May the 19th, 2019, Snowbird, Utah, US

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Applications of Dynamical Systems
May 19–23, 2019
Snowbird Ski and Summer Resort
Snowbird, Utah, USA

Timoteo Carletti

Dynamical instabilities in networked systems, it is a matter of time and direction

naxys
Namur Institute for Complex Systems

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Acknowledgements

"Belgian" team:

R. Lambiotte

M. Asllani, N. Kouvaris (post docs)

J. Petit (PhD),

A. Bellière, G. Planchon, R. Muolo (Master students)

Italian team:

D. Fanelli, D.M. Busiello, C. Cianci, M. Galanti, F. Miele, F. Di Patti, M. Lucas

Ph. Maini (Oxford, UK)

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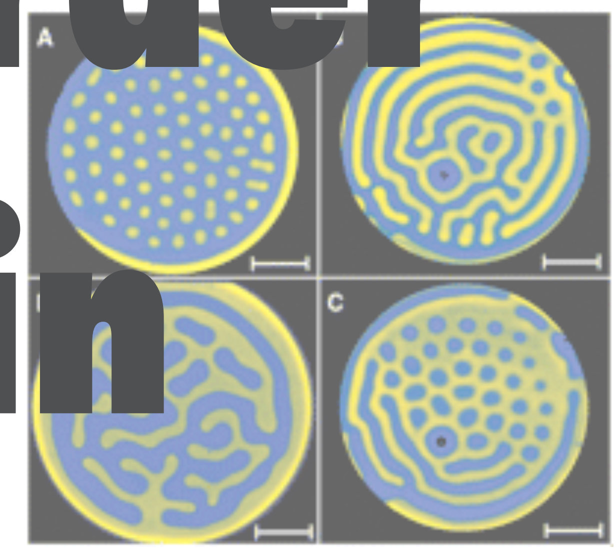
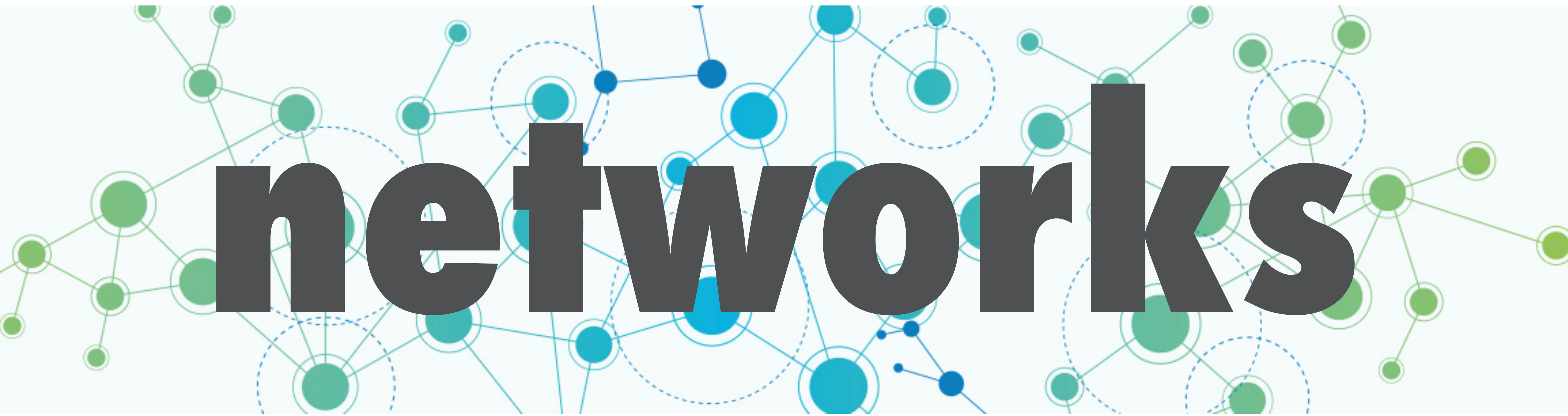
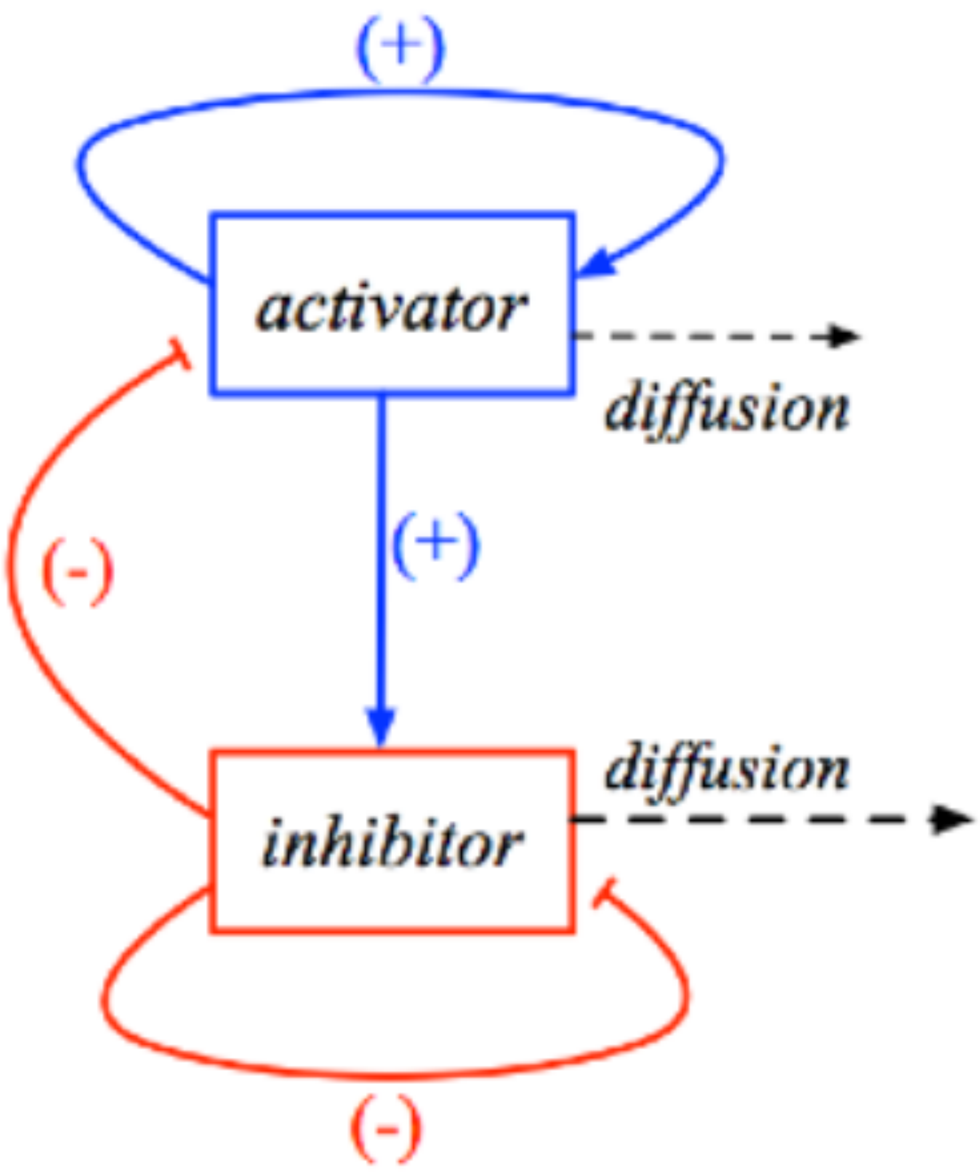
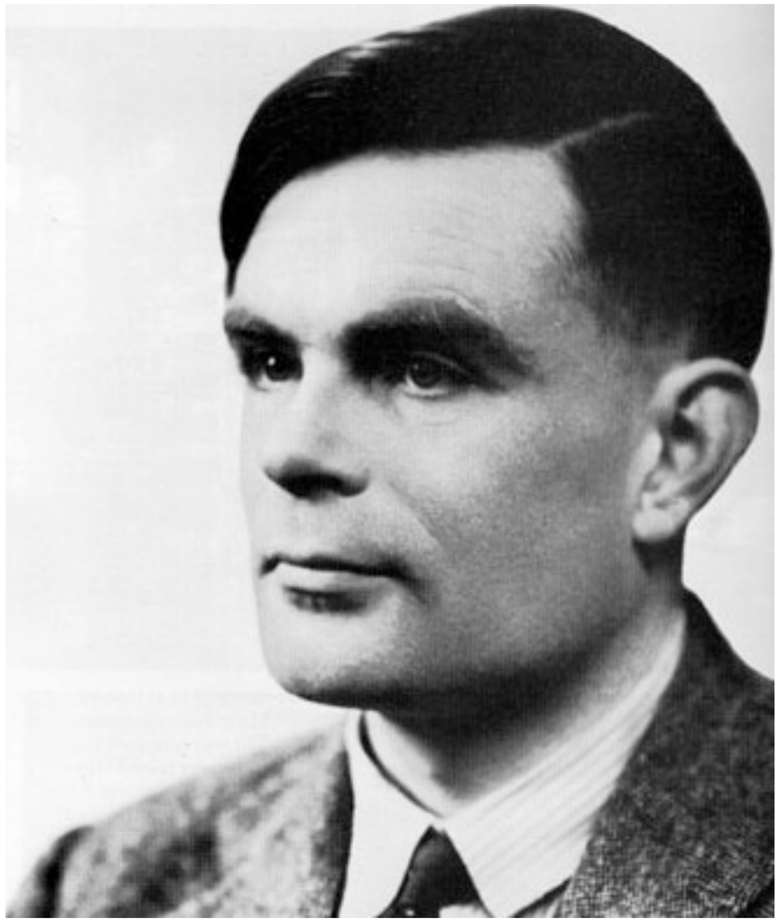
Order from disorder is a leitmotif in Nature




networks

Turing mechanism: diffusion driven instability





$u(x,y,t)$ Amount of activator in (x,y) at time t

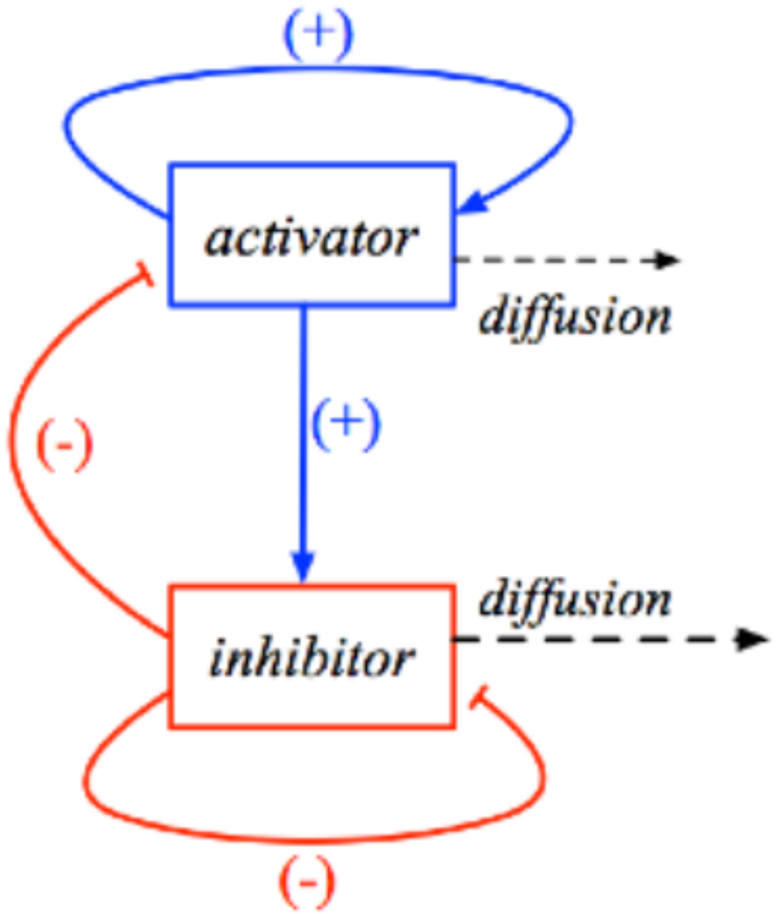
$v(x,y,t)$ Amount of inhibitor in (x,y) at time t

A.M.Turing, The chemical basis of morphogenesis, Phil. Trans. R Soc London B, 237, (1952), pp.37

Outlook

- A (short) review of Turing instability ;
- Turing instability on networks (Nakao and Mikhailov, 2010) ;
- Limitations of Turing mechanism ;
- It is a matter of time : <u>time varying networks</u> ;
- It is a matter of direction : <u>non-normal networks</u> ;

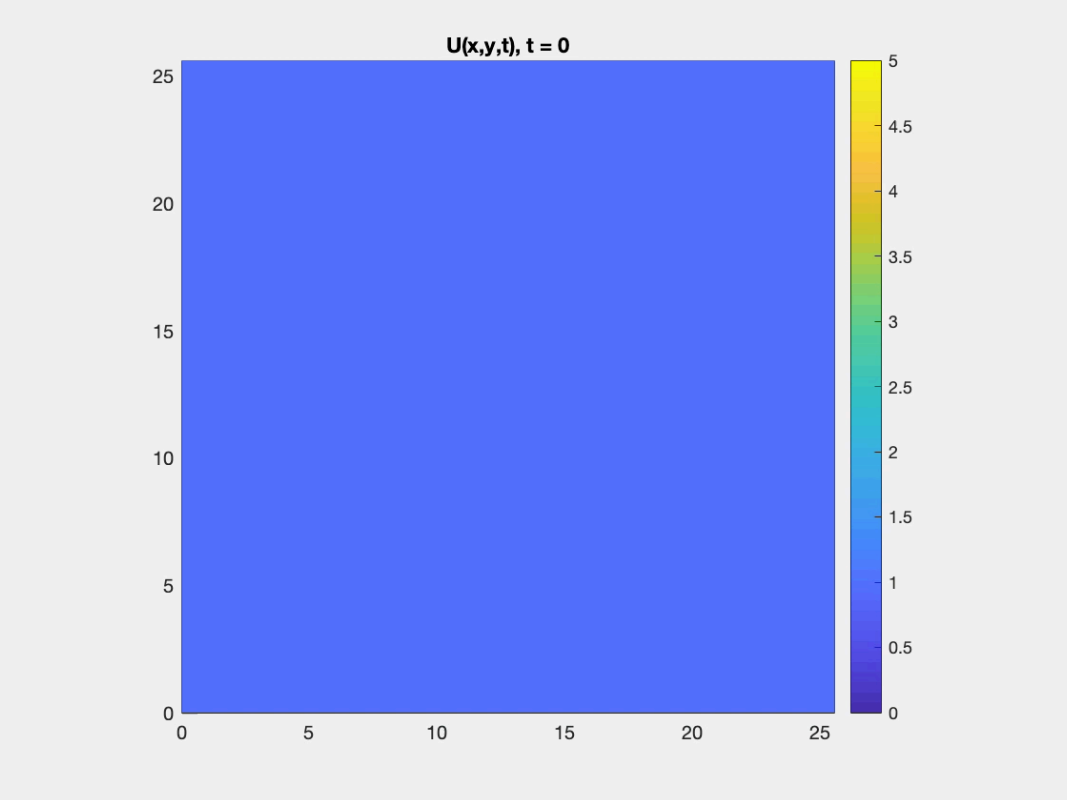
Turing mechanism: diffusion driven instability



$$\begin{cases} \frac{\partial u}{\partial t} = f(u,v) + D_u \nabla^2 u \\ \frac{\partial v}{\partial t} = g(u,v) + D_v \nabla^2 v \end{cases}$$

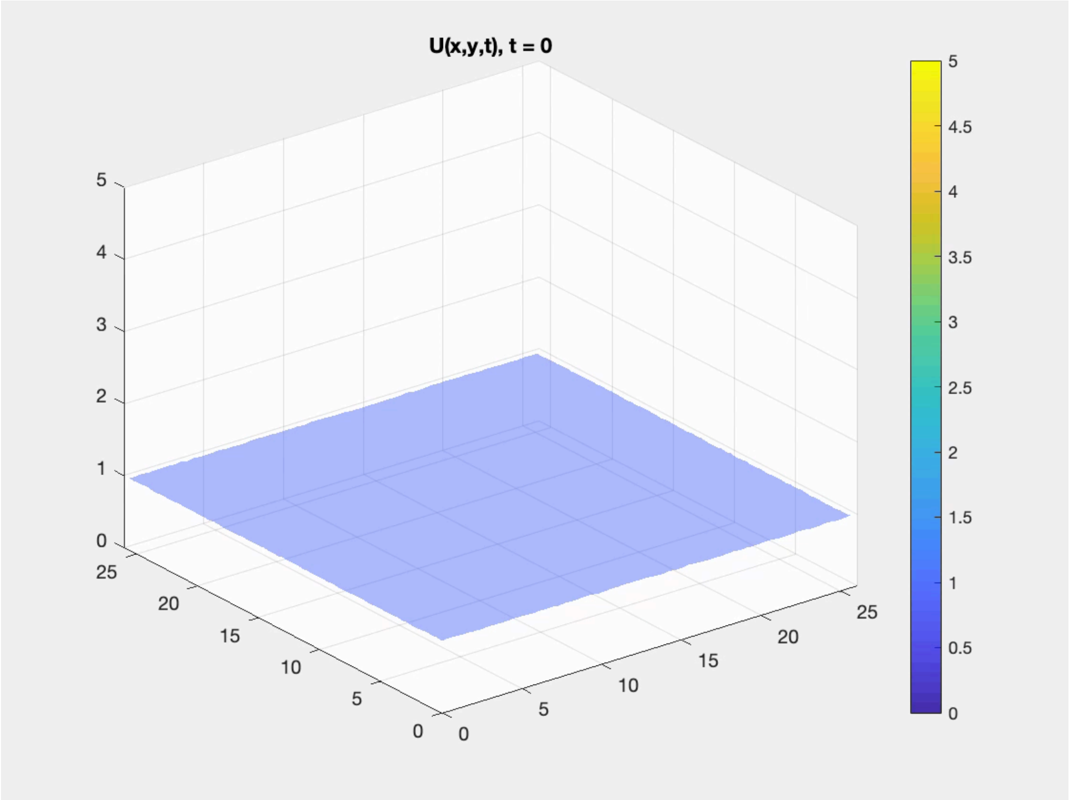
$$(x,y) \in \Omega$$

+ boundary conditions
+ initial condition

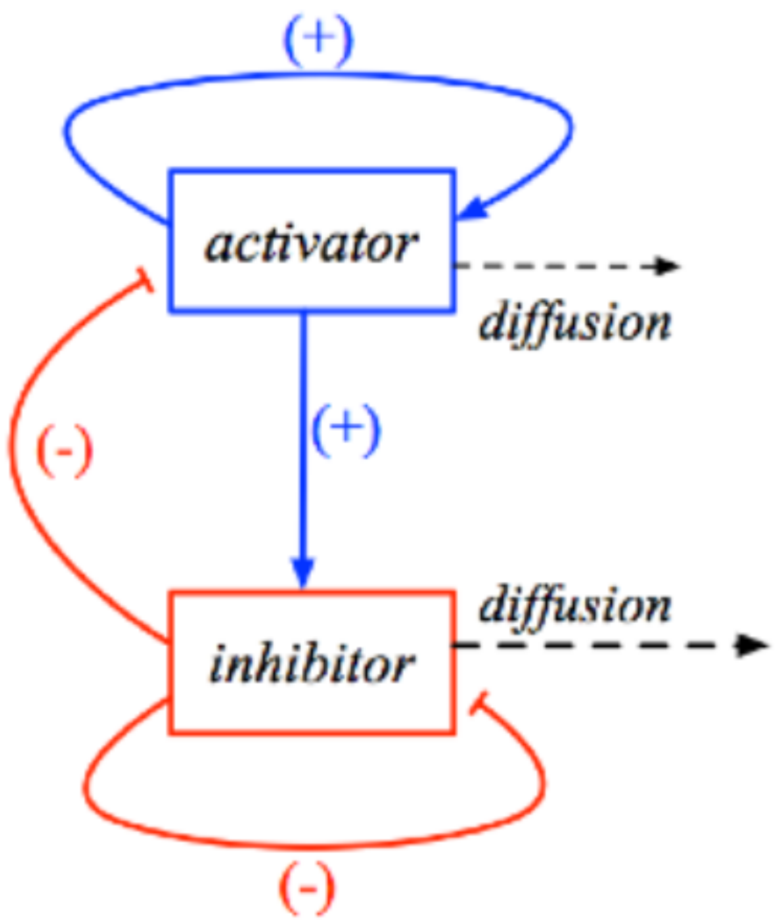


$$D_u = 0$$

$$D_v = 0$$



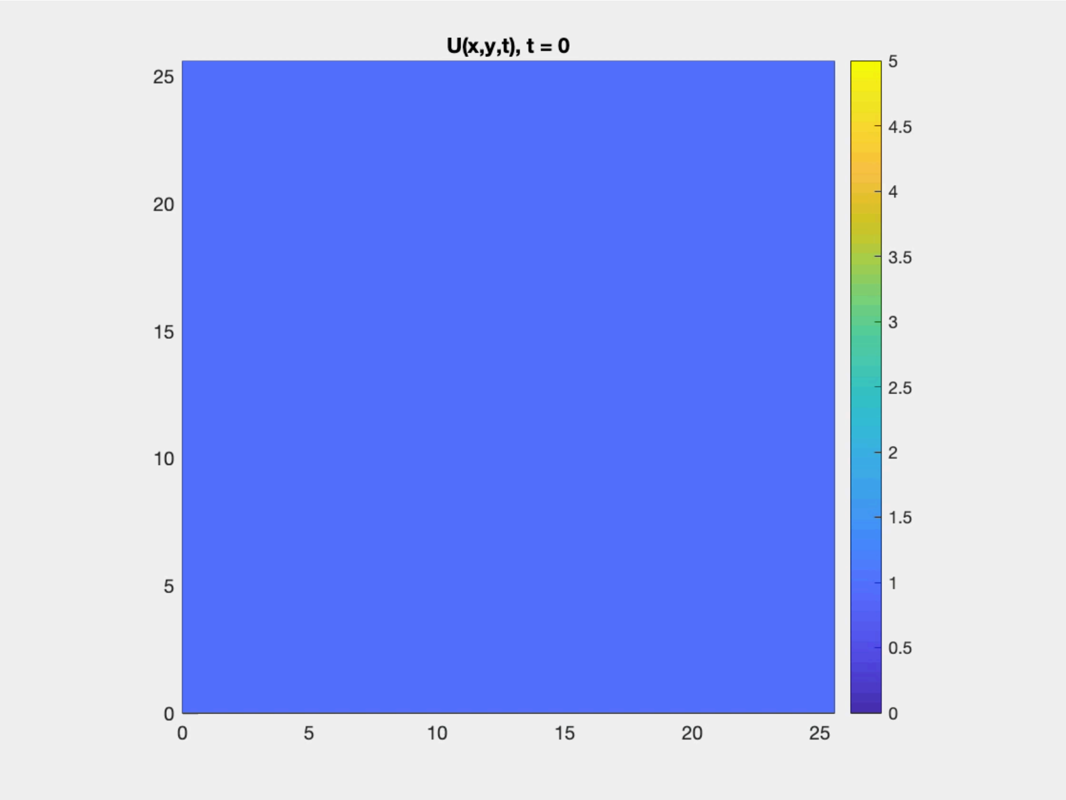
Turing mechanism: diffusion driven instability



$$\begin{cases} \frac{\partial u}{\partial t} = f(u,v) + D_u \nabla^2 u \\ \frac{\partial v}{\partial t} = g(u,v) + D_v \nabla^2 v \end{cases}$$

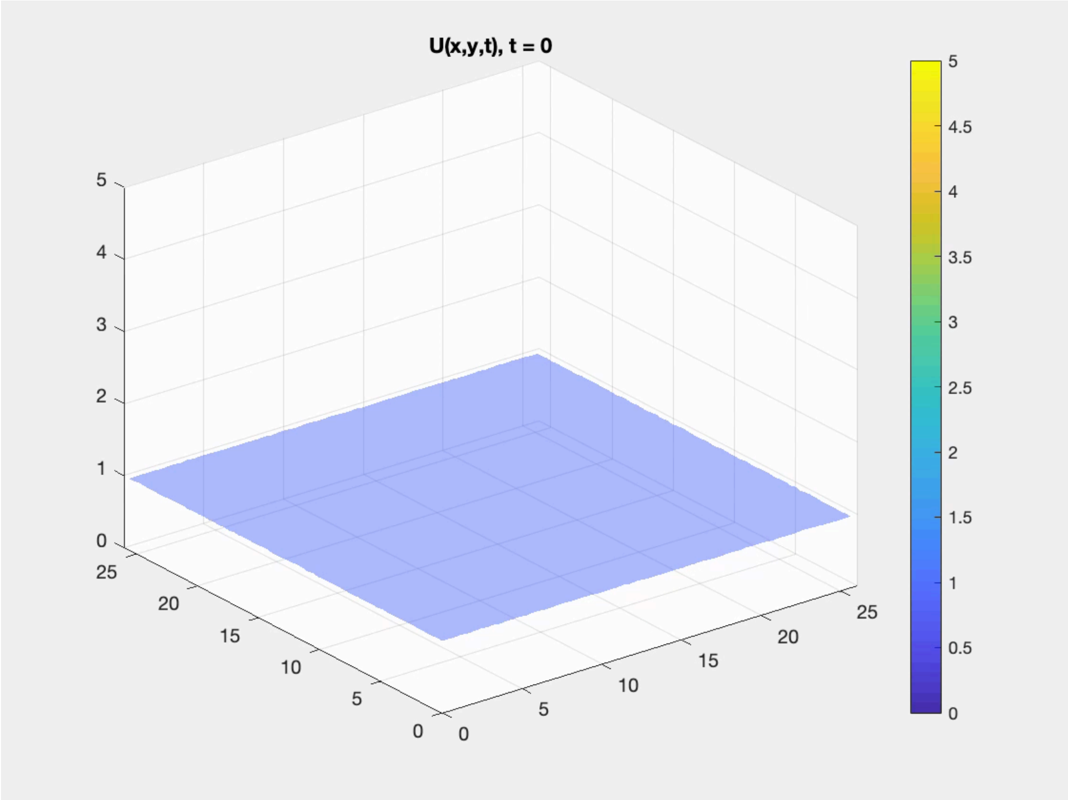
$$(x,y) \in \Omega$$

+ boundary conditions
+ initial condition



$$D_u = 1$$

$$D_v = 20$$



Turing mechanism: diffusion driven instability

1) Assume there exists a spatially homogeneous solution:

$$u(x,y,t) = \hat{u} \text{ and } v(x,y,t) = \hat{v} \quad \forall (x,y) \in \Omega \text{ and } \forall t \geq 0$$

which moreover is stable when there is no diffusion: $D_u = D_v = 0$

2) Linearise around this solution:

$$\begin{cases} u(x,y,t) = \hat{u} + \delta u(x,y,t) \\ v(x,y,t) = \hat{v} + \delta v(x,y,t) \end{cases}$$

$$\begin{pmatrix} \dot{\delta \boldsymbol{u}} \\ \dot{\delta \boldsymbol{v}} \end{pmatrix} = \tilde{\boldsymbol{\mathcal{J}}} \begin{pmatrix} \delta \boldsymbol{u} \\ \delta \boldsymbol{v} \end{pmatrix} \qquad \tilde{\boldsymbol{\mathcal{J}}} = \begin{pmatrix} f_u + D_u \nabla^2 & f_v \\ g_u & g_v + D_v \nabla^2 \end{pmatrix}$$

3) Prove that the spatially homogeneous solution:

$$u(x,y,t) = \hat{u} \text{ and } v(x,y,t) = \hat{v}$$

turns out to be <u>unstable</u> once the diffusion is in action

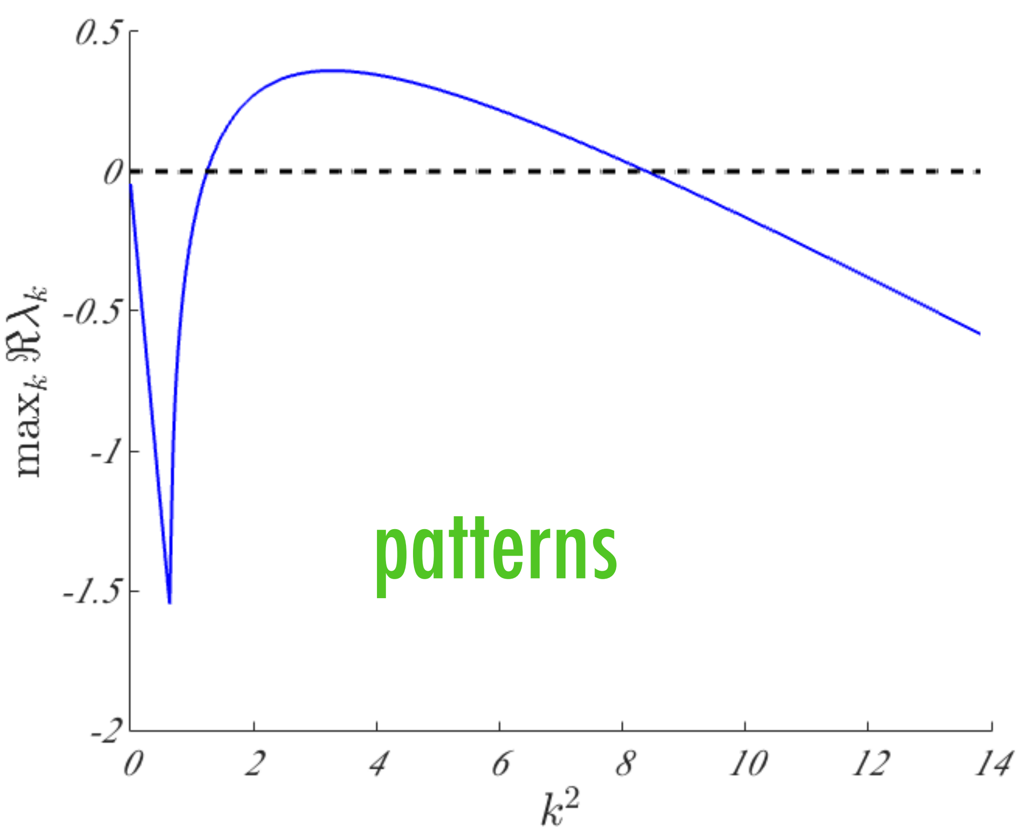
$$D_u > 0 \text{ and } D_v > 0$$

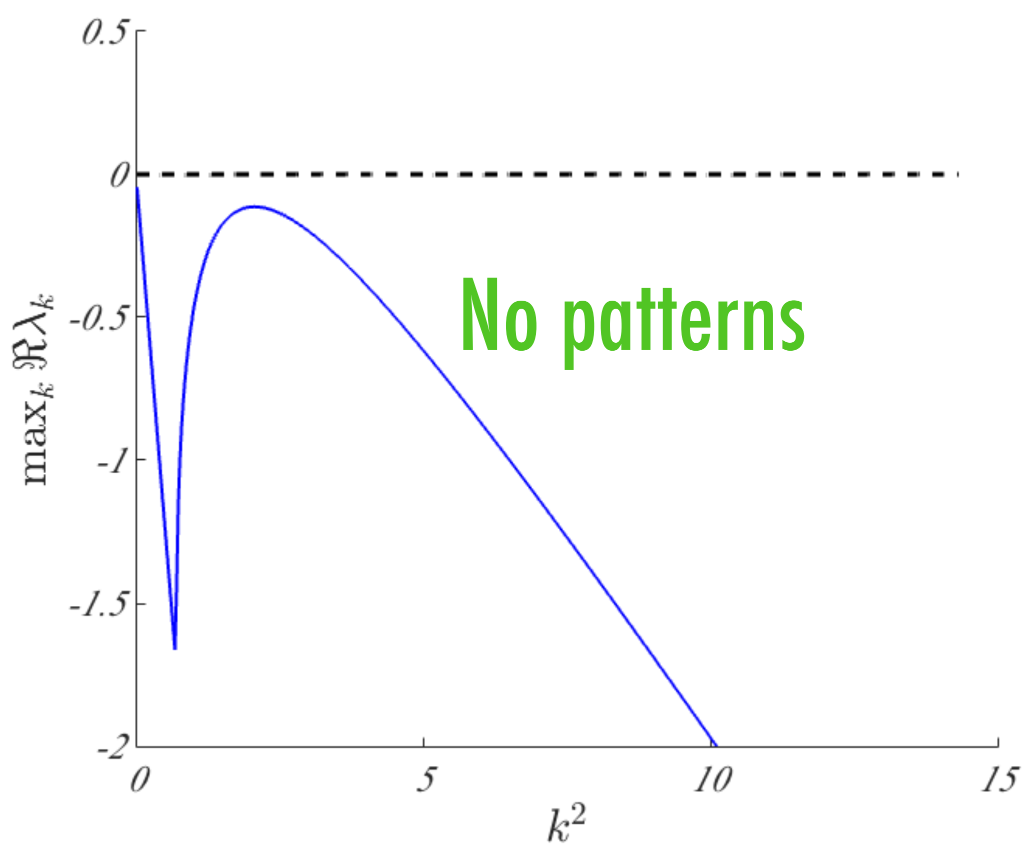
Turing mechanism: diffusion driven instability

i) decompose the solution on the Fourier modes (Laplacian eigenbasis) and use the ansatz $\delta u(x,y,t) = \sum_{k=(k_1,k_2)} c_k e^{2\pi i(k_1 x + k_2 y)} e^{\lambda_k t}$

ii) λ_k is solution of $\det\left[\lambda_k - \begin{pmatrix} f_u - D_u k^2 & f_v \\ g_u & g_v - D_v k^2 \end{pmatrix}\right] = 0$

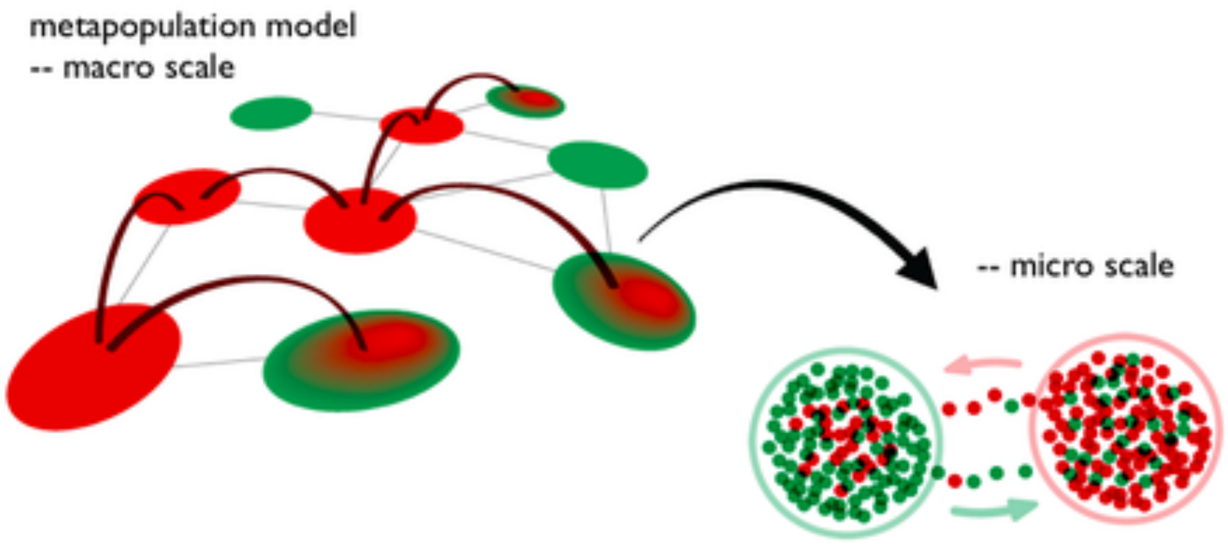
iii) if there exists $\hat{k}^2 \in (k_-^2, k_+^2)$ such that $\Re\lambda_{\hat{k}} > 0$ then Turing patterns do emerge.





Turing mechanism on complex networks

Nakao H. and Mikhailov A. S., Nature Physics, 6, pp. 544 (2010)



Metapopulation models

e.g. in the framework of ecology:
May R., *Will a large complex system be stable?* Nature, 238, pp. 413, (1972)

Reactions occur at each node. Diffusion occurs across edges.

Turing mechanism on complex networks

Nakao H. and Mikhailov A. S., Nature Physics, 6, pp. 544 (2010)

$$\begin{cases} \dot{u}_i(t) &= f(u_i(t), v_i(t)) + D_u \sum_{j=1}^{n} L_{ij} u_j(t) \\ \dot{v}_i(t) &= g(u_i(t), v_i(t)) + D_v \sum_{j=1}^{n} L_{ij} v_j(t) \end{cases} \quad \forall i = 1, \dots, n \text{ and } t > 0.$$

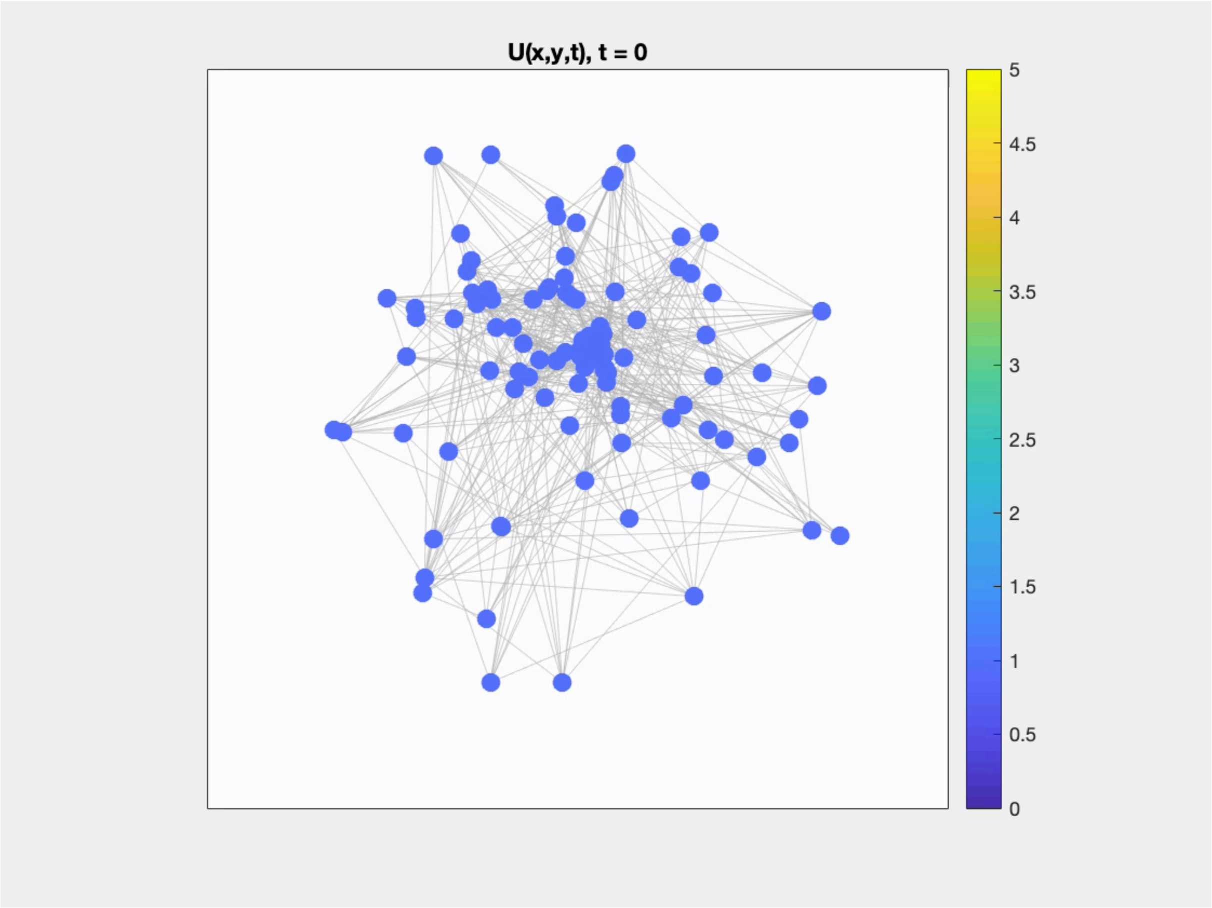
A_{ij} Adjacency matrix

$L_{ij} = A_{ij} - k_i \delta_{ij}$ Laplace matrix

$u_i(t)$ Amount of activator in node i at time t

$v_i(t)$ Amount of inhibitor in node i at time t

Patterns:
Sets of nodes whose asymptotic state is far from the homogeneous equilibrium.

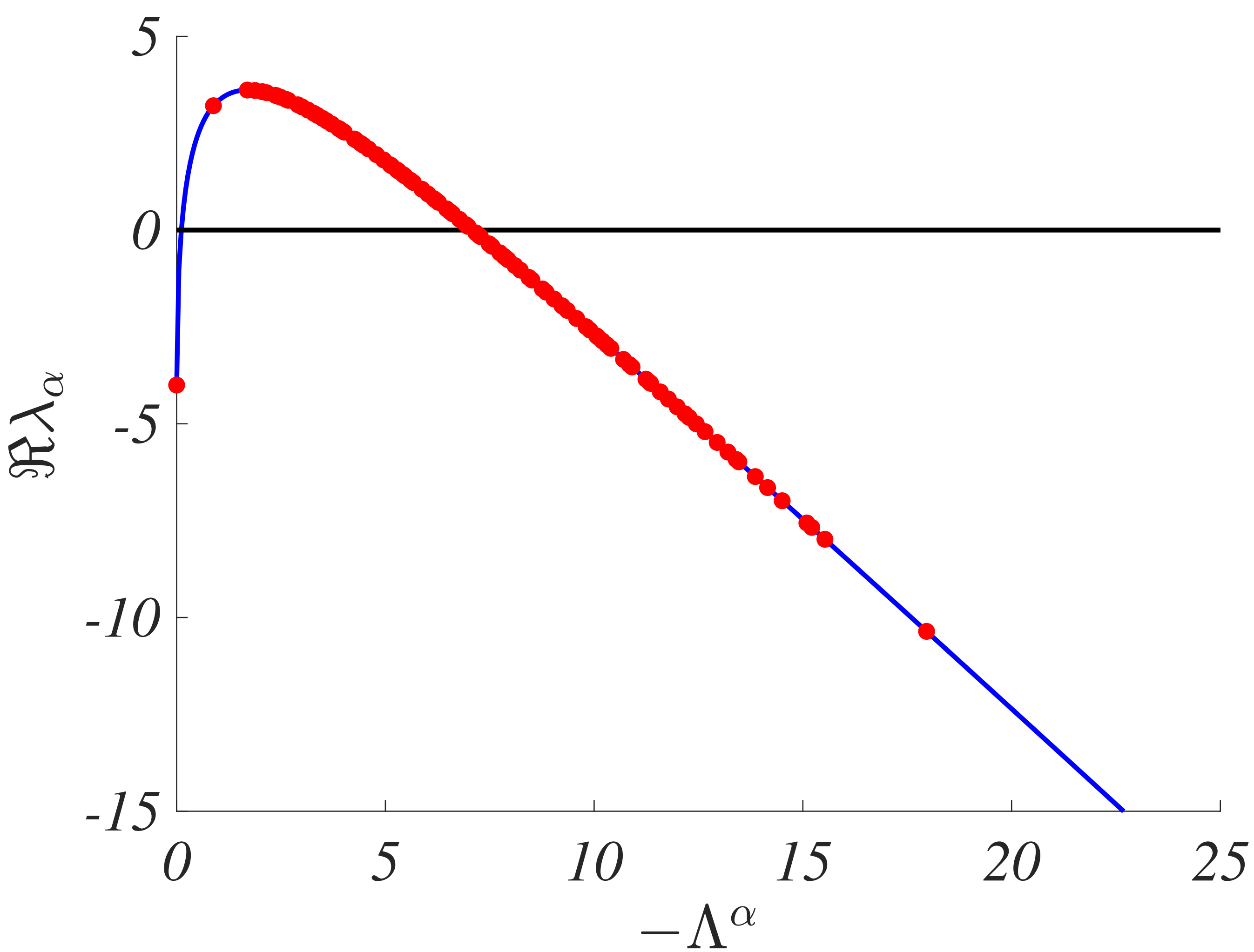


Turing mechanism on complex networks

Nakao H. and Mikhailov A. S., Nature Physics, 6, pp. 544 (2010)

Use the appropriate eigenbasis

$$\delta u_i(t) = \sum_\alpha c_\alpha \phi_i^{(\alpha)} e^{\lambda_\alpha t} \qquad \sum_j L_{ij}\phi_j^{(\alpha)} = \Lambda_\alpha \phi_i^{(\alpha)} \quad \forall i, \alpha$$



If there exists Λ_α such that

$$\Re\lambda_\alpha > 0$$

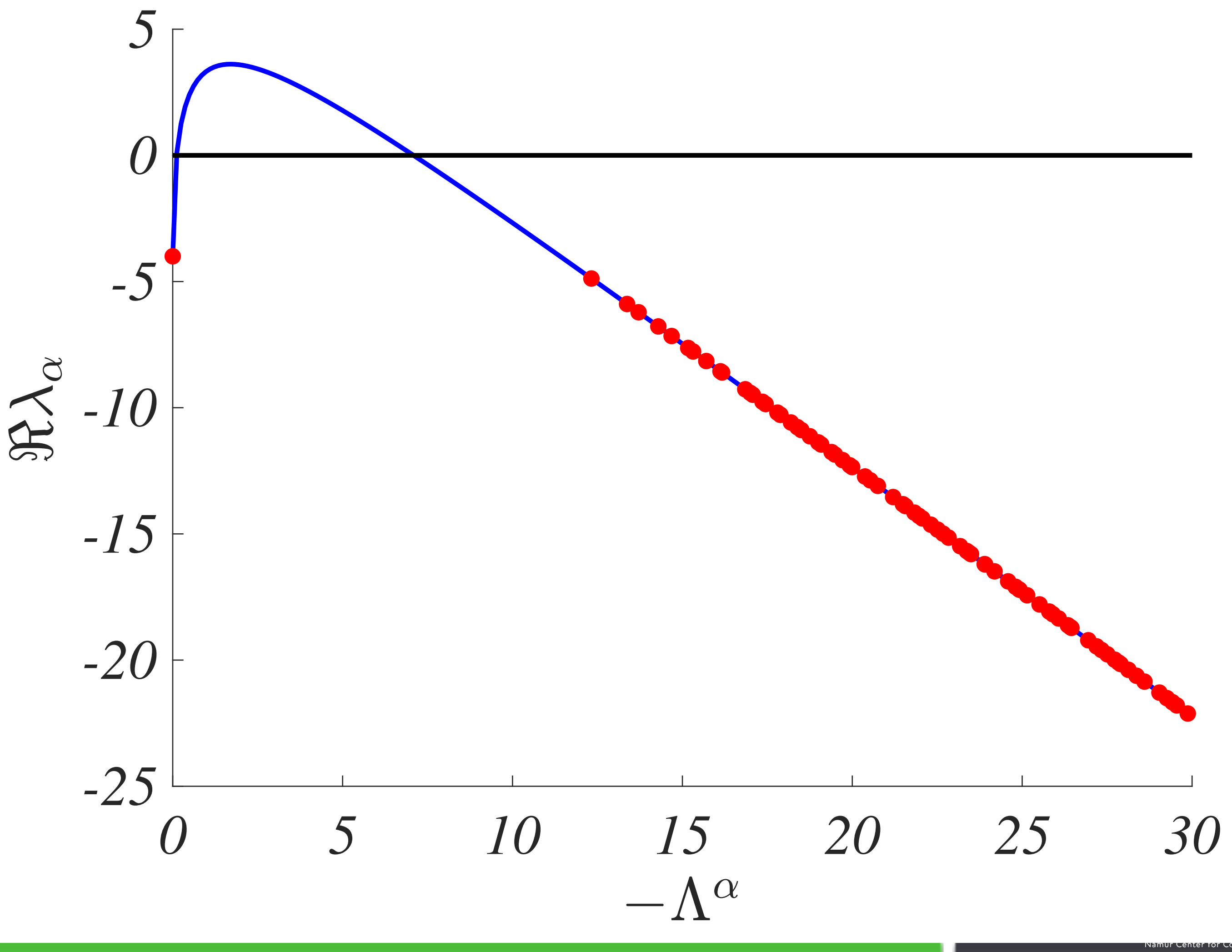
then Turing patterns do emerge.

Turing mechanism: Limitations

- At least two diffusing species are needed;
- Activator and inhibitor are both necessary;
- The inhibitor must diffuse much faster than the activator

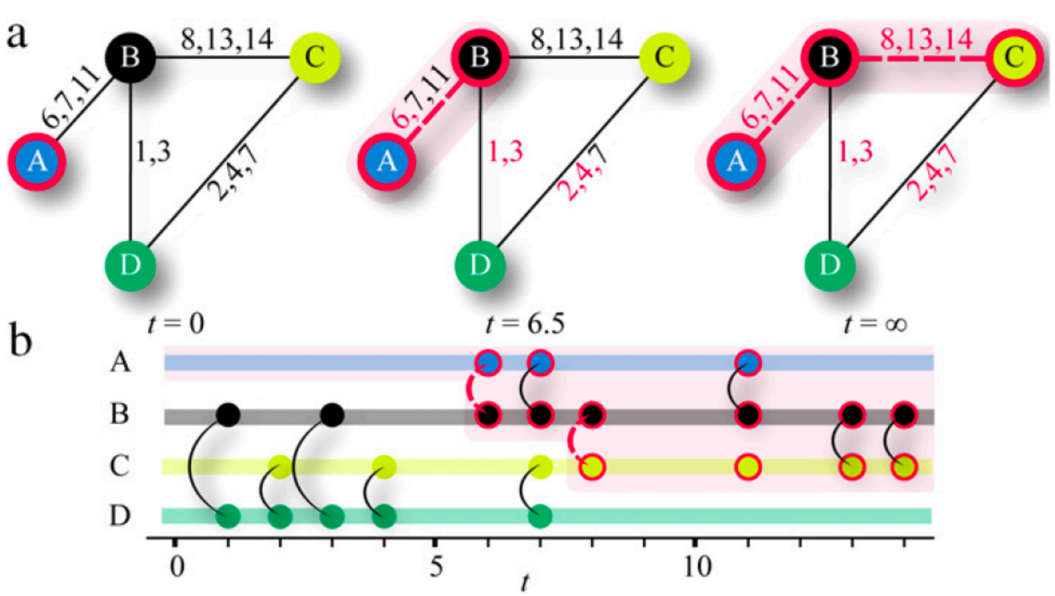
$$D_v \gg D_u$$

- "Not good topology"

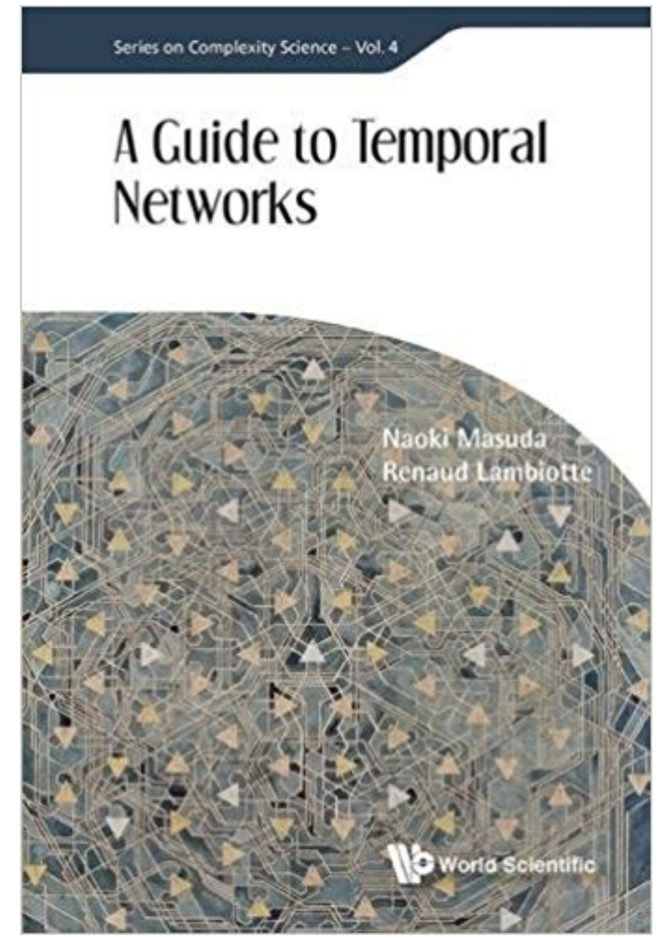


It is a matter of time: time varying networks

Contact social networks



Phone calls



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Temporal networks

Petter Holme [a,b,c,*], Jari Saramäki [d]

[a] IceLab, Department of Physics, Umeå University, 901 87 Umeå, Sweden
[b] Department of Energy Science, Sungkyunkwan University, Suwon 440–746, Republic of Korea
[c] Department of Sociology, Stockholm University, 106 91 Stockholm, Sweden
[d] Department of Biomedical Engineering and Computational Science, School of Science, Aalto University, 00076 Aalto, Espoo, Finland

It is a matter of time: time varying networks

$$A_{ij}(t) = \begin{cases} w_{ij}(t) > 0 & \text{if } i \text{ and } j \text{ are linked at time t} \\ 0 & \text{otherwise} \end{cases}$$

possible different time scale

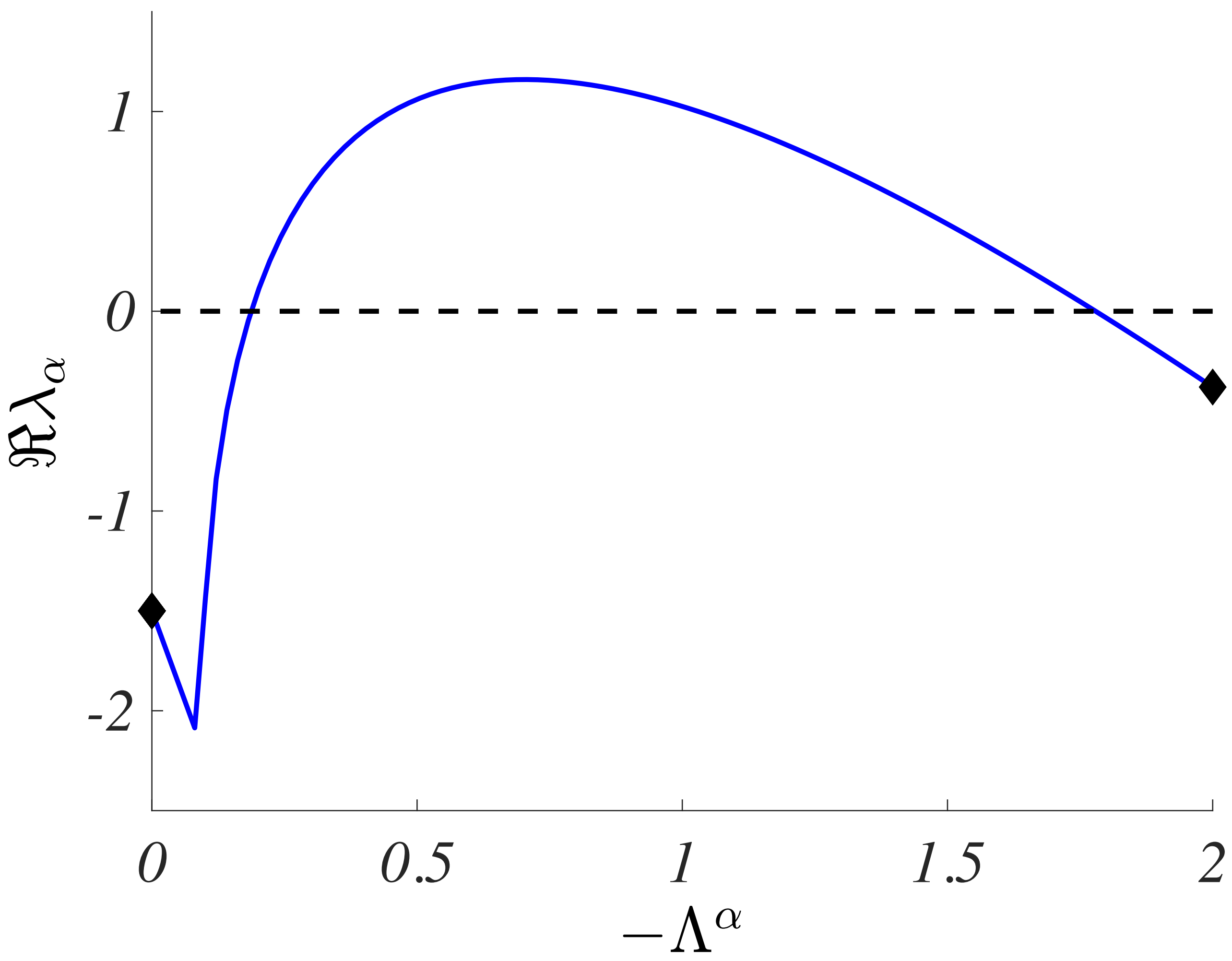
$$(1) \quad \begin{cases} \dot{u}_i(t) & = f(u_i, v_i) + D_u \sum_{j=1}^{N} L_{ij}(t/\epsilon) u_j(t) \\ \dot{v}_i(t) & = g(u_i, v_i) + D_v \sum_{j=1}^{N} L_{ij}(t/\epsilon) v_j(t) \end{cases}$$

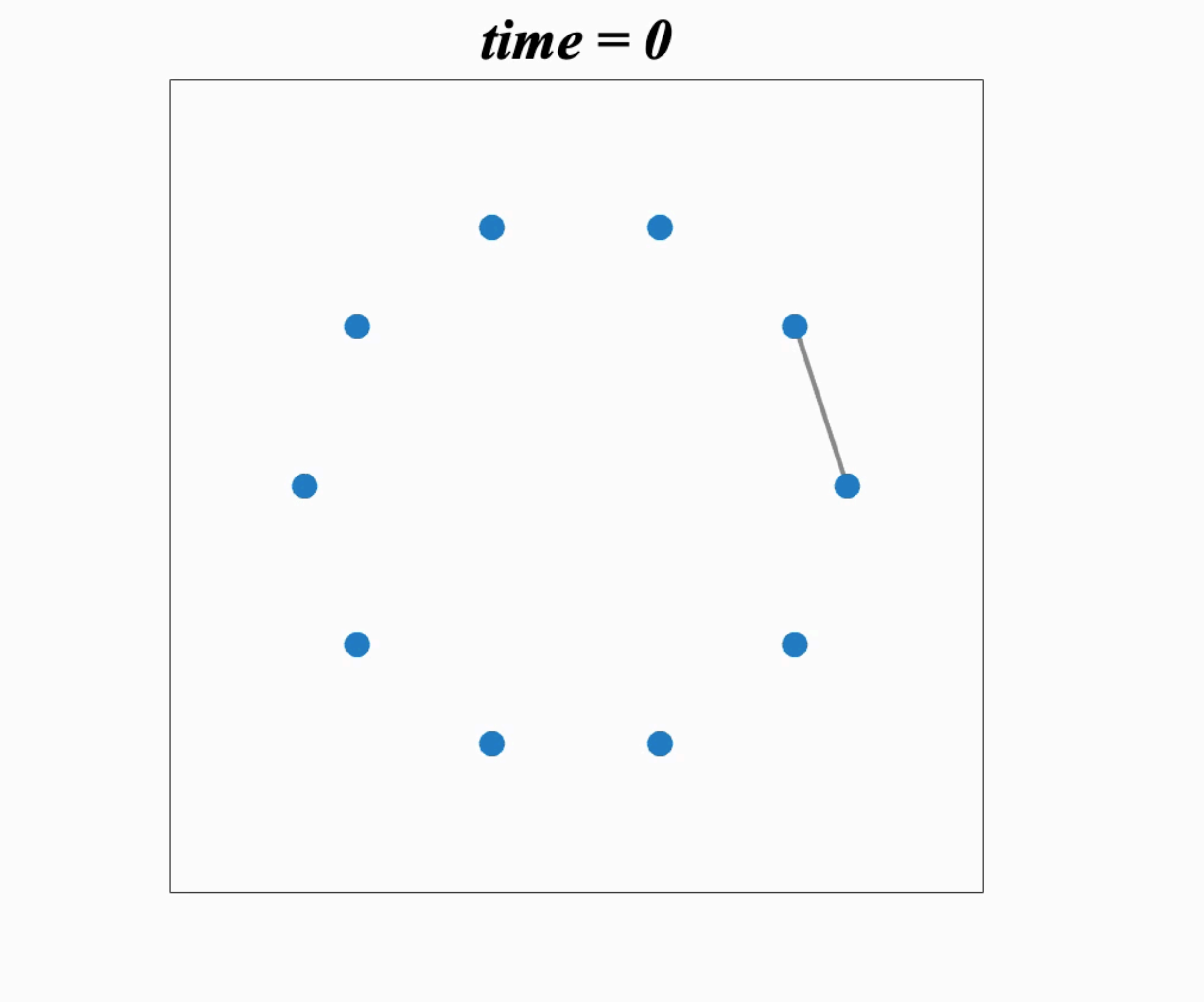
Can the system exhibit Turing patterns, once the static version doesn't?

Which is the role of network time scale?

It is a matter of time: time varying networks

"Chinese whispers" (telephone game)



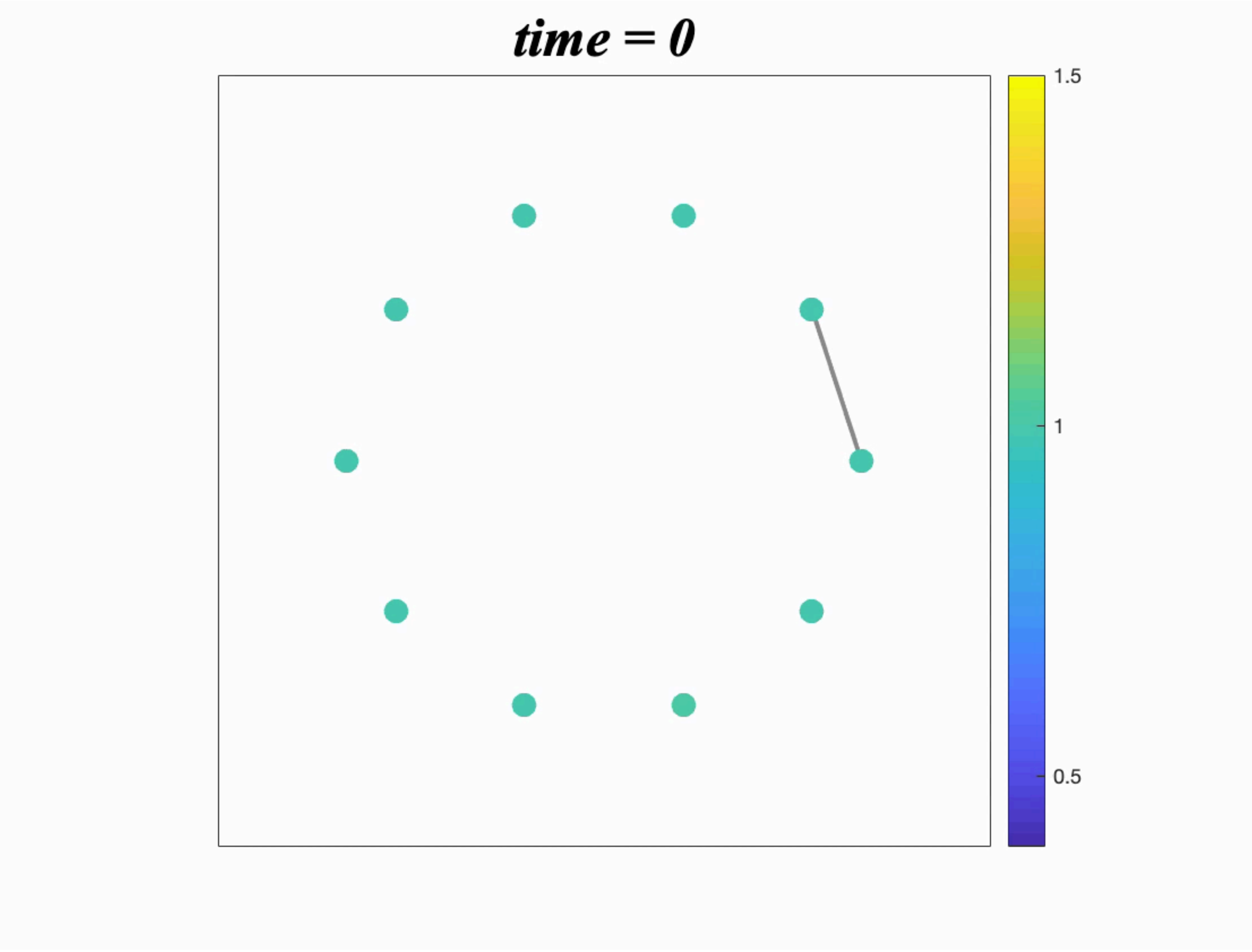


It is a matter of time: time varying networks

“Chinese whispers” (telephone game)

Natural dynamics

$\epsilon = 1$

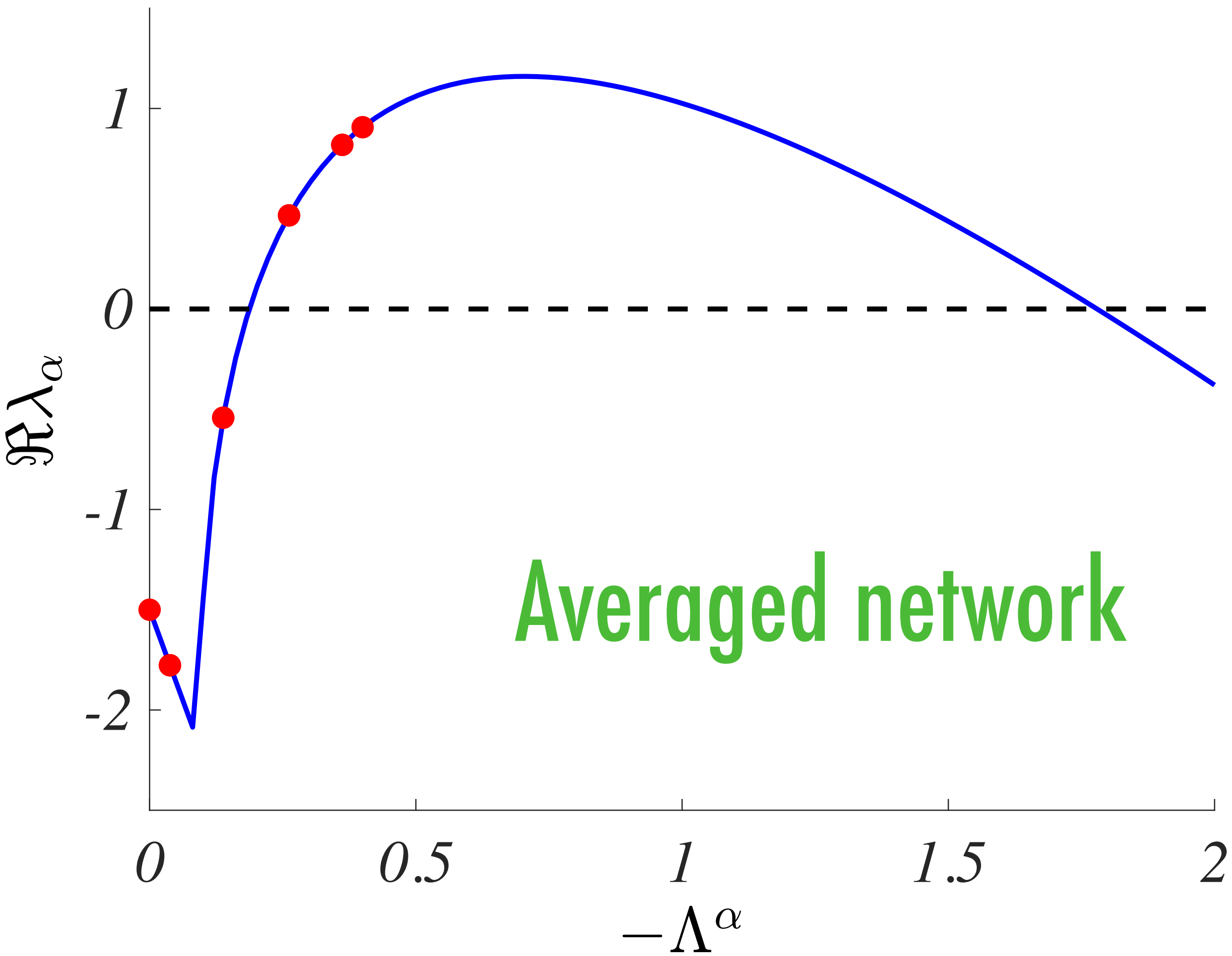


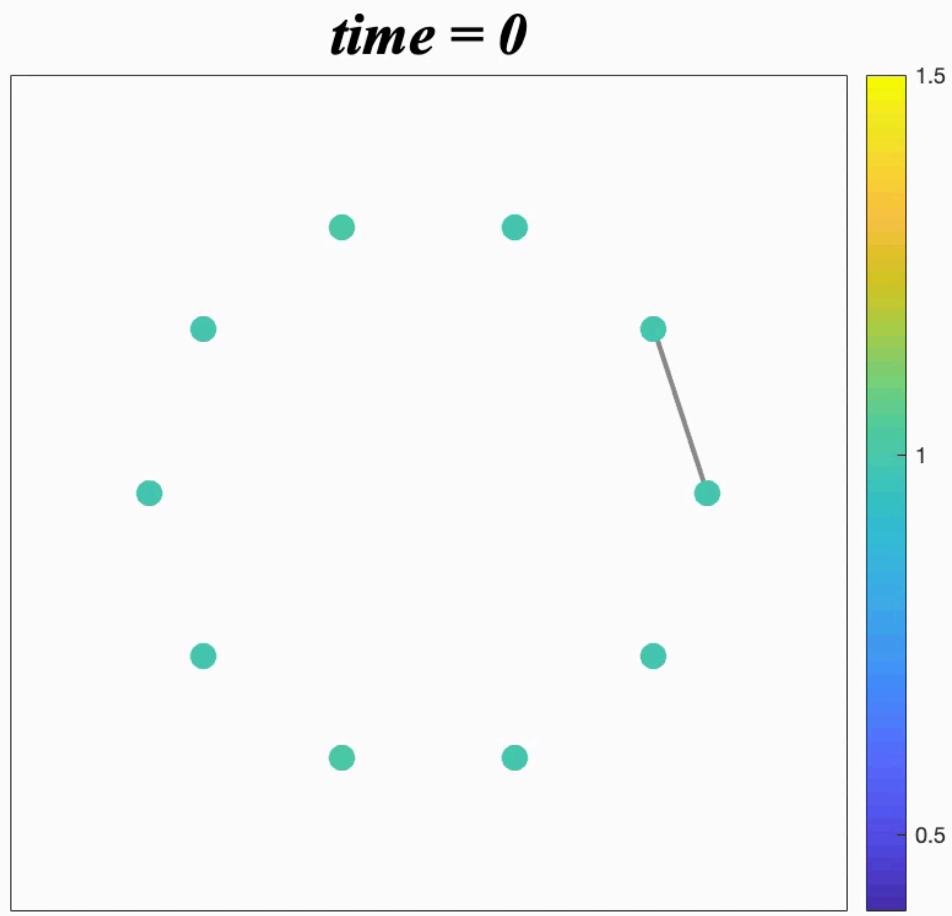
It is a matter of time: time varying networks

"Chinese whispers" (telephone game)

"fast" switch

$$\epsilon = 0.25$$





It is a matter of time: time varying networks

Assume the existence of the averaged Laplacian:

$$\langle \mathbf{L} \rangle = \frac{1}{T}\int_0^T \mathbf{L}(t)dt$$

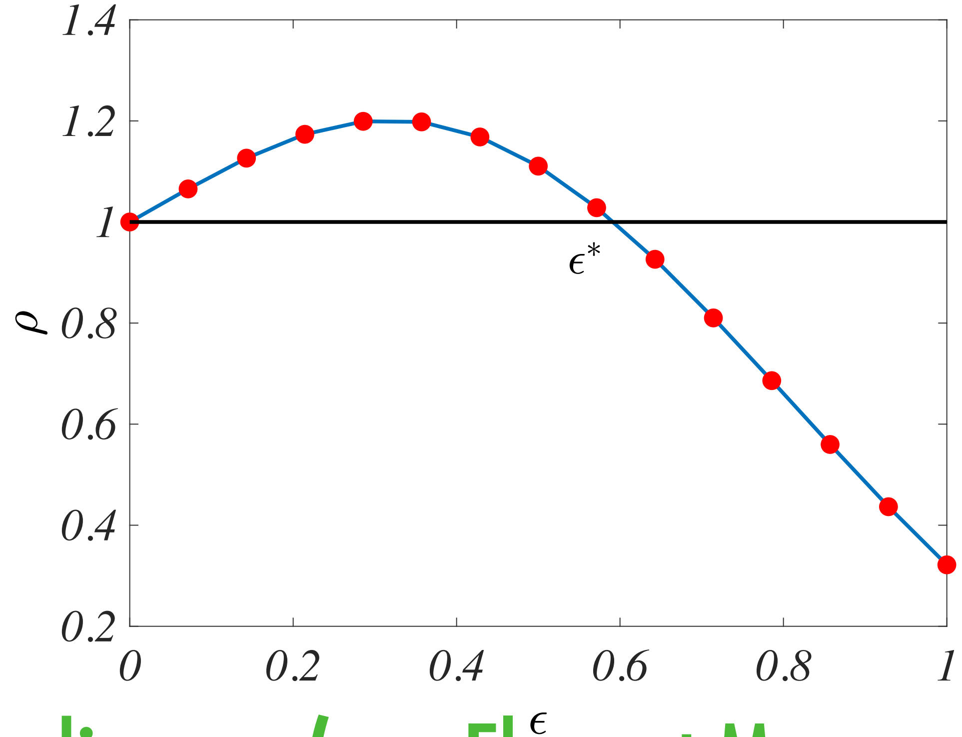
And let us introduce the averaged reaction-diffusion system:

(2) $$\begin{cases} \dot{u}_i(t) & = f(u_i, v_i) + D_u \sum_{j=1}^N \langle L_{ij} \rangle u_j \\ \dot{v}_i(t) & = g(u_i, v_i) + D_v \sum_{j=1}^N \langle L_{ij} \rangle v_j \end{cases}$$

Then, if (2) exhibits Turing patterns, there exists $\epsilon^* > 0$ such that (1) also exhibits Turing patterns $\forall \epsilon : \; 0 < \epsilon < \epsilon^*$

It is a matter of time: time varying networks

i) The threshold $\epsilon^* > 0$ can be analytically computed.



ii) Continuous non-periodic case (use Floquet-Magnus expansion)

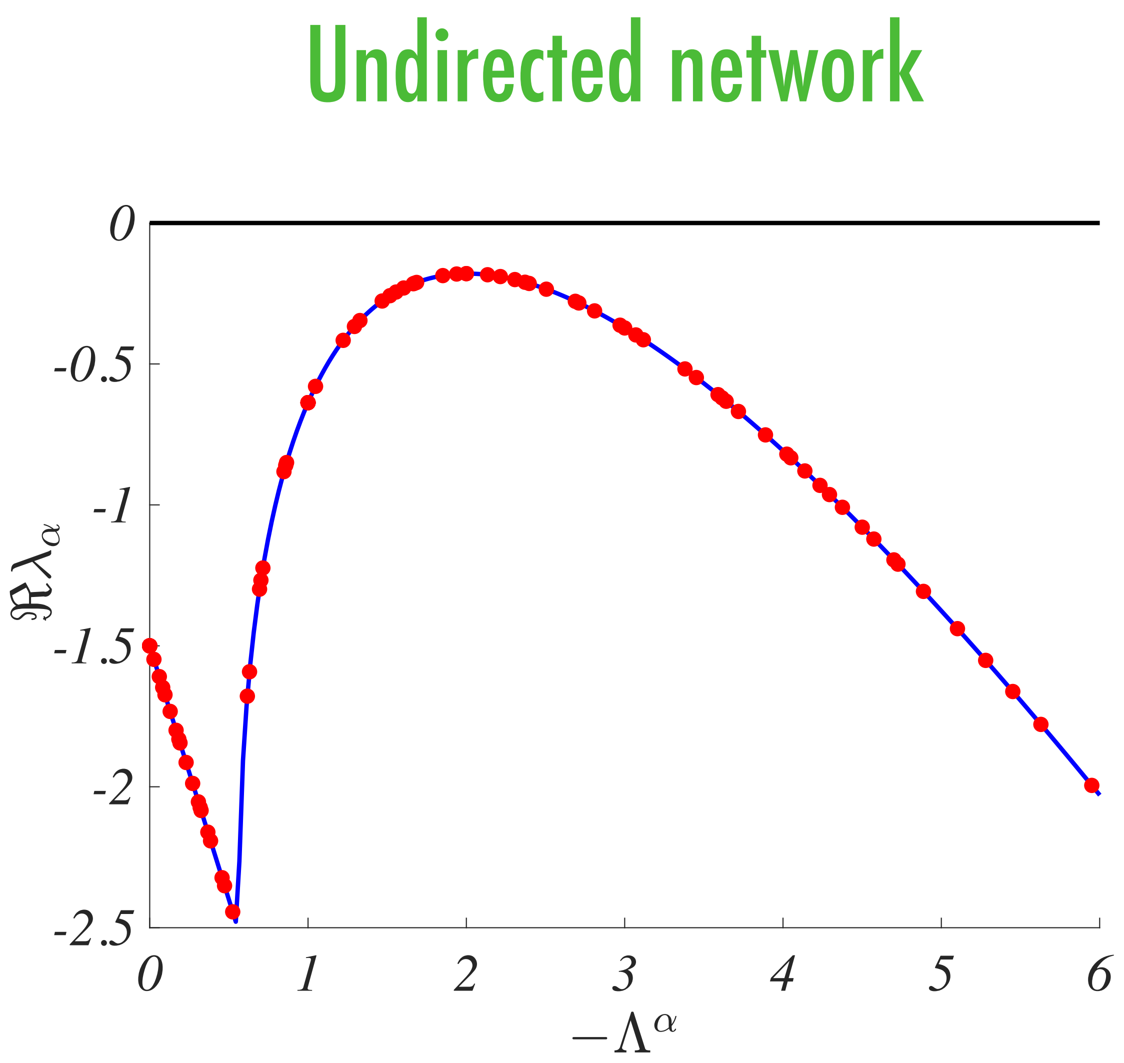
$$\langle \mathbf{L} \rangle = \lim_{T\to\infty} \frac{1}{T}\int_0^T \mathbf{L}(t)\,dt$$

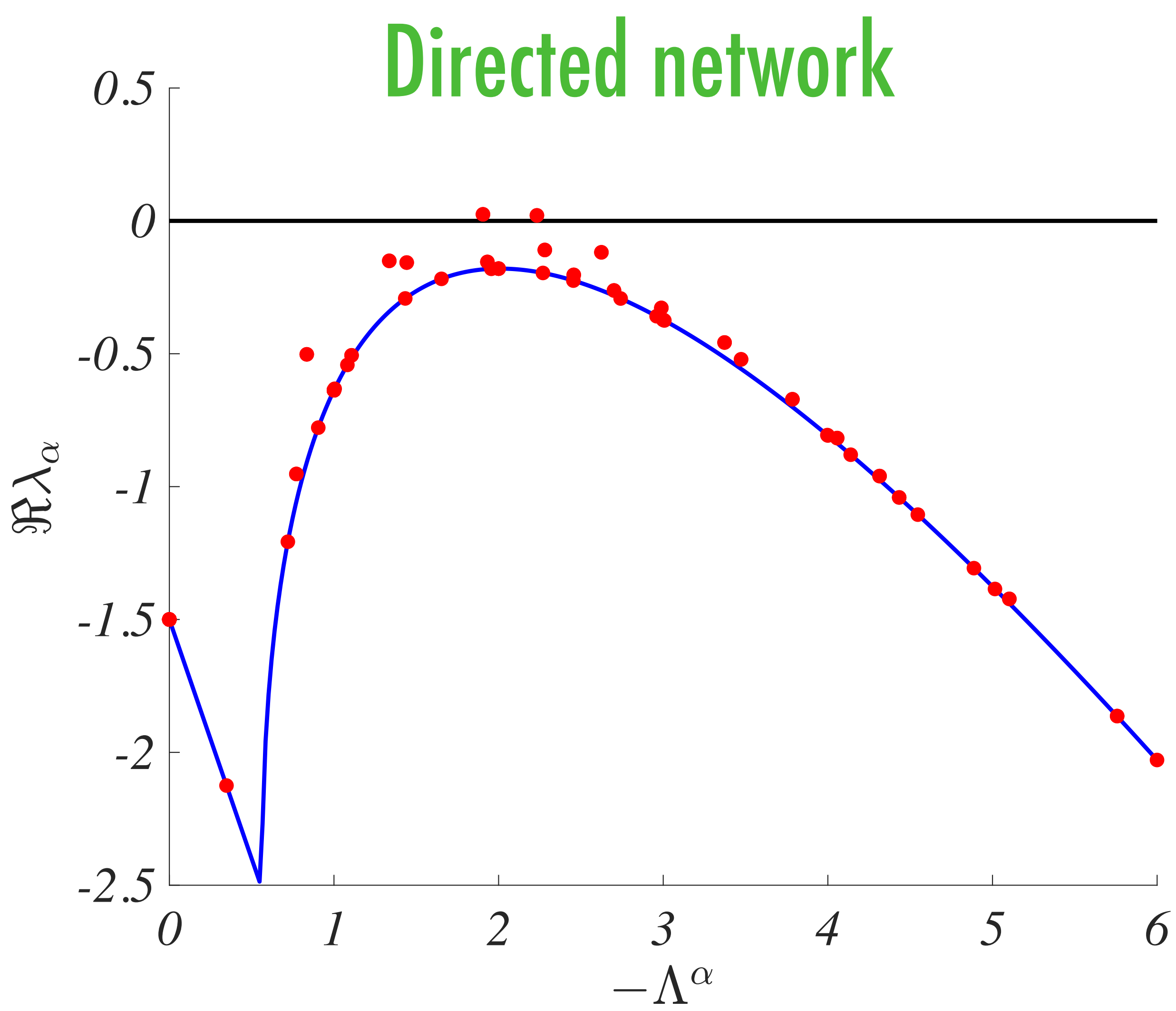
iii) Discrete time random switching:

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \mathbf{L}(t)dt = \mathbb{E}\left[\mathbf{L}(t)\right] \quad \text{(almost surely)}$$

It is a matter of direction: non-normal networks

Asllani M. et al, Nature Comm., 5, 4517,(2014)

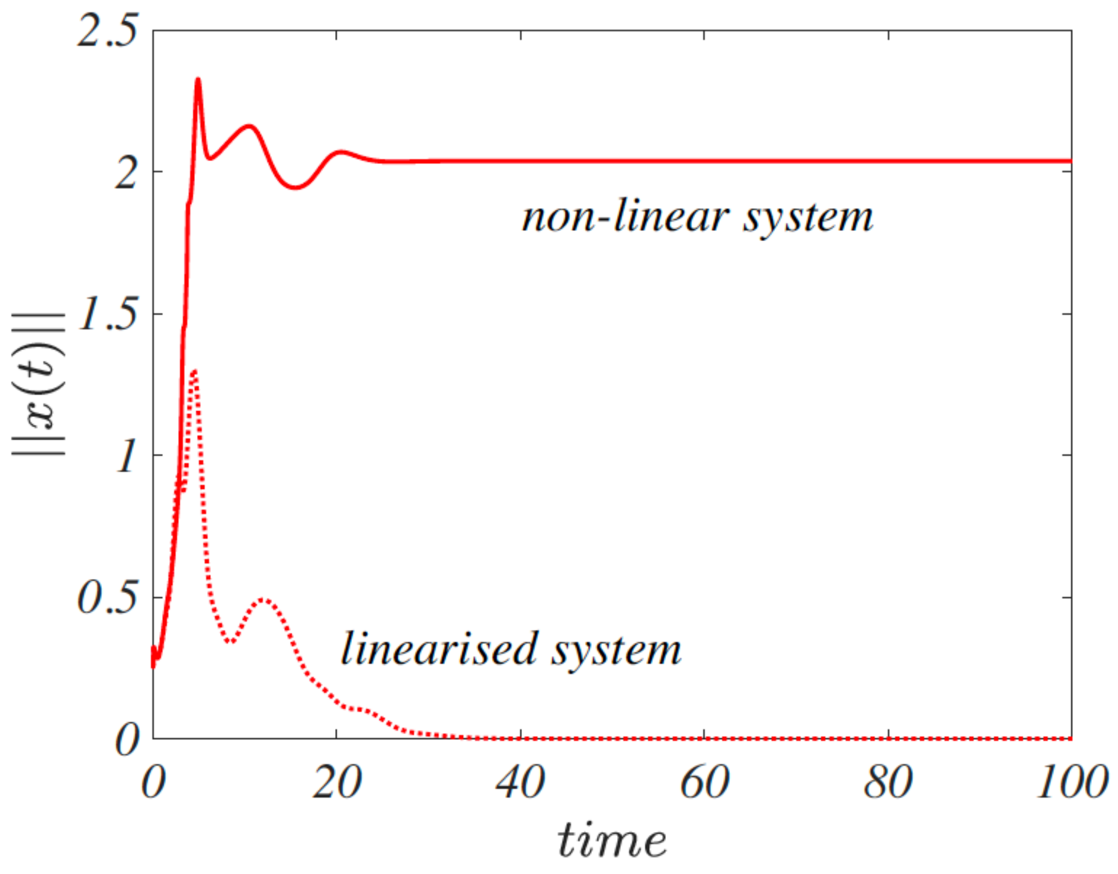


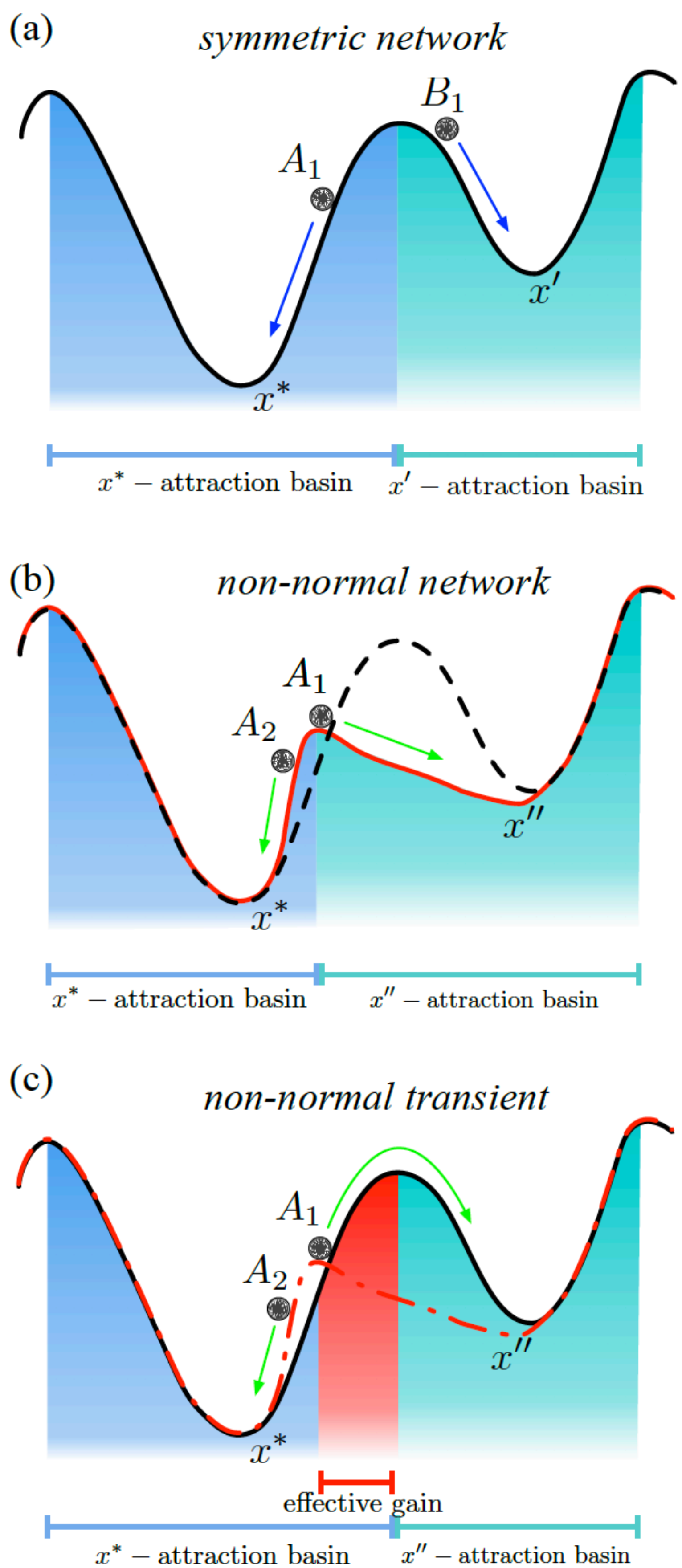


It is a matter of direction: non-normal networks

Definition (mathematical). A <u>network</u> is <u>non-normal</u>, if its adjacency matrix A_{ij} does satisfy $AA^T \neq A^T A$.

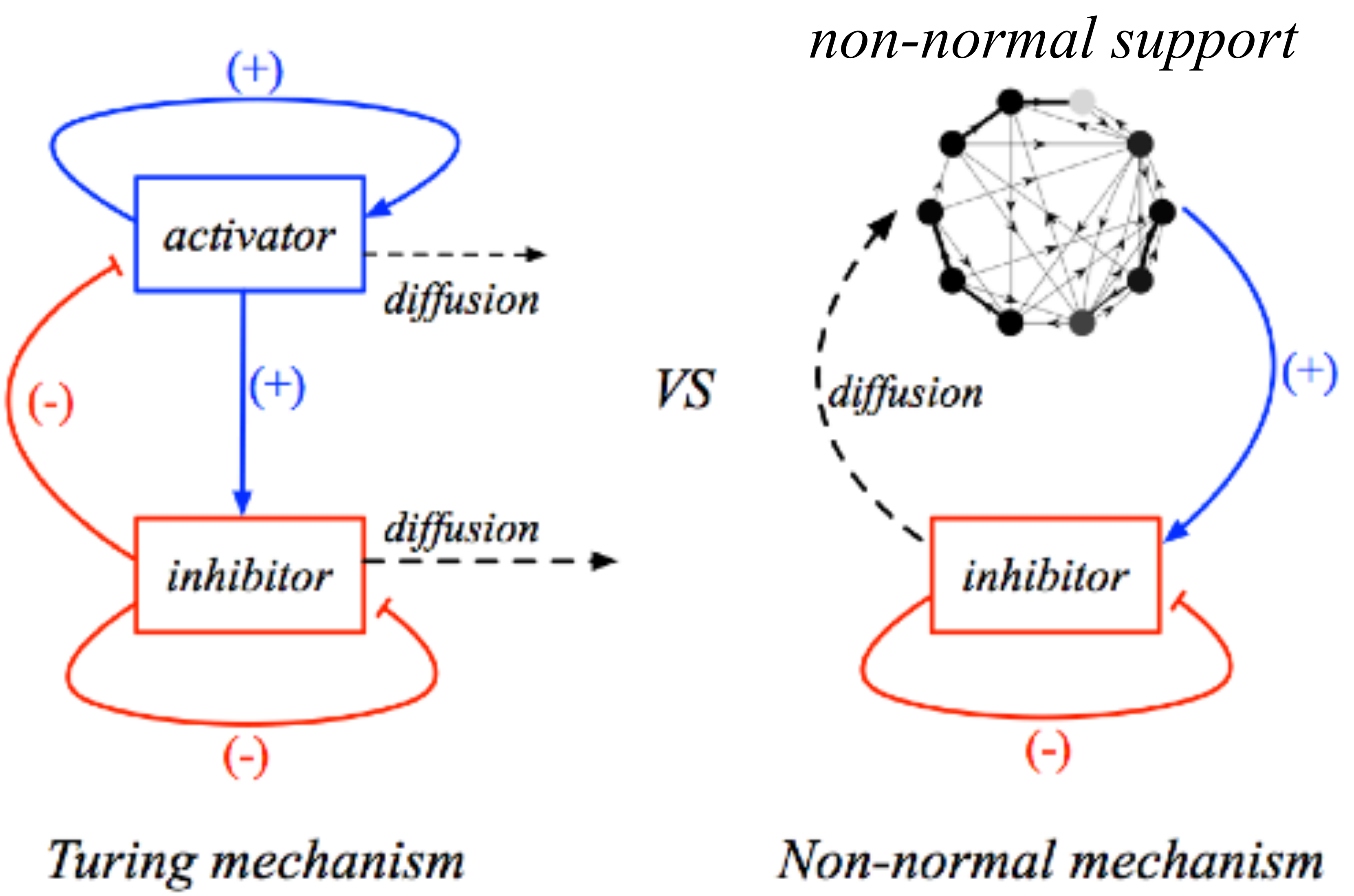
Definition (operational). It must contain a large Direct Acyclic Graph (DAG).





etti@unamur.be

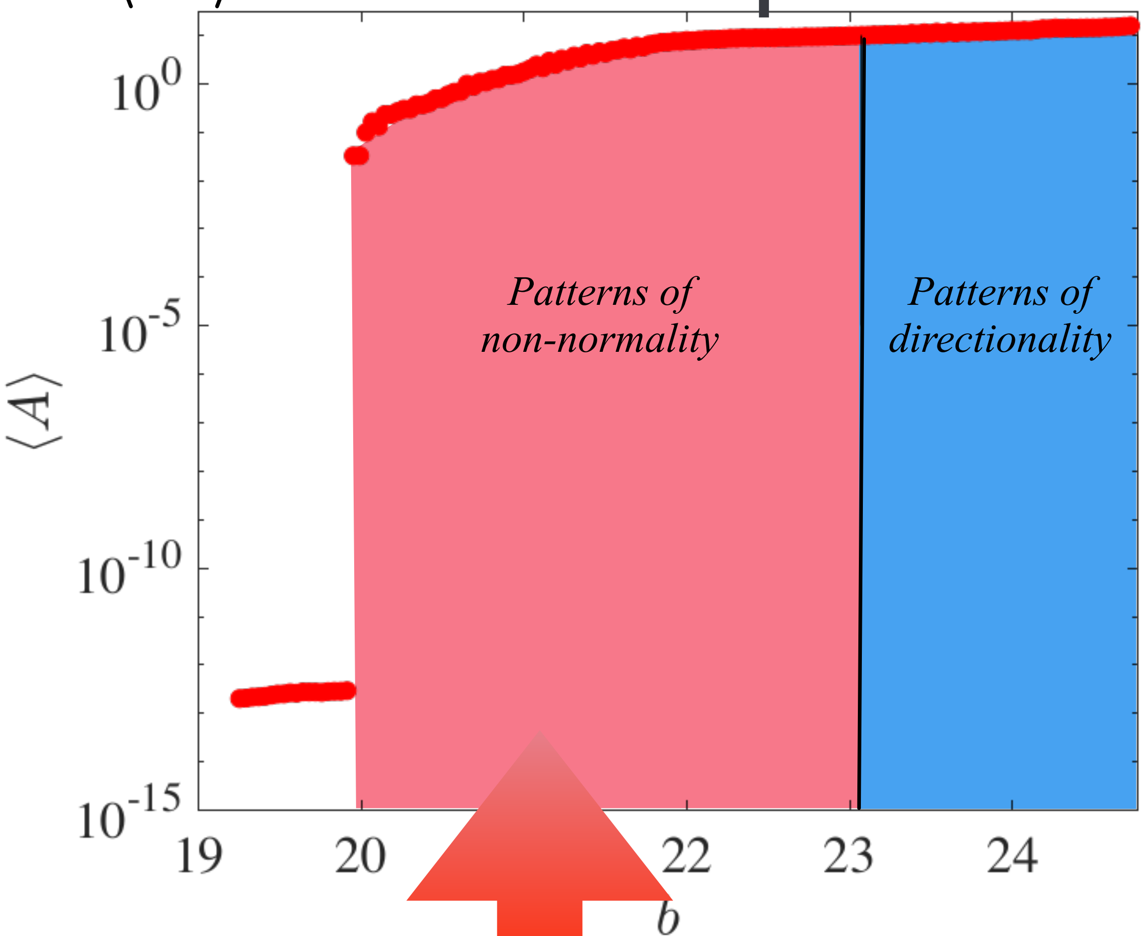
It is a matter of direction: non-normal networks

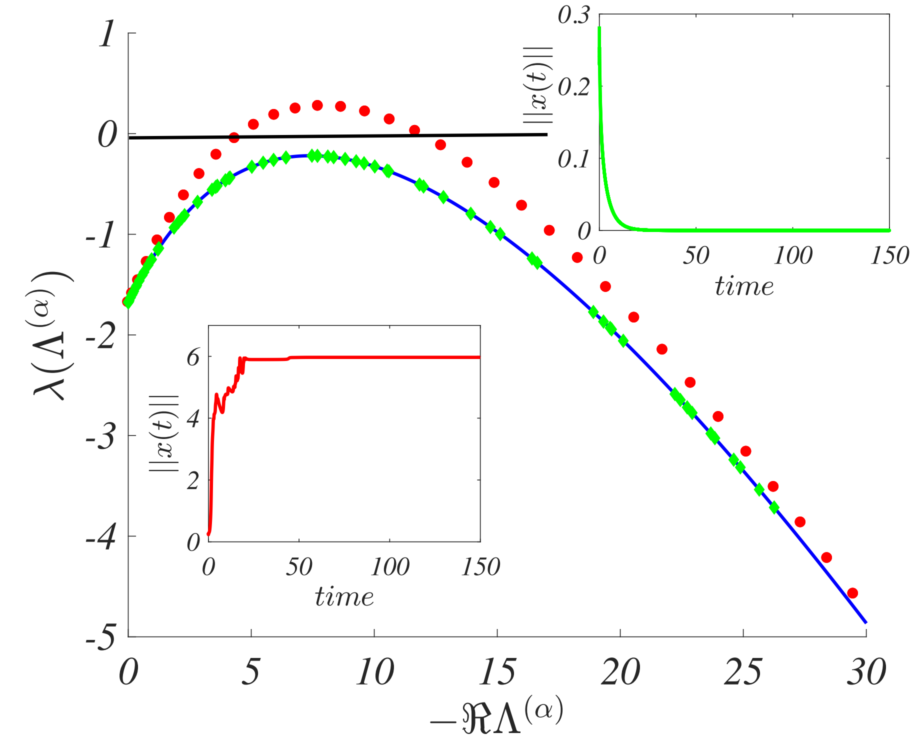


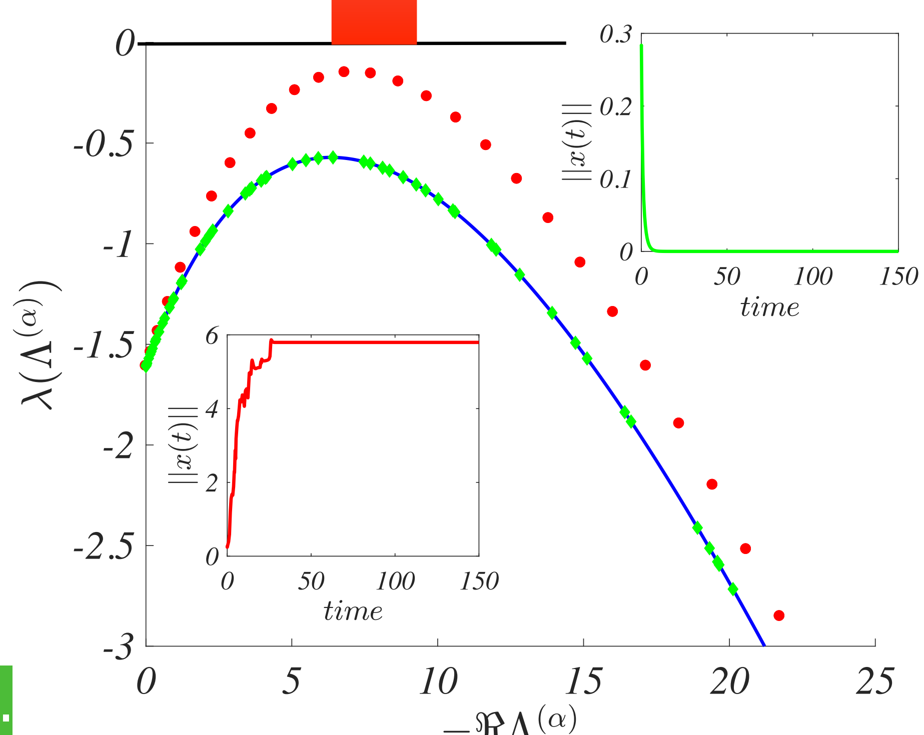
- The homogeneous equilibrium remains stable;
- One species is enough;
- No need for separate diffusion scales: $D_u \sim D_v$
- No need for activator/inhibitor.

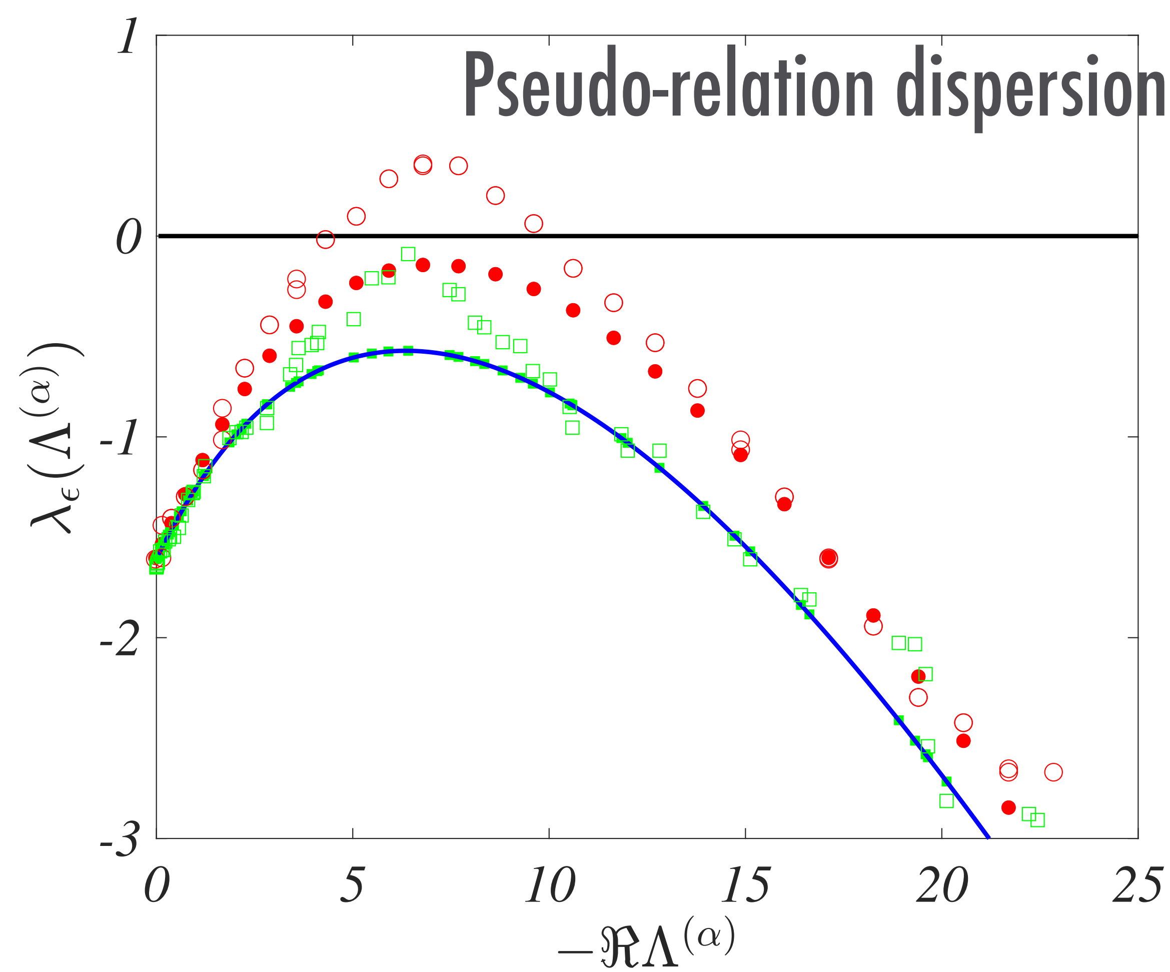
It is a matter of direction: non-normal networks

$\langle A \rangle$ Pattern amplitude









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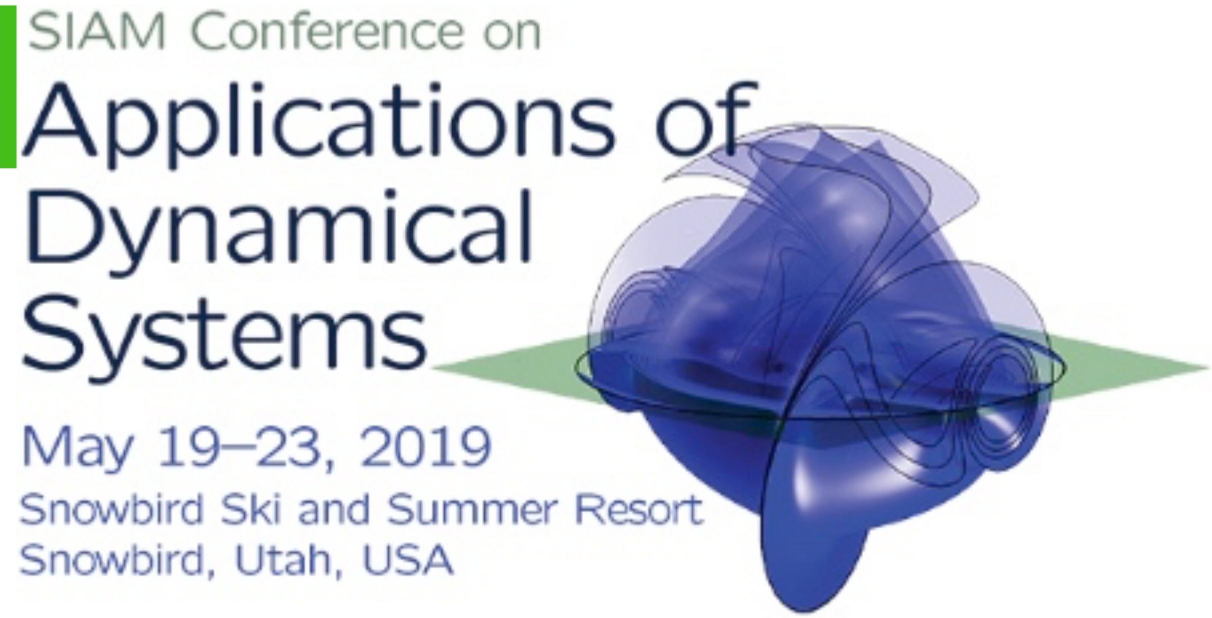

Timoteo Carletti

Dynamical instabilities in networked systems, it is a matter of time and direction

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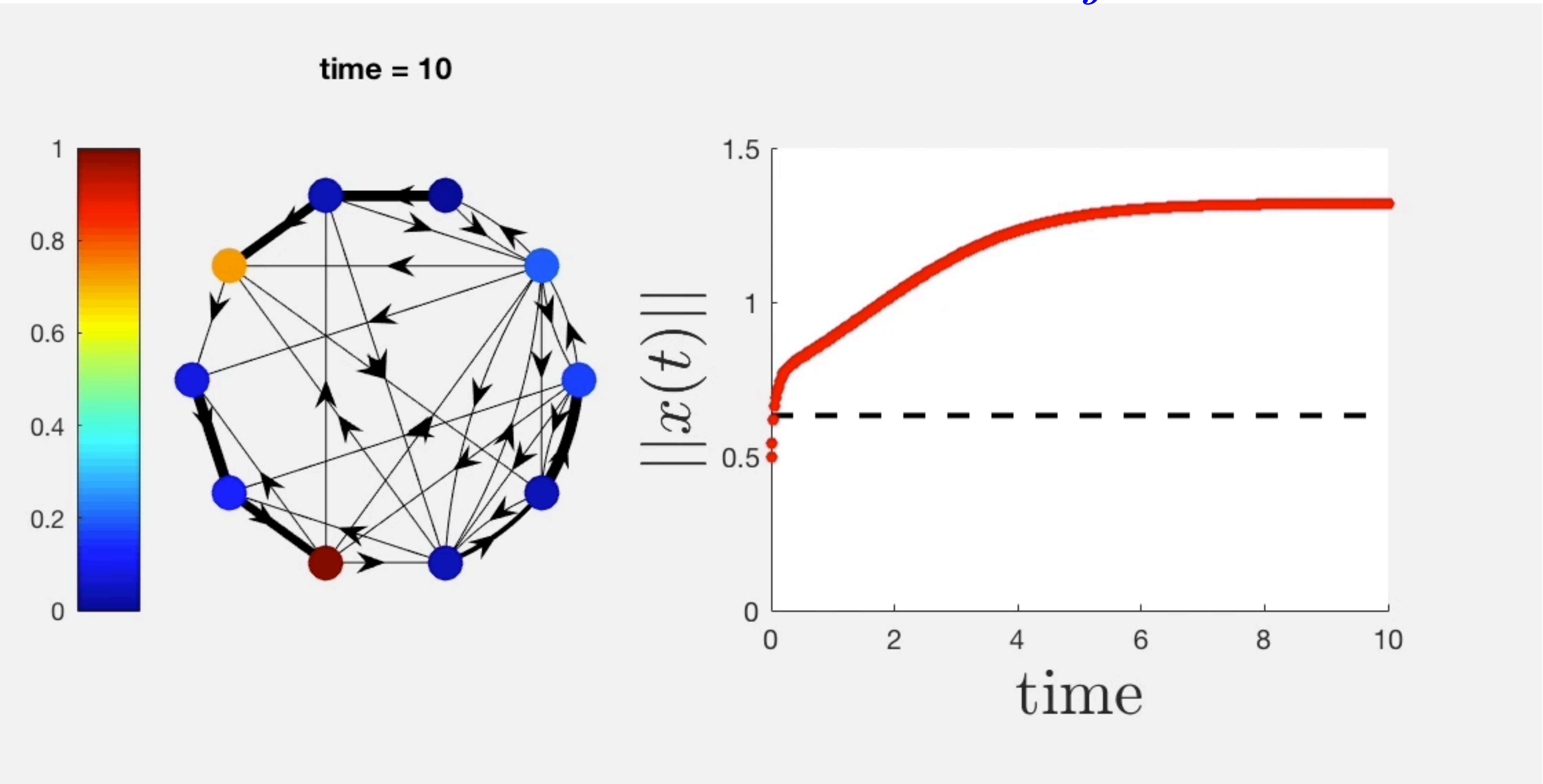
Allee effect with diffusion on a non-normal network (1 species)

$$\dot{x}_i = r x_i (1 - x_i)(x_i - a) + D \sum_{j=1}^{M} L_{ij} x_j$$



The origin remains stable but small perturbations can be macroscopically amplified and create patterns.

- J. Petit, B. Lauwens, D. Fanelli, T. Carletti, Theory of Turing Patterns on Time Varying Networks, Physical Review Letters, **119**, pp. 148301-1-5, (2017)

- M. Asllani and T. Carletti, Topological resilience in non-normal networked systems, Physical Review E, **97**, (2018), 042302

- M. Asllani, R. Lambiotte and T. Carletti, Structure and dynamics of non-normal networks, Science Adv., **4**, eaau9403, (2018), pp.1-8

- R. Muolo, M. Asllani, D. Fanelli, Ph. Maini and T. Carletti, Patterns of non-normality in networked systems, submitted

Non-normal networks are "the rule"

Network name	nodes	links	ω	$\omega - \alpha$	α_ϵ	Δ	$\hat{d}_F$
Foodwebs							
Cypress wetlands South Florida (wet)	128	2016	296.71	132.11	167.46	0.83	1.00
Cypress wetlands South Florida (dry)	128	2137	217.60	152.50	82.20	0.89	1.00
Little Rock Lake (Wisconsin, US)	183	2494	21.69	14.69	10.02	0.95	0.93
Biological							
Transcriptional regulation network (*E. coli*)	423	578	5.11	4.11	2.52	0.81	0.93
Metabolic network (*C. Elegans*)	453	4596	13.44	12.44	6.89	0.98	1.00
Pairwise proteins interaction (*Homo sapiens*)	2239	6452	15.79	13.02	4.01	0.99	0.99
Transport							
US airport 2010	1574	28236	$1.19\,10^7$	79.30	$1.19\,10^7$	0.01	1.00
Road transportation network (Rome)	3353	8870	$2.40\,10^4$	120.05	$2.39\,10^4$	0.08	0.28
Road transportation network (Chicago)	12982	39018	4.23	$4.29\,10^{-4}$	4.54	0.04	0.19
Communication							
e-mails network DNC	2029	39264	28.00	2.00	26.37	0.53	0.89
Enron email network (1999-2003)	87273	1148072	85.14	14.54	71.05	0.30	0.99(∗)
e-mails network European institution	265214	420045	76.02	6.09	70.30	0.30	0.84(∗)
Citation							
Citations to Milgram's 1967 paper (2002)	395	1988	10.48	10.48	4.49	1.00	1.00
Articles from Scientometrics (1978-2000)	3084	10416	10.32	8.32	5.28	0.98	1.00
Citation network DBLP	12591	49743	21.50	16.82	8.45	0.87	1.00
Social							
Hyper-network of 2004 US election blogs	1224	19025	45.37	10.95	34.95	0.72	0.98
Reply network of the news website Digg	30398	87627	15.92	6.56	10.18	0.61	0.97
Trust network from the website Epinions	75879	508837	123.00	16.47	106.96	0.13	0.80(∗)