \tag{5.5.5}$$

This yields

$$\begin{aligned} \dot{L}(t) = \int_a^b \begin{bmatrix} q^T \mathcal{H}_1 & p^T \mathcal{H}_2 + \gamma r^T \end{bmatrix} \begin{bmatrix} 0 & D\partial_\zeta + D_0 & 0 & 0 \\ D^T \partial_\zeta - D_0^T & -R & 0 & B\tilde{k} \end{bmatrix} \\ \begin{bmatrix} \mathcal{H}_1 & 0 & 0 & 0 \\ 0 & \mathcal{H}_2 & 0 & 0 \\ 0 & 0 & \frac{1}{L} & 0 \\ 0 & 0 & 0 & \frac{1}{C} \end{bmatrix} \begin{bmatrix} q \\ p \\ \Phi \\ Q \end{bmatrix} d\zeta + \int_a^b \gamma p^T \mathcal{H}_2 p d\zeta \\ + \begin{bmatrix} \Phi^T \frac{1}{L} & Q^T \frac{1}{C} \end{bmatrix} \begin{bmatrix} 0 & 0 & -R_1 & -I \\ 0 & -\tilde{k}^T B^* & I & -R_2 \end{bmatrix} \begin{bmatrix} \mathcal{H}_1 & 0 & 0 & 0 \\ 0 & \mathcal{H}_2 & 0 & 0 \\ 0 & 0 & \frac{1}{L} & 0 \\ 0 & 0 & 0 & \frac{1}{C} \end{bmatrix} \begin{bmatrix} q \\ p \\ \Phi \\ Q \end{bmatrix} \\ \underbrace{+\gamma \dot{\Phi}^T Q + \gamma \Phi^T \dot{Q} + \gamma \Phi^T \tilde{k}^T B^* \mathcal{H}_2 p + \gamma \dot{\Phi}^T \tilde{k}^T B^* r}_{(\sharp)}. \end{aligned} \tag{5.5.6}$$

Moreover, from (5.3.7), the term $(\sharp)$ becomes

$$\begin{aligned}(\sharp) =&\gamma\left[-R_1\frac{1}{L}\Phi-\frac{Q}{C}\right]^T Q+\gamma\Phi^T\left[-\tilde{k}^T B^*\mathcal{H}_2 p+\frac{1}{L}\Phi-R_2\frac{1}{C}Q\right]\\ &+\gamma\Phi^T\tilde{k}^T B^*\mathcal{H}_2 p-\gamma(R_1\frac{1}{L}\Phi)^T\tilde{k}^T B^* r-\gamma(\frac{1}{C}Q)^T\tilde{k}^T B^* r.\end{aligned} \tag{5.5.7}$$

By plugging (5.5.7) in (5.5.6), one gets

$$\begin{aligned}\dot{L}(t)=&\int_a^b q^T(\mathcal{H}_1(D\partial_\zeta+D_0)\mathcal{H}_2 p+p^T\mathcal{H}_2(D^T\partial_\zeta-D_0^T)\mathcal{H}_1 q+\gamma r^T(D^T\partial_\zeta-D_0^T)\mathcal{H}_1 q\\ &-(p^T\mathcal{H}_2+\gamma r^T)\mathcal{R}\mathcal{H}_2 p+(p^T\mathcal{H}_2+\gamma r^T)B\tilde{k}\frac{1}{C}Q)d\zeta+\int_a^b\gamma p^T\mathcal{H}_2 p d\zeta\\ &-\Phi^T\frac{1}{L}R_1\frac{1}{L}\Phi-\Phi^T\frac{1}{L}\frac{1}{C}Q-Q^T\frac{1}{C}\tilde{k}^T B^*\mathcal{H}_2 p+Q^T\frac{1}{C}\frac{1}{L}\Phi\\ &-Q^T\frac{1}{C}R_2\frac{1}{C}Q+\gamma\left[-R_1\frac{1}{L}\Phi-Q\frac{1}{C}\right]^T Q+\gamma\Phi^T\left[-\tilde{k}^T B^*\mathcal{H}_2 p+\frac{1}{L}\Phi-R_2\frac{1}{C}Q\right]\\ &+\gamma\Phi^T\tilde{k}^T B^*\mathcal{H}_2 p-\gamma(R_1\frac{1}{L}\Phi)^T\tilde{k}^T B^* r-\gamma(\frac{1}{C}Q)^T\tilde{k}^T B^* r.\end{aligned} \tag{5.5.8}$$

After undergoing some simplifications and by integrating by parts, we obtain

$$\begin{aligned}\dot{L}(t)=&[(\mathcal{H}_1 q)^T D\mathcal{H}_2 p]_a^b+[r^T D^T\mathcal{H}_1 q]_a^b\\ &+\int_a^b(-\partial_\zeta(\mathcal{H}_1 q)^T D\mathcal{H}_2 p+q^T\mathcal{H}_1 D_0\mathcal{H}_2 p+p^T\mathcal{H}_2(D^T\partial_\zeta-D_0^T)\mathcal{H}_1 q\\ &-\gamma\partial_\zeta r^T D^T\mathcal{H}_1 q-\gamma r^T D_0^T\mathcal{H}_1 q-p^T\mathcal{H}_2\mathcal{R}\mathcal{H}_2 p-\gamma r^T\mathcal{R}\mathcal{H}_2 p)d\zeta+\int_a^b\gamma p^T\mathcal{H}_2 p d\zeta\\ &-\Phi^T\frac{1}{L}R_1\frac{1}{L}\Phi-Q^T\frac{1}{C}R_2\frac{1}{C}Q+\left(-\gamma(R_1\frac{1}{L}\Phi)^T Q-\gamma Q^T\frac{1}{C}Q+\gamma\Phi^T\frac{1}{L}\Phi\right)\\ &-\gamma\Phi^T R_2\frac{1}{C}Q-\gamma(R_1\frac{1}{L}\Phi)^T\tilde{k}^T B^* r.\end{aligned} \tag{5.5.9}$$

From the boundary conditions (4.3.6) and (4.3.8), it follows that

$$[(\mathcal{H}_1 q)^T D\mathcal{H}_2 p]_a^b=0 \qquad \text{and} \qquad [r^T D^T\mathcal{H}_1 q]_a^b=0. \tag{5.5.10}$$

This leads to

$$\begin{aligned}\dot{L}(t)=&\int_a^b(-\gamma q^T\mathcal{H}_1 q-\gamma r^T\mathcal{R}\mathcal{H}_2 p+p^T\mathcal{H}_2(\gamma I-\mathcal{R}\mathcal{H}_2)p)d\zeta-\Phi^T\frac{1}{L}(R_1\frac{1}{L}-\gamma I)\Phi\\ &-Q^T\frac{1}{C}\left(R_2\frac{1}{C}+\gamma I\right)Q-\gamma\Phi^T(\frac{1}{L}R_1+R_2\frac{1}{C})Q-\gamma\tilde{k}^T B^* r R_1\frac{1}{L}\Phi.\end{aligned} \tag{5.5.11}$$

Therefore, by using the conditions $\gamma I \leq R_1 \frac{1}{L}$ and $\gamma I \leq \mathcal{H}_2 R$ with (5.5.4), there exists a positive constant K for sufficiently small γ such that

$$\dot{L}(t) \leq -K(E(t) + E_{RLC}(t)) \leq -\frac{K}{\lambda_2} L(t). \tag{5.5.12}$$

Finally, this leads to

$$L(t) \leq e^{-\frac{K}{\lambda_2} t} L(0), \tag{5.5.13}$$

which proves the exponential stability. □

Remark 5.5.1. As already stressed in [RZG17], some feedthrough term is necessary to damp the high frequency modes of the plant when no internal dissipation is occurring $(G_0 = 0)$. When some internal damping inducing the exponential stability of the plant (5.4.1) is present, the situation is more favorable. Indeed, these dissipation effects enable to damp the high frequency modes so that the controller has only to act on the low frequency modes.

From Remark 4.3.1, Theorem 5.5.1 obviously implies the exponential stabilizability and detectability of the interconnected system (5.4.1). Hence, since the assumptions of stabilizability and detectability hold, we can derive the FORE and the CORE for the interconnected system (5.4.1). Towards this end, some system and measurement noises are added to (5.4.1), which yields

$$\begin{aligned} d\mathbf{x}(t) &= ((\mathbf{J} - \mathbf{R})\mathbf{Q}\mathbf{x} + \mathbf{B}u(t))dt + \mathbf{H}dw(t), \\ d\mathbf{y}(t) &= \mathbf{B}^*\mathbf{Q}\mathbf{x}(t)dt + F dv(t). \end{aligned} \tag{5.5.14}$$

$w(t)$ is a Z-valued Wiener process with intensity operator $\mathbf{H} \in \mathcal{L}(Z, \mathbf{X})$ and covariance $Q \in \mathcal{L}(Z)$, and $v(t)$ is a $\mathbb{R}^m$-valued Wiener process with intensity $F \in \mathbb{R}^{m\times m}$ and covariance $V \in \mathbb{R}^{m\times m}$.

- There exists a unique stabilizing nonnegative self-adjoint solution $P_f \in \mathcal{L}(X)$ of the FORE given by

$$[(\mathbf{J} - \mathbf{R})\mathbf{Q}P_f + P_f((\mathbf{J} - \mathbf{R})\mathbf{Q})^* - P_f\mathbf{Q}\mathbf{B}(FVF^*)^{-1}\mathbf{B}^*\mathbf{Q}P_f + \mathbf{H}Q\mathbf{H}^*]\mathbf{y} = 0, \tag{5.5.15}$$

  for all $\mathbf{y} \in D(((\mathbf{J} - \mathbf{R})\mathbf{Q})^*)$ with $P_f(D(((\mathbf{J} - \mathbf{R})\mathbf{Q})^*)) \subset D((\mathbf{J} - \mathbf{R})\mathbf{Q})$.

- There exists a unique stabilizing nonnegative self-adjoint solution $P_c \in \mathcal{L}(X)$ of the CORE given by

$$[((\mathbf{J} - \mathbf{R})\mathbf{Q})^* P_c + P_c(\mathbf{J} - \mathbf{R})\mathbf{Q} - P_c\mathbf{B}\tilde{R}^{-1}\mathbf{B}^* P_c + \bar{R}]\mathbf{x} = 0, \tag{5.5.16}$$

  for all $\mathbf{x} \in D((\mathbf{J} - \mathbf{R})\mathbf{Q})$ $P_c(D((\mathbf{J} - \mathbf{R})\mathbf{Q})) \subset D(((\mathbf{J} - \mathbf{R})\mathbf{Q})^*)$.

The dynamics of the LQG controller are described by

$$u_c(t) = -K\hat{\mathbf{x}}(t) := -\tilde{R}^{-1}\mathbf{B}^* P_c \hat{\mathbf{x}}(t), \tag{5.5.17}$$

$$\frac{d\hat{\mathbf{x}}}{dt}(t) = [(\mathbf{J}-\mathbf{R})\mathbf{Q} - L\mathbf{B}^*\mathbf{Q} - \mathbf{B}K]\,\hat{\mathbf{x}}(t) + Lu_c(t), \tag{5.5.18}$$
$$\hat{\mathbf{x}}(0) = \mathbf{x}(0), \tag{5.5.19}$$
$$\gamma_c(t) = K\hat{\mathbf{x}}(t). \tag{5.5.20}$$

The Riccati equations (5.5.15) and (5.5.16) cannot be exactly solved. Therefore, a suitable finite-dimensional approximation must be developed. The discretization method used here is a mixed-finite element discretization as proposed in [GTvdSM04]. To reduce the Timoshenko beam model to a finite-dimensional system, we take advantage of the port-Hamiltonian structure. This method consists of an approximation of the effort and flow variables by means of differential forms related to their physical and geometrical natures. In the case of a Timoshenko beam, on one hand, the torque and the force (effort variables) correspond to zero-forms and, on the other hand, the translational and angular velocities (flow variables) correspond to one-forms. Let us consider N infinitesimal subsections of the spatial domain $[a,b]$. The discretized port-Hamiltonian model of the Timoshenko beam with internal frictions, governed by (5.2.3), is given by

$$\dot{x}_{ab}(t) = (J_{ab} - R_{ab})\mathcal{H}_{ab}x_{ab}(t) + Bu_{ab}(t), \tag{5.5.21}$$

where $x_{ab} \in \mathbb{R}^{4N}$ such that $J_{ab} = -J_{ab}^T \in \mathbb{R}^{4N\times 4N}$, $R_{ab} = R_{ab}^T \in \mathbb{R}^{4N\times 4N}$ with R_{ab} positive semi-definite, and $B_{ab} \in \mathbb{R}^{4N\times N}$. The corresponding Hamiltonian approximation is given by $E_{ab}(t) = \frac{1}{2}x_{ab}^T(t)\mathcal{H}_{ab}x_{ab}(t)$, where $\mathcal{H}_{ab}$ is the approximated matrix of the matrix operator $\mathcal{H}$ which is given by

$$\mathcal{H}_{ab} = \begin{bmatrix} K_{ab} & 0 & 0 & 0 \\ 0 & \frac{1}{\rho_{ab}} & 0 & 0 \\ 0 & 0 & EI_{ab} & 0 \\ 0 & 0 & 0 & \frac{1}{I_{\rho,ab}} \end{bmatrix}. \tag{5.5.22}$$

According to [RL13, Section 4], the discretization of the Timoshenko beam model (preserving the port-Hamiltonian structure) yields the following matrices:

the skew-symmetric structure matrix J_{ab} is given by

$$J_{ab} = \begin{bmatrix} 0 & M & 0 & S \\ -M^T & 0 & 0 & 0 \\ 0 & 0 & 0 & M \\ -S^T & 0 & -M^T & 0 \end{bmatrix}, \tag{5.5.23}$$

where

$$M = \begin{bmatrix} -2 & 0 & 0 & \dots & 0 \\ 4 & -2 & 0 & \dots & 0 \\ -4 & 4 & -2 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ (-1)^{N-1}4 & (-1)^{N-2}4 & \dots & -2 & 0 \\ (-1)^{N}4 & (-1)^{N-1}4 & \dots & 4 & -2 \end{bmatrix} \tag{5.5.24}$$

(5.5.25)

and

$$S = \mathrm{diag}[-(b-a),\dots,-(b-a)]; \tag{5.5.26}$$

the symmetric matrix R_{ab} is expressed as

$$R_{ab} = \mathrm{diag}\begin{bmatrix} 0 & R_{t,ab} & 0 & R_{a,ab} \end{bmatrix}, \tag{5.5.27}$$

where

$$R_{t,ab} = R_t \frac{b-a}{N} \begin{bmatrix} 1 & \dots & 1 \end{bmatrix} \in \mathbb{R}^N, \tag{5.5.28}$$

$$R_{a,ab} = R_a \frac{b-a}{N} \begin{bmatrix} 1 & \dots & 1 \end{bmatrix} \in \mathbb{R}^N. \tag{5.5.29}$$

The approximated matrix B_{ab} corresponding to the boundary conditions is given by

$$B_{ab} = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ (-1)^N 2 & 0 & 0 & 0 \\ 0 & (-1)^{N+1} 2 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & (-1)^N 2 & 0 & 0 \\ 0 & 0 & (-1)^{N+1} 2 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & (-1)^N 2 & 0 \\ 0 & 0 & 0 & (-1)^{N+1} 2 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & (-1)^N 2 \end{bmatrix}. \tag{5.5.30}$$

Eventually, the control operator (5.2.11) is approximated by

$$B_{c,ab} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ b_{ab} \end{bmatrix}, \qquad \text{with } b_{ab} \in \mathbb{R}^N. \tag{5.5.31}$$

Hence, the port-Hamiltonian structure preserving approximation of the interconnected system is described by

$$\dot{\mathbf{x}}_{ab}(t) = (\mathbf{J}_{ab} - \mathbf{R}_{ab})\mathbf{Q}_{ab}\mathbf{x}_{ab}(t) + \mathbf{B}_{ab}u(t), \tag{5.5.32}$$

$$\mathbf{y}_{ab}(t) = \mathbf{B}_{ab}^T \mathbf{Q}_{ab}\mathbf{x}_{ab}(t), \tag{5.5.33}$$

where

$$\mathbf{J}_{ab} = -\mathbf{J}_{ab}^T = \begin{bmatrix} J_{ab} & 0 & B_{c,ab}\tilde{k} \\ 0 & 0 & -I \\ -\tilde{k}B_{c,ab}^T & I & 0 \end{bmatrix} \in \mathbb{R}^{(4N+2m)\times(4N+2m)},$$

$$\mathbf{R}_{ab} = \mathbf{R}_{ab}^T = \begin{bmatrix} R_{ab} & 0 & 0 \\ 0 & R_1 & 0 \\ 0 & 0 & R_2 \end{bmatrix} \in \mathbb{R}^{(4N+2m)\times(4N+2m)}.$$

Also,

$$\mathbf{Q}_{ab} = \mathbf{Q}_{ab}^T = \text{diag} \begin{bmatrix} \mathcal{H}_{ab} & \frac{1}{L} & \frac{1}{C} \end{bmatrix} \in \mathbb{R}^{(4N+2m)\times(4N+2m)}$$

and

$$\mathbf{B}_{ab} = \begin{bmatrix} 0 \\ I \\ 0 \end{bmatrix} \in \mathbb{R}^{(4N+2m)\times m}.$$

Remark 5.5.2. Note that there is no need to approximate the IPMC dynamics since they are already finite-dimensional.

According to Theorem 4.2.1, the port-Hamiltonian framework can be preserved in the LQG controller dynamics by considering

$$\tilde{Q}_{ab} = \mathbf{Q}_{ab}\mathbf{B}_{ab}\mathbf{B}_{ab}^T\mathbf{Q}_{ab}, \tag{5.5.34}$$

$$P_f = \mathbf{Q}_{ab}^{-1}P_c\mathbf{Q}_{ab}^{-1}, \tag{5.5.35}$$

$$\mathbf{R}_c = \mathbf{R}_{ab} + \mathbf{B}_{ab}\mathbf{B}_{ab}^T P_c \mathbf{Q}_{ab}^{-1} + P_f\mathbf{Q}_{ab}\mathbf{B}_{ab}\mathbf{B}_{ab}^T. \tag{5.5.36}$$

The control Riccati equation of the finite-dimensional approximated system (5.5.32)-(5.5.33) is given by

$$[((\mathbf{J}_{ab} - \mathbf{R}_{ab})\mathbf{Q}_{ab})^T P_c + P_c(\mathbf{J}_{ab} - \mathbf{R}_{ab})\mathbf{Q}_{ab} - P_c\mathbf{B}_{ab}\tilde{R}_{ab}^{-1}\mathbf{B}_{ab}^T P_c + \bar{R}_{ab}]\mathbf{x}_{ab} = 0, \tag{5.5.37}$$

where $\mathbf{x}_{ab} \in \mathbb{R}^{4N+2m}$. This Riccati equation (5.5.37) is solved with the Matlab® function `care`.

The finite-dimensional approximation of the closed-loop system, i.e. the IPMC-actuated flexible beam with the Hamiltonian LQG controller, is described by

$$\begin{pmatrix} \mathbf{x}_{ab} \\ \hat{\mathbf{x}}_{ab} \end{pmatrix}(t) = \begin{pmatrix} \mathbf{J}_{ab} & -\mathbf{B}_{ab}\mathbf{B}_{ab}^T P_c \mathbf{Q}_{ab}^{-1} \\ P_f\mathbf{Q}_{ab}\mathbf{B}_{ab}\mathbf{B}_{ab}^T & \mathbf{J}_{ab} \end{pmatrix} - \begin{pmatrix} \mathbf{R}_{ab} & 0 \\ 0 & \mathbf{R}_c \end{pmatrix}\begin{pmatrix} \mathbf{Q}_{ab}\mathbf{x}_{ab}(t) \\ \mathbf{Q}_{ab}\hat{\mathbf{x}}_{ab}(t) \end{pmatrix} + \begin{pmatrix} \mathbf{B}_{ab} \\ 0 \end{pmatrix} u(t). \tag{5.5.38}$$

By using condition (5.5.34), the functional cost to be minimized is given by

$$J(u) = \lim_{T\to\infty}\int_0^T \langle \mathbf{x}_{ab}(t), \tilde{Q}_{ab}\mathbf{x}_{ab}(t)\rangle_{\mathbb{R}^{4N+2m}} + \langle u(t), \tilde{R}_{ab}u(t)\rangle_{\mathbb{R}^m} dt \tag{5.5.39}$$

and consists in finding a trade-off between the cost in energy of the input $u(t)$ and the intensity of the output current of the interconnected system (5.4.1).

5.6 Experimental validation of the model

This section is devoted to the validation of the proposed model of an actuated flexible beam setup described in Section 5.1. It is shown that this model reproduces the main properties of the experimental setup mounted in the AS2M department of FEMTO-ST Institute in Besançon (France) and represented in Figure 5.1.

Let us recall that the flexible beam made of polyethylene plastic is assumed to be fixed at the origin. In the experimental setup, an IPMC is patched along the beam near its fixed extremity. In addition, we measure the beam displacement by means of a laser sensor (LK-G152). The laser sensor is placed so that we measure the position of the beam at 5 mm from its tip at the equilibrium position. In order to discretize the flexible beam, we consider 100 infinitesimal subsections of beam's length (160 mm). The discretized port-Hamiltonian model of the Timoshenko beam is given by $\Sigma((J_{ab}-R_{ab})\mathcal{H}_{ab}, B_{ab}, B_{ab}^T\mathcal{H}_{ab})$.

The simulations were conducted in Matlab® and the experimentations were performed several times to ensure the reproducibility of the obtained results. The parameters used for the numerical simulations and the experimentations are given in Tables 5.1 and 5.2. The parameter values of the IPMC are taken from [NTMO11].

Some parameters are unknown, in particular the Young and shear modulus, the transversal and angular frictions, and the coupling constant $\tilde{k}$ of the employed IPMC. The identification procedure is mainly based on [MWR$^+$]. For the identification process, we used optimal algorithms for nonlinear model identification (`nlgreyest` function) implemented in the Matlab Identification Toolbox®. Besides, sequential quadratic programming (SQP) and trust-region-reflective algorithms were used with the Matlab® function `fmincon`. The results of the identification procedure are depicted in Figure 5.5. One can observe that the fitting percentage between the model with the optimally estimated parameter values (red dashed line) and the experimental data (black line) is of 89.67%. The identified parameters are listed in Table 5.3. In addition, the fitting procedure with the optimally estimated parameter values was undertaken again with the laser sensor placed at 10 *mm* from the beam tip at the equilibrium position. In this case, the fitting percentage is of 85.55%, which strengthens the validity of the numerical values obtained for the parameters.

To complete the experimental validation of the model, a comparison of the displacement of the beam at 155mm between the interconnected model implemented in SIMULINK® and the experimental data was performed. The parameters of the experimental setup given in Tables 5.1, 5.2 and 5.3 were used for the numerical simulations. The experimental and simulated data are represented as a dark and a red dashed line, respectively. Figure 5.6 shows that the flexible beam displacement reaches (the desired equilibrium position of) 5 mm when applying a voltage of 1.5 V. Moreover, one can observe that the interconnected model reproduces accurately the behaviour of the experimental setup.

Table 5.1 – Parameters of the endoscope

Notation	Description	Value	Unit
L	Length	1.6×10^{-1}	m
W	Width	7×10^{-3}	m
T	Thickness	2.2×10^{-4}	m
ρ	Mass density	936	kg/m^3
I	Inertia moment	4.7×10^{-15}	m^4
I_ρ	Angular moment	4.34×10^{-12}	$kg\ m$

Table 5.2 – Parameters of the IPMC actuator

Notation	Description	Value	Unit
r_1	Resistance 1	30	Ω
r_2	Resistance 2	700	Ω
C	Capacitance	1.2×10^{-1}	F
L_{IP}	Length of IPMC	3×10^{-2}	m

Table 5.3 – Identified parameters

Notation	Description	Value	Unit
E_e	Young modulus	4.14×10^9	Pa
K	Shear modulus	1.4178×10^9	Pa
R_a	Angular friction	10^{-5}	$kg\ m/s$
R_t	Transversal friction	2×10^{-5}	$kg\ m^3/s$
$\tilde{k}$	Coupling constant	3×10^{-5}	$N\ M/V$

Figure 5.7 shows the current responses to an input voltage of 1.5 V. The solid black and dash red lines correspond to the output current simulated from the IPMC-beam model and the output current measured from the experimental setup, respectively. The current responses reach the peak of 0.0408 mA and decay rapidly afterwards. The differences between the two curves might be caused by the RLC approximation of the IPMC.

In [NTMO11], a more accurate but also more complex model of the IPMC is presented. Indeed, IPMCs are described as interconnected distributed port-Hamiltonian systems on multiple spatial scales. The proposed model reproduces the coupling between the electrical dynamics, the dynamics of the polymer gel (inducing the swelling) and the mechanical beam dynamics of the IPMCs.

5.7 Control of the IPMC-actuated flexible beam

We shall now design a control law for the proposed model that has been validated in Section 5.6. As already mentioned, the control objective consists in setting the IPMC

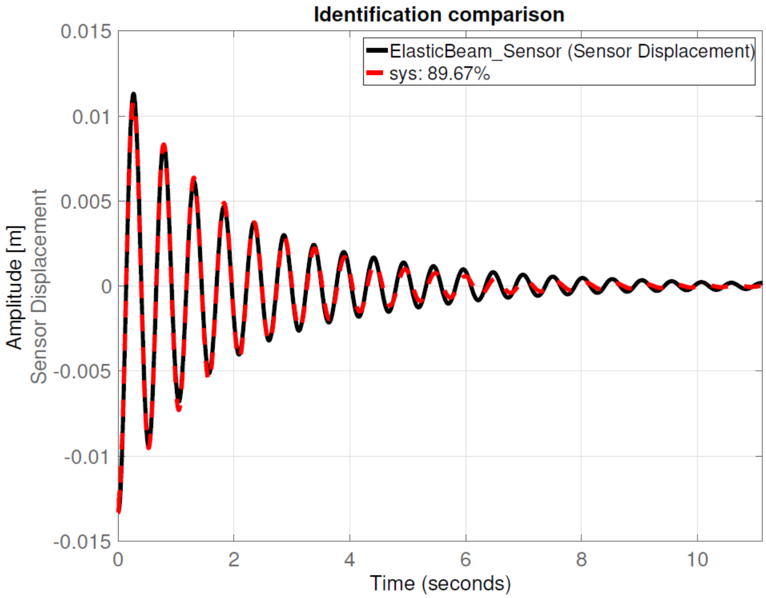

Figure 5.5 – Parameter estimation with displacement measure taken at 155 mm

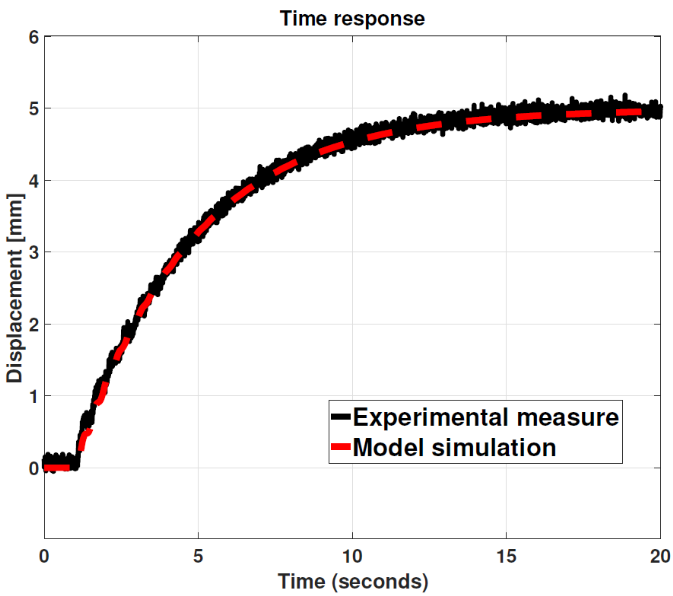

Figure 5.6 – IPMC actuated beam model vs experimental data: beam tip displacement comparison

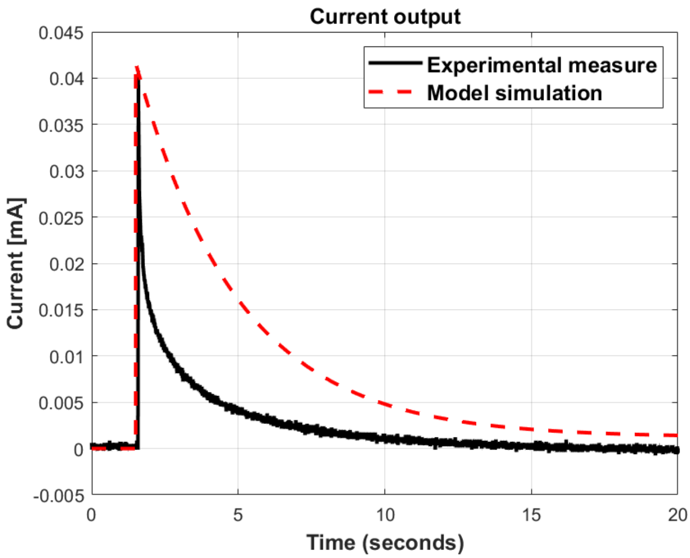

Figure 5.7 – IPMC actuated beam model vs experimental data: current intensity comparison

actuated flexible beam in a specific configuration by means of electric stimulus, while reducing vibrations. The time response of the system is improved by positive damping injection (see 5.7.2) and the induced vibrations are damped out by the Hamiltonian LQG controller. A trade-off between the time response of the system and the vibrations has to be found. A schematic diagram of the IPMC actuated beam interconnected to a LQG control and subject to damping injection is represented in Figure 5.8. We recall that the interconnection between the beam and the IPMC is given by (5.3.10) and that the interconnection with the Hamiltonian LQG controller is given by

$$u_c = \mathbf{y} \qquad \text{and} \qquad u = -y_c. \tag{5.7.1}$$

On one hand, we consider a positive damping injection given by

$$u(t) = -r_c \mathbf{y}(t) \tag{5.7.2}$$

in order to improve the time response of the system, where $\mathbf{y}(t)$ is the output current and $r_c < 0$ is a control parameter. In order to preserve the stability of the interconnected system, r_c must satisfy

$$r_c > -r_1. \tag{5.7.3}$$

The damping injection method is based on [OGC04].

On the other hand, we design an Hamiltonian LQG controller to reduce the vibrations induced by the damping injection. According to Theorem 4.2.1, the parameters

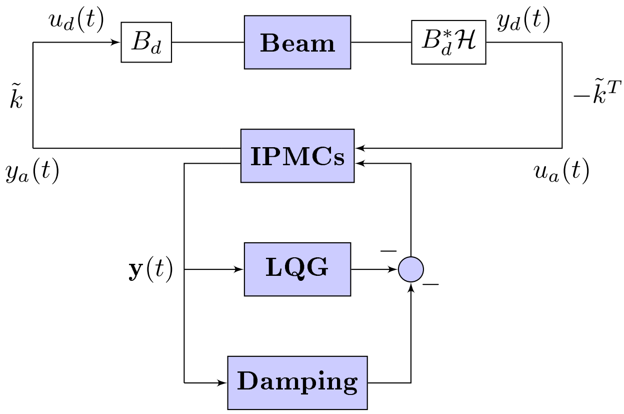

Figure 5.8 – Schematic diagram of the IPMC actuated beam interconnected to a LQG control and subject to damping injection

of the LQG control problem have to be set appropriately in order to preserve the port-Hamiltonian framework in the LQG controller dynamics. Towards this end, let us consider the optimal control first, and thereby the covariance matrices must be interpreted as further control parameters and do not have any statistical meaning anymore. To be more specific, the control parameters are chosen as

$$\tilde{Q}_{ab} = 1700(\mathbf{Q}_{ab}\mathbf{B}_{ab})\mathbf{B}_{ab}^T\mathbf{Q}_{ab}, \tag{5.7.4}$$

$$\tilde{R}_{ab} = 1000. \tag{5.7.5}$$

The discretization method of mixed-finite elements presented in Section 5.5 was applied for 100 subsections of the beam, which entails that $\mathbf{x}_{ab}(t) \in \mathbb{R}^{402}$. The control Riccati equation was solved with the Matlab® function `care` to obtain P_c, and the LQG controller described by (5.5.38) was numerically implemented in SIMULINK®. Figure 5.9 shows the open-loop time response of the IPMC-actuated beam model in blue, the closed-loop response with damping injection in red, and the closed-loop response with positive damping injection and LQG control in dashed black. In addition, the reference is plotted in yellow. The time response of the open-loop system is drastically improved by the positive damping injection. One can observe that the positive damping injection improves the time response of the system and that the LQG controller reduces the vibrations. As depicted as a dashed black line in Figure 5.9, a trade-off between time response and oscillations can be found. The feedback control law was then tested when applying several references, see Figure 5.10. Again, one can observe that the Hamiltonian LQG controller with positive damping injection provides a good trade-off between time response and oscillations.

Spatial discretization methods of infinite-dimensional systems often lead to high dimensional systems, and then to high-dimensional controllers. This motivates the

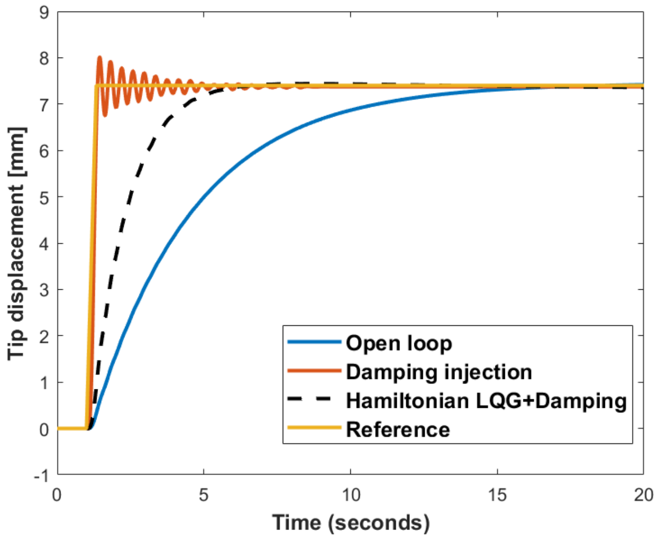

Figure 5.9 – Comparison between open-loop system and positive damping injection with Hamiltonian LQG control system

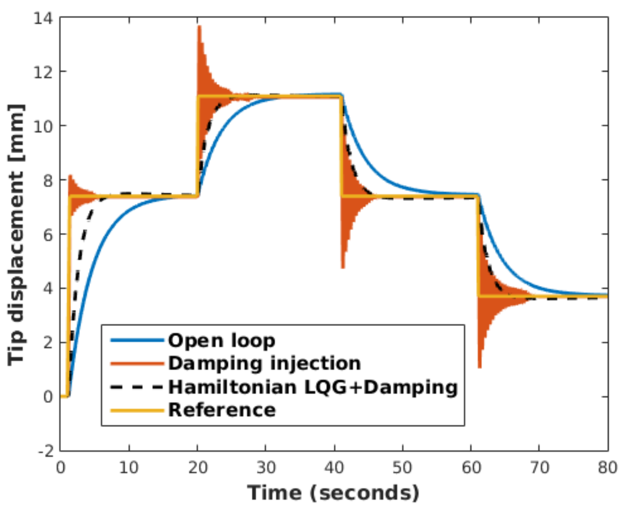

Figure 5.10 – Comparison between open-loop system and positive damping injection with Hamiltonian LQG control system

study of model-reduction methods, in particular for the class of port-Hamiltonian systems. Open-loop reduction methods for high dimensional port-Hamiltonian systems can be found in [BCL+09]. Nevertheless, when the reduced-order controller is directly applied to the full order system, the control will act on the low frequency modes but also on the residual modes (from the unmodeled dynamics), which can induce some undesired vibrations in the case of a flexible endoscope. This is called the spillover effect, which can cause performance degradation of the controller or even lead to instability. The spillover effect can be addressed in the closed-loop dynamics. We refer to [WHGM18], where a closed-loop reduction method was proposed to design a reduced-order LQG controller with good closed-loop performances.

Taking advantage of the power-preserving interconnection of port-Hamiltonian systems, it would be interesting to establish the exponential stability of the closed-loop system composed of an exponentially stable infinite-dimensional dissipative port-Hamiltonian system with a finite-dimensional port-Hamiltonian controller. Guaranteeing the exponential stability when interconnecting the plant to a finite-dimensional port-Hamiltonian controller would allow us to avoid the spillover effect. However, this stability result has so far remained to be proved, leading to Conjecture 5.7.1. Notice that a similar situation was studied in [RZG17] where the plant is controlled at the boundary by an exponentially stable nonlinear dynamical system with a feedthrough term. When compared to the stabilization problem (5.4.1) with a finite-dimensional port-Hamiltonian controller, the situation in [RZG17] is less favorable since the controller has to damp both high and low frequency modes, which is quite challenging from a control perspective.

Conjecture 5.7.1. *Let us consider the infinite-dimensional dissipative port-Hamiltonian system described by*

$$\begin{aligned} \dot{x}(t) &= (\mathcal{J}-\mathcal{R})\mathcal{H}x(t)+Bu(t), \\ y(t) &= B^*\mathcal{H}x(t), \end{aligned} \tag{5.7.6}$$

where $\mathcal{J}: D(\mathcal{J}) \to \mathcal{X}$ is skew-adjoint, $\mathcal{R} \in \mathcal{L}(\mathcal{X})$ is self-adjoint and $B \in \mathcal{L}(\mathbb{R}^m, \mathcal{X})$. Assume that the system (5.7.6) is exponentially stable. In addition, let us consider a finite-dimensional control system given by

$$\begin{aligned} x_c(t) &= (J_c - R_c)Q_c x_c(t) + B_c(t)u_c(t), \\ y_c(t) &= B^T Q_c x_c(t), \end{aligned} \tag{5.7.7}$$

where $J_c = -J_c^T$, $R_c = R_c^T \geq 0$, $Q_c = Q_c^T > 0$ and B_c is full rank.
Then the power-preserving interconnection, which is given by

$$u = -y_c \qquad \text{and} \qquad y = u_c,$$

of (5.7.6) with (5.7.7) is again an exponentially stable port-Hamiltonian system.

In a similar manner as in Theorem 5.5.1, cross terms would have to be included in the candidate of Lyapunov functional due to the domain interconnection.

5.8 Conclusion and perspectives

This chapter was devoted to the study of the LQG control problem of an IPMC actuated endoscope. The IPMC actuated endoscope was modeled as an interconnected Timoshenko beam model with a RLC circuit. In Section 5.5, the LQG control problem was addressed for this specific application. Next, in Section 5.6, the validity of the proposed model for the IPMC-actuated endoscope was studied on the experimental setup mounted in the AS2M department of the FEMTO-ST Institute in Besançon, France. It was shown that the main mechanical characteristics of the experimental setup have been reproduced by the proposed model. In Section 5.7, a feedback control law is implemented as the sum of a positive damping injection to improve the time response, and of the stabilizing output of an Hamiltonian LQG controller to reduce the mechanic vibrations. Note that the feedback control presented in Section 5.7 is prospective. Further works would be to implement and evaluate the performance of the Hamiltonian LQG controller on the experimental setup. In addition, notice that the theory presented in this chapter is developed for an arbitrary number of IPMCs. Future works would be to implement and experiment it for several IPMCS patched along the beam to shape it in a desired manner.

As already pointed out in [NTMO11], the actuation and the corresponding bending force of an IPMC depend on its relative humidity. As a matter of fact, a humid IPMC is more efficient than a dryer one. This is due to the working mechanism of an IPMC. The mechanical bending depends on the transport and the swelling of water and cations molecules in the polyelectrolyte gel. Thus, a better humidity will induce a better bending of the IPMC. A generalization of the RLC model proposed for the IPMC would be to consider the fatigue behaviour of the IPMC. Besides, a simplified model of the IPMC as a RLC circuit was considered in this work. Hence, the polymer gel dynamics would have to be taken into consideration for future research.

The Riccati equation is the crux when solving a LQ optimal control problem. The rationale in this chapter was to solve the Riccati equations related to a LQG control of a finite-dimensional approximation of the controlled system. Nevertheless, a finite-dimensional approximation leads to some errors when solving the operator Riccati equations. It would be of great interest to carry out a study of this error. Besides, a suggestion would be to add a further term in the functional cost to minimize this error.

Finally, in this last chapter, stochastic disturbances are not considered in the plant dynamics. The covariance operators of the LQG controller are used as further parameters to design an Hamiltonian LQG controller. While undertaking manipulations of the experimental setup, it was noticed that it was highly sensible to environmental disturbances such as acoustic noises or even people walking into the lab. Measurement noises could also occur. The latter depends on the quality of the sensors used. Ongoing research would be to take under consideration the nature of these neglected effects (in this chapter) for the design of a feedback control law.

Conclusion and perspectives

General conclusion

In this thesis stochastic and deterministic port-Hamiltonian systems with both distributed and boundary controls along with distributed and boundary observations were explored. From a mathematical point of view, the stochastic integration theory and the semigroup approach of infinite-dimensional systems theory are employed together for the study and the analysis of the stochastic partial differential equation governing this new class of stochastic systems.

As a first step, the class of first-order linear port-Hamiltonian systems with distributed and boundary control was presented in a similar way as in [JZ12]. The Riesz basis property of this class of distributed parameter systems has been investigated. A subclass of linear port-Hamiltonian systems, namely nice port-Hamiltonian systems, has been introduced and such systems have been proved to be Riesz-spectral systems. As a direct consequence, the growth bound condition has been proved to hold for nice port-Hamiltonian systems. Further contributions are the author's willingness of giving a clarification of [Vil07, Chapter 4] and proposing a state of the art on the basis of the existing literature and current works on this topic.

As a second step, the new class of stochastic port-Hamiltonian systems with distributed and boundary control and observation is investigated. The passivity property has been introduced for infinite-dimensional stochastic systems and has been proved to be not preserved for SPHSs, due to the energy increments induced by the Wiener process. A well-posedness concept has been presented for boundary controlled and observed stochastic systems in the spirit of the deterministic well-posedness definition. Furthermore, theoretical results were illustrated on an example of a stochastic vibrating string by means of a modal representation via a Riesz basis.

In the second part of this thesis, we have studied the control aspects of stochastic port-Hamiltonian systems. More particularly, we have introduced a first attempt to treat the question of LQG control of stochastic port-Hamiltonian systems on infinite-dimensional spaces with bounded control, observation and noise operators. The generalization of the port-Hamiltonian LQG controller for infinite-dimensional systems has been performed in Theorem 4.2.2. To be more specific, this result allows one to represent the LQG control problem of a stochastic port-Hamiltonian system as the power-preserving interconnection of SPHSs. We have then extended the result to strongly stabilizable port-Hamiltonian systems based on results borrowed from [Oos00]. In Section 4.3, by setting conditions on matrices P_1, P_0 and G_0, Theorem 4.3.1 states that port-Hamiltonian systems with dissipative effects on the momentum components of the state are exponentially stable. The exponential stabilizability and detectability of the controlled plant are then easily deduced.

Finally, the port-Hamiltonian formalism has been used for the modeling of a flexible beam interconnected in a power-preserving manner with several IPMC actuators patched along the beam. This formalism has been used advantageously to take under consideration the physical dynamics of the actuator in the control process. Moreover, the proposed interconnected port-Hamiltonian model has been proved to be exponentially stable under some conditions on the parameters. Once a finite-dimensional approximation obtained via a mixed-finite element approach [GTvdSM04], a validation of the proposed model has been undertaken by using the physical parameters of an experimental setup. The main physical characteristics have been proved to be reproduced by the simulated model. Furthermore, the feedback control law, which consists of a damping injection (DI) method with a LQG controller, has been implemented on an IPMC-actuated flexible beam model. Numerical simulations demonstrate the efficiency of the feedback DI-LQG control law designed to improve the time response while reducing the induced vibrations.

This thesis has been mainly intended to propose new perspectives by taking into account stochastic effects in the port-Hamiltonian modeling for distributed parameter systems operating in a random environment. This work constitutes a first attempt to study this new class of stochastic systems and to investigate some of their properties. The author hopes that the work presented in this thesis will bring some attention to the study of control and observation of SPDEs. Furthermore, the port-Hamiltonian formalism has been considered for the modeling and the analysis of the deterministic dynamics of distributed parameter systems as well. The author hopes to have convinced the reader of the interest of this powerful and efficient formalism, which continues to attract more and more attention from mathematicians and engineers.

The new concepts and results presented in this thesis entail inexorably further questions and lead to further investigations that could complete the work conducted throughout this thesis. Some of them are identified and resumed in the next section.

Suggestions for future work

In the deterministic setting, a complete characterization of the Riesz-spectral property of port-Hamiltonian systems is still an open question. As a matter of fact, the assumptions of simple eigenvalues and of a uniform gap within the eigenvalues considered for nice port-Hamiltonian systems need to be relaxed in order to include a wider range of applications. For instance, coupled vibrating strings may have Jordan blocks or a two-dimensional vibrating string will not satisfy the uniform gap condition. Moreover, the Riesz basis property has not been studied in the context of dissipation effects from internal friction within the port-Hamiltonian framework.

Clearly, introducing and studying the new class of stochastic port-Hamiltonian systems open the way to further considerations. In this thesis we have only considered additive Gaussian white noises occurring along the domain. Following the natural tendency of mathematicians of greater generality, it would be relevant to allow the noise intensity operator to depend on the state process, which would yield the following SPDE

$$d\varepsilon(t) = (P_1 \frac{\partial}{\partial \zeta}(\mathcal{H}\varepsilon(t)) + P_0 \mathcal{H}\varepsilon(t))dt + (H\varepsilon(t))dw(t), \tag{5.8.1}$$

where the noise port $H \in \mathcal{L}(\mathcal{X}, \mathcal{L}(Z, \mathcal{X}))$. Adding stochastic disturbances on boundary and distributed controls would also be recommended for future research. In addition, Wiener processes considered in this work could be generalized to jump processes such as Lévy processes. A theory of integration with respect to jump processes can be found in [App04].

In Chapter 3, the passivity property has been proved to be unpreserved for the considered SPHSs. This is also known in the finite-dimensional case, see [SF13]. The recovery of the passivity property of infinite-dimensional SPHSs would in any doubt be a major research topic. The author believes that a generalization of the stochastic generalized canonical transformations proposed in [SF13] would enable a passivity recovery in the infinite-dimensional case. Nevertheless, such generalization is not an easy task. As already mentioned, the main difficulty lies in the identification of a set of transformations when the Hamiltonian is spatially dependent. Moreover, notice that in [CF70], a version of the Itô's formula (see Theorem 2.4.2) was proposed in a Hilbert space context, i.e. for any function

$$f : [0,T] \times \mathcal{X} \to \mathcal{X}, \tag{5.8.2}$$

where $\mathcal{X}$ is a Hilbert space.

In the last part of this thesis, the case of a specific application, which consists of an ionic polymer metal composite actuated endoscope, has been considered. Some perspectives of work are listed herebelow.

- The resolution of the operator Riccati equations is the core issue when LQG control problems are considered. In this work this has been done by using a finite-dimensional approximation of the plant. As already mentioned, it would be interesting to conduct a rigorous error analysis when solving the operator Riccati equations, and eventually consider a further term in the cost to minimize the error due to the approximation.

- In the experimental setup described in Section 5.1, we have only considered one IPMC patched at the fixed extremity of the plastic beam. In order to be able to shape the beam configuration as desired, it would be interesting to consider several IPMCs in future works.
- The LQG controller has been designed on a finite-dimensional approximation of the system in Chapter 5. Even though this LQG controller has been proved to be efficient to damp out the vibrations, there is no theoretical guarantee that the controller will still be working on the infinite-dimensional system, i.e. the experimental setup. Nevertheless, a similar approach as in [RZG17] could be developed. With a Lyapunov approach as in Theorem 5.5.1, it would be interesting to prove that the finite-dimensional LQG Hamiltonian controller proposed in Section 5.4 exponentially stabilizes the interconnected system (5.4.1). In contrast to [RZG17], the plant is exponentially stable, and thus there is no need to consider a feedthrough term to damp the high frequencies.
- A natural future control method to consider would be H_∞ control, which has been developed to provide robust control and allows disturbances rejection, see [vK12].

Eventually, one area of investigation that has been neglected in this work is the study of controllability and observability of stochastic systems. As far as known, there is so far no clear definitions of these concepts for infinite-dimensional stochastic systems.

Appendix

Appendix A

Power-preserving discretization

In this appendix we give some details on the finite-dimensional approximation of the Timoshenko beam introduced in Section 5.5, while preserving the port-Hamiltonian structure. The method is based on an approximation of the effort and flow variables by means of differential forms. We refer the reader to [GTvdSM04] for a complete description of the method and to [RL13] for its application to the Timoshenko beam.

Let us consider a spatial domain $[a,b]$ with N infinitesimal subsections. We first detail the Dirac structure approximation. Let us denote the flow variables and the effort variables by f_{x_i} and e_{x_i}, $i = 1,\dots,4$, respectively. Their approximations on an infinitesimal section $[\alpha,\beta]$ are given by

$$f_{x_i}(t,\zeta) = f_{x_i}^{\alpha\beta}(t)w_{x_i}^{\alpha\beta}(\zeta), \tag{A.0.1}$$

$$e_{x_i}(t,\zeta) = e_{x_i}^{\alpha}(t)w_{x_i}^{\alpha}(\zeta) + e_{x_i}^{\beta}(t)w_{x_i}^{\beta}(\zeta), \tag{A.0.2}$$

where the forms $w_{x_i}^{\alpha\beta}$, $w_{x_i}^{\alpha}$ and $w_{x_i}^{\beta}$ satisfy

$$w_{x_i}^{\alpha\beta} = \frac{1}{\beta-\alpha}, \qquad w_{x_i}^{\alpha} = \frac{\beta-\zeta}{\beta-\alpha}, \qquad w_{x_i}^{\beta} = \frac{\zeta-\alpha}{\beta-\alpha}. \tag{A.0.3}$$

The PDE (5.2.3) describing the dynamics of the Timoshenko beam can then be approximated by

$$f_{x_1}^{\alpha\beta}(t)w_{x_1}^{\alpha\beta}(\zeta) = e_{x_2}^{\alpha}(t)\frac{d}{d\zeta}w_{x_2}^{\alpha}(\zeta) + e_{x_2}^{\beta}(t)\frac{d}{d\zeta}w_{x_2}^{\beta}(\zeta) - e_{x_4}^{\alpha}(t)w_{x_4}^{\alpha}(\zeta) - e_{x_4}^{\beta}(t)w_{x_4}^{\beta}(\zeta), \tag{A.0.4}$$

$$\begin{aligned} f_{x_2}^{\alpha\beta}(t)w_{x_2}^{\alpha\beta}(\zeta) =& e_{x_1}^{\alpha}(t)\frac{d}{d\zeta}w_{x_1}^{\alpha}(\zeta) + e_{x_1}^{\beta}(t)\frac{d}{d\zeta}w_{x_1}^{\beta}(\zeta) - R_t e_{x_2}^{\alpha}(t)w_{x_2}^{\alpha}(\zeta) \\ & - R_t e_{x_2}^{\beta}(t)w_{x_2}^{\beta}(\zeta), \end{aligned} \tag{A.0.5}$$

$$f_{x_3}^{\alpha\beta}(t)w_{x_3}^{\alpha\beta}(\zeta) = e_{x_4}^{\alpha}(t)\frac{d}{d\zeta}w_{x_4}^{\alpha}(\zeta) + e_{x_4}^{\beta}(t)\frac{d}{d\zeta}w_{x_4}^{\beta}(\zeta), \tag{A.0.6}$$

$$\begin{aligned} f_{x_4}^{\alpha\beta}(t)w_{x_4}^{\alpha\beta}(\zeta) =& e_{x_3}^{\alpha}(t)\frac{d}{d\zeta}w_{x_3}^{\alpha}(\zeta) + e_{x_3}^{\beta}(t)\frac{d}{d\zeta}w_{x_3}^{\beta}(\zeta) + e_{x_1}^{\alpha}(t)w_{x_1}^{\alpha}(\zeta) + e_{x_1}^{\beta}(t)w_{x_1}^{\beta}(\zeta) \\ & - R_a e_{x_4}^{\alpha}(t)w_{x_4}^{\alpha}(\zeta) - R_a e_{x_4}^{\beta}(t)w_{x_4}^{\beta}(\zeta). \end{aligned} \tag{A.0.7}$$

By taking the derivatives of the density functions, which satisfy (A.0.3), we get that

$$f_{x_1}^{\alpha\beta}(t)w_{x_1}^{\alpha\beta}(\zeta) = -e_{x_2}^{\alpha}(t)w_{x_2}^{\alpha\beta}(\zeta) + e_{x_2}^{\beta}(t)w_{x_2}^{\alpha\beta}(\zeta) - e_{x_4}^{\alpha}(t)w_{x_4}^{\alpha}(\zeta) - e_{x_4}^{\beta}(t)w_{x_4}^{\beta}(\zeta) \tag{A.0.8}$$

$$f_{x_2}^{\alpha\beta}(t)w_{x_2}^{\alpha\beta}(\zeta) = -e_{x_1}^{\alpha}(t)w_{x_1}^{\alpha\beta}(\zeta) + e_{x_1}^{\beta}(t)w_{x_1}^{\alpha\beta}(\zeta) - R_t e_{x_2}^{\alpha}(t)w_{x_2}^{\alpha}(\zeta) - R_t e_{x_2}^{\beta}(t)w_{x_2}^{\beta}(\zeta) \tag{A.0.9}$$

$$f_{x_3}^{\alpha\beta}(t)w_{x_3}^{\alpha\beta}(\zeta) = -e_{x_4}^{\alpha}(t)w_{x_4}^{\alpha\beta}(\zeta) + e_{x_4}^{\beta}(t)w_{x_4}^{\alpha\beta}(\zeta) \tag{A.0.10}$$

$$\begin{aligned} f_{x_4}^{\alpha\beta}(t)w_{x_4}^{\alpha\beta}(\zeta) =& -e_{x_3}^{\alpha}(t)w_{x_3}^{\alpha\beta}(\zeta) + e_{x_3}^{\beta}(t)w_{x_3}^{\alpha\beta}(\zeta) + e_{x_1}^{\alpha}(t)w_{x_1}^{\alpha}(\zeta) + e_{x_1}^{\beta}(t)w_{x_1}^{\beta}(\zeta) \\ & - R_a e_{x_4}^{\alpha}(t)w_{x_4}^{\alpha}(\zeta) - R_a e_{x_4}^{\beta}(t)w_{x_4}^{\beta}(\zeta). \end{aligned} \tag{A.0.11}$$

By integrating along the subsection domain $[\alpha, \beta]$, we obtain the following expression

$$f_{x_1}^{\alpha\beta}(t) = -e_{x_2}^{\alpha}(t) + e_{x_2}^{\beta}(t) - \frac{\beta-\alpha}{2}e_{x_4}^{\alpha}(t) - \frac{\beta-\alpha}{2}e_{x_4}^{\beta}(t) \tag{A.0.12}$$

$$f_{x_2}^{\alpha\beta}(t) = -e_{x_1}^{\alpha}(t) + e_{x_1}^{\beta}(t) - R_t\frac{\beta-\alpha}{2}e_{x_2}^{\alpha}(t) - R_t\frac{\beta-\alpha}{2}e_{x_2}^{\beta}(t) \tag{A.0.13}$$

$$f_{x_3}^{\alpha\beta}(t) = -e_{x_4}^{\alpha}(t) + e_{x_4}^{\beta}(t) \tag{A.0.14}$$

$$\begin{aligned} f_{x_4}^{\alpha\beta}(t) =& -e_{x_3}^{\alpha}(t) + e_{x_3}^{\beta}(t) + \frac{\beta-\alpha}{2}e_{x_1}^{\alpha}(t) + \frac{\beta-\alpha}{2}e_{x_1}^{\beta}(t) - R_a\frac{\beta-\alpha}{2}e_{x_4}^{\alpha}(t) \\ & - R_a\frac{\beta-\alpha}{2}e_{x_4}^{\beta}(t). \end{aligned} \tag{A.0.15}$$

The efforts on the interval are defined as

$$\begin{aligned} e_{x_1}^{\alpha\beta} &= \frac{1}{2}e_{x_1}^{\alpha} + \frac{1}{2}e_{x_1}^{\beta}, \qquad & e_{x_2}^{\alpha\beta} &= \frac{1}{2}e_{x_2}^{\alpha} + \frac{1}{2}e_{x_2}^{\beta}, \\ e_{x_3}^{\alpha\beta} &= \frac{1}{2}e_{x_3}^{\alpha} + \frac{1}{2}e_{x_3}^{\beta}, \qquad & e_{x_4}^{\alpha\beta} &= \frac{1}{2}e_{x_4}^{\alpha} + \frac{1}{2}e_{x_4}^{\beta}. \end{aligned} \tag{A.0.16}$$

By expressing the approximated flows $f_{x_i}^{\alpha\beta}$ and efforts $e_{x_i}^{\alpha\beta}$ under the boundary port-variables $f_{\partial}^{\alpha\beta}$ and $e_{\partial}^{\alpha\beta}$, we get that

$$\begin{bmatrix} f_{x_1}^{\alpha\beta} \\ f_{x_2}^{\alpha\beta} \\ f_{x_3}^{\alpha\beta} \\ f_{x_4}^{\alpha\beta} \end{bmatrix} = \begin{bmatrix} -1 & -\frac{\beta-\alpha}{2} & 1 & -\frac{\beta-\alpha}{2} & 0 & 0 & 0 & 0 \\ -R_t\frac{\beta-\alpha}{2} & 0 & -R_t\frac{\beta-\alpha}{2} & 0 & -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -R_a\frac{\beta-\alpha}{2} & 0 & -R_a\frac{\beta-\alpha}{2} & \frac{\beta-\alpha}{2} & -1 & \frac{\beta-\alpha}{2} & 1 \end{bmatrix} \begin{bmatrix} f_{\partial}^{\alpha\beta} \\ e_{\partial}^{\alpha\beta} \end{bmatrix}, \tag{A.0.17}$$

$$\begin{bmatrix} e_{x_1}^{\alpha\beta} \\ e_{x_2}^{\alpha\beta} \\ e_{x_3}^{\alpha\beta} \\ e_{x_4}^{\alpha\beta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} f_\partial^{\alpha\beta} \\ e_\partial^{\alpha\beta} \end{bmatrix}, \tag{A.0.18}$$

where the approximated selected boundary variables satisfy

$$f_\partial^{\alpha\beta} = \begin{bmatrix} f_{\partial,x_2}^{\alpha} \\ f_{\partial,x_4}^{\alpha} \\ f_{\partial,x_2}^{\beta} \\ f_{\partial,x_4}^{\beta} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} e_{x_2}^{\alpha} \\ e_{x_4}^{\alpha} \\ e_{x_2}^{\beta} \\ e_{x_4}^{\beta} \end{bmatrix}, \tag{A.0.19}$$

$$e_\partial^{\alpha\beta} = \begin{bmatrix} e_{\partial,x_1}^{\alpha} \\ e_{\partial,x_3}^{\alpha} \\ e_{\partial,x_1}^{\beta} \\ e_{\partial,x_3}^{\beta} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} e_{x_1}^{\alpha} \\ e_{x_3}^{\alpha} \\ e_{x_1}^{\beta} \\ e_{x_3}^{\beta} \end{bmatrix}. \tag{A.0.20}$$

Similarly, the approximation of the Hamiltonian on an infinitesimal section $[\alpha, \beta]$ is given by

$$E_{\alpha\beta} = \frac{1}{2}\int_\alpha^\beta K(x_1^{\alpha\beta} w_{x_1}^{\alpha\beta})^2 + \frac{1}{\rho}(x_2^{\alpha\beta} w_{x_2}^{\alpha\beta})^2 + E_e I(x_3^{\alpha\beta} w_{x_3}^{\alpha\beta})^2 + \frac{1}{I_\rho}(x_4^{\alpha\beta} w_{x_4}^{\alpha\beta})^2 d\zeta. \tag{A.0.21}$$

Hence, the equation (A.0.21) yields the following Hamiltonian matrix

$$\mathcal{H}_{\alpha\beta} = \begin{bmatrix} K_{\alpha\beta} & 0 & 0 & 0 \\ 0 & \frac{1}{\rho_{\alpha\beta}} & 0 & 0 \\ 0 & 0 & (E_e I)_{\alpha\beta} & 0 \\ 0 & 0 & 0 & \frac{1}{I_{\rho,\alpha\beta}} \end{bmatrix}, \tag{A.0.22}$$

where

$$K_{\alpha\beta} = \int_\alpha^\beta K(w_{x_1}^{\alpha\beta})^2 d\zeta = \frac{K}{\beta-\alpha}, \qquad (E_e I)_{\alpha\beta} = \int_\alpha^\beta E_e I(w_{x_3}^{\alpha\beta})^2 d\zeta = \frac{E_e I}{\beta-\alpha}$$

$$\rho_{\alpha\beta} = \int_\alpha^\beta \frac{\rho}{(w_{x_2}^{\alpha\beta})^2} d\zeta = \rho(\beta-\alpha) \qquad I_{\rho,\alpha\beta} = \int_\alpha^\beta \frac{I\rho}{(w_{x_4}^{\alpha\beta})^2} d\zeta = I_\rho(\beta-\alpha)$$

Let us define

$$f_{\alpha\beta} = [f_{x_1}^{\alpha\beta}, f_{x_2}^{\alpha\beta}, f_{x_3}^{\alpha\beta}, f_{x_4}^{\alpha\beta}, f_{\partial,x_2}^{\alpha}, f_{\partial,x_4}^{\alpha}, f_{\partial,x_2}^{\beta}, f_{\partial,x_4}^{\beta}], \tag{A.0.23}$$

$$e_{\alpha\beta} = [e_{x_1}^{\alpha\beta}, e_{x_2}^{\alpha\beta}, e_{x_3}^{\alpha\beta}, e_{x_4}^{\alpha\beta}, e_{\partial,x_1}^{\alpha}, e_{\partial,x_3}^{\alpha}, e_{\partial,x_1}^{\beta}, e_{\partial,x_3}^{\beta}]. \tag{A.0.24}$$

Rearranging (A.0.17) and (A.0.18), we deduce the following system

$$\mathcal{D} = \{(f_{\alpha\beta}, e_{\alpha\beta}) \in \mathcal{F}_{\alpha\beta} \times \mathcal{E}_{\alpha\beta} : G_{\alpha\beta} e_{\alpha\beta} + F_{\alpha\beta} f_{\alpha\beta} = 0\}, \tag{A.0.25}$$

which defines a Dirac structure. A state space representation is more convenient for simulation or control design of the finite-dimensional approximated port-Hamiltonian system. Towards this end, the boundary inputs and the outputs given by

$$u(t) = \begin{bmatrix} \frac{1}{\rho}x_2(a,t) \\ \frac{1}{I_\rho}x_4(a,t) \\ Kx_1(b,t) \\ EIx_3(b,t) \end{bmatrix} \quad \text{and} \quad y(t) = \begin{bmatrix} Kx_1(a,t) \\ EIx_3(a,t) \\ -\frac{1}{\rho}x_2(b,t) \\ -\frac{1}{I_\rho}x_4(b,t) \end{bmatrix} \tag{A.0.26}$$

must be assigned properly. We then obtain the following approximated port-Hamiltonian system on the subsection $[\alpha,\beta]$

$$\dot{x}_{\alpha\beta}(t) = (J_{\alpha\beta} - R_{\alpha\beta})\mathcal{H}_{\alpha\beta}x_{\alpha\beta}(t) + B_{\alpha\beta}u_{\alpha\beta}(t), \tag{A.0.27}$$

$$y_{\alpha\beta}(t) = B^T_{\alpha\beta}\mathcal{H}_{\alpha\beta}x_{\alpha\beta}(t) + D_{\alpha\beta}u_{\alpha\beta}(t). \tag{A.0.28}$$

Finally, the N-subsections are interconnected together to obtain the complete approximated port-Hamiltonian model for the Timoshenko beam on the spatial domain $[a,b]$. The efforts and the flows of section i at extremity β are interconnected with the efforts of section $i+1$ at extremity α. The interconnection of each subsection is performed as follows

$$\begin{bmatrix} f^{b,i}_{\partial,x_2} \\ f^{b,i}_{\partial,x_4} \end{bmatrix} = \begin{bmatrix} f^{a,i+1}_{\partial,x_2} \\ f^{a,i+1}_{\partial,x_4} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} e^{b,i}_{\partial,x_1} \\ e^{b,i}_{\partial,x_3} \end{bmatrix} = \begin{bmatrix} e^{b,i+1}_{\partial,x_1} \\ e^{b,i+1}_{\partial,x_3} \end{bmatrix}. \tag{A.0.29}$$

This leads to the complete interconnected system given by

$$\dot{x}_{ab}(t) = (J_{ab} - R_{ab})\mathcal{H}_{ab}x_{ab}(t) + B_{ab}u_{ab}(t), \tag{A.0.30}$$

$$y_{ab}(t) = B^T_{ab}\mathcal{H}_{ab}x_{ab}(t) + D_{ab}u_{ab}(t). \tag{A.0.31}$$

where the matrices J_{ab}, R_{ab}, B_{ab} and D_{ab} are given by (5.5.23), (5.5.27), (5.5.30) and by

$$D_{ab} = \begin{bmatrix} 0 & (-1)^N & 0 & 0 \\ (-1)^{N+1} & 0 & 0 & 0 \\ 0 & 0 & 0 & (-1)^{N+1} \\ 0 & 0 & (-1)^N & 0 \end{bmatrix} \tag{A.0.32}$$

Appendix B

Lyapunov stability theorem

This appendix is mainly devoted to recall the Lyapunov's direct method for stability. For further details, see [Lia47], [Rah13, Chapter 3], [Mov59] and [LGM12].

Theorem B.0.1. *Let us consider a dynamical system described by*

$$\dot{x}(t) = Ax(t), \qquad x(0) = x_0, \tag{B.0.1}$$

where A is an infinitesimal generator of a C_0-semigroup with domain $D(A)$. The state $x(t) = 0$ with $t \geq 0$ is said to be exponentially stable if there exists a functional $V(x(t)) : \mathcal{X} \to \mathbb{R}^+$ such that

$$\lambda_1 \|x\|_{\mathcal{X}}^2 \leq V(x) \leq \lambda_2 \|x\|_{\mathcal{X}}^2 \tag{B.0.2}$$

and that satisfies the condition

$$\frac{d}{dt} V(x(t)) \leq -K \|x(t)\|_{\mathcal{X}}^2 \tag{B.0.3}$$

for all $t \geq 0$ and $x \in D(A)$.

Note that the condition (B.0.2) implies that

$$\frac{d}{dt} V(x(t)) \leq -K \|x\|_{\mathcal{X}}^2 \leq -\frac{K}{\lambda_2} V(x(t)), \tag{B.0.4}$$

which yields

$$V(x(t)) \leq e^{-\frac{K}{\lambda_2} t} V(x(0)). \tag{B.0.5}$$

We have the following variant of Theorem B.0.1.

Theorem B.0.2. *Let us consider a dynamical system described by*

$$\dot{x}(t) = Ax(t), \qquad x(0) = x_0, \tag{B.0.6}$$

where A is an infinitesimal generator of a C_0-semigroup with domain $D(A)$. The state $x(t) = 0$ with $t \geq 0$ is said to be exponentially stable if there exists a functional $V(x(t)) : \mathcal{X} \to \mathbb{R}^+$ such that

$$V(x) \geq \lambda_1 \|x\|^2_{\mathcal{X}} \geq 0 \tag{B.0.7}$$

and that satisfies the condition

$$\frac{d}{dt} V(x(t)) \leq -KV(x(t)) \tag{B.0.8}$$

for all $t \geq 0$ and $x \in D(A)$.

We end this appendix by recalling some inequalities that enable bounding candidate of Lyapunov functionals and their time derivatives.

Lemma B.0.3. *For any $w_1, w_2 \in \mathbb{R}^n$ and $\alpha > 0$, the following inequality*

$$-\alpha^2 \|w_1\|^2 - \frac{1}{\alpha^2} \|w_2\|^2 \leq w_1^T w_2 + w_2^T w_1 \leq \alpha^2 \|w_1\|^2 + \frac{1}{\alpha^2} \|w_2\|^2 \tag{B.0.9}$$

holds.

Lemma B.0.4. *[Rah13, Lemma 10] With $\mathcal{G} = \{r | r \in H^1([a,b]; \mathbb{R}^n), r(a) = 0\}$, if $r(\zeta, t) \in \mathcal{G}$, then*

$$r^T(\zeta,t) r(\zeta,t) \leq (b-a) \int_a^b (\partial_\zeta r)^T(\zeta,t) \, (\partial_\zeta r)(\zeta,t) d\zeta \tag{B.0.10}$$

holds for $\zeta \in [a,b]$.

Bibliography

[App04] D. Applebaum. *Lévy Processes and Stochastic Calculus*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2004.

[Aug16] B. Augner. *Stabilisation of Infinite-dimensional Port-Hamiltonian Systems*. PhD thesis, University of Wuppertal, 2016.

[Bak63] V.V. Baklan. On the existence of solutions of stochastic equations in Hilbert space,. *Depov. Akad. Nauk. Ukr. USR.*, 10:1299–1303, 1963.

[Bal76] A.V. Balakrishnan. *Applied functional analysis*. Applications of Mathematics Series. SPRINGER VERLAG GMBH, 1976.

[Bar95] R. G. Bartle. *The Elements of Integration and Lebesgue Measure*. John Wiley & Sons, New York, 1995.

[BCE+09] A. Baaiu, F. Couenne, D. Eberard, C. Jallut, L. Lefevre, Y. Le Gorrec, and B. Maschke. Port-based modelling of mass transport phenomena. *Mathematical and Computer Modelling of Dynamical Systems*, 15(3):233–254, 2009.

[BCL+09] A. Baaiu, F. Couenne, L. Lefevre, Y. Le Gorrec, and M. Tayakout. Structure-preserving infinite dimensional model reduction: Application to adsorption processes. *Journal of Process Control*, 19(3):394 – 404, 2009.

[BDPDM06] A. Bensoussan, G. Da Prato, M.C. Delfour, and S.K. Mitter. *Representation and Control of Infinite Dimensional Systems*. Systems & Control: Foundations & Applications. Birkhäuser Boston, 2006.

[BHOC+12] E. Butler, R. Hammond-Oakley, S. Chawarski, A. Gosline, P. Codd, T. Anor, J. Madsen, P. Dupont, and J. Lock. Robotic neuro-endoscope with concentric tube augmentation. *Proceedings of the IEEE/RSJ International Conference on Intelligent Robots and Systems*, 2012.

[Bre68] L. Breiman. *Probability*. Classics in Applied Mathematics. Society for Industrial and Applied Mathematics (SIAM, 3600 Market Street, Floor 6, Philadelphia, PA 19104), 1968.

[BSR+11] J. Burgner, P. J. Swaney, D. C. Rucker, H. B. Gilbert, S. T. Nill, P. T. Russell, K. D. Weaver, and R. J. Webster. A bimanual teleoperated system for endonasal skull base surgery. In *2011 IEEE/RSJ International Conference on Intelligent Robots and Systems*, pages 2517–2523, Sep. 2011.

[CF70] R.F. Curtain and P.L. Falb. Ito's lemma in infinite dimensions. *Journal of Mathematical Analysis and Applications*, 31(2):434 – 448, 1970.

[Cho14] P.L. Chow. *Stochastic Partial Differential Equations, Second Edition*. Advances in Applied Mathematics. Taylor & Francis, 2014.

[CI77a] R.F. Curtain and A. Ichikawa. The separation principle for stochastic evolution equations. *SIAM Journal on Control and Optimization*, 15(3):367–383, may 1977.

[CI77b] R.F. Curtain and A. Ichikawa. The separation principle for stochastic evolution equations. 15(3):367–383, May 1977.

[CP78] R.F. Curtain and A.J. Pritchard. *Infinite Dimensional Linear Systems Theory*. Lecture notes in control and information sciences. Springer-Verlag, 1978.

[CRA14] M.T. Chikhaoui, K. Rabenorosoa, and N. Andreff. Kinematic modeling of an EAP actuated continuum robot for active micro-endoscopy. In *Advances in Robot Kinematics*, pages 457 – 465. Springer, 1st edition, may 2014.

[Cur78] R.F. Curtain. A semigroup approach to the LQG problem for infinite-dimensional systems. *IEEE Transactions on Circuits and Systems*, 25:713–721, 1978.

[CW92] F.M. Callier and J.J. Winkin. LQ-optimal control of infinite-dimensional systems by spectral factorization. *Automatica*, 28(4):757 – 770, 1992.

[CZ95] R.F. Curtain and H.J. Zwart. *An Introduction to Infinite-Dimensional Linear Systems Theory*. Springer edition, 1995.

[DDW03] C. Delattre, D. Dochain, and J. Winkin. Sturm-Liouville systems are Riesz-spectral systems. *International Journal of Applied Mathematics and Computer Science*, 13(4):481–484, 2003.

[DGV+12] P. Dupont, A. Gosline, N. Vasilyev, J. Lock, E. Butler, C. Folk, A. Cohen, R. Chen, G. Schmitz, and H. Ren. Concentric tube robots for minimally invasive surgery. *Neurosurgery*, 7, 01 2012.

[DM11] C. Dellacherie and P.A. Meyer. *Probabilities and Potential, B: Theory of Martingales*. North-Holland Mathematics Studies. Elsevier Science, 2011.

[DMSB09] V. Duindam, A. Macchelli, S. Stramigioli, and H. Bruyninckx. *Modeling and Control of Complex Physical Systems: The Port-Hamiltonian Approach*. Springer Berlin Heidelberg, 2009.

[Doo90] J.L. Doob. *Stochastic processes*. Wiley publications in statistics. Wiley, 1990.

[DPI85] G. Da Prato and A. Ichikawa. Riccati equations with unbounded coefficients. *Annali di Matematica Pura ed Applicata*, 140(1):209–221, Dec 1985.

[DPZ14] G. Da Prato and J. Zabczyk. *Stochastic Equations in Infinite Dimensions*. Encyclopedia of Mathematics and its Applications. Cambridge University Press, 2014.

[DS88] N. Dunford and J.T. Schwartz. *Linear Operators, Part 2: Spectral Theory, Self Adjoint Operators in Hilbert Space*. Wiley Classics Library. Wiley, 1988.

[DW14] J. Duan and W. Wang. *Effective Dynamics of Stochastic Partial Differential Equations*. Boston, 2014.

[Fat68] H. Fattorini. Boundary control systems. *SIAM J. Control and Optimization*, 6(3):349–385, 1968.

[FG17] Z. Fang and C. Gao. Stabilization of input-disturbed stochastic port-Hamiltonian systems via passivity. *IEEE Transactions on Automatic Control*, 62(8):4159–4166, Aug 2017.

[Flo99] P. Florchinger. A passive system approach to feedback stabilization of nonlinear control stochastic systems. *SIAM J. Control Optim.*, 37(6):1848–1864, sep 1999.

[Fra86] F. Franco. Riccati equation arising in the boundary control of stochastic hyperbolic systems. *Stochastic Analysis and Applications*, 4:131–150, 01 1986.

[FS01] K. Fujimoto and T. Sugie. Canonical transformation and stabilization of generalized Hamiltonian systems. *Systems and Control Letters*, 42(3):217 – 227, 2001.

[GLTY09] S. Gutta, J.S. Lee, M.B. Trabia, and W. Yim. Modeling of ionic polymer metal composite actuator dynamics using a large deflection beam model. *Smart Materials and Structures*, 18(11):115023, 2009.

[GTvdSM04] G. Golo, V. Talasila, A.J. van der Schaft, and B. Maschke. Hamiltonian discretization of boundary control systems. *Automatica*, 40(5):757 – 771, 2004.

[HLDWon] A. Hastir, F. Lamoline, D. Dochain, and J.J. Winkin. Analysis of the existence of equilibrium profiles in nonisothermal axial dispersion tubular reactors. *IEEE Transaction of Automatic Control*, under revision.

[HLM06] H. Hamroun, L. Lefevre, and E. Mendes. Port-based modelling for open channel irrigation systems. *Transaction on Fluid Mechanics*, 1(12):999–1009, 2006.

[HM76] D. Hill and P. Moylan. The stability of nonlinear dissipative systems. *IEEE Transactions on Automatic Control*, 21(5):708–711, October 1976.

[Hol08] B. J Hollingsworth. *Stochastic Differential Equations: A Dynamical Systems Approach.* PhD thesis, University of Auburn, 2008.

[HRJ18] W.M. Haddad, T. Rajpurohit, and X. Jin. Energy-based feedback control for stochastic port-controlled Hamiltonian systems. *Automatica*, 97:134 – 142, 2018.

[HS01] E. Hausenblas and J. Seidler. A note on maximal inequality for stochastic convolutions. *Czechoslovak Mathematical Journal*, 51(4):785–790, 2001.

[HW97] S. Hansen and G. Weiss. New results on the operator carleson measure criterion. *IMA Journal of Mathematical Control and Information*, 14(1):3–32, 1997.

[Ich79] A. Ichikawa. Dynamic programming approach to stochastic evolution equations. *SIAM Journal on Control and Optimization*, 17(1):152–174, 1979.

[Ito44] K. Ito. Stochastic integral. *Proc. Imp. Acad.*, 20(8):519–524, 1944.

[JK18] B. Jacob and J. Kaiser. Riesz Bases and Controllability for Port-Hamiltonian Systems. In *Poster presented at 2nd Workshop on Stability and Control of Infinite-Dimensional Systems*, 2018.

[JS83] E. Jonckheere and L. Silverman. A new set of invariants for linear systems–application to reduced order compensator design. *IEEE Transactions on Automatic Control*, 28(10):953–964, 1983.

[JZ12] B. Jacob and H.J. Zwart. *Linear Port-Hamiltonian Systems on Infinite-dimensional Spaces*. Springer edition, 2012.

[KMX$^+$14] N. Kottenstette, M.J. McCourt, M. Xia, V. Gupta, and P.J. Antsaklis. On relationships among passivity, positive realness, and dissipativity in linear systems. *Automatica*, 50(4):1003 – 1016, 2014.

[Kuo75] H.H. Kuo. *Gaussian Measures in Banach Spaces*. Lecture Notes in Mathematics. Springer-Verlag, 1975.

[Kus62] H.J. Kushner. Optimal stochastic control. *IRE Trans. Auto. Control*, 7:120–122, 1962.

[LCO08] J. Lázaro-Camí and J. Ortega. Stochastic Hamiltonian dynamical systems. *Reports on Mathematical Physics*, 61(1):65 – 122, 2008.

[LGM12] Z.H. Luo, B.Z. Guo, and Ö. Morgül. *Stability and Stabilization of Infinite Dimensional Systems with Applications*. Communications and Control Engineering. Springer London, 2012.

[Lia47] A.M. Liapounoff. *Probleme General de La Stabilite Du Mouvement. (Am-17)*. Annals of mathematics studies. Princeton University Press, 1947.

[Lio71] J.L. Lions. *Optimal control of systems governed by partial differential equations*. Grundlehren der mathematischen Wissenschaften. Springer-Verlag, 1971.

[Liu05] K. Liu. *Stability of Infinite Dimensional Stochastic Differential Equations with Applications*. Monographs and Surveys in Pure and Applied Mathematics. CRC Press, 2005.

[LPS14] G.J. Lord, C.E. Powell, and T. Shardlow. *An Introduction to Computational Stochastic PDEs*. Cambridge Texts in Applied Mathematics. Cambridge University Press, 2014.

[LT14] I. Lasiecka and R. Triggiani. *Differential and Algebraic Riccati Equations with Application to Boundary/Point Control Problems: Continuous Theory and Approximation Theory*. Lecture Notes in Control and Information Sciences. Springer Berlin Heidelberg, 2014.

[Lü15] Q. Lü. Stochastic Well-Posed Systems and Well-Posedness of Some Stochastic Partial Differential Equations with Boundary Control and Observation. *SIAM J. Control and Optimization*, 53(6):3457–3482, 2015.

[LW17a] F. Lamoline and J.J. Winkin. Nice port-Hamiltonian systems are Riesz-spectral systems. *Preprints of the 20th World Congress, The International Federation of Automatic Control*, pages 695–699, 2017.

[LW17b] F. Lamoline and J.J. Winkin. On stochastic port-Hamiltonian systems with boundary control and observation. In *Proceedings of the 56th IEEE Conference on Decision and Control*, pages 2492–2497, 2017.

[LZM05] Y. Le Gorrec, H.J. Zwart, and B. Maschke. Dirac structures and boundary control systems associated with skew-symmetric differential operators. *SIAM journal on control and optimization*, 44(5):1864–1892, December 2005.

[Mac12] A. Macchelli. Boundary energy shaping of linear distributed port-Hamiltonian systems. *IFAC Proceedings Volumes*, 45(19):120 – 125, 2012. 4th IFAC Workshop on Lagrangian and Hamiltonian Methods for Non Linear Control.

[MGR17] A. Macchelli, Y. Le Gorrec, and H. Ramírez. Boundary control of distributed port-Hamiltonian systems via generalised canonical transformations. In *2017 IEEE 56th Annual Conference on Decision and Control (CDC)*, pages 70–75, Dec 2017.

[MH13] D. Matignon and T. Hélie. A class of damping models preserving eigenspaces for linear conservative port-Hamiltonian systems. *European Journal of Control*, 19(6):486 – 494, 2013.

[MM03] R. Mennicken and M. Möller. *Non-Self-Adjoint Boundary Eigenvalue Problems*. North-Holland Mathematics Studies. Elsevier Science, 2003.

[MM04] A. Macchelli and C. Melchiorri. Modeling and control of the timoshenko beam. The distributed port Hamiltonian approach. *SIAM Journal on Control and Optimization*, 43(2):743–767, 2004.

[Mov59] A.A. Movchan. The direct method of Liapunov in stability problems of elastic systems. *Journal of Applied Mathematics and Mechanics*, 23(3):686 – 700, 1959.

[MPBL14] M. Metivier, J. Pellaumail, Z.W. Birnbaum, and E. Lukacs. *Stochastic Integration*. Probability and Mathematical Statistics. Elsevier Science, 2014.

[MvdS92] B. Maschke and A.J. van der Schaft. Port-controlled Hamiltonian systems: modelling origins and system theoretic properties. In *2nd IFAC Symp. Nonlinear Control Systems*, pages 282–288, 1992.

[MWR$^+$] A. Mattioni, Y. Wu, H. Ramirez, Y. Le Gorrec, and A. Macchelli. Modelling and control of an IPMC actuated flexible structure: A port Hamiltonian approach. *private communication*.

[NTMO11] G. Nishida, K. Takagi, B. Maschke, and T. Osada. Multi-scale distributed parameter modeling of ionic polymer-metal composite soft actuator. *Control Engineering Practice*, 19(4):321 – 334, 2011.

[OGC04] R. Ortega and E. García-Canseco. Interconnection and damping assignment passivity-based control: A survey. *European Journal of Control*, 10(5):432 – 450, 2004.

[Oos00] J. Oostveen. *Strongly Stabilizable Distributed Parameter Systems*. Frontiers in Applied Mathematics. Society for Industrial and Applied Mathematics, 2000.

[OvdSME99] R. Ortega, A.J. van der Schaft, B. Maschke, and G. Escobar. Energy-shaping of port-controlled Hamiltonian systems by interconnection. In *Proceedings of the 38th IEEE Conference on Decision and Control (Cat. No.99CH36304)*, volume 2, pages 1646–1651 vol.2, Dec 1999.

[OvdSME02] R. Ortega, A.J. van der Schaft, B. Maschke, and G. Escobar. Interconnection and damping assignment passivity-based control of port-controlled Hamiltonian systems. *Automatica*, 38(4):585 – 596, 2002.

[PS87] A. J. Pritchard and D. Salamon. The linear quadratic control problem for infinite dimensional systems with unbounded input and output operators. *SIAM J. Control Optim.*, 25(1):121–144, January 1987.

[Rah13] C.D. Rahn. *Mechatronic Control of Distributed Noise and Vibration: A Lyapunov Approach*. Springer Berlin Heidelberg, 2013.

[RCJW09] D Rucker, Jordan Croom, and R J. Webster. Aiming a surgical laser with an active cannula. *Journal of Medical Devices-transactions of The Asme*, 3, 06 2009.

[RDW16] J. R. Dehaye and J. Winkin. Lq-optimal boundary control of infinite-dimensional systems with yosida-type approximate boundary observation. *Automatica*, 67:94–106, 05 2016.

[RL13] H. Ramirez and Y. Le Gorrec. Exponential stability of a class of PDE's with dynamic boundary control. In *2013 American Control Conference*, pages 3290–3295, June 2013.

[RLGMZ14] H. Ramirez, Y. Le Gorrec, A. Macchelli, and H. Zwart. Exponential stabilization of boundary controlled port-Hamiltonian systems with dynamic feedback. *IEEE Transactions on Automatic Control*, 59(10):2849–2855, 2014.

[RZG17] H. Ramirez, H. Zwart, and Y. Le Gorrec. Stabilization of infinite dimensional port-Hamiltonian systems by nonlinear dynamic boundary control. *Automatica*, 85:61 – 69, 2017.

[Sal89] D. Salamon. Realization theory in Hilbert space. *Math. Systems Theory*, 1989.

[Sat17] S. Satoh. Input-to-state stability of stochastic port-Hamiltonian systems using stochastic generalized canonical transformations. *International Journal of Robust and Nonlinear Control*, 27(17):3862–3885, 2017.

[Sch57] L. Schwartz. *Théorie des distributions*. Number vol. 1 à 2 in Actualités scientifiques et industrielles. Hermann et Cie, 1957.

[SF13] S. Satoh and K. Fujimoto. Passivity Based Control of Stochastic Port-Hamiltonian Systems. *IEEE Transactions on Automatic Control*, 58(5):1139–1153, May 2013.

[Shk86] A.A. Shkalikov. Boundary problems for ordinary differential equations with parameter in the boundary conditions. *Journal of Soviet Mathematics*, 33(6):1311–1342, Jun 1986.

[SK01] M. Shahinpoor and K.J. Kim. Ionic polymer-metal composites: I. fundamentals. *Smart Materials and Structures*, 10(4):819–833, aug 2001.

[SN53] B. Sz.-Nagy. Sur les contractions de l'espace de Hilbert. *Acta Sci. Math. Szeged*, 15:87–92, 1953.

[SS14] S. Satoh and M. Saeki. Bounded stabilisation of stochastic port-Hamiltonian systems. *International Journal of Control*, 87(8):1573–1582, 2014.

[Sta95] O. Staffans. Quadratic optimal control of stable systems through spectral factorization. *Math. Control, Signals, Systems*, 8:167–197, 01 1995.

[Sta05] O. Staffans. *Well-Posed Linear Systems*. Cambridge University Press, 2005.

[Tre00] C. Tretter. Spectral problems for systems of differential equations $y' + a_0 y = \lambda a_1 y$ with polynomial boundary conditions. *Mathematische Nachrichten*, 214(1):129–172, 2000.

[TW09] M. Tucsnak and G. Weiss. *Observation and Control for Operator Semigroups*. Birkhäuser Advanced Texts Basler Lehrbücher. Birkhäuser Basel, 2009.

[TW14] M. Tucsnak and G. Weiss. Well-posed systems–the LTI case and beyond. *Automatica*, 50(7):1757–1779, 2014.

[vdS16] A.J. van der Schaft. *L2-Gain and Passivity Techniques in Nonlinear Control*. Communications and Control Engineering. Springer International Publishing, 2016.

[vdSJ14] A.J. van der Schaft and D. Jeltsema. *Port-Hamiltonian Systems Theory: An Introductory Overview*. Foundations and Trends(r) in Systems and Control Series. Now Publishers, 2014.

[vdSM02] A.J. van der Schaft and B.M. Maschke. Hamiltonian formulation of distributed-parameter systems with boundary energy flow. *Journal of Geometry and Physics*, 42:166 – 194, 2002.

[Vil07] J.A. Villegas. *A Port-Hamiltonian Approach to distributed parameter systems*. PhD thesis, University of Twente, 2007.

[vK12] B. van Keulen. *H-Control for Distributed Parameter Systems: A State-Space Approach*. Systems & Control: Foundations & Applications. Birkhäuser Boston, 2012.

[WHGM18] Y. Wu, B. Hamroun, Y. Le Gorrec, and B. Maschke. Reduced order lqg control design for port Hamiltonian systems. *Automatica*, 95:86 – 92, 2018.

[Wil72] J.C. Willems. Dissipative dynamical systems, parts i and ii. *Archive for Rational Mechanics and Analysis*, 45(5):321–351, January 1972.

[WLGW19] Y. Wu, F. Lamoline, Y. Le Gorrec, and J.J. Winkin. Modelling and control of an IPMC actuated flexible beam under the port-Hamiltonian framework. In *Proceedings of the 3rd Control of Systems Governed by Partial Differential Equations*, 2019.

[Won68a] W. M. Wonham. On a matrix Riccati equation of stochastic control. *SIAM Journal on Control*, 6(4):681–697, 1968.

[Won68b] W. M. Wonham. On the separation theorem of stochastic control. *SIAM Journal on Control*, 6(2):312–326, 1968.

[WST01] G. Weiss, O.J. Staffans, and M. Tucsnak. Well-posed linear systems a survey with emphasis on conservative systems. *Int. J. Appl. Math. Comput. Sci.*, 11(1):7–33, 2001.

[WW97a] M. Weiss and G. Weiss. Optimal control of stable weakly regular linear systems. *Mathematics of Control, Signals and Systems*, 10(4):287–330, 1997.

[WW97b] Martin Weiss and George Weiss. Optimal control of stable weakly regular linear systems. *Mathematics of Control Signals and Systems*, 10:287–330, 11 1997.

[XF02] gen qi Xu and De-Xing Feng. The Riesz basis property of a timoshenko beam with boundary feedback and application. *Ima Journal of Applied Mathematics - IMA J APPL MATH*, 67:357–370, 08 2002.

[Xu05] G. Q. Xu. Boundary feedback exponential stabilization of a Timoshenko beam with both ends free. *International Journal of Control*, 2005.

[Yos95] K. Yosida. *Functional Analysis*. Classics in Mathematics. Springer Berlin Heidelberg, 1995.

[You01] R.M. Young. *An Introduction to Nonharmonic Fourier Series*. Academic Press, 2001.

[ZGMV10] H.J. Zwart, Y. Le Gorrec, B. Maschke, and J.A. Villegas. Well-posedness and regularity of hyperbolic boundary control systems on a one-dimensional spatial domain. *ESAIM: Control, Optimisation and Calculus of Variations*, 16:1077–1093, 2010.