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# Adaptive Regularization for Nonconvex Optimization Using Inexact Function Values and Randomly Perturbed Derivatives

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## Abstract

A regularization algorithm allowing random noise in derivatives and inexact function values is proposed for computing approximate local critical points of any order for smooth unconstrained optimization problems. For an objective function with Lipschitz continuous  $p$ -th derivative and given an arbitrary optimality order  $q \leq p$ , it is shown that this algorithm will, in expectation, compute such a point in at most  $O\left(\left(\min_{j \in \{1, \dots, q\}} \epsilon_j\right)^{-\frac{p+1}{p-q+1}}\right)$  inexact evaluations of  $f$  and its derivatives whenever  $q \in \{1, 2\}$ , where  $\epsilon_j$  is the tolerance for  $j$ th order accuracy. This bound becomes at most  $O\left(\left(\min_{j \in \{1, \dots, q\}} \epsilon_j\right)^{-\frac{q(p+1)}{p}}\right)$  inexact evaluations if  $q > 2$  and all derivatives are Lipschitz continuous. Moreover these bounds are sharp in the order of the accuracy tolerances. An extension to “inexpensively constrained” problems is also outlined.

*Keywords:* evaluation complexity, regularization methods, inexact functions and derivatives, stochastic analysis.

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## 1. Introduction

We consider the evaluation complexity of an adaptive regularization algorithm for computing approximate *local* minimizers of arbitrary order for the unconstrained minimization problem of the form

$$\min_{x \in \mathbb{R}^n} f(x), \quad (1.1)$$

where the objective function  $f$  is sufficiently smooth and the values of its  $j$ -th derivatives  $\nabla_x^j f$  are subject to random noise and can only be computed inexactly. Inexact values of the objective function are also allowed, but their inaccuracy is assumed to be deterministically controllable.

**Motivation and context.** Without further comments, this statement of the paper’s purpose may be difficult to interpret, and we start by clarifying the notion of “evaluation complexity”. Since the computational cost of nonlinear optimization algorithms for the local solution of (1.1) is typically dominated by that of evaluating  $f(x)$  and its derivatives, and since these evaluations are fully independent of the algorithms (and unknown to the algorithm designers), classical concepts of total computational complexity are difficult to apply to such algorithms, except for the simplest cases<sup>1</sup>. This difficulty is at the origin of the now standard concept of “evaluation complexity” (sometimes called “oracle complexity”) where the cost of running an algorithm is approximated by the total cost of evaluating the objective function and its relevant derivatives (the oracle), which is then measured by counting the total number of such evaluations. This concept has a long history in the optimization research community, and has generated a vast literature covering many kinds of algorithms and problems types (see , , , , , , , and the reference therein for a limited sample of this vibrant research area). Of course, like the classical notion of complexity, this approach only applies when a class of problems is well identified and when the algorithm’s termination rules are clear. In our case, these rules need to reflect the notion of approximate local solution of problem (1.1), and will be detailed in due course (in Section 2).

The class of problems of interest here (as summarized above) is the approximate local minimization of smooth functions whose values (and that of its derivatives) can only be computed inexactly. Before providing more technical detail on exactly what this means, it is useful to briefly review the existing contributions in this domain<sup>2</sup>. Indeed, solving optimization problems involving inexact evaluations is not a new topic and has already been investigated in two different frameworks. The first is that of (deterministic) explicit dynamic accuracy, where it is assumed that the accuracy of  $f$  and its derivatives can be controlled by the algorithm (see [, Section 10.6], , or for example). In this context, accuracy requirements are proposed that guarantee convergence to approximate local solutions, and evaluation complexity of the resulting algorithms can be analyzed , indicating a very modest degradation of the worst-case performance compared with the case where evaluations are exact , . A drawback of this approach is that nothing is said for the case where the requested accuracy requirement cannot be met or, as is often the case, cannot even be measured. This problem does not occur in the second framework, in which the inexactness in the function (and possible derivatives) values can be seen as caused by some random noise, in which case the algorithm/user is not able to specify an accuracy level

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<sup>1</sup>For example, consider the typical optimization problem in deep learning applications, where the cost of the objective function depends on the size, depth and structure of the neural network used, all things that are not known by the algorithm. Or optimization of large inverse problems like weather forecasting, where evaluating the objective function depends on solving a complicated multidimensional time-dependent partial-differential equation, whose dimension, domain shape, horizon, level of discretization and nonlinearity are also unknown and may vary from application of the algorithm to the next.

<sup>2</sup>We focus here on algorithms whose definition only involves quantities that are known to the user. In particular, we explicitly avoid requiring the knowledge of the problem Lipschitz constants for the definition of the algorithm.

and poor derivative approximations might result. The available analysis for this case differ by the assumptions made on the distribution of this noise. In , the authors consider the unbiased case and estimate the evaluation complexity a linesearch method for finding approximate first-order critical points. Here, we assume that derivatives values can be approximated within a prescribed accuracy with a fixed, sufficiently high probability, conditioned to the past. A similar context is considered in , where the objective function values are inexact but computed with accuracy guaranteed with probability one, and in , where the authors consider the unbiased case. A trust-region method (see for a full coverage of such methods) is also proposed in , where it is proved to converge almost-surely to first-order critical points. Using similar assumptions, the approach of includes the use of random first-order models and directions within line search method as well as probabilistic second-order models in Adaptive Cubic Regularization (ARC) algorithms. In both cases, the authors employ exact function evaluations. A general theory for global convergence rate analysis is also provided. More recently, proposed an evaluation complexity analysis for a trust-region method (covering convergence to second-order points) using elegant properties of sub-martingales, and making no assumption on bias. A recent overview of this active research area is proposed in .

**Contributions.** As suggested above, this paper deliberately considers the well-established concept of evaluation complexity, measuring the number of calls to user-supplied procedures for computing approximate function and derivatives values, irrespective of internal computations within the algorithm itself. Of course, the authors are fully aware that total computational complexity (as opposed to evaluation complexity) is a different question. Fortunately, the difference is well-understood when searching for first- or second-order approximate local minimizers, in that moderately costly methods are available for handling the algorithm’s internal calculations (see , , ). In other situations, the results presented here give an admittedly idealized but hopefully interesting estimation.

Having set the scene and clarified our objective, we now make the contributions of this paper more precise.

- We consider finding approximate local solution of problem (1.1) assuming that the objective function values can be computed within a prescribed accuracy, while at the same time allowing randomly inexact evaluations of its derivatives, thereby using a mix of the two frameworks described above. This work thus extends the analysis provided in , , for adaptive regularization algorithms.

Our assumptions on the type of inaccuracy allowed for the objective function complements that of , , allowing for more inexactness, but in a deterministic context. This is a realistic request in applications such as those where the objective function value is approximated by using smoothing operators and the derivatives are approximated by randomized finite differences , , .

- As in , , , we propose a regularization algorithm which is based on polynomial models of arbitrary degree. This obviously allows us to seek for first- and second-order critical points, as is standard, but we may also seek critical points of arbitrary order

(we define what we mean by that in Section 2). In this respect we improve upon the algorithms with stochastic models such as , , , , .

- We establish sharp worst-case bounds (in expectation) on the evaluation complexity of computing these (possibly high-order) approximate critical points, depending on the order and on the degree of the polynomial model used. Remarkably, these bounds correspond in order to the best known bounds for regularization algorithms using exact evaluations.

These results are obtained by a novel combination of the probabilistic framework of , the approximation results of and the proof techniques of .

**Outline.** The paper is organized as follows. Section 2 discusses optimality measures for arbitrary order and introduces the regularization algorithm and the associated probabilistic assumptions. Its evaluation complexity is then studied in Section 3. We finally present some conclusions and perspectives in Section 5.

**Notations.** Unless otherwise specified,  $\|\cdot\|$  denotes the standard Euclidean norm for both vectors and matrices. For a general symmetric tensor  $S$  of order  $p$ , we define

$$\|S\| \stackrel{\text{def}}{=} \max_{\|v\|=1} |S[v]^p| = \max_{\|v_1\|=\dots=\|v_p\|=1} |S[v_1, \dots, v_p]| \quad (1.2)$$

the induced Euclidean norm (see [ , Theorem 2.1] for a proof of the second equality). We denote by  $\nabla_x^\ell f(x)$  the  $\ell$ -th order derivative of  $f$  evaluated at  $x$ , noting that such a tensor is always symmetric for any  $\ell \geq 2$ . The notation  $\nabla_x^\ell f(x)[s]^\ell$  denotes this  $\ell$ -th derivative tensor applied to  $\ell$  copies of the vector  $s$ . All inexact quantities are indicated by an overbar. For a symmetric matrices  $M$ ,  $\lambda_{\min}(M)$  is the smallest eigenvalue of  $M$ . We will also use the function

$$\chi_j(t) \stackrel{\text{def}}{=} \sum_{\ell=1}^j \frac{t^\ell}{\ell!} \quad (t \geq 0), \quad (1.3)$$

where  $j \geq 1$ . We use the notation  $\mathbb{E}[X]$  to indicate the expected value of a random variable  $X$ . In addition, given a random event  $A$ ,  $\Pr(A)$  denotes the probability of  $A$ , while  $\mathbb{1}_A$  refers to the indicator of the random event  $A$  occurring. The notation  $A^c$  indicates that event  $A$  does not occur.

## 2. A regularization algorithm with inexact evaluations

### 2.1. The problem class

We first make our framework more formal by detailing our assumptions on problem (1.1).

**AS.1** The function  $f$  is  $p$  times continuously differentiable in  $\mathbb{R}^n$ . Moreover, its  $j$ -th order derivative tensor is Lipschitz continuous for  $j \in \{1, \dots, p\}$  in the sense that there exist constants  $L_{f,\ell} \geq 0$  such that, for all  $\ell \in \{1, \dots, p\}$  and all  $x, y \in \mathbb{R}^n$ ,

$$\|\nabla_x^\ell f(x) - \nabla_x^\ell f(y)\| \leq L_{f,j} \|x - y\|. \quad (2.1)$$

**AS.2**  $f$  is bounded below in  $\mathbb{R}^n$ , that is there exists a constant  $f_{\text{low}}$  such that  $f(x) \geq f_{\text{low}}$  for all  $x \in \mathbb{R}^n$ .

Because of AS.1, the  $\ell$ -th derivative of  $f$  exists for  $\ell \in \{1, \dots, p\}$  and is a symmetric tensor of dimension  $\ell$ , which we denote by

$$\nabla_x^\ell f(x) \stackrel{\text{def}}{=} \left( \frac{\partial^\ell f}{\partial x_{i_1} \dots \partial x_{i_\ell}} \right)_{i_j \in \{1, \dots, n\}, j \in \{1, \dots, \ell\}} (x).$$

Moreover, the  $p$ -th degree Taylor series of  $f$  at a point  $x$  and evaluated for a step  $s$  is well-defined and can be written as

$$T_{f,p}(x, s) \stackrel{\text{def}}{=} f(x) + \sum_{\ell=1}^p \frac{1}{\ell!} \nabla_x^\ell f(x) [s]^\ell, \quad (2.2)$$

where  $\nabla_x^\ell f(x) [s]^\ell$  denotes the scalar obtained by applying the  $\ell$ -dimensional tensor  $\nabla_x^\ell f(x)$  to  $\ell$  copies of the vector  $s$ . Because we will reuse this notation later, note that the first subscript in  $T_{f,p}(x, s)$  is the function whose Taylor expansion is being considered, while the second is the degree of the expansion. The argument  $x$  is the point at which derivatives of  $f$  are computed and  $s$  is a step from  $x$  so that  $T_{f,p}(x, s)$  approximates the value of  $f$  at the point  $x + s$ . We will also make frequent use of the *Taylor decrement* defined as

$$\Delta T_{f,p}(x, s) = T_{f,p}(x, 0) - T_{f,p}(x, s) = - \sum_{\ell=1}^p \frac{1}{\ell!} \nabla_x^\ell f(x) [s]^\ell \quad (2.3)$$

We will also rely on the following well-known but important consequence of AS.1.

**Lemma 2.1.** Suppose that AS.1 holds. Then, for all  $x, s \in \mathbb{R}^n$ ,

$$f(x + s) \leq T_{f,p}(x, s) + \frac{L_{f,p}}{(p+1)!} \|s\|^{p+1} \quad (2.4)$$

and

$$\|\nabla_x^\ell f(x + s) - \nabla_s^\ell T_{f,p}(x, s)\| \leq \frac{L_{f,p}}{(p-\ell+1)!} \|s\|^{p-\ell+1} \text{ for all } \ell \in \{1, \dots, p\}. \quad (2.5)$$

**Proof.** See [ , Lemma 2.1]. □

## 2.2. Optimality measures

We now turn to the important question of defining what we mean by (approximate) critical points of arbitrary order but first address the motivation for considering this issue.

In the standard exact case, it has long been known that using Newton’s method (i.e. a model of degree two) practically outperforms the steepest descent method (which only uses a model of degree one), even for computing first-order approximate critical points. More recently, it was shown in [1] that using a model of degree  $p > 2$  (if possible) results in further improvements in evaluation complexity. However, if an algorithm uses a model of degree  $p > 2$ , why should it be constrained to seek only for first- or second-order approximate critical points? As it turns out, this question raises a number of issues, the first being to define what is meant by an approximate critical point of general order  $q \leq p$ . In the rest of this paper, we use the concept of approximate minimizers discussed in [2]. Specifically, given “accuracy requests”  $\epsilon = (\epsilon_1, \dots, \epsilon_q)$  and “optimality radii”  $\delta = (\delta_1, \dots, \delta_q)$  with

$$\epsilon_j \in (0, 1] \text{ and } \delta_j \in (0, 1] \text{ for } j \in \{1, \dots, q\},$$

we say that  $x$  is a  $q$ -th order  $(\epsilon, \delta)$ -approximate minimizer (or  $(\epsilon, \delta)$ -approximate critical point) for problem (1.1) if

$$\phi_{f,j}^{\delta_j}(x) \leq \epsilon_j \frac{\delta_j^j}{j!} \text{ for } j \in \{1, \dots, q\}, \quad (2.6)$$

where

$$\phi_{f,j}^{\delta_j}(x) \stackrel{\text{def}}{=} f(x) - \min_{\|d\| \leq \delta_j} T_{f,j}(x, d) = \max_{\|d\| \leq \delta_j} \Delta T_{f,j}(x, d), \quad (2.7)$$

where, as is standard, the min and max are considered global. Note that  $\phi_{f,j}^{\delta_j}(x)$  is nothing but the largest decrease obtainable on the  $j$ -th degree Taylor expansion of  $f$  in a neighbourhood of size  $\delta_j$ . As such, it is always well-defined for functions satisfying AS.1 and is always non-negative. Also note that, because of the Cauchy-Schwarz inequality,

$$\phi_{f,1}^{\delta_1}(x) = \max_{\|d\| \leq \delta_1} (-\nabla_x^1 f(x)[d]) = \|\nabla_x^1 f(x)\| \delta_1, \quad (2.8)$$

and we immediately see that (2.6), when specialized to first-order, is identical to the classical condition asking that  $\|\nabla_x^1 f(x)\| \leq \epsilon_1$ . Similarly, it is easy to verify that, when  $\nabla_x^1 f(x) = 0$ , the second-order version of (2.6) gives that

$$\phi_{f,2}^{\delta_2}(x) = \max_{\|d\| \leq \delta_2} (-\frac{1}{2} \nabla_x^2 f(x)[d]^2) = \frac{1}{2} \max [0, -\lambda_1[\nabla_x^2 f(x)]] \delta_2^2$$

where  $\lambda_1[\nabla_x^2 f(x)]$  is the leftmost eigenvalue of the Hessian  $\nabla_x^2 f(x)$ , so we obtain that (2.6) is the same as the classical condition that the absolute value of this eigenvalue is less than  $\epsilon_2$  in this case. For example, the origin is both an  $(\epsilon_1, 1)$ -approximate first-order and an  $(\epsilon_2, 1)$ -approximate second-order minimizer of function  $\frac{1}{6}x^3$  for any  $\epsilon_1, \epsilon_2 \in (0, 1]$ , but is not an  $(\epsilon_3, \delta_3)$ -approximate first-order one for any  $\epsilon_3, \delta_3 \in (0, 1]$ . We refer the reader to [3] for a more extensive discussion.

The condition (2.6) has clear advantages over the more usual definitions for first- and second-order approximate critical points: it is well defined for all orders and it is a con-

tinuous<sup>3</sup> function of  $x$ . Moreover, its evaluation is straightforward for  $j = 1$  (see (2.8)) and easy for  $q = 2$  (it then reduces to the standard trust-region subproblem whose cost is comparable to that of computing the Hessian's leftmost eigenvalue, see [ , Chapter 7]). However, its evaluation may actually be extremely costly for  $j > 2$ . From a formal point of view, this does not affect the evaluation complexity of an algorithm using it because it does not involve any new evaluation of  $f$  and its derivatives. We also note that we could consider an approximate version of (2.6), where we would require that, for each  $j \in \{1, \dots, q\}$ , there exists a  $d_j$  such that  $\|d_j\| \leq \delta_j$  and

$$\nu \phi_{f,j}^{\delta_j}(x) \leq \Delta T_{f,j}(x, d_k) \leq \epsilon_j \frac{\delta_j^j}{j!}, \quad (2.9)$$

where  $\nu$  is a constant in  $(0, 1]$ . Note that (2.9) does not assume the knowledge of the global minimizer or  $\phi_{f,j}^{\delta_j}(x)$ , but merely that we can ensure the second part of (2.9) (see , , for research in this direction). Note also that, by definition,

$$\Delta T_{f,j}(x, d_j) \leq \nu \epsilon_j \frac{\delta_j^j}{j!} \quad \text{implies} \quad \phi_{f,j}^{\delta_j}(x) \leq \epsilon_j \frac{\delta_j^j}{j!}$$

and this approximate and potentially less costly variant of (2.6) could thus replace it at the price of multiplying every  $\epsilon_j$  by the constant  $\nu$ . We will however ignore this possibility in our analysis, keeping (2.6) for simplicity of exposition.

### 2.3. The regularization algorithm

We are now in a position to describe our adaptive regularization algorithm IAR $_{qp}$  whose purpose is to compute a  $q$ -th order  $(\epsilon, \delta)$ -approximate minimizer of  $f$  in problem (1.1). The vector of accuracies  $\epsilon$  is given, together with a model degree  $p \geq q$ , corresponding to the maximum order of available derivatives. If values of the objective function  $f$  and its derivatives of orders ranging from one to  $p$  were known exactly, a typical adaptive regularization method could be outlined as follows. At iteration  $k$ , a local model of the objective function's variation would first be defined by regularizing the Taylor series of degree  $p$  at the current iterate  $x_k$ , namely

$$m_k(s) = -\Delta T_{f,p}(x_k, s) + \frac{\sigma_k}{(p+1)!} \|s\|^{p+1}, \quad (2.10)$$

where  $\sigma_k$  is a regularization parameter to be specified later. A step  $s_k$  would next be computed by approximately minimizing  $m_k(s)$  in the sense that  $m_k(s_k) \leq m_k(0) = 0$  and

$$\phi_{m_k,j}^{\delta_{k,j}}(s_k) \leq \theta \epsilon_j \frac{\delta_{k,j}^j}{j!}, \quad (2.11)$$

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<sup>3</sup>Difficulties with the standard definition already start with order three because the nullspace of  $\nabla_x^2 f(x)$  is not a continuous function of  $x$ .



for some  $\theta \in (0, \frac{1}{2})$  and  $\delta_k \in (0, 1]^q$ . In this condition,

$$\phi_{m_k, j}^{\delta_k, j}(s_k) = \max_{\|d\| \leq \delta_k, j} \Delta T_{m_k, j}(s_k, d)$$

is the  $j$ -th order optimality measure (2.7) for the model (2.10) computed at  $s_k$ , in which, for  $j \in \{1, \dots, q\}$ ,

$$T_{m_k, j}(s_k, d) = m_k(s_k) + \sum_{\ell=1}^j \frac{1}{\ell!} \nabla_s^\ell T_{f, p}(x_k, s_k)[d]^\ell + \frac{\sigma_k}{(p+1)!} \sum_{\ell=1}^j \frac{1}{\ell!} \nabla_s^\ell \|s_k\|^{p+1}[d]^\ell \quad (2.12)$$

for  $d \in \mathbb{R}^n$  and thus

$$\Delta T_{m_k, j}(s_k, d) = - \sum_{\ell=1}^j \frac{1}{\ell!} \nabla_s^\ell T_{f, p}(x_k, s_k)[d]^\ell - \frac{\sigma_k}{(p+1)!} \sum_{\ell=1}^j \frac{1}{\ell!} \nabla_s^\ell \|s_k\|^{p+1}[d]^\ell$$

(note the reuse of the notations introduced in (2.2) and (2.3), but for the function  $m_k(s)$  instead of  $f(x)$ ). The values of  $f(x_k + s_k)$  and  $\{\nabla_x^\ell f(x_k + s_k)\}_{\ell=q+1}^p$  would then be computed and the trial point  $x_k + s_k$  would then be accepted as the next iterate, provided the ratio

$$\rho_k = \frac{f(x_k) - f(x_k + s_k)}{\Delta T_{f, p}(x_k, s_k)},$$

is sufficiently positive. The regularization parameter  $\sigma_k$  would then be adapted/updated before a new iteration is started, providing the ‘‘adaptive regularization’’ suggested by the name of the method. (See for the complete description of such an algorithm using exact function and derivatives values.) The IAR $qp$  algorithm follows the same lines, except that the values of  $f(x_k)$ ,  $f(x_k + s_k)$  and  $\Delta T_{f, p}(x_k, s_k)$  are not known exactly, the inexactness in the latter resulting from the inexactness of the derivatives  $\{\nabla_x^\ell f(x_k)\}_{\ell=1}^p$ . Instead, inexact values  $\bar{f}(x_k)$ ,  $\bar{f}(x_k + s_k)$  and  $\bar{\Delta T}_{f, p}(x_k, s_k)$  are now computed and used to (re)-define the model

$$m_k(s) = -\bar{\Delta T}_{f, p}(x_k, s) + \frac{\sigma_k}{(p+1)!} \|s\|^{p+1}. \quad (2.13)$$

In particular, setting

$$0 < \omega < \min \left[ \frac{1-\eta}{3}, \frac{\eta}{2} \right], \quad (2.14)$$

the approximations  $\bar{f}(x_k)$  and  $\bar{f}(x_k + s_k)$  are required to satisfy the accuracy conditions

$$|\bar{f}(x_k) - f(x_k)| \leq \omega \bar{\Delta T}_{f, p}(x_k, s_k), \quad (2.15)$$

$$|\bar{f}(x_k + s_k) - f(x_k + s_k)| \leq \omega \bar{\Delta T}_{f, p}(x_k, s_k). \quad (2.16)$$

In what follows, we will consistently denote inexact values by an overbar.

The model (2.13) is then approximately minimized by the feasible step  $s_k$  in the sense that the trial point  $x_k + s_k$  satisfies

$$m_k(s_k) \leq m_k(0) = 0 \quad (2.17)$$

and

$$\bar{\phi}_{m_k, j}^{\delta_{k, j}}(s_k) = \max_{\|d\| \leq \delta_{k, j}} \bar{\Delta T}_{m_k, j}(s_k, d) \leq \theta \epsilon_j \frac{\delta_{k, j}^j}{j!}, \quad (2.18)$$

for  $j \in \{1, \dots, q\}$  and some  $\theta \in (0, \frac{1}{2})$  and  $\delta_k \in (0, 1]^q$ . The values  $\bar{f}(x_k)$ ,  $\bar{f}(x_k + s_k)$  and  $\bar{\Delta T}_{f, p}(x_k, s_k)$  are also used to compute the ratio  $\rho_k$ , the value of which decides of the acceptance of the trial point. The IAR $qp$  algorithm is detailed as Algorithm 2.1 on the following page.

### Algorithm 2.1: The IAR $qp$ Algorithm

**Step 0: Initialization.** An initial point  $x_0 \in \mathbb{R}^n$ , an initial regularization parameter  $\sigma_0 > 0$  and a sought optimality order  $q \in \{1, \dots, p\}$  are given, as well as a vector of accuracies  $\epsilon \in (0, 1]^q$ . The constants  $\theta \in (0, \frac{1}{2})$ ,  $\eta \in (0, 1)$ ,  $\gamma > 1$ ,  $\alpha \in (0, 1)$ ,  $0 < \omega < \min[\frac{1-\eta}{3}, \frac{\eta}{2}]$  and  $\sigma_{\min} \in (0, \sigma_0)$  are also given. Set  $k = 0$ .

**Step 1: Model construction.** Compute approximate derivatives  $\{\bar{\nabla}_x^\ell f(x_k)\}_{\ell \in \{1, \dots, p\}}$  and form the model  $m_k(s)$  defined in (2.13).

**Step 2: Step calculation.** Compute a step  $s_k$  satisfying (2.17) and (2.18) for  $j \in \{1, \dots, q\}$  and some  $\delta_k \in (0, 1]^q$ . If  $\bar{\Delta T}_{f, p}(x_k, s_k) = 0$ , go to Step 4.

**Step 3: Function estimates computation.** Compute the approximations  $\bar{f}(x_k)$  and  $\bar{f}(x_k + s_k)$  of  $f(x_k)$  and  $f(x_k + s_k)$ , respectively, such that (2.15)–(2.16) are satisfied.

**Step 4: Acceptance test.** Set

$$\rho_k = \begin{cases} \frac{\bar{f}(x_k) - \bar{f}(x_k + s_k)}{\bar{\Delta T}_{f, p}(x_k, s_k)} & \text{if } \bar{\Delta T}_{f, p}(x_k, s_k) > 0, \\ -\infty & \text{otherwise.} \end{cases} \quad (2.19)$$

If  $\rho_k \geq \eta$  (*successful iteration*), then define  $x_{k+1} = x_k + s_k$ ; otherwise (*unsuccessful iteration*) define  $x_{k+1} = x_k$ .

**Step 5: Regularization parameter update.** Set

$$\sigma_{k+1} = \begin{cases} \max\left[\sigma_{\min}, \frac{1}{\gamma}\sigma_k\right], & \text{if } \rho_k \geq \eta, \\ \gamma\sigma_k, & \text{if } \rho_k < \eta. \end{cases} \quad (2.20)$$

Increment  $k$  by one and go to Step 1.

We first verify that the algorithm is well-defined.

**Lemma 2.2.** A step  $s_k$  satisfying (2.17) and (2.18) for  $j \in \{1, \dots, q\}$  and some  $\delta_k \in (0, 1]^q$  always exists.

**Proof.** The proof is a direct extension of that of [ , Lemma 4.4] using inexact models. It is given in appendix for completeness.  $\square$

Some comments on this algorithm are useful at this stage.

1. It is important to observe that the algorithm is fully implementable with existing computational technology in the very frequent cases where  $q = 1$  or  $q = 2$ . Indeed the value of  $\bar{\phi}_{m_k,1}^{\delta_k,1}$  can easily be obtained analytically. When  $q = 2$ , the same comment obviously applies for  $\bar{\phi}_{m_k,1}^{\delta_k,1}$ , while the value  $\bar{\phi}_{m_k,2}^{\delta_k,2}$  can be computed by a standard trust-region solver (whose cost is comparable to that of the more usual calculation of the most negative eigenvalue), again making the algorithm practical. We refer the interested reader to for the presentation of numerical results in the framework of finite sum optimization for automatic learning.

In other cases, the computation  $\bar{\phi}_{m_k,j}^{\delta_k,j}$  may be extremely expensive, making our approach mostly theoretical at this stage. However, we recall that, since evaluations of the objective function and its derivatives do not occur in this computation (once the approximate derivatives are known), its cost has no impact on the evaluation complexity of the  $\text{IAR}_{qp}$  algorithm.

2. We assume in what follows that, once the inexact model  $m_k(s)$  is determined, then the computation of the pair  $(s_k, \delta_k)$  (and thus of the trial point  $x_k + s_k$ ) is deterministic. Moreover, we assume that the mechanism which ensures (2.15)-(2.16) in Step 3 of the algorithm is also deterministic, so that  $\rho_k$  and the fact that iteration  $k$  is successful are deterministic outcomes of the realization of the inexact model.
3. Observe that, because we have chosen  $m_k$  to be a model of the local variation in  $f$  rather than a model of  $f$  itself,  $\bar{f}(x_k)$  is not needed (and not computed) in Steps 1 and 2 of the algorithm. This distinguishes the  $\text{IAR}_{qp}$  algorithm from the approaches of , .

In what follows, all random quantities are denoted by capital letters, while the use of small letters is reserved for their realization. In particular, let us denote a random model at iteration  $k$  as  $M_k$ , while we use the notation  $m_k$  for its realizations. Given  $x_k$ , the source of randomness in  $m_k$  comes from the random approximation of the derivatives. Similarly, the iterates  $X_k$ , as well as the regularization parameters  $\Sigma_k$  and the steps  $S_k$  are random variables (except for initial values  $x_0$  and  $\sigma_0$  for the former two) and  $x_k$ ,  $\sigma_k$  and  $s_k$  denote their realizations. Moreover,  $\delta_k$  denotes a realization of the random vector  $\Delta_k$  arising in (2.18). Hence, the  $\text{IAR}_{qp}$  Algorithm generates a random process

$$\{X_k, S_k, M_k, \Sigma_k, \Delta_k\}. \quad (2.21)$$

where  $X_0 = x_0$  and  $\Sigma_0 = \sigma_0$  are deterministic.

#### 2.4. The probabilistic setting

We now make our probabilistic assumptions on the IAR $qp$  algorithm explicit. For  $k \geq 0$ , our assumption on the past is formalized by considering  $\mathcal{A}_{k-1}^M$  the  $\hat{\sigma}$ -algebra induced by the random variables  $M_0, M_1, \dots, M_{k-1}$ , with  $\mathcal{A}_{-1}^M = \hat{\sigma}(x_0)$ . In order to formalize our probabilistic assumptions we need a few more definitions. We define, at iteration  $k$  of an arbitrary realization,

$$d_{k,j} = \arg \max_{\|d\| \leq \delta_{k,j}} \Delta T_{m_{k,j}}(s_k, d) \quad (2.22)$$

the argument of the maximum in the definition of  $\phi_{m_{k,j}}^{\delta_{k,j}}(x_k)$ , and

$$\bar{d}_{k,j} = \arg \max_{\|d\| \leq \delta_{k,j}} \overline{\Delta T}_{m_{k,j}}(s_k, d) \quad (2.23)$$

that in the definition of  $\bar{\phi}_{m_{k,j}}^{\delta_{k,j}}(s_k)$ . We also define, at the end of Step 2 of iteration  $k$ , the events

$$\mathcal{M}_k = \begin{cases} \mathcal{M}_k^{(1)} \cap \bigcap_{j=1}^q \left( \mathcal{M}_{k,j}^{(2)} \cap \mathcal{M}_{k,j}^{(3)} \right) & \text{if } q \in \{1, 2\} \\ \mathcal{M}_k^{(1)} \cap \mathcal{M}_k^{(4)} \cap \bigcap_{j=1}^q \left( \mathcal{M}_{k,j}^{(2)} \cap \mathcal{M}_{k,j}^{(3)} \right) & \text{otherwise,} \end{cases} \quad (2.24)$$

with

$$\begin{aligned} \mathcal{M}_k^{(1)} &= \{ |\overline{\Delta T}_{f,p}(X_k, S_k) - \Delta T_{f,p}(X_k, S_k)| \leq \omega \overline{\Delta T}_{f,p}(X_k, S_k) \}, \\ \mathcal{M}_{k,j}^{(2)} &= \{ |\overline{\Delta T}_{m_{k,j}}(S_k, D_{k,j}) - \Delta T_{m_{k,j}}(S_k, D_{k,j})| \leq \omega \overline{\Delta T}_{m_{k,j}}(S_k, D_{k,j}) \}, \\ \mathcal{M}_{k,j}^{(3)} &= \{ |\overline{\Delta T}_{m_{k,j}}(S_k, \bar{D}_{k,j}) - \Delta T_{m_{k,j}}(S_k, \bar{D}_{k,j})| \leq \omega \overline{\Delta T}_{m_{k,j}}(S_k, \bar{D}_{k,j}) \}, \\ \mathcal{M}_k^{(4)} &= \{ \max_{\ell \in \{2, \dots, p\}} \|\overline{\nabla}_x^\ell f(X_k)\| \leq \Theta \}, \end{aligned}$$

for some  $\Theta > 0$ . Note that  $\Theta$  is independent of  $k$  and does not need to be known explicitly. Moreover,  $\mathcal{M}_k^{(4)}$  is not involved in the definition of  $\mathcal{M}_k$  if  $q \in \{1, 2\}$ . In what follows, we will say that iteration  $k$  is *accurate*, if  $\mathbb{1}_{\mathcal{M}_k} = 1$ , and iteration  $k$  is *inaccurate*, if  $\mathbb{1}_{\mathcal{M}_k} = 0$ .

The conditions defining  $\mathcal{M}_k$  may seem abstract at first sight, but we now motivate them by looking at what kind of accuracy on each derivative  $\overline{\nabla}_x^\ell f(x_k)$  ensures that they hold.

**Lemma 2.3.** For each  $k \geq 0$ , we have the following.

1. Let

$$\tau_k \stackrel{\text{def}}{=} \max \left[ \|S_k\|, \max_{j \in \{1, \dots, q\}} [\|D_{k,j}\|, \|\overline{D}_{k,j}\|] \right] \quad (2.25)$$

and

$$\overline{\Delta T}_{k,\min} \stackrel{\text{def}}{=} \min \left[ \overline{\Delta T}_{f,p}(X_k, S_k), \min_{j \in \{1, \dots, q\}} \left[ \overline{\Delta T}_{m_{k,j}}(S_k, D_{k,j}), \overline{\Delta T}_{m_{k,j}}(S_k, \overline{D}_{k,j}) \right] \right]. \quad (2.26)$$

Then  $\mathcal{M}_k^{(1)}$ ,  $\{\mathcal{M}_{k,j}^{(2)}\}_{j=1}^q$  and  $\{\mathcal{M}_{k,j}^{(3)}\}_{j=1}^q$  occur if

$$\|\overline{\nabla_x^\ell f}(X_k) - \nabla_x^\ell f(X_k)\| \leq \omega \frac{\overline{\Delta T}_{k,\min}}{6\tau_k^\ell} \quad \text{for } \ell \in \{1, \dots, p\}. \quad (2.27)$$

2. Suppose that AS.1 holds. Then  $\mathcal{M}_k^{(4)}$  occurs if

$$\|\overline{\nabla_x^\ell f}(X_k) - \nabla_x^\ell f(X_k)\| \leq \Theta_0 \quad \text{for } \ell \in \{2, \dots, p\} \quad (2.28)$$

and some constant  $\Theta_0 \geq 0$  independent of  $k$  and  $\ell$ .

**Proof.** Consider the first assertion. That  $\mathcal{M}_k^{(1)}$  occurs follows from the inequalities

$$\begin{aligned} |\overline{\Delta T}_{f,p}(X_k, S_k) - \Delta T_{f,p}(X_k, S_k)| &\leq \sum_{\ell=1}^p \frac{\|S_k\|^\ell}{\ell!} \|\overline{\nabla_x^\ell f}(X_k) - \nabla_x^\ell f(X_k)\| \\ &\leq \sum_{\ell=1}^p \frac{\tau_k^\ell}{\ell!} \|\overline{\nabla_x^\ell f}(X_k) - \nabla_x^\ell f(X_k)\| \\ &\leq \sum_{\ell=1}^p \frac{\omega}{6\ell!} \overline{\Delta T}_{k,\min} \\ &\leq \sum_{\ell=1}^p \frac{\omega}{6\ell!} \overline{\Delta T}_{f,p}(X_k, S_k) \\ &\leq \frac{1}{6} \chi_p(1) \omega \overline{\Delta T}_{f,p}(X_k, S_k) \\ &< \omega \overline{\Delta T}_{f,p}(X_k, S_k). \end{aligned}$$

where we have used (2.25), (2.27), (2.26) and the fact that  $\chi_p(1) \leq 2$ . The verification that  $\{\mathcal{M}_{k,j}^{(2)}\}_{j=1}^q$  and  $\{\mathcal{M}_{k,j}^{(3)}\}_{j=1}^q$  also occur uses a very similar argument, with one additional ingredient: employing the triangle inequality, (2.13), we have that, for all

$\ell \in \{1, \dots, p\}$ ,

$$\left\| \overline{\nabla_d^\ell T}_{m_k, j}(S_k, 0) - \nabla_d^\ell T_{m_k, j}(S_k, 0) \right\| \leq \sum_{t=\ell}^p \left\| \overline{\nabla_x^t f}(X_k) - \nabla_x^t f(X_k) \right\| \frac{\|S_k\|^{t-\ell}}{(t-\ell)!}.$$

Considering now  $D = D_{k, j}$  or  $D = \overline{D}_{k, j}$  and using the above inequality, (2.25), (2.27), (2.26) and the facts that  $\chi_j(1) \leq 2$  and  $\chi_{p-\ell}(1) \leq 2$ , we have that

$$\begin{aligned} & \left| \overline{\Delta T}_{m_k, j}(S_k, D) - \Delta T_{m_k, j}(S_k, D) \right| \\ & \leq \sum_{\ell=1}^j \frac{\|D\|^\ell}{\ell!} \left\| \overline{\nabla_d^\ell T}_{m_k, j}(S_k, 0) - \nabla_d^\ell T_{m_k, j}(S_k, 0) \right\| \\ & \leq \sum_{\ell=1}^j \frac{\|D\|^\ell}{\ell!} \sum_{t=\ell}^p \left\| \overline{\nabla_x^t f}(X_k) - \nabla_x^t f(X_k) \right\| \frac{\|S_k\|^{t-\ell}}{(t-\ell)!} \\ & \leq \sum_{\ell=1}^j \frac{1}{\ell!} \sum_{t=\ell}^p \left\| \overline{\nabla_x^t f}(X_k) - \nabla_x^t f(X_k) \right\| \frac{\tau_k^t}{(t-\ell)!} \\ & \leq \sum_{\ell=1}^j \frac{1}{\ell!} \sum_{t=\ell}^p \frac{1}{(t-\ell)!} \omega \frac{\overline{\Delta T}_{k, \min}}{6} \\ & \leq \frac{1}{6} \omega \overline{\Delta T}_{k, \min} \sum_{\ell=1}^j \frac{1}{\ell!} (1 + \chi_{p-\ell}(1)) \\ & \leq \omega \overline{\Delta T}_{m_k, j}(S_k, D), \end{aligned}$$

as desired. To prove the second assertion, observe that AS.1 implies that  $\|\nabla_x^\ell f(X_k)\| \leq L_{f, \ell-1}$  for  $j \in \{2, \dots, p\}$ , and thus, using (2.28), that, for  $\ell \in \{2, \dots, p\}$ ,

$$\begin{aligned} \left\| \overline{\nabla_x^\ell f}(X_k) \right\| & \leq \left\| \nabla_x^\ell f(X_k) \right\| + \left\| \overline{\nabla_x^\ell f}(X_k) - \nabla_x^\ell f(X_k) \right\| \\ & \leq L_{f, \ell-1} + \Theta_0. \end{aligned}$$

This gives the desired conclusion with the choice  $\Theta = \max_{\ell \in \{2, \dots, p\}} L_{f, \ell-1} + \Theta_0$ .  $\square$

Of course, the conditions stated in Lemma 2.3 are sufficient but by no means necessary to ensure  $\mathcal{M}_k$ . In particular, they make no attempt to exploit a possible favourable balance between the errors made on derivatives at different degrees, nor do they take into account that  $\mathcal{M}_k^{(1)}$ ,  $\mathcal{M}_{k, j}^{(2)}$  and  $\mathcal{M}_{k, j}^{(3)}$  only specify conditions on model accuracy in a finite, dimension-independent subset of directions. Despite these limitations, (2.27) and (2.28) allow the crucial conclusion that  $\mathcal{M}_k$  does occur if the derivatives  $\overline{\nabla_x^j f}(X_k)$  are sufficiently accurate compared to the model decrease. Moreover, since one would expect that, as an approximate minimizer is approached,  $\|S_k\|$ ,  $\|D_{k, j}\|$  and  $\|\overline{D}_{k, j}\|$  (and thus  $\tau_k$ ) become small, they also show the accuracy requirement becomes looser for derivatives of higher degree.

We now formalize our assumption on the stochastic process generated by the IARqp algorithm.

### AS.3

For all  $k \geq 0$ , the event  $\mathcal{M}_k$  satisfies the condition

$$p_{\mathcal{M},k} = \mathbb{P}r(\mathcal{M}_k | \mathcal{A}_{k-1}^M) = \mathbb{E}[\mathbb{1}_{\mathcal{M}_k} | \mathcal{A}_{k-1}^M] \geq p_* \quad (2.29)$$

for some  $p_* \in (\frac{1}{2}, 1]$  independent of  $k$ .

AS.3 is realistic for instance in cases where derivatives are approximated by randomized finite differences or by subsampling in the context of finite sum minimization.

We observe that, in contrast with , , the definition of  $\mathcal{M}_k$  does not require the model to be “linearly/quadratically” accurate everywhere in a ball around  $x_k$  of radius at least  $\|s_k\|$ , but merely that their variation is accurate enough along  $s_k$  (as specified in  $\mathcal{M}_k^{(1)}$ ) and along  $d_{k,j}$  and  $\bar{d}_{k,j}$  (as specified in  $\mathcal{M}_{k,j}^{(2)}$  and  $\mathcal{M}_{k,j}^{(3)}$ )<sup>4</sup> for all  $j \in \{1, \dots, q\}$ . The need to consider  $\mathcal{M}_{k,j}^{(2)}$  and  $\mathcal{M}_{k,j}^{(3)}$  for  $j \in \{1, \dots, q\}$  in the definition of  $\mathcal{M}_k$  results from our insistence that  $q$ -th order approximate optimality must include  $j$ -th order approximate optimality for all such  $j$ . AS.3 also parallels assumptions in , , , where accuracy in derivatives’ values is measured using the guaranteed model decrease or proxies given by the  $(p + 1)$ -st power of the trust-region radius or the steplength. Finally, the conditions imposed by  $\mathcal{M}_{k,j}^{(2)}$  and  $\mathcal{M}_{k,j}^{(3)}$  are only used whenever considering the value of  $\bar{\phi}_{m_{k,j}}^{\delta_{k,j}}(s_k)$ , that is in Lemma 3.1, itself only called upon in Lemma 3.3 in the case where  $\|S_k\| \leq 1$ . As a consequence, they are irrelevant when long steps are taken ( $\|S_k\| > 1$ ).

### 3. Worst-case evaluation complexity

Having set the stage and stated our assumptions, we may now consider the worst-case evaluation complexity of the IAR $qp$  algorithm. Our aim is to derive a bound on the expected number of iterations  $\mathbb{E}(N_\epsilon)$  which is needed, in the worst-case, to reach an  $(\epsilon, \delta)$ -approximate  $q$ -th-order-necessary minimizer. Specifically,  $N_\epsilon$  is the number of iterations required until (2.6) holds for the first time, i.e.,

$$N_\epsilon = \inf \left\{ k \geq 0 \mid \phi_{f,j}^{\Delta_{k-1,j}}(X_k) \leq \epsilon_j \frac{\Delta_{k-1,j}^j}{j!} \text{ for } j \in \{1, \dots, q\} \right\}. \quad (3.1)$$

Note that  $\phi_{f,j}^{\Delta_{k-1,j}}(X_k)$ , the  $j$ -th order optimality measure at iteration  $k$ , uses the optimality radii  $\Delta_{k-1,j}$  resulting from the step computation at iteration  $k - 1$ , as is the case in , . Now recall that the trial point  $X_{k-1} + S_{k-1}$  and the vector of radii  $\Delta_{k-1}$  are deterministic once the inexact model at iteration  $k - 1$  is known. Thus these variables are measurable for  $\mathcal{A}_{k-1}^M$  and because of our deterministic assumptions on the accuracy of  $f$ , the event  $\{X_k = X_{k-1} + S_{k-1}\}$  (which occur when iteration  $k - 1$  is successful) is also measurable

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<sup>4</sup>A slightly stronger assumption would be to require a sufficient relative accuracy along  $s_k$  and in a (typically small) neighbourhood of  $s_k$ .

for  $\mathcal{A}_{k-1}^M$ . As a consequence and since  $\phi_{f,j}^{\Delta_{k-1,j}}(X_k)$  uses exact derivatives of  $f$ , the event  $\{N_\epsilon = k\}$  is measurable with respect to  $\mathcal{A}_{k-1}^M$ . The definition (3.1) can thus be viewed as that of a family of  $\epsilon$ -dependent stopping times for the stochastic process generated by the IAR<sub>qp</sub> algorithm (see, e.g., [ , section 2.3]).

### 3.1. General properties of the IAR<sub>qp</sub> algorithm

We first consider properties of “accurate” iterations, in the sense that  $\mathcal{M}_k$  occurs, and start with the relation between  $\phi_{m_k,j}^{\delta_{k,j}}(s_k)$  and its approximation. The next lemma is inspired by Lemma 3.2 in [ ], but significantly differs in that it now requires considering both directions  $d_{k,j}$  and  $\bar{d}_{k,j}$ .

**Lemma 3.1.** Consider any realization of the algorithm and assume that  $\mathcal{M}_k$  occurs. Then, for  $j \in \{1, \dots, q\}$ ,

$$(1 - \omega)\bar{\phi}_{m_k,j}^{\delta_{k,j}}(s_k) \leq \phi_{m_k,j}^{\delta_{k,j}}(s_k) \leq (1 + \omega)\bar{\phi}_{m_k,j}^{\delta_{k,j}}(s_k) \quad (3.2)$$

**Proof.** Let  $j \in \{1, \dots, q\}$ . Consider  $d_{k,j}$  defined in (2.22). From (2.18), we have that

$$\begin{aligned} \Delta T_{m_k,j}(s_k, d_{k,j}) &\leq \overline{\Delta T}_{m_k,j}(s_k, d_{k,j}) + |\Delta T_{m_k,j}(s_k, d_{k,j}) - \overline{\Delta T}_{m_k,j}(s_k, d_{k,j})| \\ &\leq (1 + \omega)\overline{\Delta T}_{m_k,j}(s_k, d_{k,j}) \\ &\leq (1 + \omega) \max_{\|d\| \leq \delta_{k,j}} \overline{\Delta T}_{m_k,j}(s_k, d) \\ &= (1 + \omega)\overline{\Delta T}_{m_k,j}(s_k, \bar{d}_{k,j}) \end{aligned}$$

where we used the fact that  $\mathcal{M}_k$  occurs to derive the second inequality and considered  $\bar{d}_{k,j}$  defined in (2.23). Therefore

$$\phi_{m_k,j}^{\delta_{k,j}}(s_k) = \Delta T_{m_k,j}(s_k, d_{k,j}) \leq (1 + \omega)\bar{\phi}_{m_k,j}^{\delta_{k,j}}(s_k).$$

This proves the rightmost inequality of (3.2). Similarly, using our assumption that  $\mathcal{M}_k$  occurs, we obtain that

$$\begin{aligned} \Delta T_{m_k,j}(s_k, \bar{d}_{k,j}) &\geq \overline{\Delta T}_{m_k,j}(s_k, \bar{d}_{k,j}) - |\Delta T_{m_k,j}(s_k, \bar{d}_{k,j}) - \overline{\Delta T}_{m_k,j}(s_k, \bar{d}_{k,j})| \\ &\geq (1 - \omega)\overline{\Delta T}_{m_k,j}(s_k, \bar{d}_{k,j}) \end{aligned}$$

and hence, from (2.7) and (2.18), that

$$(1 - \omega)\bar{\phi}_{m_k,j}^{\delta_{k,j}}(s_k) \leq \max_{\|d\| \leq \delta_{k,j}} \Delta T_{m_k,j}(s_k, d) = \phi_{m_k,j}^{\delta_{k,j}}(s_k),$$

which concludes the proof of (3.2).  $\square$



The next step is to adapt an important property of  $\Delta_{k,j}$  in the exact case to our inexact framework.

**Lemma 3.2.** Suppose that AS.1 holds. Then, for any  $j \in \{1, \dots, q\}$ ,

1. if  $j \in \{1, 2\}$ ,  $\Delta_{k,j}$  can always be chosen equal to one;
2. in the other cases, and assuming that  $\mathcal{M}_k$  occurs, then, either  $\|s_k\| > 1$  or  $\Delta_{k,j} \leq 1$  can be chosen such that

$$\Delta_{k,j} \geq \kappa_\delta(\sigma_k)\epsilon_j, \quad (3.3)$$

where  $\kappa_\delta(\sigma) \in (0, 1)$  is independent of  $\epsilon$  and decreasing when  $\sigma$  grows.

**Proof.** The proof broadly follows the developments of [ , Lemmas 4.3 and 4.4], except that it now uses the model involving approximate derivatives and that  $L_f$ , the upper bound of the derivatives of  $f$  at  $x_k$  derived from AS.1 is now replaced by  $\Theta$ , as guaranteed by  $\mathcal{M}_k^{(4)}$ . The details (including the reason for the dichotomy between the two cases in the lemma's statement) are provided in appendix.  $\square$

In what follows, we will assume that, whenever  $q > 2$ , the IAR $qp$  algorithm computes a pair  $(s_k, \delta_k)$  such that, for each  $j \in \{1, \dots, q\}$ ,  $\delta_{k,j}$  is always within a fraction of its maximal value, thereby ensuring (3.3). We now prove a crucial inequality relating the step length to the accuracy requirements.

**Lemma 3.3.** Consider any realization of the algorithm. Assume that  $\mathcal{M}_k$  occurs, that iteration  $k$  is successful and that, for some  $j \in \{1, \dots, q\}$ , (2.6) fails for  $(x_{k+1}, \delta_{k,j})$ . Then either  $\|s_k\| > 1$  or

$$(1 - 2\theta)\epsilon_j \frac{\delta_{k,j}^j}{j!} \leq \frac{L_{f,p} + \sigma_k}{(p - q + 1)!} \sum_{\ell=1}^j \frac{\delta_{k,j}^\ell}{\ell!} \|s_k\|^{p-\ell+1} \quad (3.4)$$

**Proof.** [See [ , Lemma 5.3] for the composite unconstrained Lipschitz continuous case.] Suppose that  $\|s_k\| \leq 1$ . Since (2.6) fails at  $(x_{k+1}, \delta_{k,j})$ , we must have that

$$\phi_{f,j}^{\delta_{k,j}}(x_{k+1}) > \epsilon_j \frac{\delta_{k,j}^j}{j!} > 0 \quad (3.5)$$

for some  $j \in \{1, \dots, q\}$ . Define  $d$  to be the argument of the minimum in the definition

of  $\phi_{f,j}^{\delta_{k,j}}(x_{k+1})$ . Hence,

$$0 < \|d\| \leq \delta_{k,j}. \quad (3.6)$$

Using (3.5), (2.7) and the triangle inequality, we thus obtain that

$$\phi_{f,j}^{\delta_{k,j}}(x_{k+1}) = \Delta T_{f,j}(x_{k+1}, d) \leq |\Delta T_{f,j}(x_{k+1}, d) - \Delta T_{m_k,j}(s_k, d)| + \Delta T_{m_k,j}(s_k, d). \quad (3.7)$$

Recalling now from [ , Lemma 2.4]) that

$$\|\nabla_s^\ell \|s_k\|^{p+1}\| = \frac{(p+1)!}{(p-\ell+1)!} \|s_k\|^{p-\ell+1},$$

we may now use the fact that  $x_{k+1} = x_k + s_k$  since iteration  $k$  is successful, (2.5) in Lemma 2.1, (2.12), (3.6) and the triangle inequality to obtain that

$$\begin{aligned} |\Delta T_{f,j}(x_{k+1}, d) - \Delta T_{m_k,j}(s_k, d)| &\leq \sum_{\ell=1}^j \frac{\delta_{k,j}^\ell}{\ell!} \|\nabla_x^\ell f(x_{k+1}) - \nabla_s^\ell T_{f,p}(x_k, s_k)\| \\ &\quad + \frac{\sigma_k}{(p+1)!} \sum_{\ell=1}^j \frac{\delta_{k,j}^\ell}{\ell!} \|\nabla_s^\ell \|s_k\|^{p+1}\| \quad (3.8) \\ &\leq \frac{L_{f,p} + \sigma_k}{(p-q+1)!} \sum_{\ell=1}^j \frac{\delta_{k,j}^\ell}{\ell!} \|s_k\|^{p-\ell+1} \end{aligned}$$

Moreover, using (2.18), (3.2) and the fact that  $\omega < 1$  (see (2.14)), we deduce that

$$\Delta T_{m_k,j}(s_k, d) \leq \phi_{m_k,j}^{\delta_{k,j}}(s_k) \leq (1 + \omega) \bar{\phi}_{m_k,j}^{\delta_{k,j}}(s_k) \leq 2\theta \epsilon_j \frac{\delta_{k,j}^j}{j!}. \quad (3.9)$$

Substituting (3.8) and (3.9) into (3.7) and using (3.6) and (3.5), we obtain (3.4).  $\square$

**Lemma 3.4.** Suppose that AS.1 holds and consider any realization of the algorithm. Suppose also that  $\mathcal{M}_k$  occurs, that iteration  $k$  is successful and that, for some  $j \in \{1, \dots, q\}$ , (2.6) fails for  $(x_{k+1}, \delta_{k,j})$ . Then

$$\|s_k\|^{p+1} \geq \psi(\sigma_k) \epsilon_j^\varpi \quad (3.10)$$

where

$$\varpi = \begin{cases} \frac{p+1}{p-q+1} & \text{if } q \in \{1, 2\} \text{ ,} \\ \frac{q(p+1)}{p} & \text{otherwise.} \end{cases} \quad (3.11)$$

and

$$\psi(\sigma) = \begin{cases} \min \left[ 1, \left( \frac{(1-2\theta)(p-q+1)!}{q!(L_{f,p} + \sigma)} \right)^\varpi \right] & \text{if } q \in \{1, 2\} \text{ ,} \\ \min \left[ 1, \left( \frac{(1-2\theta)(p-q+1)! \kappa_\delta(\sigma)^{q-1}}{q!(L_{f,p} + \sigma)} \right)^\varpi \right] & \text{otherwise.} \end{cases} \quad (3.12)$$

**Proof.** [See [, Lemma 5.4].] If  $\|s_k\| > 1$ , the conclusion immediately follows. Suppose therefore that  $\|s_k\| \leq 1$  and consider  $j$  such that (3.4) holds. Recalling the definition of  $\chi_j$  in (1.3), (3.4) can be rewritten as

$$\alpha_k \epsilon_j \delta_{k,j}^j \leq \|s_k\|^{p+1} \chi_j \left( \frac{\delta_{k,j}}{\|s_k\|} \right) \quad (3.13)$$

where we have set

$$\alpha_k = \frac{(1-2\theta)(p-q+1)!}{q!(L_{f,p} + \sigma_k)}.$$

In particular, since  $\chi_j(t) \leq 2t^j$  for  $t \geq 1$ , we have that, when  $\|s_k\| \leq \delta_{k,j}$ ,

$$\alpha_k \epsilon_j \leq 2 \|s_k\|^{p+1} \left( \frac{1}{\|s_k\|} \right)^j = 2 \|s_k\|^{p-j+1}. \quad (3.14)$$

Suppose first that  $q \in \{1, 2\}$ . Then, from our assumptions and Lemma 3.2,  $\delta_{k,j} = 1$  and  $\|s_k\| \leq 1 = \delta_{k,j}$ . Thus (3.14) yields the first case of (3.11)–(3.12). Suppose now that  $q > 2$ . Then our assumptions imply that (3.3) holds. If  $\|s_k\| \leq \delta_{k,j}$ , we may again deduce from (3.14) that the first case of (3.11)–(3.12) holds, which implies, because  $\kappa_\delta(\sigma) < 1$  and  $1/(p-j+1) \leq j/p$ , that the second case also holds. Consider therefore the case where  $\|s_k\| > \delta_{k,j}$ . Then (3.13) and the fact that  $\chi_j(t) < 2t$  for  $t \in [0, 1]$  give that

$$\alpha_k \epsilon_j \delta_{k,j}^j \leq 2 \|s_k\|^{p+1} \left( \frac{\delta_{k,j}}{\|s_k\|} \right),$$

which, with (3.3), implies the second case of (3.11)–(3.12) as requested.  $\square$

Note that  $\psi(\sigma)$  is decreasing as a function of  $\sigma$  in both cases of (3.12). We now investigate the decrease of the exact objective function values at successful iterations.

**Lemma 3.5.** Suppose that AS.1 holds and consider any realization of the algorithm. Then

$$\overline{\Delta T}_{f,p}(x_k, s_k) \geq \frac{\sigma_k}{(p+1)!} \|s_k\|^{p+1} \geq \frac{\sigma_{\min}}{(p+1)!} \|s_k\|^{p+1} \geq 0, \quad (3.15)$$

where  $\sigma_{\min}$  is defined in Step 0 of the IAR $qp$  algorithm. Moreover, if iteration  $k$  is successful, then

$$f(x_k) - f(x_{k+1}) \geq \frac{(\eta - 2\omega)\sigma_{\min}}{(p+1)!} \|s_k\|^{p+1} > 0. \quad (3.16)$$

**Proof.** The inequality (3.15) immediately follows from (2.13), (2.17), (2.20). Now the fact that iteration  $k$  is successful, together with (2.14) and (2.15)–(2.16), imply that

$$\begin{aligned} f(x_k) - f(x_{k+1}) &\geq \bar{f}(x_k) - \bar{f}(x_{k+1}) - 2\omega \overline{\Delta T}_{f,p}(x_k, s_k) \\ &\geq \eta \overline{\Delta T}_{f,p}(x_k, s_k) - 2\omega \overline{\Delta T}_{f,p}(x_k, s_k), \end{aligned}$$

yielding (3.16) using (3.15) and (2.14).  $\square$

We finally conclude our analysis of “accurate” iterations by proving a standard result in the analysis of adaptive regularization methods. A similar version of this result was presented in [ , Lemma 4.2] for the case where both function values and models are sufficiently accurate.

**Lemma 3.6.** Suppose that AS.1 holds and let  $\beta > 1$  be given. Then, for any realization of the algorithm, if iteration  $k$  is such that  $\mathcal{M}_k$  occurs and

$$\sigma_k \geq \sigma_s \stackrel{\text{def}}{=} \max \left[ \beta \sigma_0, \frac{L_{f,p}}{1 - \eta - 3\omega} \right], \quad (3.17)$$

then iteration  $k$  is successful.

**Proof.** Suppose that (3.17) holds. Thus, using successively (2.19), the triangle inequality, the fact that  $\mathcal{M}_k$  occurs, (2.4), (3.15), (2.14), (2.15)–(2.16) and (3.17), we

deduce that

$$\begin{aligned}
|\rho_k - 1| &\leq \frac{1}{\overline{\Delta T}_{f,p}(x_k, s_k)} \left[ \left( \bar{f}(x_k) - f(x_k) \right) + \left( f(x_k + s_k) - \bar{f}(x_k + s_k) \right) \right. \\
&\quad \left. + \left( -f(x_k + s_k) + f(x_k) - \Delta T_{f,p}(x_k, s_k) \right) \right. \\
&\quad \left. + \left( \Delta T_{f,p}(x_k, s_k) - \overline{\Delta T}_{f,p}(x_k, s_k) \right) \right] \\
&\leq \frac{1}{\overline{\Delta T}_{f,p}(x_k, s_k)} \left[ |f(x_k + s_k) - T_{f,p}(x_k, s_k)| + 3\omega |\overline{\Delta T}_{f,p}(x_k, s_k)| \right] \\
&\leq \frac{1}{\overline{\Delta T}_{f,p}(x_k, s_k)} \left[ \frac{L_{f,p}}{(p+1)!} \|s_k\|^{p+1} + 3\omega |\overline{\Delta T}_{f,p}(x_k, s_k)| \right] \\
&\leq \frac{L_{f,p}}{\sigma_k} + 3\omega \\
&\leq 1 - \eta.
\end{aligned}$$

Therefore  $\rho_k \geq \eta$  and iteration  $k$  is successful.  $\square$

### 3.2. Bounding the expected number of steps with $\Sigma_k \geq \sigma_s$

We now return to the general stochastic process generated by the  $\text{IAR}_{qp}$  algorithm aiming at bounding from above the expected number of steps in the process generated by the algorithm with  $\Sigma_k \geq \sigma_s$ . To this purpose, for all  $0 \leq k \leq \ell$ , given  $\ell \in \{0, \dots, N_\epsilon - 1\}$ , let us define the events

$$\begin{aligned}
\Lambda_k &= \{ \text{iteration } k \text{ is such that } \Sigma_k < \sigma_s \}, & \Lambda_k^c &= \{ \text{iteration } k \text{ is such that } \Sigma_k \geq \sigma_s \} \\
\mathcal{S}_k &= \{ \text{iteration } k \text{ is successful} \},
\end{aligned}$$

and let

$$N_\Lambda \stackrel{\text{def}}{=} \sum_{k=0}^{N_\epsilon-1} \mathbb{1}_{\Lambda_k}, \quad N_{\Lambda^c} \stackrel{\text{def}}{=} \sum_{k=0}^{N_\epsilon-1} \mathbb{1}_{\Lambda_k^c}, \quad (3.18)$$

be the number of steps, in the stochastic process induced by the  $\text{IAR}_{qp}$  algorithm, with  $\Sigma_k < \sigma_s$  and  $\Sigma_k \geq \sigma_s$ , before iteration  $N_\epsilon$  is reached, respectively. In what follows we suppose that AS.1–AS.3 hold.

We may now follow the argument of to derive an upper bound on  $\mathbb{E}[N_{\Lambda^c}]$ . In particular, the argument unfolds as follows:

- (i) we apply [, Lemma 2.2] to deduce that, for any  $\ell \in \{0, \dots, N_\epsilon - 1\}$  and for all realizations of the  $\text{IAR}_{qp}$  algorithm, one has that

$$\sum_{k=0}^{\ell} \mathbb{1}_{\Lambda_k^c} \mathbb{1}_{\mathcal{S}_k} \leq \frac{\ell + 1}{2}; \quad (3.19)$$

(ii) as in , we note that both  $\hat{\sigma}(\mathbb{1}_{\Lambda_k})$  and  $\hat{\sigma}(\mathbb{1}_{\Lambda_k^c})$  belong to  $\mathcal{A}_{k-1}^M$ , as the random variable  $\Lambda_k$  is fully determined by the first  $k - 1$  iterations of the IARqp algorithm. Then, setting  $\ell = N_\epsilon - 1$  we can rely on [, Lemma 2.1] (with  $W_k = \mathbb{1}_{\Lambda_k^c}$ ) and (2.29) to deduce that

$$\mathbb{E} \left[ \sum_{k=0}^{N_\epsilon-1} \mathbb{1}_{\Lambda_k^c} \mathbb{1}_{\mathcal{M}_k} \right] \geq \mathbb{E} \left[ \sum_{k=0}^{N_\epsilon-1} p_{\mathcal{M},k} \mathbb{1}_{\Lambda_k^c} \right] \geq p_* \mathbb{E} \left[ \sum_{k=0}^{N_\epsilon-1} \mathbb{1}_{\Lambda_k^c} \right]; \quad (3.20)$$

(iii) as a consequence, given that Lemma 3.6 ensures that each iteration  $k$  where  $\mathcal{M}_k$  occurs and  $\sigma_k \geq \sigma_s$  is successful, we have that

$$\sum_{k=0}^{N_\epsilon-1} \mathbb{1}_{\Lambda_k^c} \mathbb{1}_{\mathcal{M}_k} \leq \sum_{k=0}^{N_\epsilon-1} \mathbb{1}_{\Lambda_k^c} \mathbb{1}_{\mathcal{S}_k} \leq \frac{N_\epsilon}{2},$$

in which the last inequality follows from (3.19), with  $\ell = N_\epsilon - 1$ . Taking expectation in the above inequality, using (3.20) and recalling the rightmost definition in (3.18), we obtain, as in [, Lemma 2.3], that, for any realization,

$$\mathbb{E}[N_{\Lambda^c}] \leq \frac{1}{2p_*} \mathbb{E}[N_\epsilon]. \quad (3.21)$$

The remaining upper bound on  $\mathbb{E}[N_\Lambda]$  will be the focus of the next subsection.

### 3.3. Bounding the expected number of steps with $\Sigma_k < \sigma_s$

For analyzing  $\mathbb{E}[N_\Lambda]$ , where  $N_\Lambda$  is defined in (3.18), we now introduce the following variables.

**Definition 1.** *With reference to the process (2.21) generated by the IARqp algorithm, let us define:*

- $\bar{\Lambda}_k = \{ \text{iteration } k \text{ is such that } \Sigma_k \leq \sigma_s \};$
  - $N_I = \sum_{k=0}^{N_\epsilon-1} \mathbb{1}_{\bar{\Lambda}_k} \mathbb{1}_{\mathcal{M}_k^c} : \text{the number of inaccurate iterations with } \Sigma_k \leq \sigma_s;$
  - $N_A = \sum_{k=0}^{N_\epsilon-1} \mathbb{1}_{\bar{\Lambda}_k} \mathbb{1}_{\mathcal{M}_k} : \text{the number of accurate iterations with } \Sigma_k \leq \sigma_s;$
  - $N_{AS} = \sum_{k=0}^{N_\epsilon-1} \mathbb{1}_{\bar{\Lambda}_k} \mathbb{1}_{\mathcal{M}_k} \mathbb{1}_{\mathcal{S}_k} : \text{the number of accurate successful iterations with } \Sigma_k \leq \sigma_s;$
  - $N_{AU} = \sum_{k=0}^{N_\epsilon-1} \mathbb{1}_{\bar{\Lambda}_k} \mathbb{1}_{\mathcal{M}_k} \mathbb{1}_{\mathcal{S}_k^c} : \text{the number of accurate unsuccessful iterations with } \Sigma_k < \sigma_s;$
  - $N_{IS} = \sum_{k=0}^{N_\epsilon-1} \mathbb{1}_{\bar{\Lambda}_k} \mathbb{1}_{\mathcal{M}_k^c} \mathbb{1}_{\mathcal{S}_k} : \text{the number of inaccurate successful iterations with } \Sigma_k \leq \sigma_s;$
  - $N_S = \sum_{k=0}^{N_\epsilon-1} \mathbb{1}_{\bar{\Lambda}_k} \mathbb{1}_{\mathcal{S}_k} : \text{the number of successful iterations with } \Sigma_k \leq \sigma_s;$
  - $N_U = \sum_{k=0}^{N_\epsilon-1} \mathbb{1}_{\bar{\Lambda}_k} \mathbb{1}_{\mathcal{S}_k^c} : \text{the number of unsuccessful iterations with } \Sigma_k < \sigma_s.$
- (3.22)

Observe that  $\bar{\Lambda}_k$  is the “closure” of  $\Lambda_k$  in that the inequality in its definition is no longer strict.

We immediately notice that an upper bound on  $\mathbb{E}[N_\Lambda]$  is available, once an upper bound on  $\mathbb{E}[N_I] + \mathbb{E}[N_A]$  is known, since

$$\mathbb{E}[N_\Lambda] \leq \mathbb{E} \left[ \sum_{k=0}^{N_\epsilon-1} \mathbb{1}_{\bar{\Lambda}_k} \right] = \mathbb{E} \left[ \sum_{k=0}^{N_\epsilon-1} \mathbb{1}_{\bar{\Lambda}_k} \mathbb{1}_{\mathcal{M}_k^c} + \sum_{k=0}^{N_\epsilon-1} \mathbb{1}_{\bar{\Lambda}_k} \mathbb{1}_{\mathcal{M}_k} \right] = \mathbb{E}[N_I] + \mathbb{E}[N_A]. \quad (3.23)$$

Using again [1, Lemma 2.1] (with  $W_k = \mathbb{1}_{\bar{\Lambda}_k}$ ) to give an upper bound on  $\mathbb{E}[N_I]$ , we obtain the following result.

**Lemma 3.7.** [1, Lemma 2.6] Let  $\mathcal{M}_k$  be the sequence of events in (2.24) and assume that (2.29) holds. Let  $N_I, N_A$  be defined as in Definition 1 in the context of the stochastic process (2.21) generated by the IARqp algorithm. Then

$$\mathbb{E}[N_I] \leq \frac{1 - p_*}{p_*} \mathbb{E}[N_A]. \quad (3.24)$$

Turning to the upper bound for  $\mathbb{E}[N_A]$ , we observe that

$$\mathbb{E}[N_A] = \mathbb{E}[N_{AS}] + \mathbb{E}[N_{AU}] \leq \mathbb{E}[N_{AS}] + \mathbb{E}[N_U]. \quad (3.25)$$

Hence, bounding  $\mathbb{E}[N_I]$  can be achieved by providing upper bounds on  $\mathbb{E}[N_{AS}]$  and  $\mathbb{E}[N_U]$ . Regarding the latter, we first note that the process induced by the  $\text{IAR}_{qp}$  algorithm ensures that  $\Sigma_k$  is decreased by a factor  $\gamma$  on successful steps and increased by the same factor on unsuccessful ones. Consequently, by virtue of [ , Lemma 2, 5], we obtain the following bound.

**Lemma 3.8.** [ , Lemma 2.5] For any  $\ell \in \{0, \dots, N_\epsilon - 1\}$  and for all realisations of the  $\text{IAR}_{qp}$  algorithm, we have that

$$\sum_{k=0}^{\ell} \mathbb{1}_{\Lambda_k} \mathbb{1}_{\mathcal{S}_k^c} \leq \sum_{k=0}^{\ell} \mathbb{1}_{\bar{\Lambda}_k} \mathbb{1}_{\mathcal{S}_k} + \left\lceil \log_{\gamma} \left( \frac{\sigma_s}{\sigma_0} \right) \right\rceil.$$

From this inequality with  $\ell = N_\epsilon - 1$ , recalling Definition 1 and taking expectations, we therefore obtain that

$$\mathbb{E}[N_U] \leq \mathbb{E}[N_S] + \left\lceil \log_{\gamma} \left( \frac{\sigma_s}{\sigma_0} \right) \right\rceil = \mathbb{E}[N_{AS}] + \mathbb{E}[N_{IS}] + \left\lceil \log_{\gamma} \left( \frac{\sigma_s}{\sigma_0} \right) \right\rceil. \quad (3.26)$$

An upper bound on  $\mathbb{E}[N_{AS}]$  is given by the following lemma.

**Lemma 3.9.** Let Assumption AS.1 and AS.2 hold. For all realizations of the  $\text{IAR}_{qp}$  algorithm we have that

$$\mathbb{E}[N_{AS}] \leq \frac{(f_0 - f_{\text{low}})(p+1)!}{(\eta - 2\omega)\sigma_{\min}\psi(\sigma_s)} \left( \min_{j \in \{1, \dots, q\}} \epsilon_j \right)^{-\varpi} + 1, \quad (3.27)$$

where  $\varpi$ ,  $\psi(\sigma)$  and  $\sigma_s$  are defined in (3.11), (3.12) and (3.17), respectively.

**Proof.** For all realizations of the  $\text{IAR}_{qp}$  algorithm we have that:

- if iteration  $k$  is successful, then (3.16) holds;
- if iteration  $k$  is successful and accurate (i.e.,  $\mathbb{1}_{\mathcal{S}_k} \mathbb{1}_{\mathcal{M}_k} = 1$ ) and (2.6) fails for  $(x_{k+1}, \delta_{k,j})$ , then (3.10) holds;
- if iteration  $k$  is unsuccessful, the mechanism of the  $\text{IAR}_{qp}$  algorithm guarantees that  $x_k = x_{k+1}$  and, hence, that  $f(x_{k+1}) = f(x_k)$ .



Therefore, for any  $\ell \in \{0, \dots, N_\epsilon - 1\}$ ,

$$\begin{aligned}
f_0 - f_{\text{low}} &\geq f_0 - f(X_{\ell+1}) = \sum_{k=0}^{\ell} \mathbb{1}_{S_k} (f(X_k) - f(X_{k+1})) \geq \sum_{k=0}^{\ell} \mathbb{1}_{S_k} \frac{(\eta - 2\omega)\sigma_{\min}}{(p+1)!} \|S_k\|^{p+1} \\
&\geq \sum_{k=0}^{\ell-1} \mathbb{1}_{S_k} \mathbb{1}_{\mathcal{M}_k} \frac{(\eta - 2\omega)\sigma_{\min}}{(p+1)!} \|S_k\|^{p+1} \tag{3.28} \\
&\geq \sum_{k=0}^{\ell-1} \mathbb{1}_{S_k} \mathbb{1}_{\mathcal{M}_k} \frac{(\eta - 2\omega)\sigma_{\min}}{(p+1)!} \psi(\Sigma_k) \left( \min_{j \in \{1, \dots, q\}} \epsilon_j \right)^\varpi \\
&\geq \sum_{k=0}^{\ell-1} \mathbb{1}_{S_k} \mathbb{1}_{\mathcal{M}_k} \mathbb{1}_{\bar{\Lambda}_k} \frac{(\eta - 2\omega)\sigma_{\min}}{(p+1)!} \psi(\Sigma_k) \left( \min_{j \in \{1, \dots, q\}} \epsilon_j \right)^\varpi \\
&\geq \frac{(\eta - 2\omega)\sigma_{\min}}{(p+1)!} \psi(\sigma_s) \left( \min_{j \in \{1, \dots, q\}} \epsilon_j \right)^\varpi \left( \sum_{k=0}^{\ell-1} \mathbb{1}_{S_k} \mathbb{1}_{\mathcal{M}_k} \mathbb{1}_{\bar{\Lambda}_k} \right), \tag{3.29}
\end{aligned}$$

having set  $f_0 \stackrel{\text{def}}{=} f(X_0)$  and where the last inequality is due to fact that  $\psi(\sigma)$  is a decreasing function. We now notice that, by Definition 1,

$$N_{AS} - 1 \leq \sum_{k=0}^{N_\epsilon - 2} \mathbb{1}_{\bar{\Lambda}_k} \mathbb{1}_{\mathcal{M}_k} \mathbb{1}_{S_k}.$$

Hence, letting  $\ell = N_\epsilon - 1$  and taking expectations in (3.29), we conclude that

$$f_0 - f_{\text{low}} \geq (\mathbb{E}[N_{AS}] - 1) \frac{(\eta - 2\omega)\sigma_{\min}}{(p+1)!} \psi(\sigma_s) \left( \min_{j \in \{1, \dots, q\}} \epsilon_j \right)^\varpi,$$

which is equivalent to (3.27).  $\square$

While inequalities (3.27) and (3.26) provide upper bounds on  $\mathbb{E}[N_{AS}]$  and  $\mathbb{E}[N_U]$ , as desired, the latter still depends on  $\mathbb{E}[N_{IS}]$ , which has to be bounded from above as well. This can be done by following once more: Definition 1, (3.24) and (3.25) directly imply that

$$\mathbb{E}[N_{IS}] \leq \mathbb{E}[N_I] \leq \frac{1 - p_*}{p_*} \mathbb{E}[N_A] \leq \frac{1 - p_*}{p_*} (\mathbb{E}[N_{AS}] + \mathbb{E}[N_U]) \tag{3.30}$$

and hence

$$\mathbb{E}[N_{IS}] \leq \frac{1 - p_*}{2p_* - 1} \left( 2\mathbb{E}[N_{AS}] + \left\lceil \log_\gamma \left( \frac{\sigma_s}{\sigma_0} \right) \right\rceil \right) \tag{3.31}$$

follows from (3.26) (remember that  $\frac{1}{2} < p_* \leq 1$ ). Thus, the right-hand side in (3.24) is in

turn upper bounded by virtue of (3.25), (3.26), (3.31) and (3.27), giving

$$\begin{aligned}
\mathbb{E}[N_A] &\leq \mathbb{E}[N_{AS}] + \mathbb{E}[N_U] \leq 2\mathbb{E}[N_{AS}] + \mathbb{E}[N_{IS}] + \left\lceil \log_\gamma \left( \frac{\sigma_s}{\sigma_0} \right) \right\rceil \\
&\leq \left( \frac{1-p_*}{2p_*-1} + 1 \right) \left( 2\mathbb{E}[N_{AS}] + \left\lceil \log_\gamma \left( \frac{\sigma_s}{\sigma_0} \right) \right\rceil \right) \\
&= \frac{p_*}{2p_*-1} \left[ 2\mathbb{E}[N_{AS}] + \left\lceil \log_\gamma \left( \frac{\sigma_s}{\sigma_0} \right) \right\rceil \right] \\
&\leq \frac{p_*}{2p_*-1} \left[ \frac{2(f_0 - f_{\text{low}})(p+1)!}{(\eta - 2\omega)\sigma_{\min}\psi(\sigma_s)} \left( \min_{j \in \{1, \dots, q\}} \epsilon_j \right)^{-\varpi} + \left\lceil \log_\gamma \left( \frac{\sigma_s}{\sigma_0} \right) \right\rceil + 2 \right] \quad (3.32)
\end{aligned}$$

This inequality, together with (3.23) and (3.24), finally gives the desired bound on  $\mathbb{E}[N_\Lambda]$ :

$$\mathbb{E}[N_\Lambda] \leq \frac{1}{p_*} \mathbb{E}[N_A] \leq \frac{1}{2p_*-1} \left[ \frac{2(f_0 - f_{\text{low}})(p+1)!}{(\eta - 2\omega)\sigma_{\min}\psi(\sigma_s)} \left( \min_{j \in \{1, \dots, q\}} \epsilon_j \right)^{-\varpi} + \left\lceil \log_\gamma \left( \frac{\sigma_s}{\sigma_0} \right) \right\rceil + 2 \right]. \quad (3.33)$$

We can now express our final complexity result in full.

**Theorem 3.10.** Suppose that AS.1–AS.3 hold. Then the following conclusions also hold.

1. If  $q \in \{1, 2\}$ , then

$$\mathbb{E}[N_\epsilon] \leq \kappa(p_*) \left( \frac{2(f_0 - f_{\text{low}})(p+1)!}{(\eta - 2\omega)\sigma_{\min}\psi(\sigma_s)} \left( \min_{j \in \{1, \dots, q\}} \epsilon_j \right)^{-\frac{p+1}{p-q+1}} + \left\lceil \log_\gamma \left( \frac{\sigma_s}{\sigma_0} \right) \right\rceil + 2 \right),$$

2. If  $q > 2$ , then

$$\mathbb{E}[N_\epsilon] \leq \kappa(p_*) \left( \frac{2(f_0 - f_{\text{low}})(p+1)!}{(\eta - 2\omega)\sigma_{\min}\psi(\sigma_s)} \left( \min_{j \in \{1, \dots, q\}} \epsilon_j \right)^{-\frac{q(p+1)}{p}} + \left\lceil \log_\gamma \left( \frac{\sigma_s}{\sigma_0} \right) \right\rceil + 2 \right),$$

with  $\kappa(p_*) \stackrel{\text{def}}{=} \frac{2p_*}{(2p_*-1)^2}$  and  $N_\epsilon$ ,  $\psi(\sigma)$ ,  $\sigma_s$  defined as in (3.1), (3.12), (3.17), respectively.

**Proof.** Recalling the definitions (3.18) and the bound (3.21), we obtain that

$$\mathbb{E}[N_\epsilon] = \mathbb{E}[N_\Lambda^c] + \mathbb{E}[N_\Lambda] \leq \frac{\mathbb{E}[N_\epsilon]}{2p_*} + \mathbb{E}[N_\Lambda],$$

which implies, using (3.33), that

$$\frac{2p_* - 1}{2p_*} \mathbb{E}[N_\epsilon] \leq \frac{1}{2p_* - 1} \left( \frac{2(f_0 - f_{\text{low}})(p+1)!}{(\eta - 2\omega)\sigma_{\min}\psi(\sigma_s)} \left( \min_{j \in \{1, \dots, q\}} \epsilon_j \right)^{-\varpi} + \left\lceil \log_\gamma \left( \frac{\sigma_s}{\sigma_0} \right) \right\rceil + 2 \right).$$

This bound and the inequality  $\frac{1}{2} < p_* \leq 1$  yield the desired result.  $\square$

Since the  $\text{IAR}_{qp}$  algorithm requires at most two function evaluations and one evaluation of the derivatives of orders one to  $p$  per iteration, the bounds stated in the above theorem effectively provide an upper bound on the average evaluation complexity of finding  $(\epsilon, \delta)$ -approximate  $q$ -th order minimizers.

Theorem 3.10 generalizes the complexity bounds stated in [1, Theorem 5.5] to the case where evaluations of  $f$  and its derivatives are inexact, under probabilistic assumptions on the accuracies of the latter. Remarkably, the bounds of Theorem 3.10 *are essentially identical in order of the tolerance  $\epsilon$  to those obtained in [1, Theorem 5.5]*, in that they only differ by the presence of an additional term in  $|\log(\min_{j \in \{1, \dots, q\}} \epsilon_j)|$ . Moreover, it was shown in [1, Theorems 6.1 and 6.4] that the evaluation complexity bounds are sharp in  $\epsilon$  for exact evaluations and Lipschitz continuous derivatives of  $f$ . Since the  $\text{IAR}_{qp}$  algorithm reduces to the algorithm proposed in that reference when all values are exact, we deduce that the lower bound on evaluation complexity presented in this reference is also valid in our case. Thus, considering that, for small  $\epsilon_j$ , the term  $|\log(\min_{j \in \{1, \dots, q\}} \epsilon_j)|$  is much smaller than the terms in  $\min_{j \in \{1, \dots, q\}} \epsilon_j^{-(p+1)/(p-q+1)}$  or  $\min_{j \in \{1, \dots, q\}} \epsilon_j^{-q(p+1)/p}$ , we conclude that *the presence of random noise in the derivatives and of inexactness in function values does not affect the evaluation complexity* of adaptive regularization algorithms for the local solution of problem (1.1). In addition, we also deduce that *the complexity bounds of Theorem 3.10 are essentially<sup>5</sup> sharp in order of  $\epsilon$ .*

It is interesting to compare our results with those of [1]. These authors mention an “elbow effect” for algorithms using randomly perturbed derivatives in that they state a lower bound on evaluation complexity for second-order approximate minimizers of  $\mathcal{O}(\epsilon_2^{-3})$  for all  $p \geq 2$ , in contrast with our smoothly decreasing  $\mathcal{O}(\min[\epsilon_1, \epsilon_2]^{-(p+1)/(p-1)})$  bound. However, their framework is very different. Firstly, they assume the a priori knowledge of the Lipschitz constants, which makes monitoring of the function values unnecessary in an adaptive regularization algorithm, an assumption we have explicitly avoided for consistency. Most importantly, their accuracy model is significantly more permissive than ours, as it allows<sup>6</sup> derivatives’ estimates of the form  $\nabla_x^j f(x) = (z/\mu)\nabla_x^j f(x)$  where  $z$  is a  $(0, 1)$  Bernoulli random variable of parameter  $\mu$ . Although unbiased and of bounded variance under AS.1, such estimates result, for nonzero  $\nabla_x^j f(x)$ , in an infinite relative error with probability  $1 - \mu$ . Since they consider values of  $\mu$  of the order of  $\epsilon_2^2$ , this is clearly too loose for AS.3 to hold. This illustrates that, unsurprisingly, the evaluation complexity bound for algorithms using

<sup>5</sup>Modulo the negligible logarithmic term.

<sup>6</sup>See definitions (76) and (87) in [1].

inexact information strongly depends on the specific (potentially probabilistic) accuracy model considered.

We conclude this section by noting that the complexity bounds we have derived depend on the smallest of the accuracy thresholds  $\epsilon_j$ . We could therefore derive the complete theory with a single  $\epsilon$  for all optimality orders, marginally improving notation. We have refrained from doing so because users of numerical optimization algorithms very rarely makes this choice in practice, but typically uses application- and order-dependent thresholds.

#### 4. Extension to convexly constrained problems

As it turns out, it is easy to extend the above results to the case where the problem is convexly constrained, that is when (1.1) is replaced by

$$\min_{x \in \mathcal{X}} f(x), \quad (4.1)$$

where  $\mathcal{X}$  is a convex subset of  $\mathbb{R}^n$ . We have refrained from considering this problem from the start for clarity of exposition, but we now review the (limited) changes that are necessary to cover this more general problem.

1. We may first weaken AS.1 to require that  $f$  is  $p$  times continuously differentiable in an open convex neighbourhood of  $\mathcal{X}$  and that the Lipschitz conditions (2.1) only hold in that neighbourhood.
2. We must then revise our approximate criticality measure (2.7) to reflect the constrained nature of (4.1). This is done by considering the Taylor decrement which is achievable only for displacements  $d$  which preserve feasibility. We may therefore replace  $\phi_{f,j}^{\delta_j}(x)$  in (2.7) by

$$\phi_{f,j}^{\delta_j}(x) = \max_{x+d \in \mathcal{X}, \|d\| \leq \delta_j} \Delta T_{f,j}(x, d)$$

for  $x \in \mathcal{X}$ . This new definition is then used in (3.1) to obtain a new family of stopping times.

3. We next insist that feasibility is maintained throughout the execution of the algorithm, in that we require that  $x_0 \in \mathcal{X}$  and that  $s_k$  is computed such that the trial point  $x_k + s_k$  is also feasible. Moreover, our criterion for terminating the step search must also reflect its constrained nature, which is obtained by replacing  $\overline{\phi}_{m_k,j}^{\delta_{k,j}}(s_k)$  in (2.18) by

$$\overline{\phi}_{m_k,j}^{\delta_{k,j}}(s_k) = \max_{x_k + s_k + d \in \mathcal{X}, \|d\| \leq \delta_{k,j}} \overline{\Delta T}_{m_k,j}(s_k, d)$$

for  $x_k + s_k \in \mathcal{X}$ .

4. The theory is then unchanged for this new context, with one caveat. We note that the proof of Lemma 3.2 for the case where  $q \in \{1, 2\}$  does depend on the fact that  $T_{m_k,j}(s_k^*, d)$  is a convex function of  $d$  for  $j \in \{1, 2\}$  because of the unconstrained optimality conditions. Obviously, while maintaining convexity is possible in the convexly

constrained case when  $q = 1$ , it may now fail for  $q = 2$ . As a consequence, this case must be considered in the same way as for other larger values of  $q$ . This then imposes that we have to change the condition “if  $q \in \{1, 2\}$ ” or “if  $j \in \{1, 2\}$ ” to “if  $q = 1$ ” or “if  $j = 1$ ”, respectively, in (2.24), the first statement of Lemma 3.2, (3.11) and its proof, and in the first statement of Theorem 3.10.

It is remarkable that no further change is necessary for deducing Theorem 3.10 for problem (4.1). This extension to the convexly convex case is also a novel feature for algorithms considering randomly perturbed derivatives.

## 5. Conclusions and perspectives

We have shown that the  $\text{IAR}_{qp}$  algorithm, a stochastic inexact adaptive regularization algorithm using derivatives of order up to  $p$ , computes an  $(\epsilon, \delta)$ -approximate  $q$ -th order minimizer of  $f$  in problem (1.1) in at most  $O(\epsilon^{-\frac{p+1}{p-q+1}})$  iterations in expectation if  $q$  is either one or two, while it may need  $O(\epsilon^{-\frac{q(p+1)}{p}})$  iterations in expectation in the other cases<sup>7</sup>. Moreover, these bounds are essentially sharp in the order of  $\epsilon$  (see ). We therefore conclude that, if the probabilities  $p_{\mathcal{M},k}$  in AS.3 are suitably large, the evaluation complexity of the  $\text{IAR}_{qp}$  algorithm is identical (in order) to that of the exact algorithm in . We finally provided an extension of these results to the convexly constrained case.

We also note that the full power of AS.1 is only required for Lemma 3.2, while Lipschitz continuity of  $\nabla_x^p f(x)$  is sufficient for all subsequent derivations. Thus if suitable lower bounds on  $\Delta_{k,j}$  can be ensured in some other way, our development remains valid (although the precise complexity bounds will depend on the new bounds on  $\Delta_{k,j}$ ). In AS.1, we have also required (Lipschitz) continuity of  $f$  and its derivatives in  $\mathbb{R}^n$ . This can be weakened to requiring this property only on the “tree of iterates”  $\cup_{k \geq 0} [x_k, x_k + s_k]$ , but this weaker assumption is often impossible to verify *a priori*. In the same vein, it also is possible to avoid requiring that (3.3) is always ensured by the  $\text{IAR}_{qp}$  algorithm whenever  $q > 2$  by instead redefining  $\mathcal{M}_k$  to also include the satisfaction of this condition. We have preferred using an explicit assumption because this approach better differentiates deterministic requirements on the algorithm from stochastic assumptions more related to the problem itself.

We finally recall that also derives complexity bounds for the (possibly non-smooth) composite optimization problem. We expect that the theory presented here can be extended to also cover this case.

An analysis covering adaptive regularization algorithms where the objective function evaluations are also subject to general random noise, parallel to that provided for trust-region methods for low order minimizers in , remains, for now, an open and challenging question.

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<sup>7</sup>These simplified order bounds assume that  $\epsilon_j = \epsilon$  for  $j \in \{1, \dots, q\}$ .

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## Appendix

### Proof of Lemma 2.2

Let  $s_k^*$  be a global minimizer of  $m_k(s)$ . By Taylor's theorem, we have that, for all  $d$ ,

$$\begin{aligned}
0 \leq m_k(s_k^* + d) - m_k(s_k^*) &= \sum_{\ell=1}^p \frac{1}{\ell!} \nabla_s^\ell \bar{T}_{f,p}(x_k, s_k^*) [d]^\ell \\
&+ \frac{\sigma_k}{(p+1)!} \left[ \sum_{\ell=1}^p \frac{1}{\ell!} \nabla_s^\ell (\|s_k^*\|^{p+1}) [d]^\ell + \frac{1}{(p+1)!} \nabla_s^{p+1} (\|s_k^* + \tau d\|^{p+1}) [d]^{p+1} \right]
\end{aligned} \tag{A.1}$$

for some  $\tau \in (0, 1)$ . We may now use the expression of  $\nabla_s^\ell (\|s_k^*\|^{p+1})$  given by [1, Lemma 2.4] in (A.1) and deduce that, for any  $j \in \{1, \dots, p\}$  and all  $d$ ,

$$\begin{aligned}
& - \sum_{\ell=1}^j \frac{1}{\ell!} \nabla_s^\ell \bar{T}_{f,p}(x_k, s_k^*) [d]^\ell - \frac{\sigma_k}{(p+1)!} \sum_{\ell=1}^j \nabla_s^\ell \|s_k^*\|^{p+1} [d]^\ell \\
& \leq \sum_{\ell=j+1}^p \frac{1}{\ell!} \nabla_s^\ell \bar{T}_{f,p}(x_k, s_k^*) [d]^\ell + \frac{\sigma_k}{(p+1)!} \left[ \sum_{\ell=j+1}^p \frac{1}{\ell!} \nabla_s^\ell \|s_k^*\|^{p+1} [d]^\ell + \|d\|^{p+1} \right].
\end{aligned} \tag{A.2}$$

It is now possible to choose  $\delta_{k,j} \in (0, 1]$  such that, for every  $d$  with  $\|d\| \leq \delta_{k,j}$ ,

$$\begin{aligned}
& \sum_{\ell=j+1}^p \frac{1}{\ell!} \nabla_s^\ell \bar{T}_{f,p}(x_k, s_k^*) [d]^\ell + \frac{\sigma_k}{(p+1)!} \left[ \sum_{\ell=j+1}^p \frac{1}{\ell!} \nabla_s^\ell \|s_k^*\|^{p+1} [d]^\ell + \|d\|^{p+1} \right] \\
& \leq \frac{1}{2} \theta \epsilon_j \frac{\delta_{k,j}^j}{j!}.
\end{aligned} \tag{A.3}$$

We therefore obtain that if  $\delta_{k,j}$  is small enough to ensure (A.3), then (A.2) implies that

$$- \sum_{\ell=1}^j \frac{1}{\ell!} \nabla_s^\ell \bar{T}_{f,p}(x_k, s_k^*) [d]^\ell - \frac{\sigma_k}{(p+1)!} \sum_{\ell=1}^j \nabla_s^\ell \|s_k^*\|^{p+1} [d]^\ell \leq \frac{1}{2} \theta \epsilon_j \frac{\delta_{k,j}^j}{j!}. \tag{A.4}$$



and therefore that, for all  $j \in \{1, \dots, q\}$ ,

$$\max_{\|d\| \leq \delta_{k,j}} \overline{\Delta T}_{m_k,j}(s_k^*, d) \leq \frac{1}{2} \theta \epsilon_j \frac{\delta_{k,j}^j}{j!}.$$

Thus the pair  $(s_k^*, \delta_k)$  is acceptable for Step 2 of the algorithm. If we assume now that  $x_k + s_k^*$  is not an isolated feasible point, the above inequality and continuity of  $\overline{T}_{f,p}(x_k, s)$  and its derivatives with respect to  $s$  then ensure the existence of a feasible neighbourhood  $\mathcal{N}_k^*$  of  $s_k^*$  in which

$$\max_{\|d\| \leq \delta_{k,j}} \overline{\Delta T}_{m_k,j}(s, d) \leq \theta \epsilon_j \frac{\delta_{k,j}^j}{j!}. \quad (\text{A.5})$$

for all  $s \in \mathcal{N}_k^*$ . We may then choose any  $s_k$  in  $\mathcal{N}_k^*$  such that, in addition to satisfying (??) and being such that  $x_k + s_k$  is feasible, (2.17) also holds. Thus the definition of  $\overline{\phi}_{m_k,j}^{\delta_{k,j}}(s_k)$  in (2.18) gives that

$$\overline{\phi}_{m_k,j}^{\delta_{k,j}}(s_k) \leq \theta \epsilon_j \frac{\delta_{k,j}^j}{j!} \quad (\text{A.6})$$

and every such  $(s_k, \delta_k)$  is also acceptable for Step 2 of the algorithm.

### Proof of Lemma 3.2

Let  $s_k^*$  be a global minimizer of  $m_k(s)$ . We first consider the case where  $q \in \{1, 2\}$ . Then it is easy to verify that, for each  $j \in \{1, \dots, q\}$ , the optimization problem involved in the definition of  $\overline{\phi}_{m_k,j}^{\delta_{k,j}}(s_k^*)$  (in (2.18)) is convex and therefore that  $\delta_{k,j}$  can be chosen arbitrarily in  $(0, 1]$ . The first case of Lemma 3.2 then follows from the continuity of  $\overline{\phi}_{m_k,j}^{\delta_{k,j}}(s)$  with respect to  $s$ . Unfortunately, the crucial convexity property is lost for  $q > 2$

Unfortunately, the crucial convexity property is lost for  $q > 2$  and, in order to prove the second case, we now pursue the reasoning of the proof of Lemma 2.2. We start by supposing that  $\|s_k^*\| > 1$ . We may then reduce the neighbourhood of  $s_k^*$  in which  $s_k$  can be chosen enough to guarantee that  $\|s_k\| \geq 1$ , which then gives the desired result because of (??). Suppose therefore that  $\|s_k^*\| \leq 1$ . The triangle inequality then implies that

$$\|\nabla_s^\ell \overline{T}_{f,p}(x_k, s_k^*)\| \leq \sum_{i=\ell}^p \frac{1}{(i-\ell)!} \|\nabla_x^i f(x_k)\| \|s_k^*\|^{i-\ell},$$

for  $\ell \in \{q+1, \dots, p\}$ , and thus, using, AS.1 and [, Lemma 2.4], we deduce that

$$\begin{aligned} & \sum_{\ell=j+1}^p \frac{1}{\ell!} \nabla_s^\ell \overline{T}_{f,p}(x_k, s_k^*)[d]^\ell + \frac{\sigma_k}{(p+1)!} \left[ \sum_{\ell=j+1}^p \nabla_s^\ell \|s_k^*\|^{p+1} [d]^\ell \right] \\ & \leq \sum_{\ell=j+1}^p \frac{\|d\|^\ell}{\ell!} \left[ \sum_{i=\ell}^p \frac{\|s_k^*\|^{i-\ell}}{(i-\ell)!} \|\nabla_x^i f(x_k)\| + \frac{\sigma_k \|s_k^*\|^{p-\ell+1}}{(p-\ell+1)!} \right]. \end{aligned}$$

We now call upon the fact that, since  $q \geq 3$  and  $\mathcal{M}_k$  occurs by assumption,  $\mathcal{M}_k^{(4)}$  also occurs. Thus

$$\begin{aligned} & \sum_{\ell=j+1}^p \frac{1}{\ell!} \nabla_s^\ell \overline{T}_{f,p}(x_k, s_k^*)[d]^\ell + \frac{\sigma_k}{(p+1)!} \left[ \sum_{\ell=j+1}^p \nabla_s^\ell \|s_k^*\|^{p+1} [d]^\ell \right] \\ & \leq \sum_{\ell=j+1}^p \frac{\|d\|^\ell}{\ell!} \left[ \Theta \sum_{i=\ell}^p \frac{\|s_k^*\|^{i-\ell}}{(i-\ell)!} + \frac{\sigma_k \|s_k^*\|^{p-\ell+1}}{(p-\ell+1)!} \right]. \end{aligned}$$

We therefore obtain from (??) that any pair  $(s_k^*, \delta_{s,j})$  satisfies (??) for  $\|d\| \leq \delta_{s,j}$  if

$$\sum_{\ell=j+1}^p \frac{\delta_{s,j}^\ell}{\ell!} \left[ \Theta \sum_{i=\ell}^p \frac{1}{(i-\ell)!} \|s_k^*\|^{i-\ell} + \frac{\sigma_k \|s_k^*\|^{p-\ell+1}}{(p-\ell+1)!} \right] + \sigma_k \frac{\delta_{s,j}^{p+1}}{(p+1)!} \leq \frac{1}{2} \theta \epsilon_j \frac{\delta_{s,j}^j}{j!}. \quad (\text{A.7})$$

which, because  $\|s_k^*\| \leq 1$ , is in turn ensured by the inequality

$$\sum_{\ell=j+1}^p \frac{\delta_{s,j}^\ell}{\ell!} \left[ \Theta \sum_{i=\ell}^p \frac{1}{(i-\ell)!} + \sigma_k \right] + \sigma_k \frac{\delta_{s,j}^{p+1}}{(p+1)!} \leq \frac{1}{2} \theta \epsilon_j \frac{\delta_{s,j}^j}{j!}. \quad (\text{A.8})$$

Observe now that, since  $\delta_{s,j} \in [0, 1]$ ,  $\delta_{s,j}^\ell \leq \delta_{s,j}^{j+1}$  for  $\ell \in \{j+1, \dots, p\}$ . Moreover, we have that,

$$\sum_{i=\ell}^p \frac{1}{(i-\ell)!} \leq e < 3, \quad (\ell \in \{j+1, \dots, p+1\}), \quad \sum_{\ell=j+1}^{p+1} \frac{1}{\ell!} \leq e - 1 < 2$$

and therefore (??) is guaranteed by the condition

$$j!(6\Theta + 2\sigma_k) \delta_{s,j} \leq \frac{1}{2} \theta \epsilon_j, \quad (\text{A.9})$$

which means that the pair  $(s_k^*, \delta_s)$  satisfies (??) for all  $j \in \{1, \dots, q\}$  whenever,

$$\delta_{s,j} \leq \frac{1}{2} \delta_{\min,k} \stackrel{\text{def}}{=} \frac{\theta \epsilon_j}{2q!(6\Theta + 2\sigma_k)}.$$

As in the proof of Lemma 2.2, we may invoke continuity of the derivatives of  $m_k(s)$  with respect to  $s$  to deduce that there exists a neighbourhood  $\mathcal{N}_k^*$  of  $s_k^*$  such that (??) holds for every  $s \in \mathcal{N}_k^*$  and every  $\delta_{k,j} \leq \delta_{\min,k}$ . Choosing now  $s_k$  to ensure (2.17) in addition to (??), we obtain that the pair  $(s_k, \delta_{k,j})$  satisfies both (2.17) and

$$\overline{\phi}_{m_{k,j}}^{\delta_{k,j}}(s_k) \leq \theta \epsilon_j \frac{\delta_{k,j}^j}{j!}.$$

The desired conclusion then follows with

$$\kappa_\delta(\sigma) = \frac{\nu \theta}{q!(6\Theta + 2\sigma)}$$

for any constant  $\nu \in (0, 1)$ . Moreover,  $\kappa_\delta(\sigma)$  is clearly a decreasing function of  $\sigma$ .