May the 24th, 2021

HAPPENING VIRTUALLY: SIAM Conference on Applications of Dynamical Systems (DS21)
May 23 - 27, 2021
Virtual Conference | Originally scheduled in Portland, Oregon, U.S.

Timoteo Carletti

Emergent Properties in Dynamical Systems on Hypergraphs

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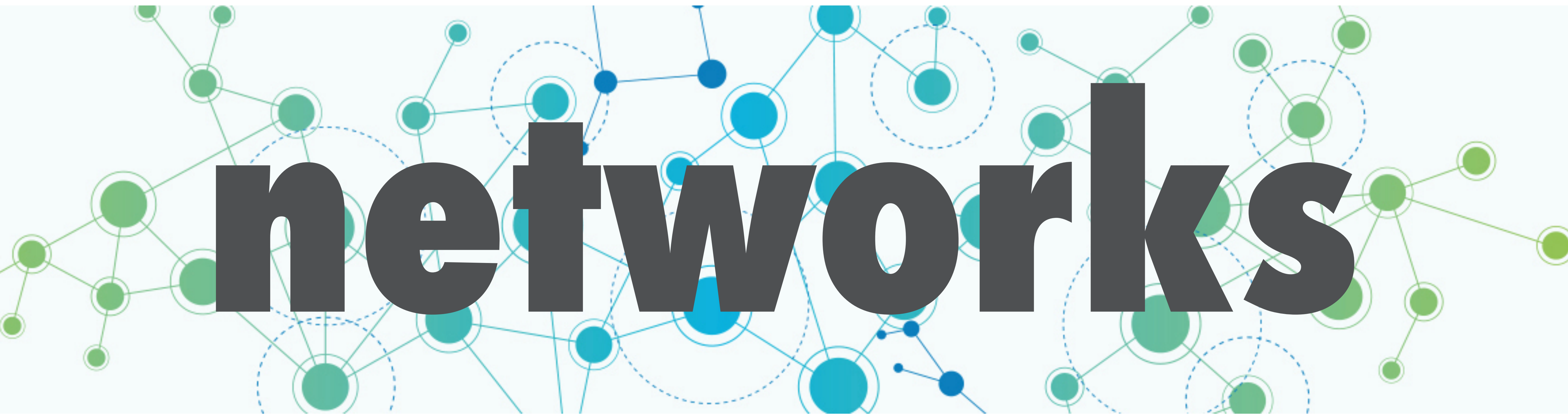

networks

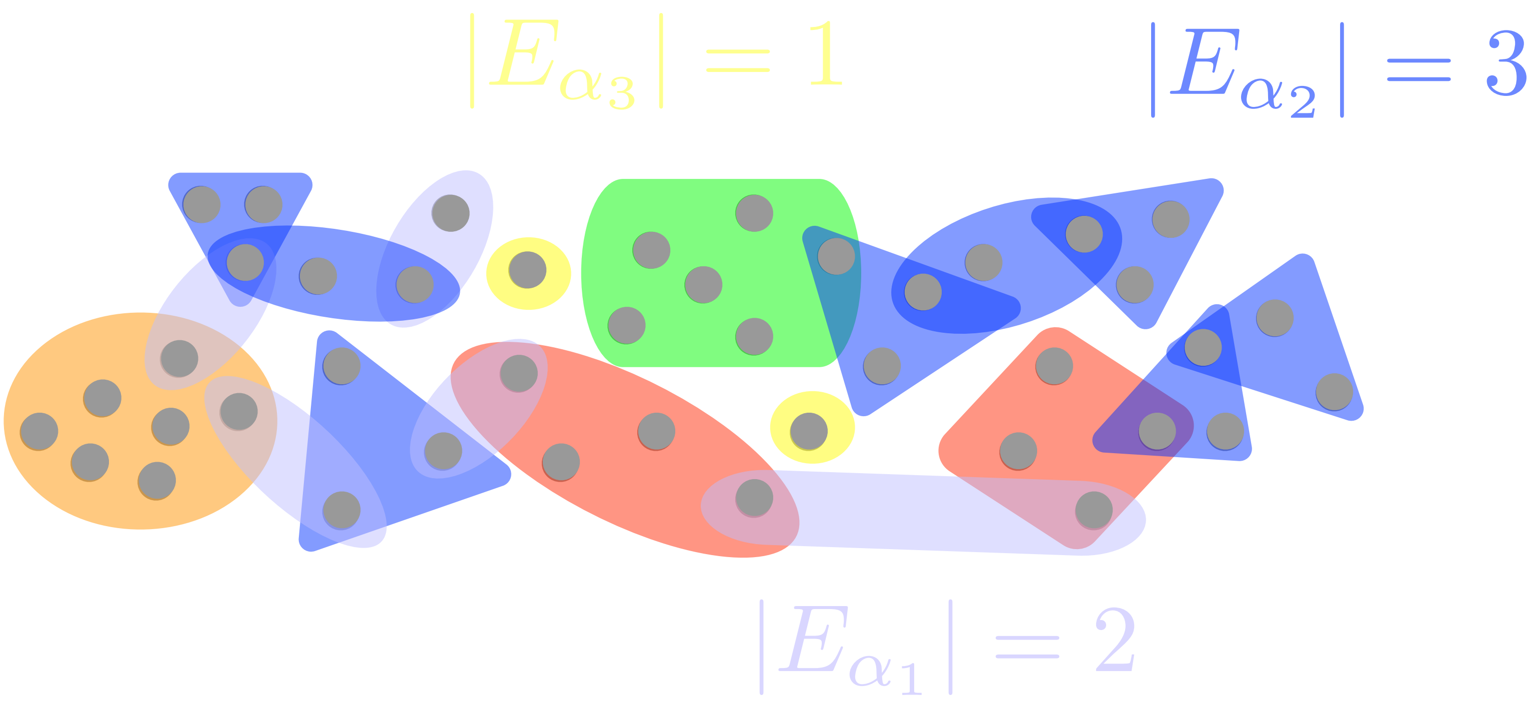
Dynamics

Structure

limitation

Hypergraphs

Hypergraphs. Some definitions.



ensemble of nodes

hyperedges

Incidence matrix

$$e_{i\alpha} = 1 \quad \text{iff } i \in E_\alpha$$

Hyperadjacency matrix

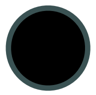
$$A = ee^{\mathrm{T}}$$

Hyperedge matrix

$$C = e^{\mathrm{T}}e$$

Dynamical systems on hypergraphs

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}) \quad \mathbf{x} \in \mathbb{R}^d$$

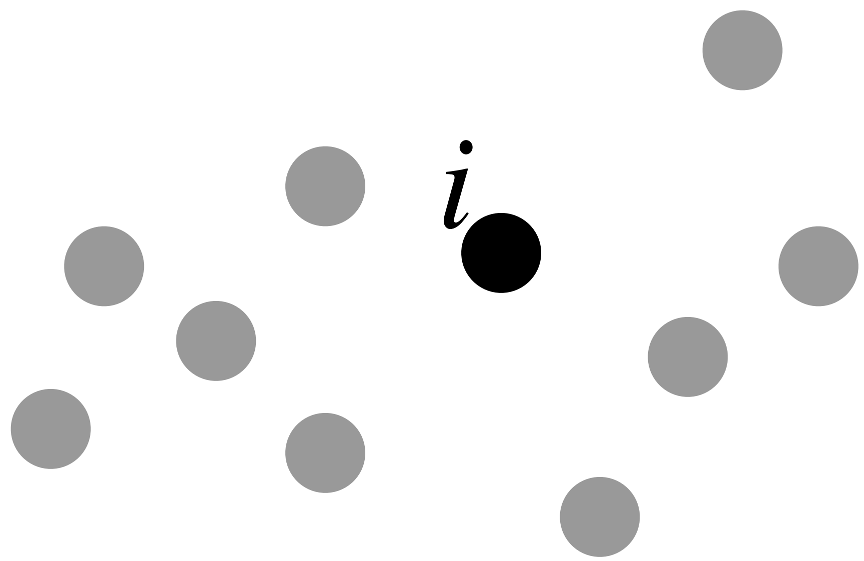


Dynamical systems on hypergraphs

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}) \quad \mathbf{x} \in \mathbb{R}^d$$

$$\frac{d\mathbf{x}^{(i)}}{dt} = \mathbf{f}(\mathbf{x}^{(i)}) \quad \mathbf{x}^{(i)} \in \mathbb{R}^d$$

$$i = 1, \dots, n$$

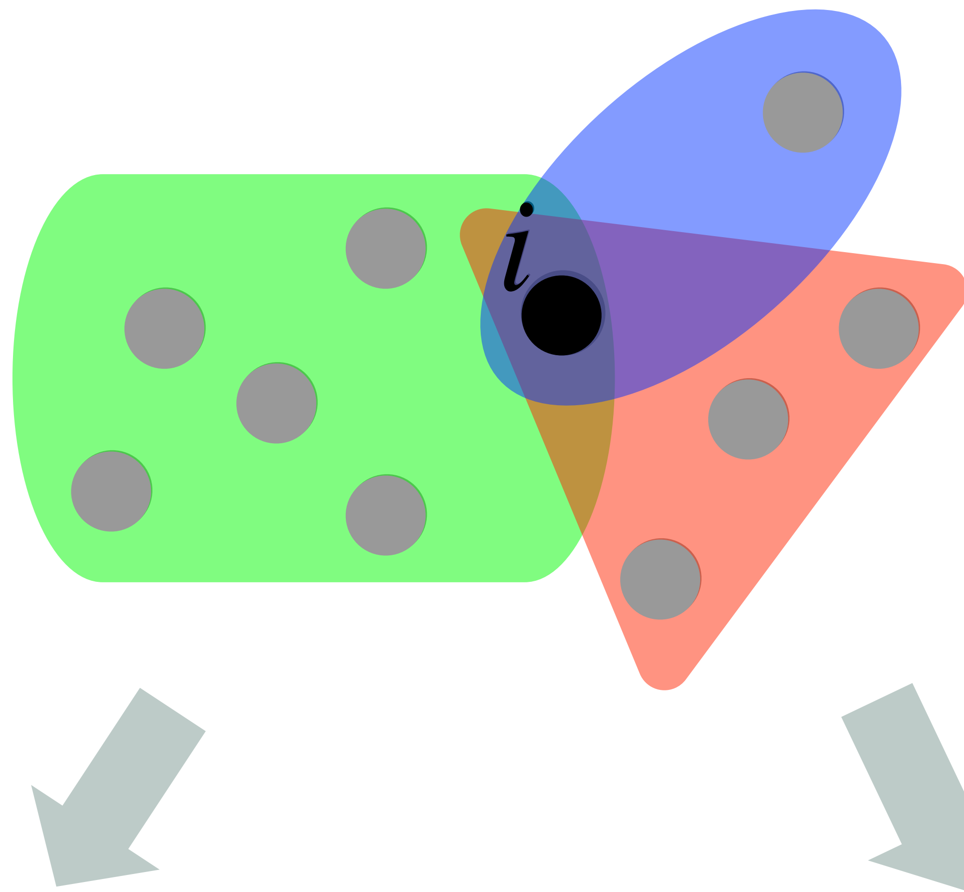


Dynamical systems on hypergraphs

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}) \quad \mathbf{x} \in \mathbb{R}^d$$

$$\frac{d\mathbf{x}^{(i)}}{dt} = \mathbf{f}(\mathbf{x}^{(i)}) \quad \mathbf{x}^{(i)} \in \mathbb{R}^d$$

$$i = 1, \dots, n$$



Mean field hyperedge coupling

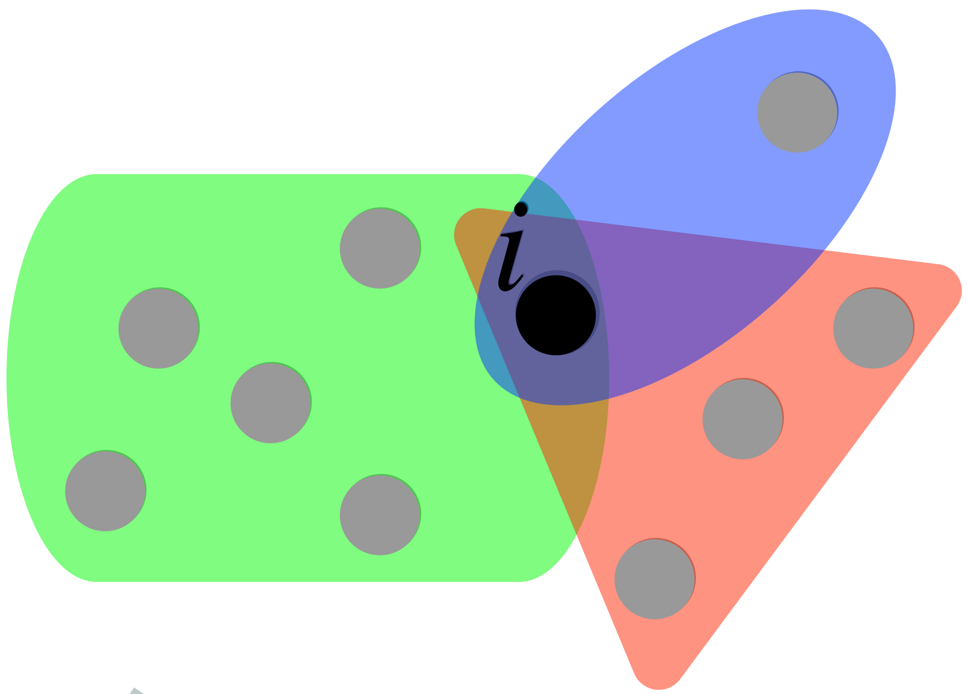
Diffusive-like coupling

Dynamical systems on hypergraphs

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}) \quad \mathbf{x} \in \mathbb{R}^d$$

$$\frac{d\mathbf{x}^{(i)}}{dt} = \mathbf{f}(\mathbf{x}^{(i)}) \quad \mathbf{x}^{(i)} \in \mathbb{R}^d$$

$$i = 1, \ldots, n$$



Mean field hyperedge coupling

Mean field hyperedge coupling

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}) \quad \mathbf{x} \in \mathbb{R}^d$$

1

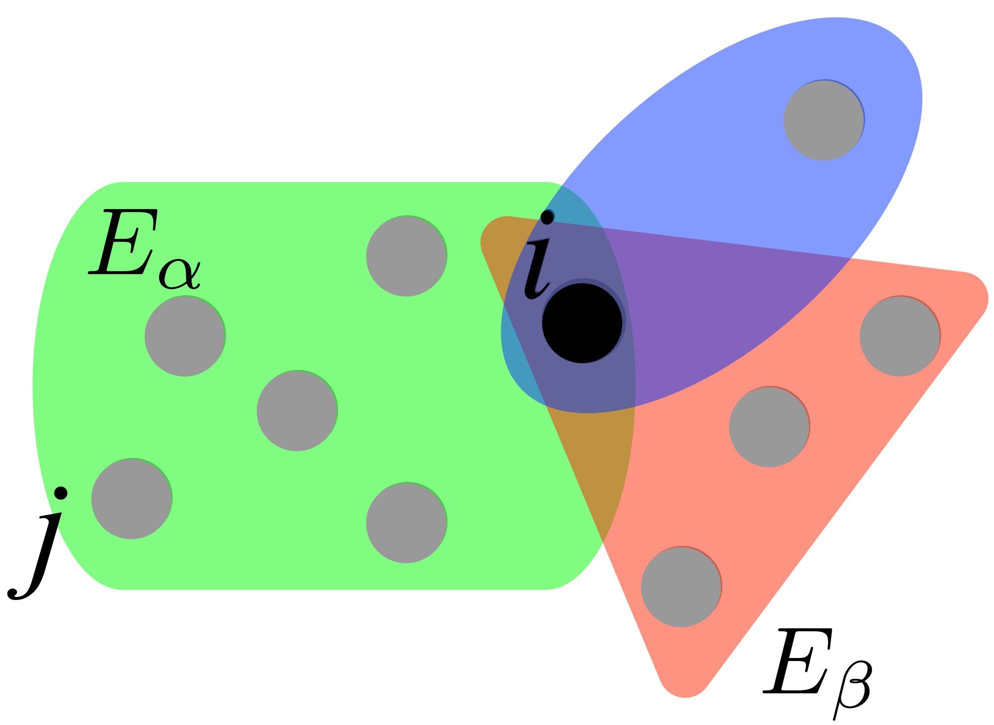
$$\frac{d\mathbf{x}^{(i)}}{dt} = \mathbf{f}(\mathbf{x}^{(i)}) \quad \mathbf{x}^{(i)} \in \mathbb{R}^d$$

$$i = 1, \dots, n$$

2

$$\frac{d\mathbf{x}^{(i)}}{dt} = \frac{\sum_\alpha e_{i\alpha} \sum_j e_{j\alpha} \varphi(C_{\alpha\alpha}) \mathbf{F}(\mathbf{x}^{(i)}, \mathbf{x}^{(j)})}{\sum_\alpha e_{i\alpha} \sum_j e_{j\alpha} \varphi(C_{\alpha\alpha})} \quad \forall i = 1, \dots, n\,,$$

$$\mathbf{F}(\mathbf{x}^{(i)}, \mathbf{x}^{(i)}) = \mathbf{f}(\mathbf{x}^{(i)})$$



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Mean field hyperedge coupling

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}) \quad \mathbf{x} \in \mathbb{R}^d$$

1

$$\frac{d\mathbf{x}^{(i)}}{dt} = \mathbf{f}(\mathbf{x}^{(i)}) \quad \mathbf{x}^{(i)} \in \mathbb{R}^d \quad i = 1, \dots, n$$

2

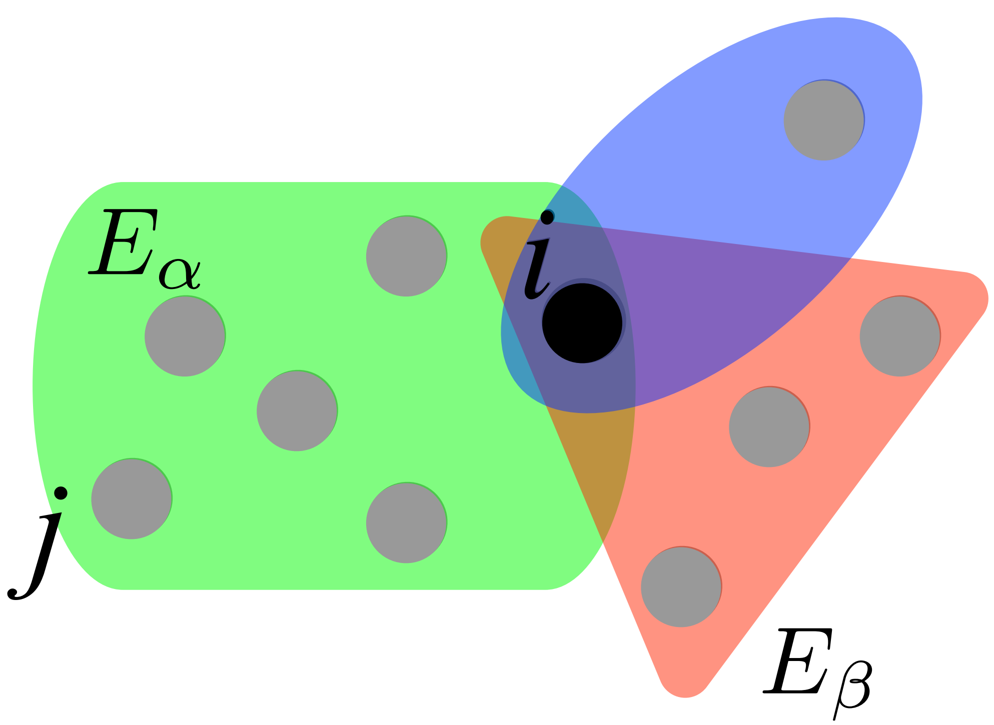
$$\frac{d\mathbf{x}^{(i)}}{dt} = \frac{\sum_\alpha e_{i\alpha} \sum_j e_{j\alpha} \varphi(C_{\alpha\alpha}) \mathbf{F}(\mathbf{x}^{(i)}, \mathbf{x}^{(j)})}{\sum_\alpha e_{i\alpha} \sum_j e_{j\alpha} \varphi(C_{\alpha\alpha})} \quad \forall i = 1, \dots, n \,,$$

$$\mathbf{F}(\mathbf{x}^{(i)}, \mathbf{x}^{(i)}) = \mathbf{f}(\mathbf{x}^{(i)})$$

3

$$D_{ij} = \sum_\alpha e_{i\alpha} \varphi(C_{\alpha\alpha}) e_{j\alpha}$$

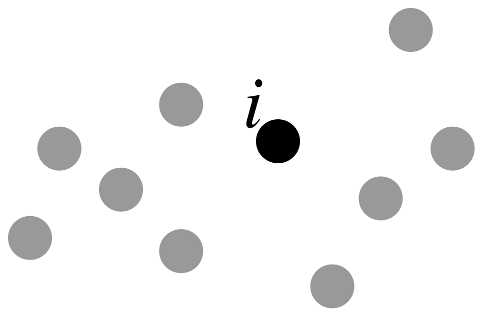
$$d_i = \sum_j D_{ij}$$



$$\frac{d\mathbf{x}^{(i)}}{dt} = \frac{1}{d_i} \sum_j D_{ij} \mathbf{F}(\mathbf{x}^{(i)}, \mathbf{x}^{(j)})$$

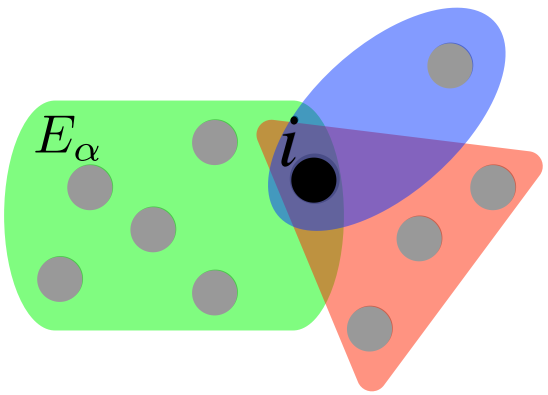
Question

Assume $\mathbf{s}(t)$ to be a stable solution of the isolated system



$$\frac{d\mathbf{x}^{(i)}}{dt} = \mathbf{f}(\mathbf{x}^{(i)}) \quad \mathbf{x}^{(i)} \in \mathbb{R}^d$$

is it possible to destabilise it using the multi-body interaction ?



$$\frac{d\mathbf{x}^{(i)}}{dt} = \frac{1}{d_i} \sum_j D_{ij} \mathbf{F}(\mathbf{x}^{(i)}, \mathbf{x}^{(j)})$$

High-order Volterra model

$$\begin{cases} \dot{x} = -dx + c_1 xy \\ \dot{y} = ry - sy^2 - c_2 xy \, , \end{cases}$$

x = density of predators

y = density of preys

$$x^* = \frac{c_1 r - sd}{c_1 c_2} \qquad y^* = \frac{d}{c_1}$$

$$c_1 r - sd > 0$$

Stable equilibrium

High-order Volterra model : sketch of the method

$$\begin{cases} \dot{x}_i = -dx_i + ac_1 y_i \frac{1}{d_i}\sum_j D_{ij}x_j + (1-a)c_1 x_i \frac{1}{d_i}\sum_j D_{ij}y_j \\ \dot{y}_i = ry_i - sy_i \frac{1}{d_i}\sum_j D_{ij}y_j - c_2 y_i \frac{1}{d_i}\sum_j D_{ij}x_j, \end{cases}$$

High-order Volterra model : sketch of the method

$$\begin{cases} \dot{x}_i = -dx_i + ac_1 y_i \frac{1}{d_i}\sum_j D_{ij}x_j + (1-a)c_1 x_i \frac{1}{d_i}\sum_j D_{ij}y_j \\ \dot{y}_i = ry_i - sy_i \frac{1}{d_i}\sum_j D_{ij}y_j - c_2 y_i \frac{1}{d_i}\sum_j D_{ij}x_j \,, \end{cases}$$

$$c_1 y_i \frac{1}{d_i}\sum_j D_{ij}x_j - c_1 y_i x_i + c_1 y_i x_i$$

$$= c_1 y_i \sum_j \left(\frac{D_{ij}}{d_i} - \delta_{ij}\right) x_j + c_1 y_i x_i$$

$$= c_1 y_i \sum_j \mathcal{L}_{ij} x_j + c_1 y_i x_i$$

(consensus) high-order Laplace matrix

$$\mathcal{L}_{ij} = \frac{D_{ij}}{d_i} - \delta_{ij} \qquad -2 \le \Lambda^{(\alpha)} \le 0$$

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High-order Volterra model : sketch of the method

$$\begin{cases} \dot{x}_i = -dx_i + ac_1 y_i \frac{1}{d_i} \sum_j D_{ij} x_j + (1-a) c_1 x_i \frac{1}{d_i} \sum_j D_{ij} y_j \\ \dot{y}_i = r y_i - s y_i \frac{1}{d_i} \sum_j D_{ij} y_j - c_2 y_i \frac{1}{d_i} \sum_j D_{ij} x_j \,, \end{cases}$$

1

$$\begin{cases} \dot{x}_i = -dx_i + c_1 y_i x_i + ac_1 y_i \sum_j \mathcal{L}_{ij} x_j + (1-a) c_1 x_i \sum_j \mathcal{L}_{ij} y_j \\ \dot{y}_i = r y_i - s y_i^2 - c_2 y_i x_i - s y_i \sum_j \mathcal{L}_{ij} y_j - c_2 y_i \sum_j \mathcal{L}_{ij} x_j \,, \end{cases}$$

High-order Volterra model : sketch of the method

$$\begin{cases} \dot{x}_i = -dx_i + ac_1 y_i \frac{1}{d_i}\sum_j D_{ij}x_j + (1-a)c_1 x_i \frac{1}{d_i}\sum_j D_{ij}y_j \\ \dot{y}_i = ry_i - sy_i \frac{1}{d_i}\sum_j D_{ij}y_j - c_2 y_i \frac{1}{d_i}\sum_j D_{ij}x_j \,, \end{cases}$$

1

$$\begin{cases} \dot{x}_i = -dx_i + c_1 y_i x_i + ac_1 y_i \sum_j \mathcal{L}_{ij}x_j + (1-a)c_1 x_i \sum_j \mathcal{L}_{ij}y_j \\ \dot{y}_i = ry_i - sy_i^2 - c_2 y_i x_i - sy_i \sum_j \mathcal{L}_{ij}y_j - c_2 y_i \sum_j \mathcal{L}_{ij}x_j \,, \end{cases}$$

2

Linearise about (x^*, y^*) and project on the eigenmodes $\phi^{(\alpha)}$

3

$$u_i = \sum_\alpha u^\alpha \phi_i^{(\alpha)} \qquad v_i = \sum_\alpha v^\alpha \phi_i^{(\alpha)}$$

$$\frac{d}{dt}\begin{pmatrix} u^\alpha \\ v^\alpha \end{pmatrix} = \left[\begin{pmatrix} 0 & c_1 x^* \\ -c_2 y^* & -sy^* \end{pmatrix} + \Lambda^{(\alpha)} \begin{pmatrix} ac_1 y^* & (1-a)c_1 x^* \\ -c_2 y^* & -sy^* \end{pmatrix}\right] \begin{pmatrix} u^\alpha \\ v^\alpha \end{pmatrix}$$

High-order Volterra model : sketch of the method

$$\frac{d}{dt}\begin{pmatrix} u^\alpha \\ v^\alpha \end{pmatrix} = \left[\begin{pmatrix} 0 & c_1 x^* \\ -c_2 y^* & -s y^* \end{pmatrix} + \Lambda^{(\alpha)} \begin{pmatrix} a c_1 y^* & (1-a) c_1 x^* \\ -c_2 y^* & -s y^* \end{pmatrix}\right] \begin{pmatrix} u^\alpha \\ v^\alpha \end{pmatrix}$$

$\lambda = \lambda(\Lambda^{(\alpha)})$ Eigenvalue with the largest real part (dispersion relation / largest Lyapunov exponent)

(x^*, y^*) is unstable if there exists $\Lambda^{(\alpha)}$ such that

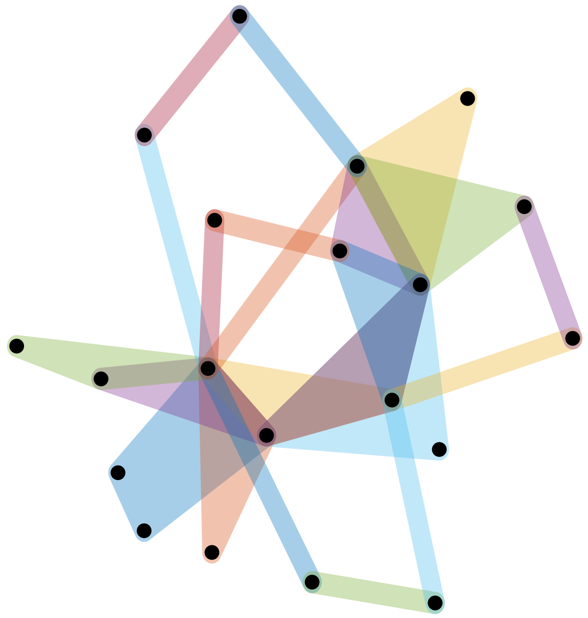
$$\lambda = \lambda(\Lambda^{(\alpha)}) > 0$$

(x^*, y^*) is stable if for all $\Lambda^{(\alpha)}$

$$\lambda = \lambda(\Lambda^{(\alpha)}) < 0$$

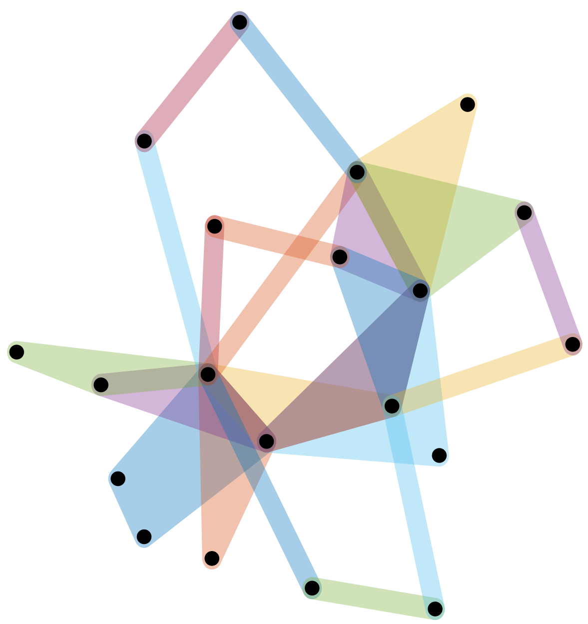
High-order Volterra model : avoid predators extinction

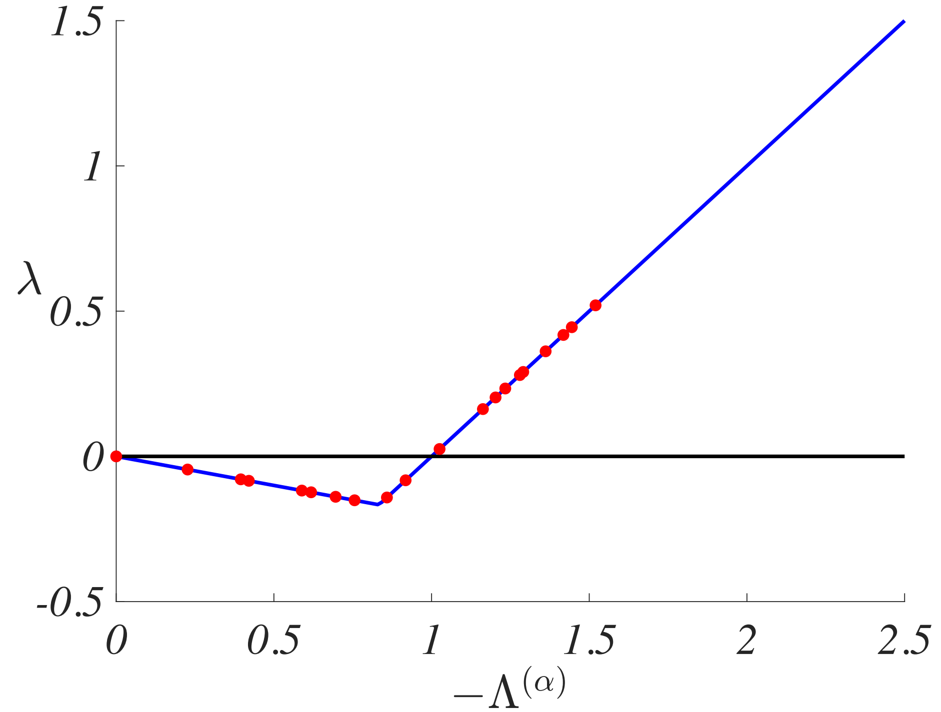
$$x^* = 0 \qquad y^* = 1$$

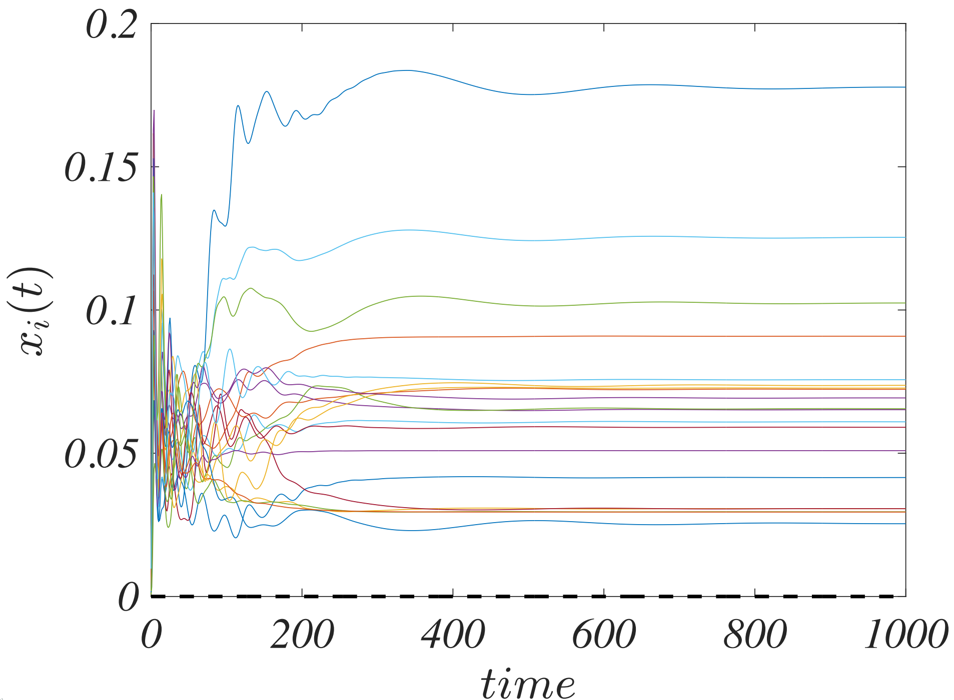


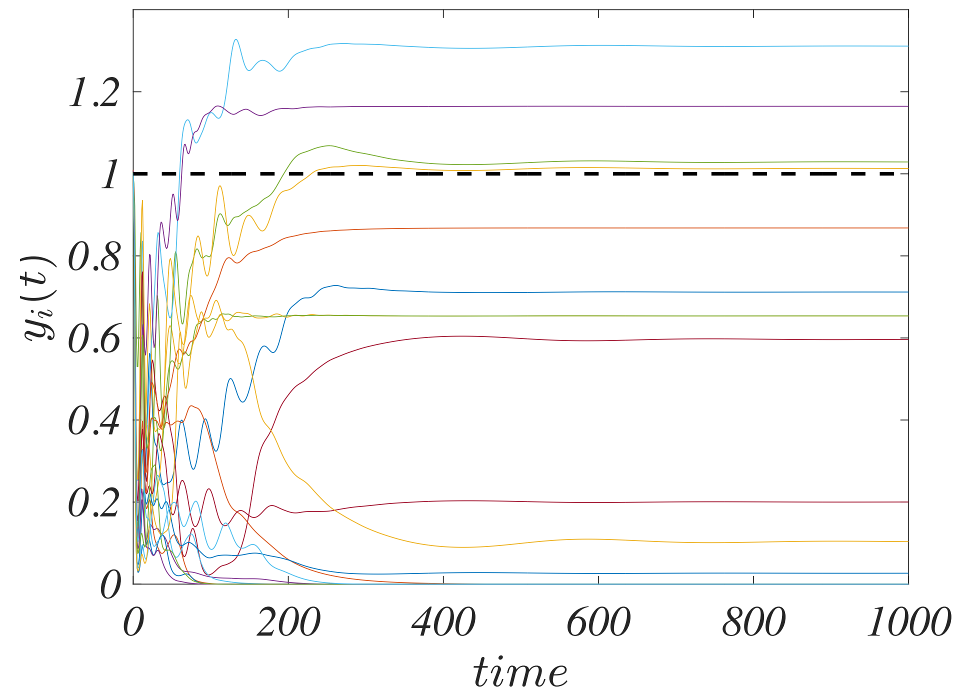
High-order Volterra model : avoid predators extinction

$$x^* = 0 \qquad y^* = 1$$







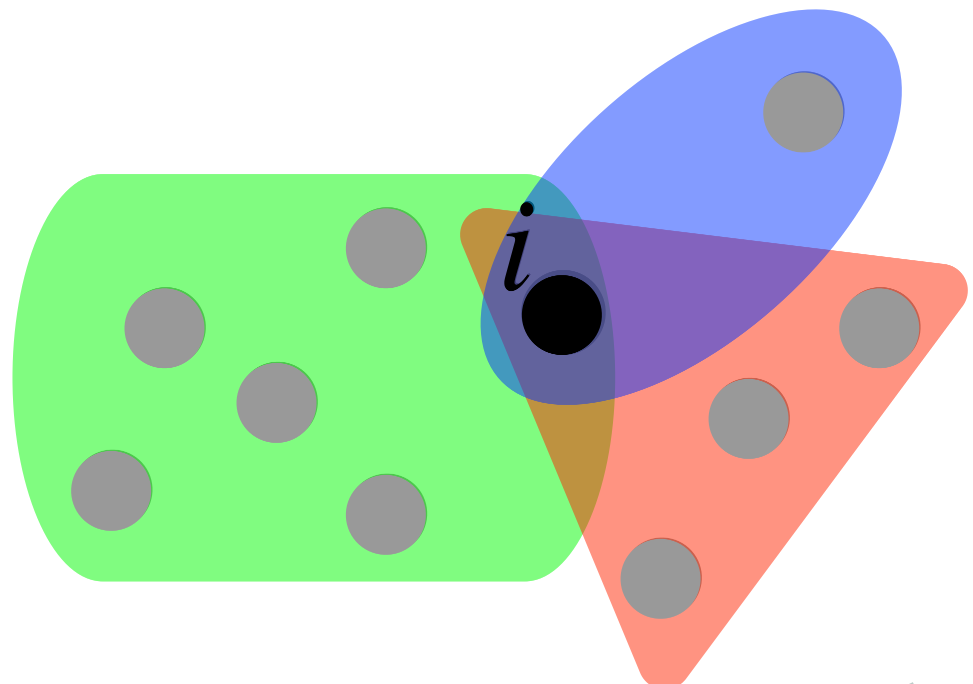


Dynamical systems on hypergraphs

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}) \quad \mathbf{x} \in \mathbb{R}^d$$

$$\frac{d\mathbf{x}^{(i)}}{dt} = \mathbf{f}(\mathbf{x}^{(i)}) \quad \mathbf{x}^{(i)} \in \mathbb{R}^d$$

$$i = 1, \dots, n$$



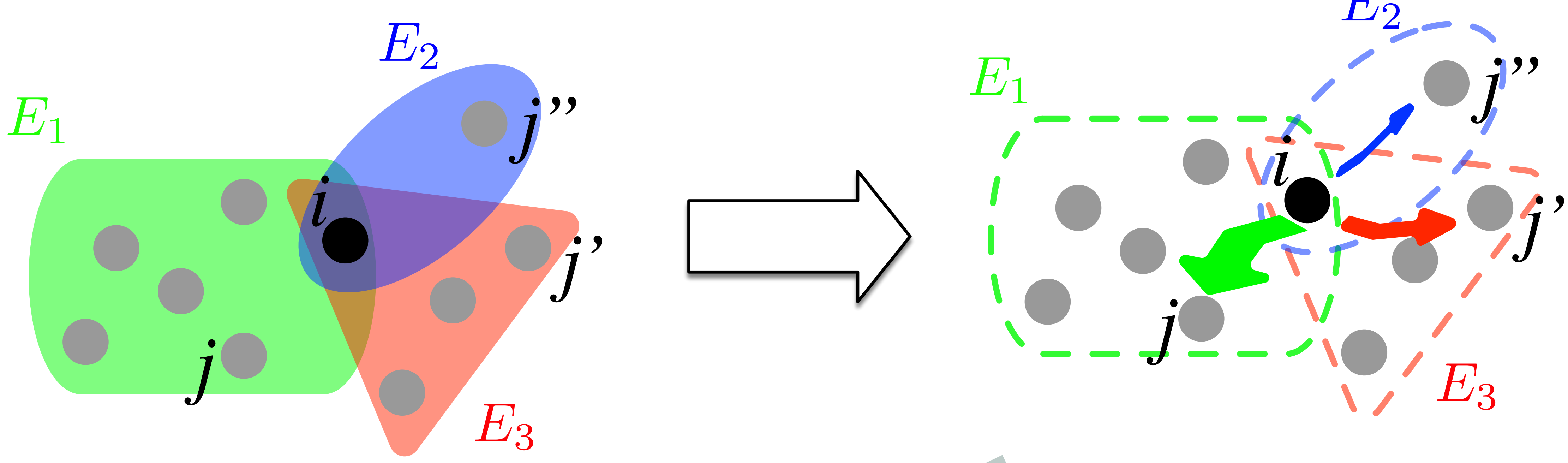
Mean field hyperedge coupling

Diffusive-like coupling

Diffusive-like hyperedges coupling

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}) \quad \mathbf{x} \in \mathbb{R}^d \quad \Rightarrow \quad \frac{d\mathbf{x}^{(i)}}{dt} = \mathbf{f}(\mathbf{x}^{(i)}) \quad \mathbf{x}^{(i)} \in \mathbb{R}^d$$

$$i = 1, \dots, n$$



Diffusive-like coupling

Diffusive-like hyperedges coupling

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}) \quad \mathbf{x} \in \mathbb{R}^d$$

1

$$\frac{d\mathbf{x}^{(i)}}{dt} = \mathbf{f}(\mathbf{x}^{(i)}) \quad \mathbf{x}^{(i)} \in \mathbb{R}^d$$

$$i = 1, \dots, n$$

2

$$\frac{d\mathbf{x}^{(i)}}{dt} = \mathbf{f}(\mathbf{x}^{(i)}) - \epsilon \sum_{\alpha} e_{i\alpha} \sum_{j} e_{j\alpha} \varphi(C_{\alpha\alpha}) \left[\mathbf{G}(\mathbf{x}^{(i)}) - \mathbf{G}(\mathbf{x}^{(j)}) \right]$$

nonlinear coupling function

Diffusive-like hyperedges coupling

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}) \quad \mathbf{x} \in \mathbb{R}^d$$

1

$$\frac{d\mathbf{x}^{(i)}}{dt} = \mathbf{f}(\mathbf{x}^{(i)}) \quad \mathbf{x}^{(i)} \in \mathbb{R}^d$$

$$i = 1, \dots, n$$

2

$$\frac{d\mathbf{x}^{(i)}}{dt} = \mathbf{f}(\mathbf{x}^{(i)}) - \epsilon \sum_\alpha e_{i\alpha} \sum_j e_{j\alpha} \varphi(C_{\alpha\alpha}) \left[\mathbf{G}(\mathbf{x}^{(i)}) - \mathbf{G}(\mathbf{x}^{(j)})\right]$$

nonlinear coupling function

3

(combinatorial) high-order Laplace matrix

$$L_{ij} = d_i \delta_{ij} - D_{ij}$$

$$\Lambda^{(\alpha)} \geq 0$$

$$\frac{d\mathbf{x}^{(i)}}{dt} = \mathbf{f}(\mathbf{x}^{(i)}) - \epsilon \sum_j L_{ij} \mathbf{G}(\mathbf{x}^{(j)})$$

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Master Stability Function on hypergraphs

$$\frac{d\mathbf{s}}{dt}(t) = \mathbf{f}(\mathbf{s}(t))$$

Homogeneous solution of the interconnected system

$$\delta\mathbf{x}_i = \mathbf{x}_i - \mathbf{s}$$

$$\frac{d\delta\mathbf{x}^{(i)}}{dt} = D\mathbf{f}(\mathbf{s}(t))\delta\mathbf{x}^{(i)} - \epsilon\sum_j L_{ij} D\mathbf{G}(\mathbf{s}(t))\delta\mathbf{x}^{(j)}$$

$$\frac{d\delta\mathbf{y}^{(\alpha)}}{dt} = \left[D\mathbf{f}(\mathbf{s}(t)) - \epsilon\Lambda^{(\alpha)} D\mathbf{G}(\mathbf{s}(t))\right]\delta\mathbf{y}^{(\alpha)}$$

Master Stability Function on hypergraphs

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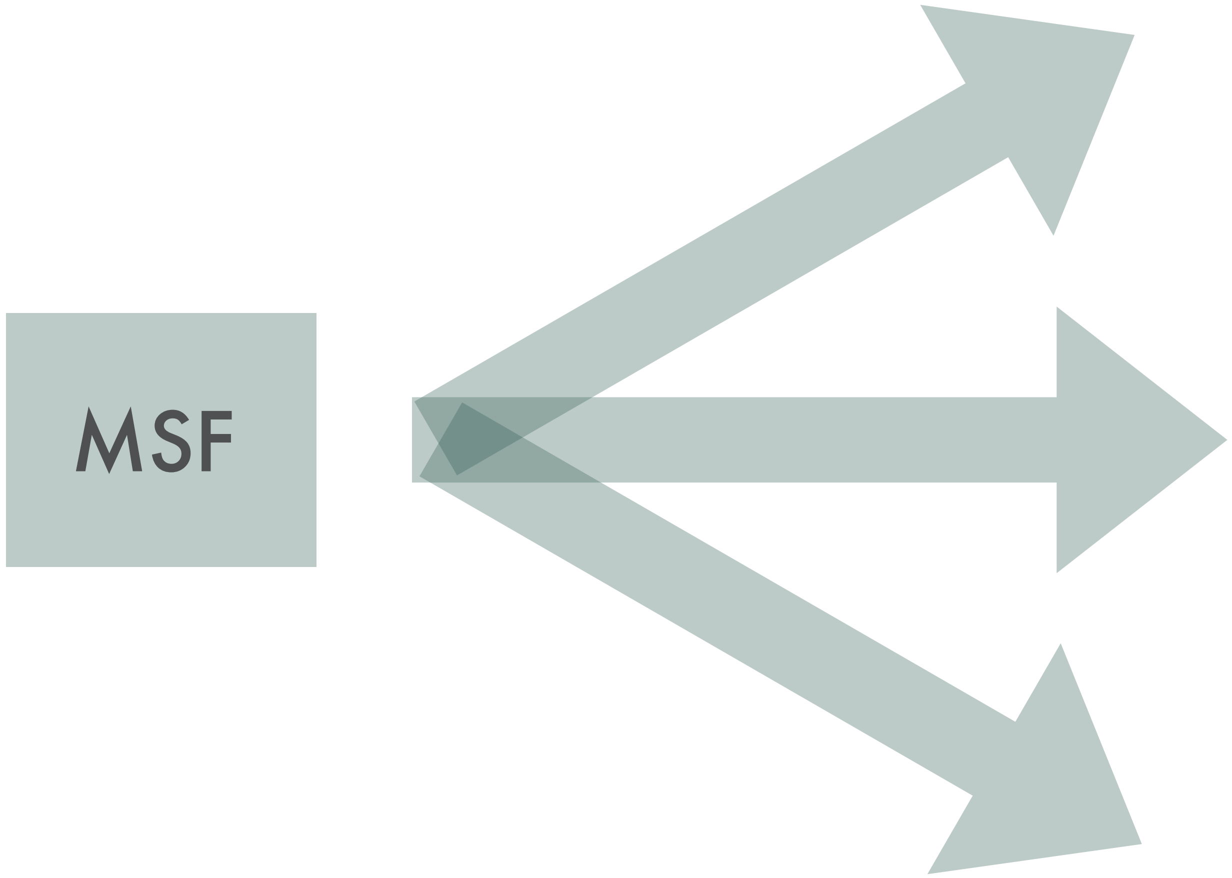

Synchronisation of nonlinear oscillators

Synchronisation of chaotic orbits

Turing patterns

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Journal of Physics: Complexity

PAPER

Dynamical systems on hypergraphs

Timoteo Carletti[1,4], **Duccio Fanelli**[2] and **Sara Nicoletti**[2,3]

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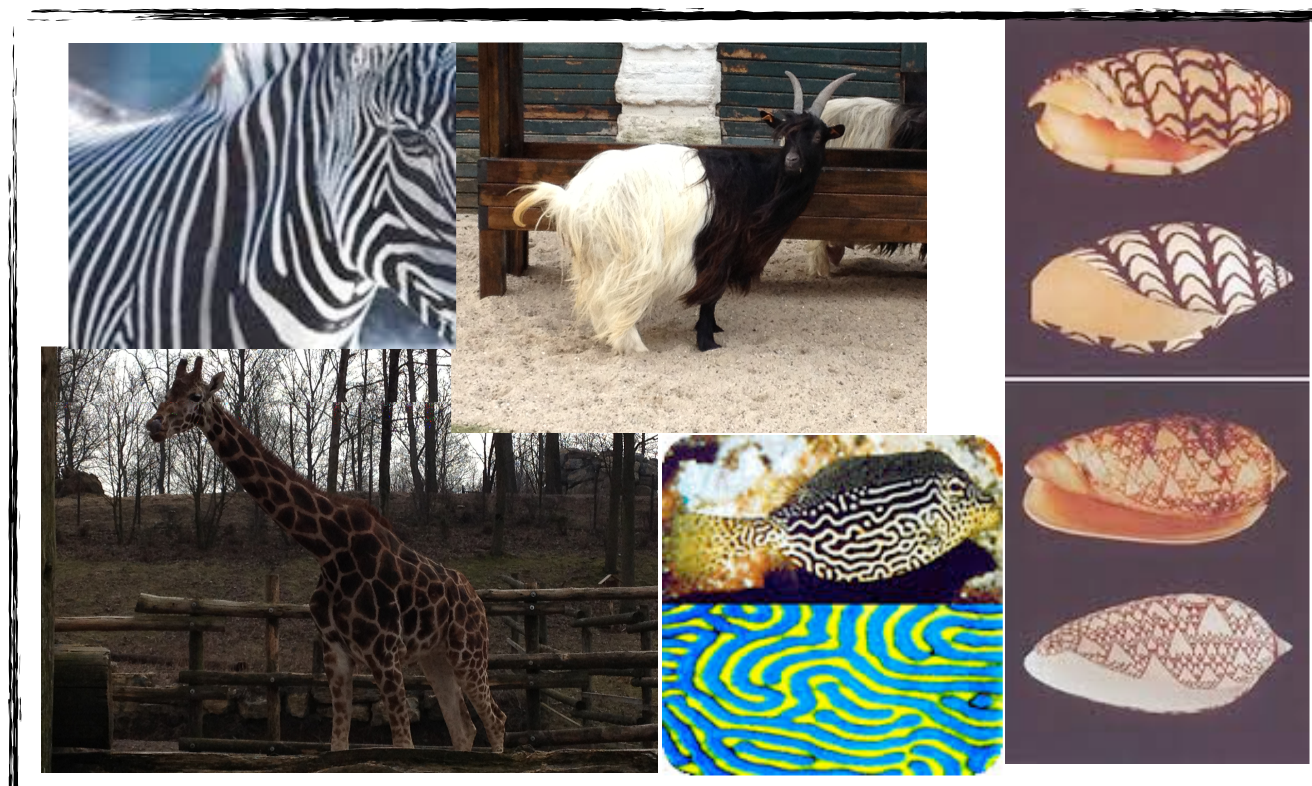
Keywords: hypergraphs, master stability function, synchronisation, Turing patterns, dynamical systems



Turing patterns on hypergraphs

$\mathbf{s}(t) = \mathbf{s}^*$ Homogeneous stationary solution of the interconnected system

$$\frac{d\delta\mathbf{y}^{(\alpha)}}{dt} = \left[D\mathbf{f}(\mathbf{s}^*) - \epsilon\Lambda^{(\alpha)} D\mathbf{G}(\mathbf{s}^*)\right]\delta\mathbf{y}^{(\alpha)}$$

Master Stability Function on hypergraphs

$\lambda = \lambda(\Lambda^{(\alpha)})$ Eigenvalue with the largest real part (dispersion relation / largest Lyapunov exponent)

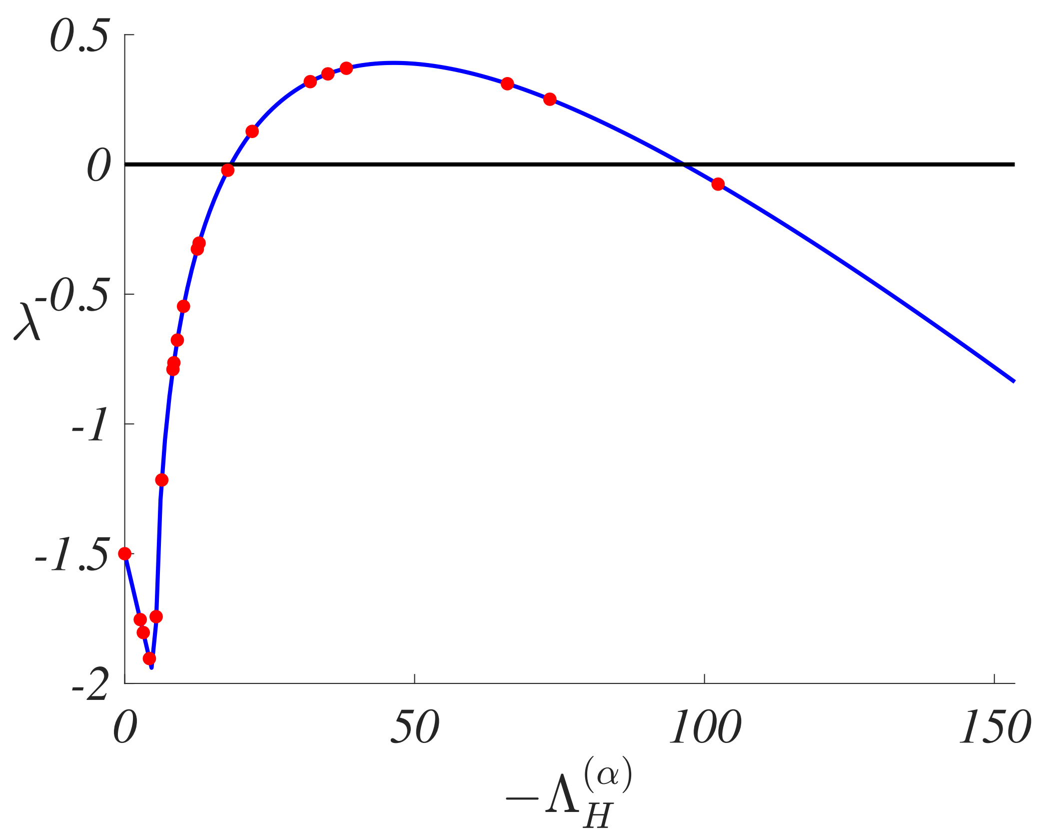
$\mathbf{s}(t) = \mathbf{s}^*$ is unstable if there exists $\Lambda^{(\alpha)}$ such that

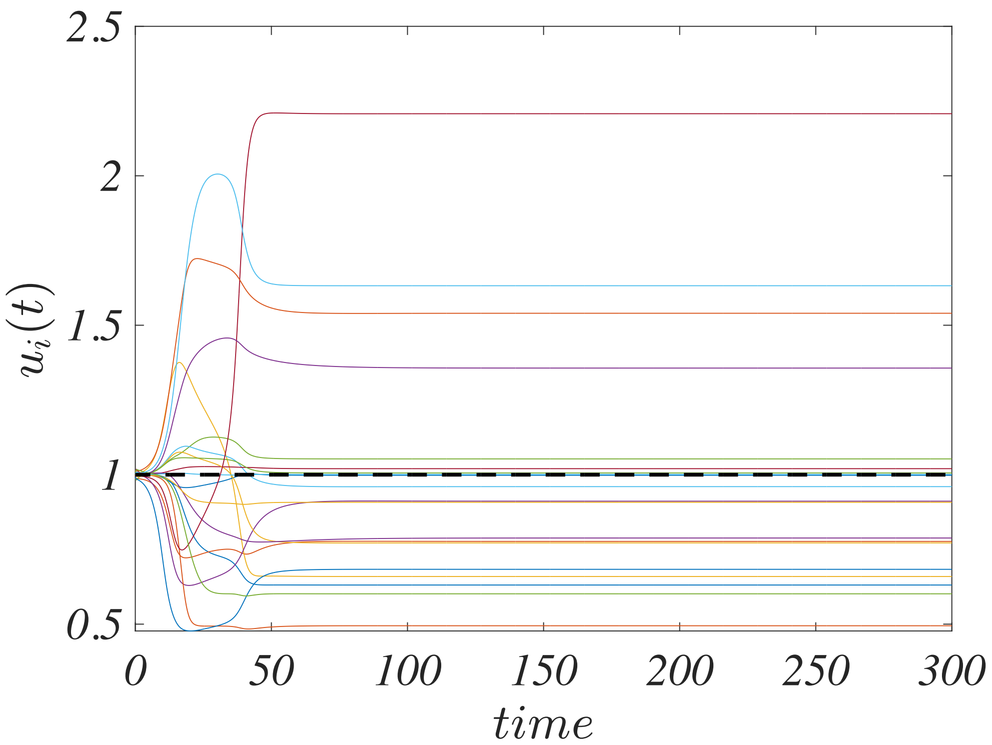
$$\lambda = \lambda(\Lambda^{(\alpha)}) > 0$$

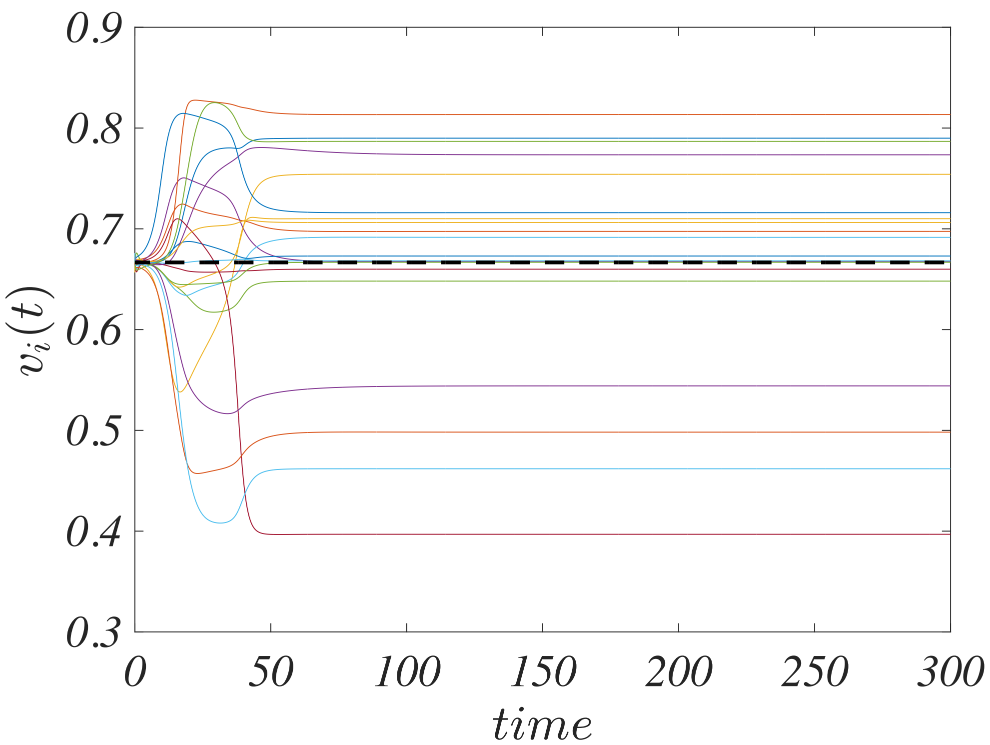
Turing patterns on hypergraphs

$$f(u,v) = 1 - (b+1)u + cu^2v$$
$$g(u,v) = bu - cu^2v$$

Brusselator model







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Random walks on hypergraphs

Timoteo Carletti,[1] Federico Battiston,[2] Giulia Cencetti,[3] and Duccio Fanelli[4]
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PAPER

Dynamical systems on hypergraphs

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Keywords: hypergraphs, master stability function, synchronisation, Turing patterns, dynamical systems

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PAPER

Random walks and community detection in hypergraphs

Timoteo Carletti[1,*], Duccio Fanelli[2] and Renaud Lambiotte[3]
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Keywords: hypergraphs, random walks, higher-order networks

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[Submitted on 5 Apr 2021]

Dynamical systems on hypergraphs

Timoteo Carletti, Duccio Fanelli

We present a general framework that enables one to model high-order interaction among entangled dynamical systems, via hypergraphs. Several relevant processes can be ideally traced back to the proposed scheme. We shall here solely elaborate on the conditions that seed the spontaneous emergence of patterns, spatially heterogeneous solutions resulting from the many-body interaction between fundamental units. In particular we will focus, on two relevant settings. First, we will assume long-ranged mean field interactions between populations, and then turn to considering diffusive-like couplings. Two applications are presented, respectively to a generalised Volterra system and the Brusselator model.

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[Submitted on 10 May 2021]

Flow-based Community Detection in Hypergraphs

Anton Eriksson, Timoteo Carletti, Renaud Lambiotte, Alexis Rojas, Martin Rosvall

To connect structure, dynamics and function in systems with multibody interactions, network scientists model random walks on hypergraphs and identify communities that confine the walks for a long time. The two flow-based community-detection methods Markov stability and the map equation identify such communities based on different principles and search algorithms. But how similar are the resulting communities? We explain both methods' machinery applied to hypergraphs and compare them on synthetic and real-world hypergraphs using various hyperedge-size biased random walks and time scales. We find that the map equation is more sensitive to time-scale changes and that Markov stability is more sensitive to hyperedge-size biases.

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Emergent Properties in Dynamical Systems on Hypergraphs

Thank you

Any questions?

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