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# Random walks on Hypergraphs

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# Acknowledgements

D. Fanelli UniFi
R. Lambiotte Oxford
F. Battiston CEU
G. Cencetti
S. Nicoletti

# networks

# Dynamics

# networks

# Structure

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naxys
Namur Center for Complex Systems

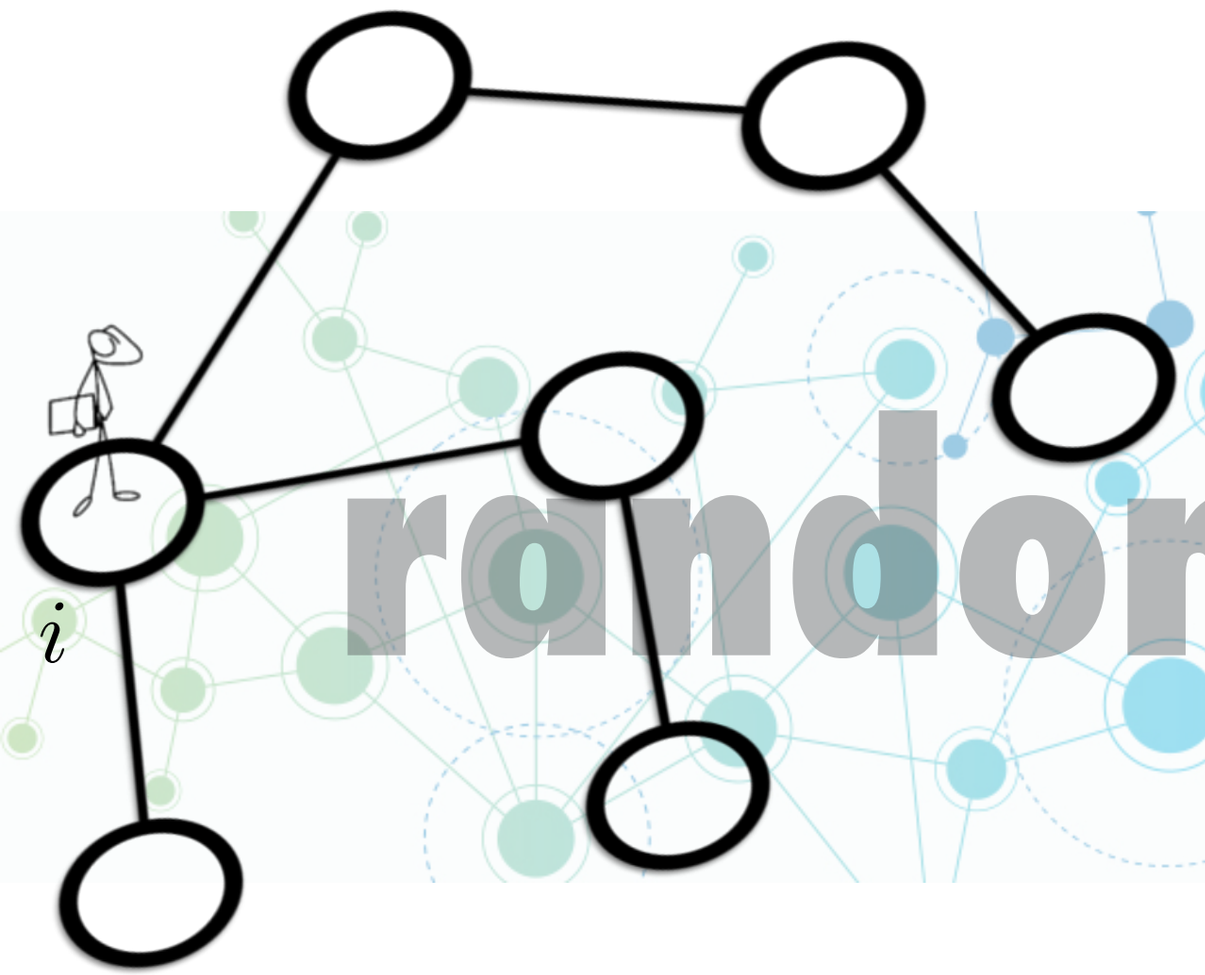
# random walks

Number of nodes: $\Omega$

Adjacency matrix: $A_{ij} = A_{ji} = 1$

Nodes degree: $k_i = \sum_j A_{ij}$

# random walks

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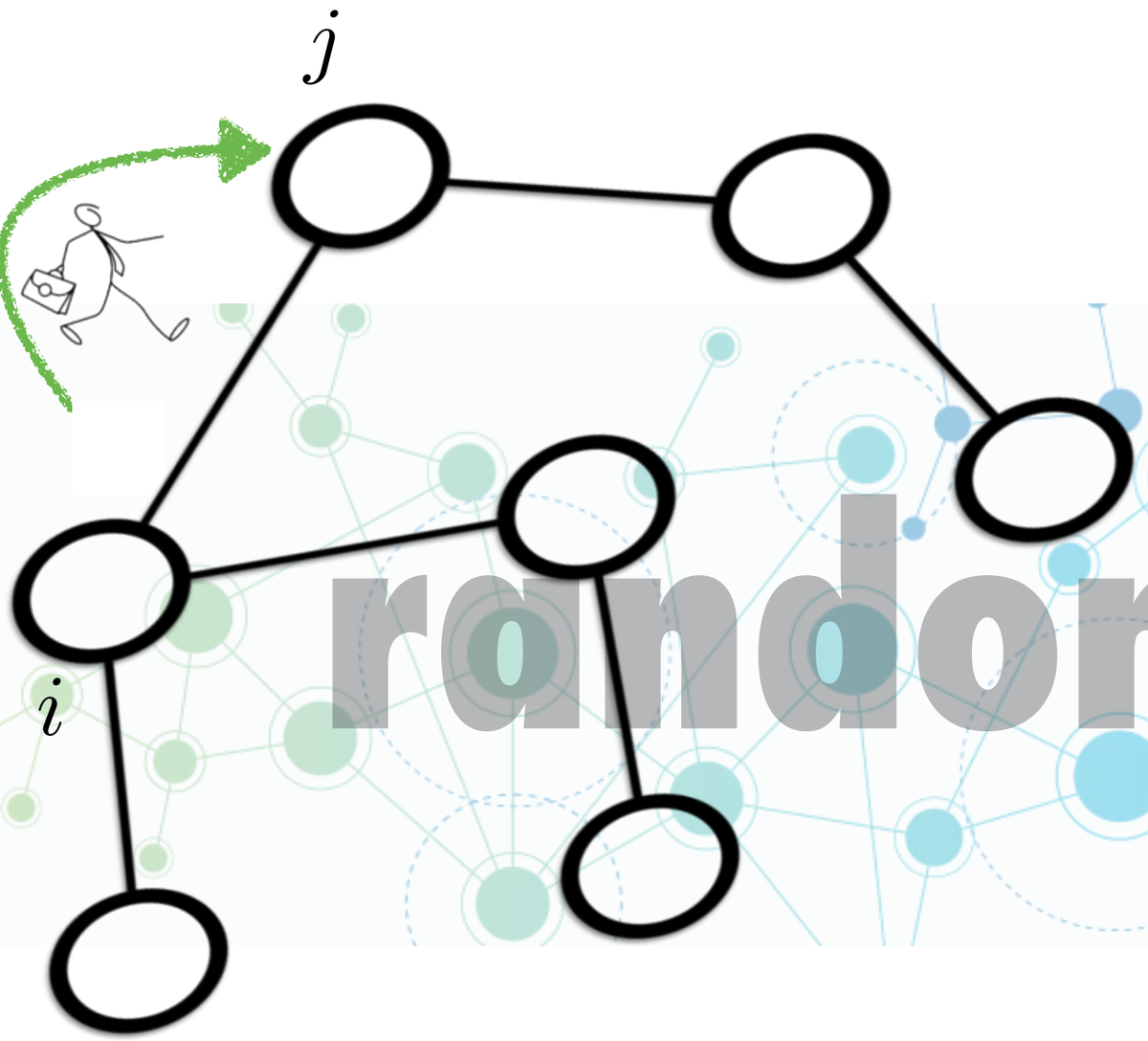
# Random walk on networks

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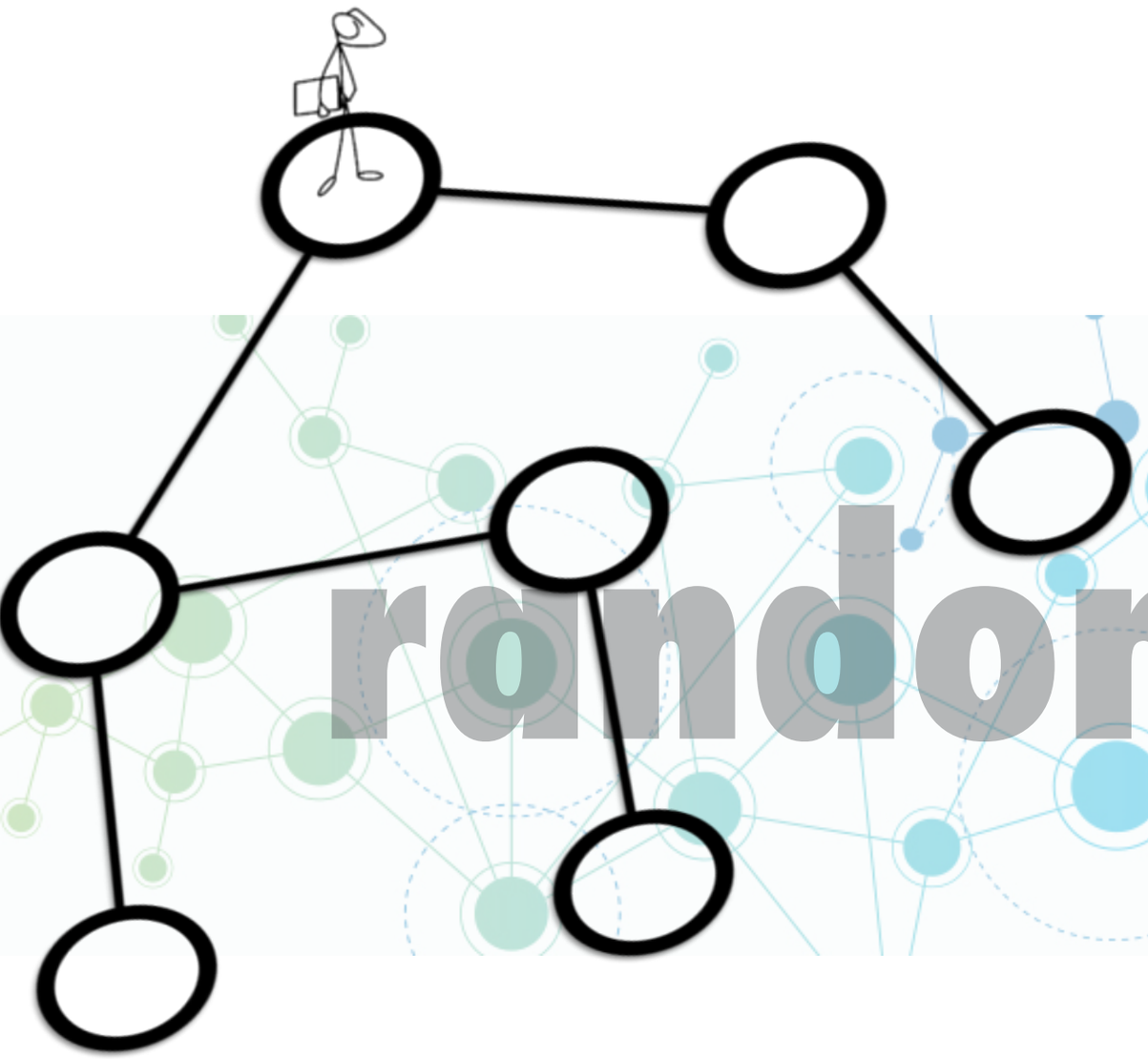
random walks

Transition probability form i to j : $T(j|i) = \frac{A_{ij}}{k_i}$

# Random walk on networks



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Nodes degree: $k_i = \sum_j A_{ij}$

random walks

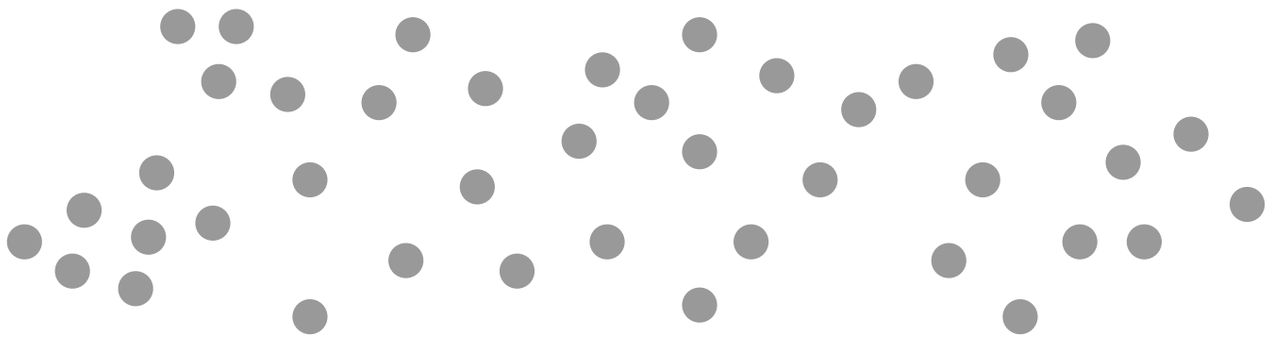
Transition probability form i to j : $T(j|i) = \frac{A_{ij}}{k_i}$

$$P_i(t) \sim k_i \qquad \text{(asymptotically)}$$

# networks limitation

# Hypergraphs

# Hypergraphs. Some definitions.



ensemble of nodes

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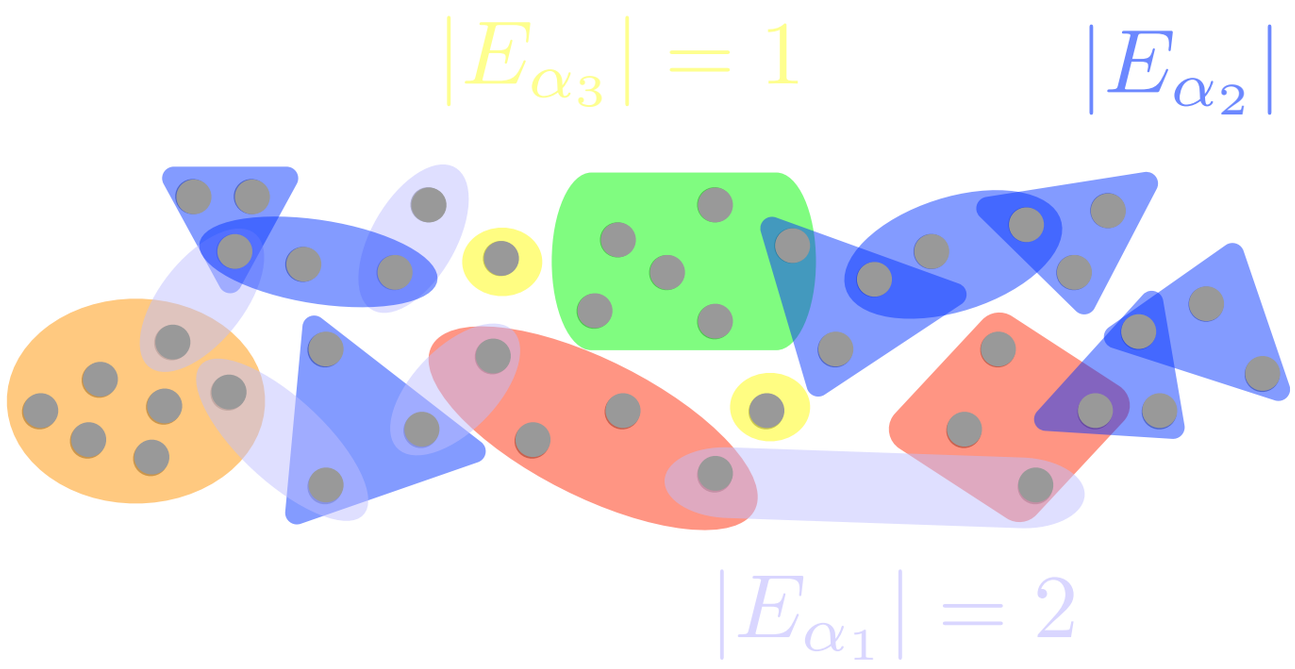
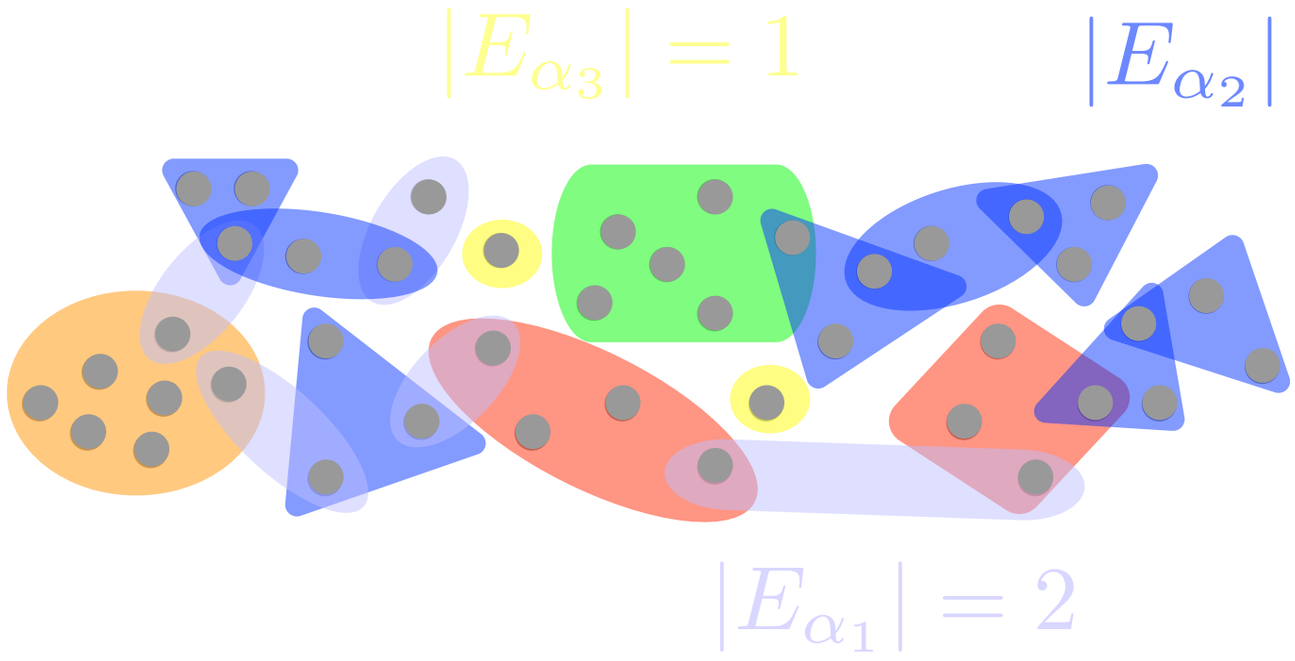



ensemble of nodes

hyperedges = set of nodes

# Hypergraphs. Some definitions.




ensemble of nodes

hyperedges

**Incidence matrix**

$$e_{i\alpha} = 1 \quad \text{iff } i \in E_\alpha$$

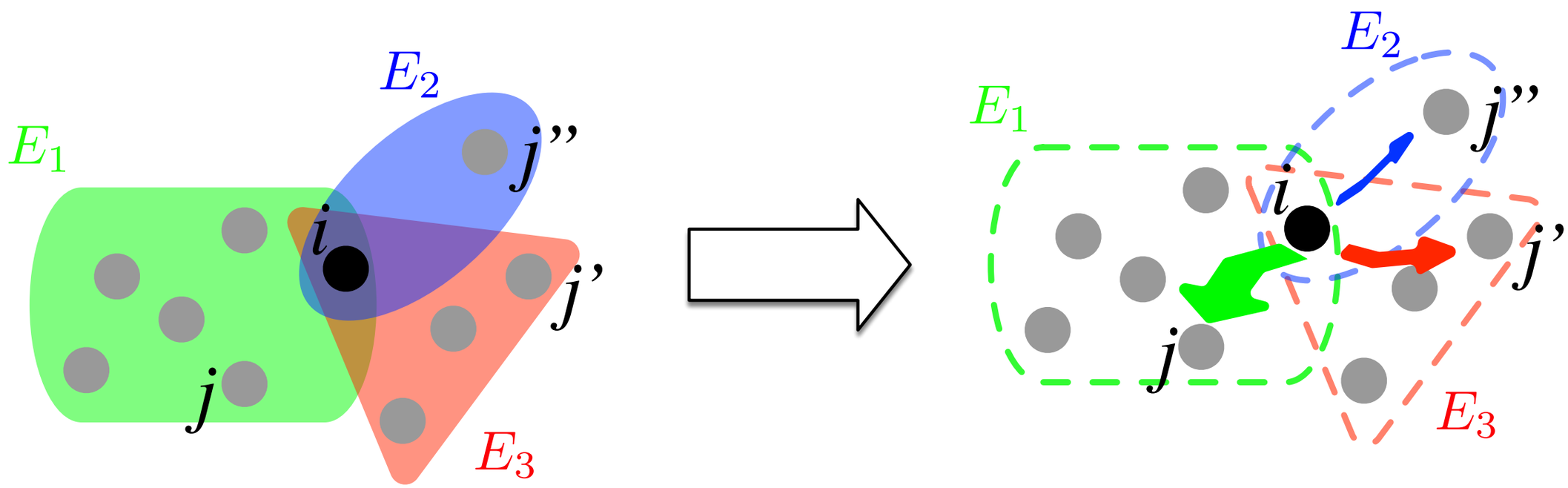
**Hyperadjacency matrix**

$$A = ee^{\mathrm{T}}$$

**Hyperedge matrix**

$$C = e^{\mathrm{T}}e$$

# Hypergraphs. Randomwalk.




$$k_{ij}^{H} = \sum_{\alpha} (C_{\alpha\alpha} - 1)^{\tau} e_{i\alpha} e_{j\alpha}$$

hyperedge size

incidence matrices

# Hypergraphs. Randomwalk.

$$k_{ij}^{H} = \sum_{\alpha} (C_{\alpha\alpha} - 1)^{\tau} e_{i\alpha} e_{j\alpha}$$

$$L_{ij}^{H} = \delta_{ij} - \frac{k_{ij}^{H}}{\sum_{\ell \neq i} k_{i\ell}^{H}} \qquad p_i^{(\infty)} = \frac{k_i^H}{\sum_{\ell} k_{\ell}^{H}}$$

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## Random walks on hypergraphs

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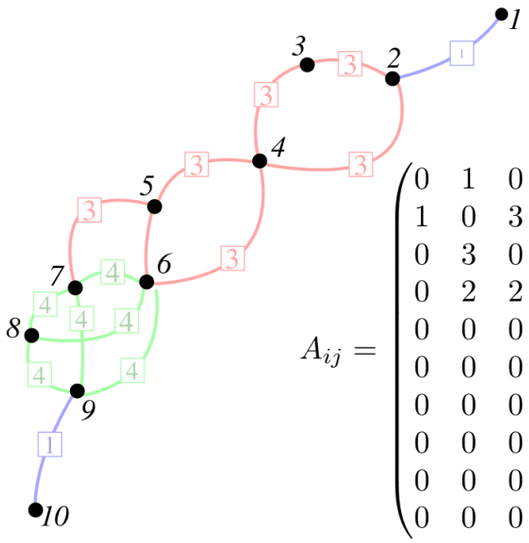
[2]*Department of Network and Data Science, Central European University, Budapest 1051, Hungary*
[3]*Mobile and Social Computing Lab, Fondazione Bruno Kessler, Via Sommarive 18, 38123 Povo, Trento, Italy*
[4]*Dipartimento di Fisica e Astronomia, Università di Firenze, INFN, and CSDC, Via Sansone 1, 50019 Sesto Fiorentino, Firenze, Italy*

(Received 14 November 2019; accepted 20 January 2020; published 18 February 2020)

# Hypergraphs & projections

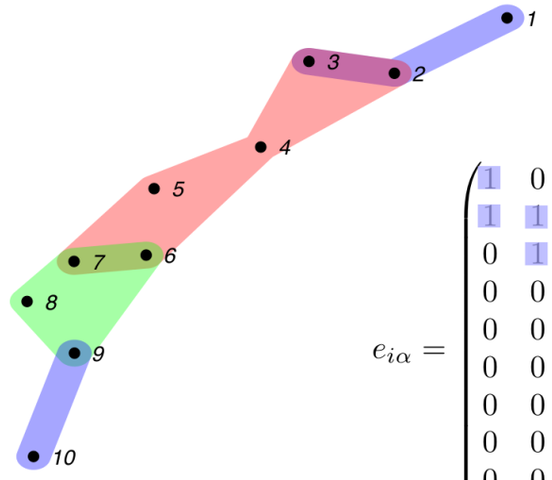
*clique reduced multigraph*




$$A_{ij} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 3 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 & 2 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 4 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 4 & 0 & 5 & 3 & 3 & 0 \\ 0 & 0 & 0 & 0 & 2 & 5 & 0 & 3 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 3 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 3 & 3 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

*A*

*hypergraph*

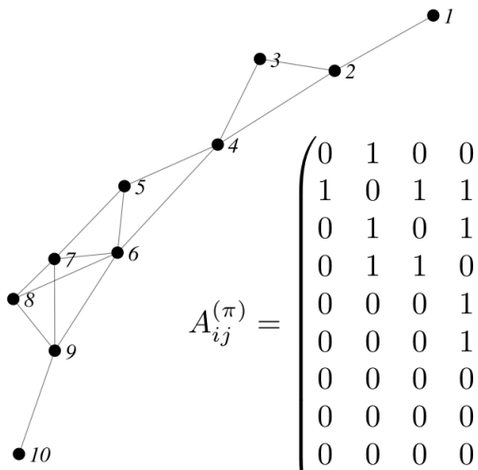



$$e_{i\alpha} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

*B*

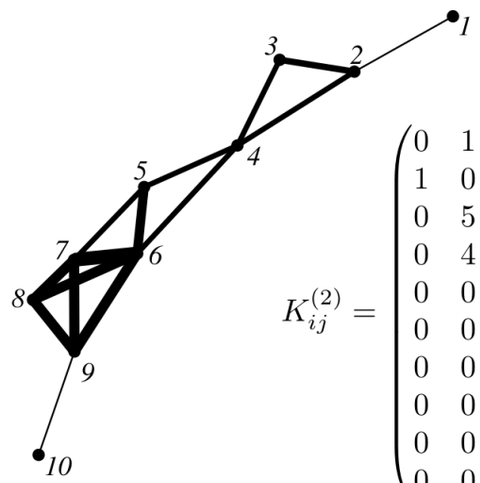
*C*

*projected network*




$$A^{(\pi)}_{ij} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

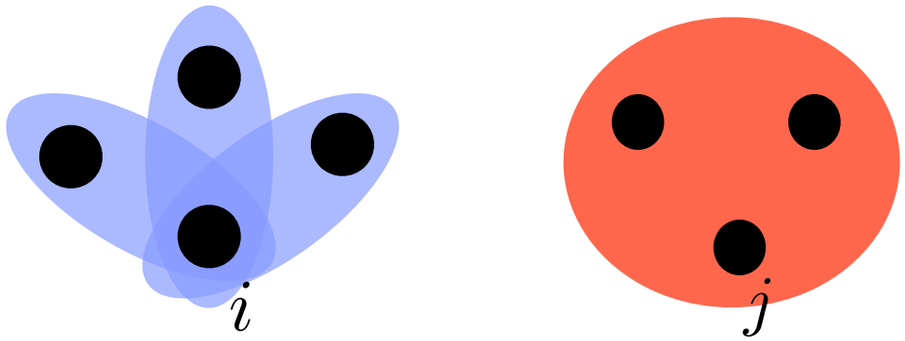
*equivalent weighted network*




$$K^{(2)}_{ij} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 5 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 4 & 0 & 4 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 8 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 8 & 0 & 13 & 9 & 9 & 0 \\ 0 & 0 & 0 & 0 & 4 & 13 & 0 & 9 & 9 & 0 \\ 0 & 0 & 0 & 0 & 0 & 9 & 9 & 0 & 9 & 0 \\ 0 & 0 & 0 & 0 & 0 & 9 & 9 & 9 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

# Ranking inversion (I)

$$k_i^H = 3 < k_j^H = 4$$



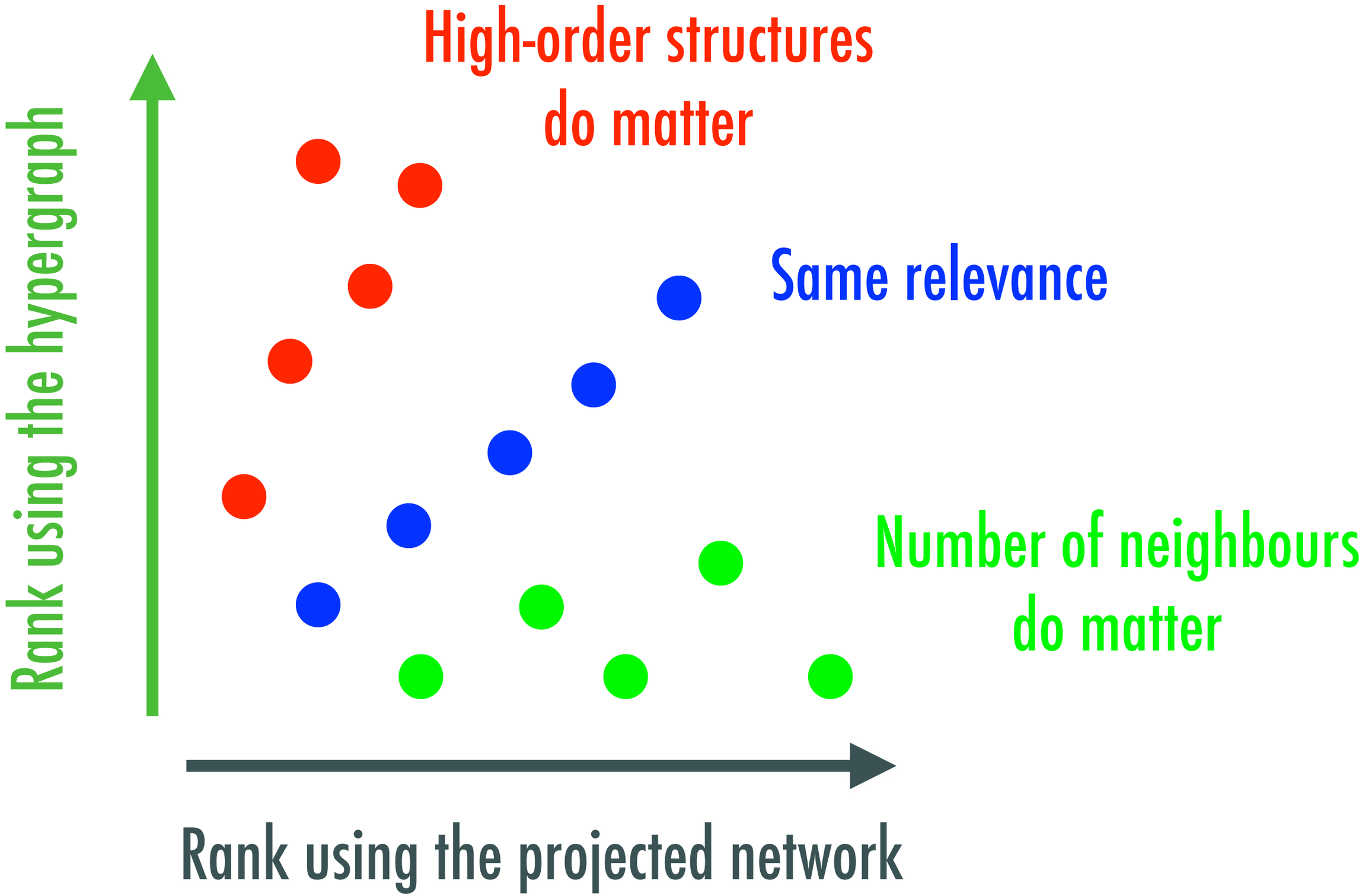

$$\mathrm{rank}(j) > \mathrm{rank}(i)$$

clique projection


$$k_i = 3 > k_j = 2$$

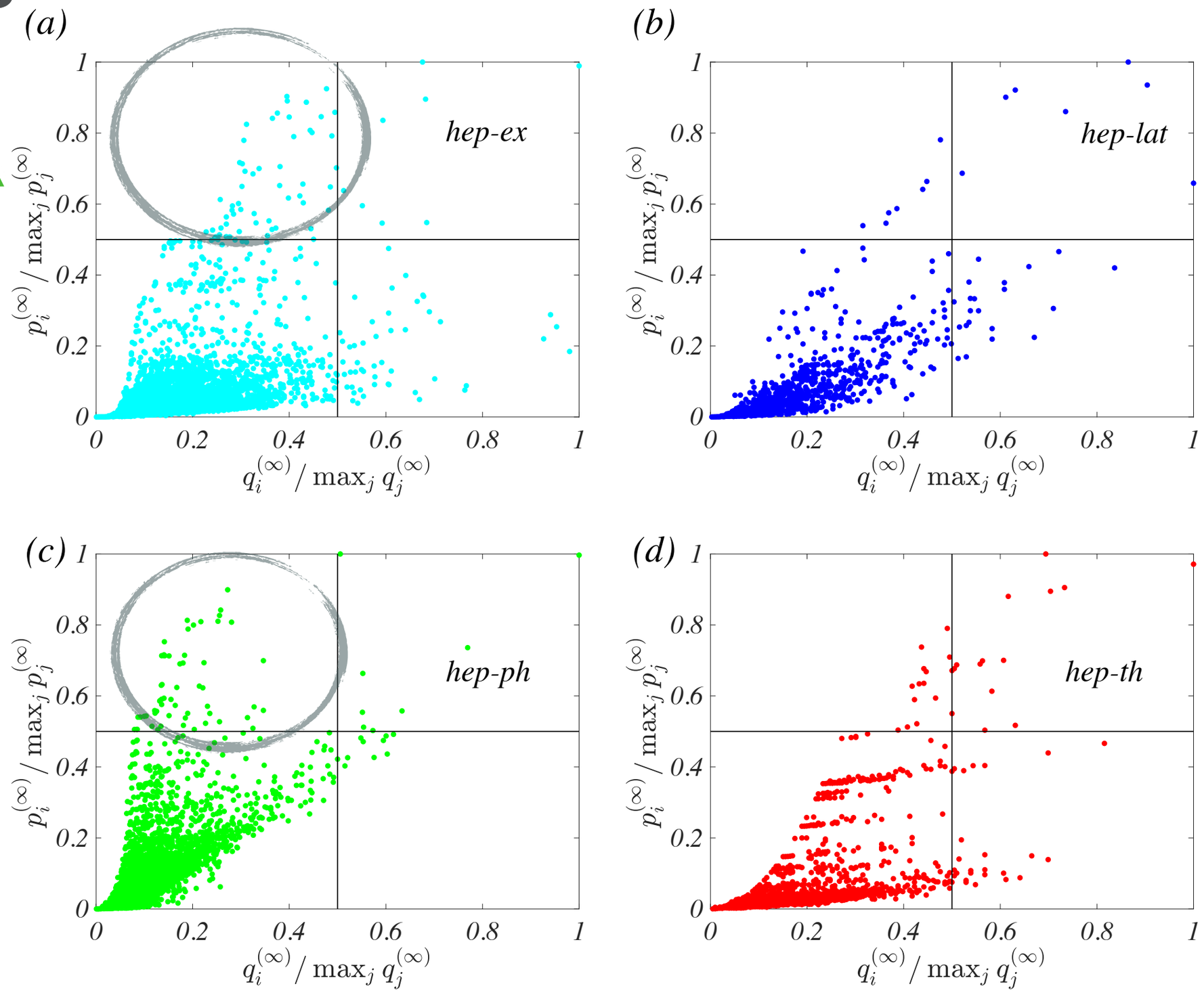
$$\mathrm{rank}(j) < \mathrm{rank}(i)$$

# Ranking on hypergraphs and on the projected network



# Ranking inversion arXiv.

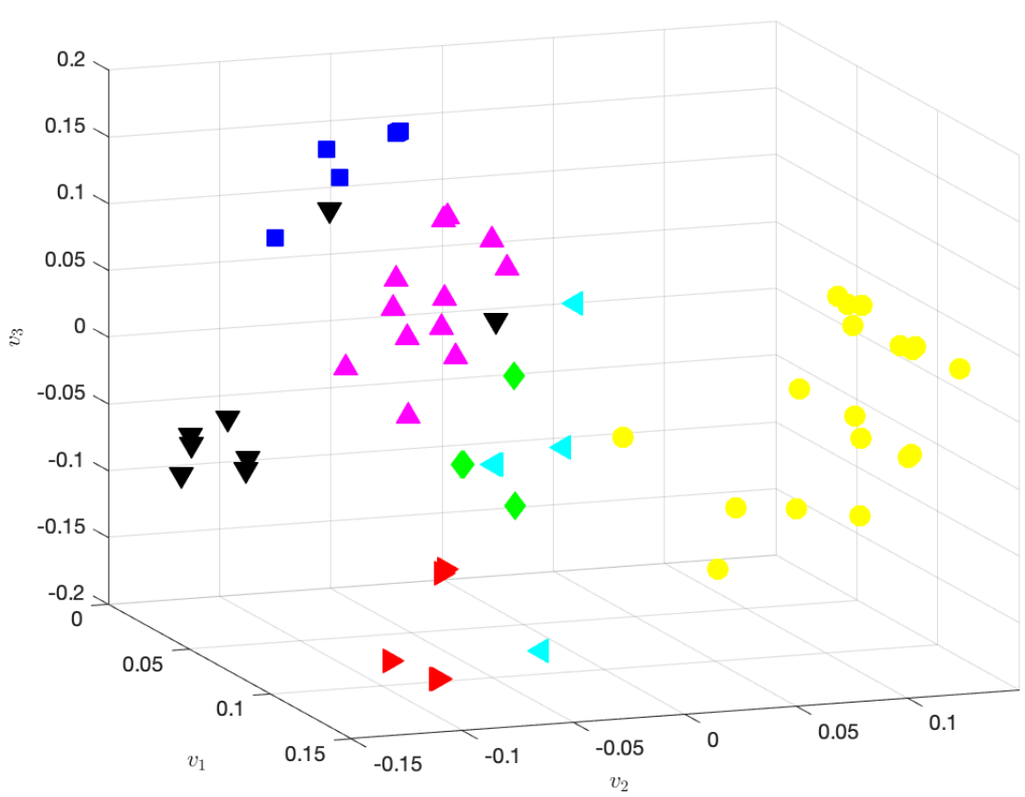
Rank using the hypergraph



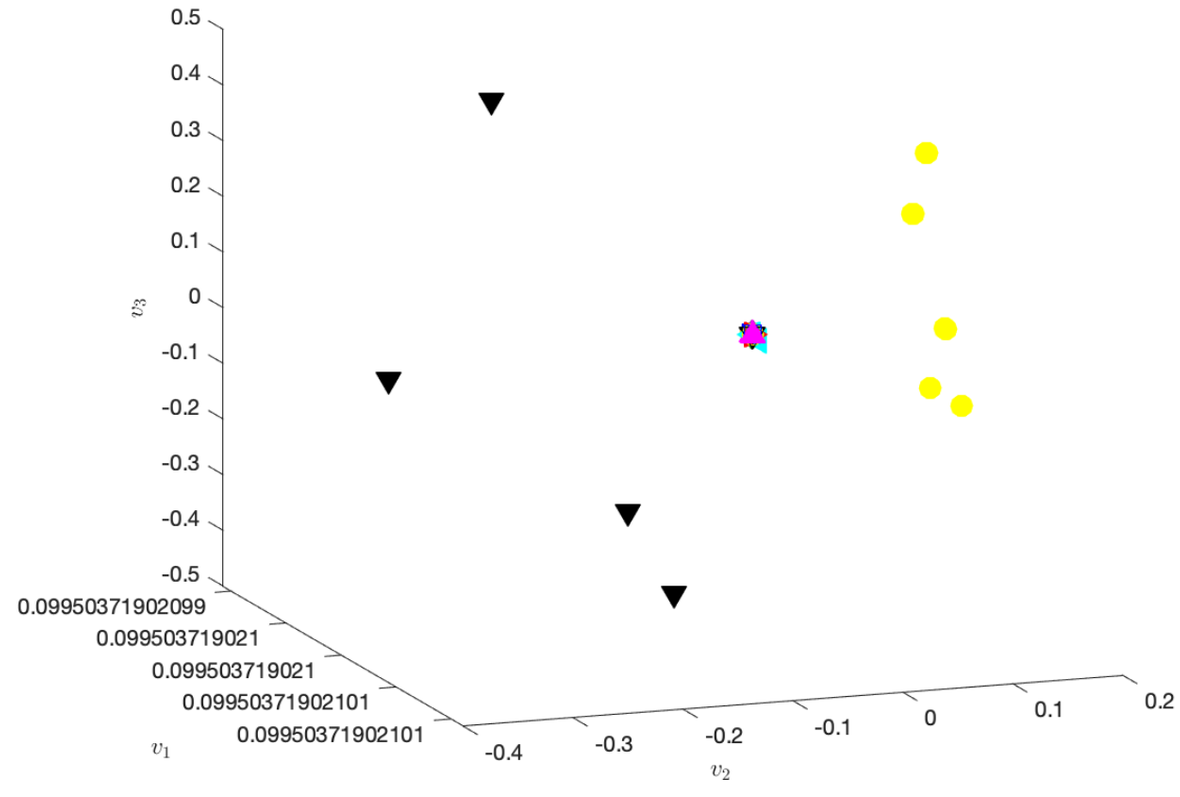

Rank using the projected network

# Classification

UCI zoo database, 101 animals, 16 features (hair, feathers, eggs, milk, airborne, aquatic, predator, toothed, backbone, breathes, venomous, fins, tail, domestic, number of legs, cat size), 7 classes.




Hypergraph




Clique projection

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**Random walks on hypergraphs**

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# Community detection

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PAPER

# Random walks and community detection in hypergraphs

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**Keywords:** hypergraphs, random walks, higher-order networks

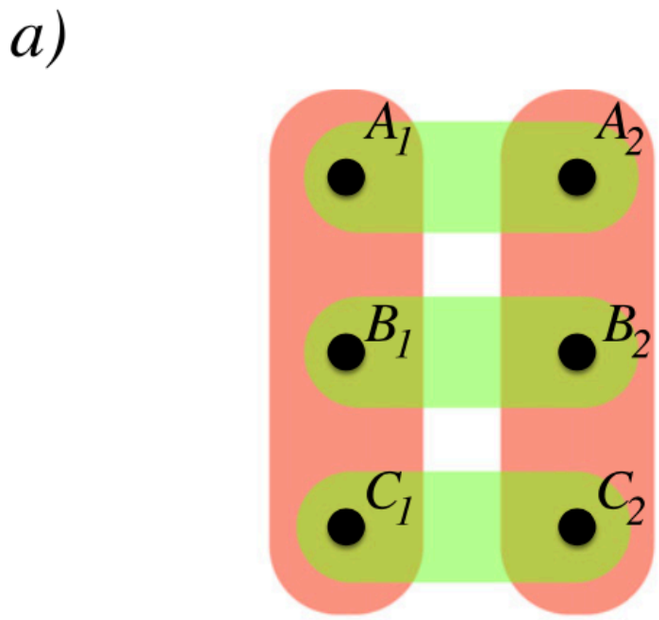


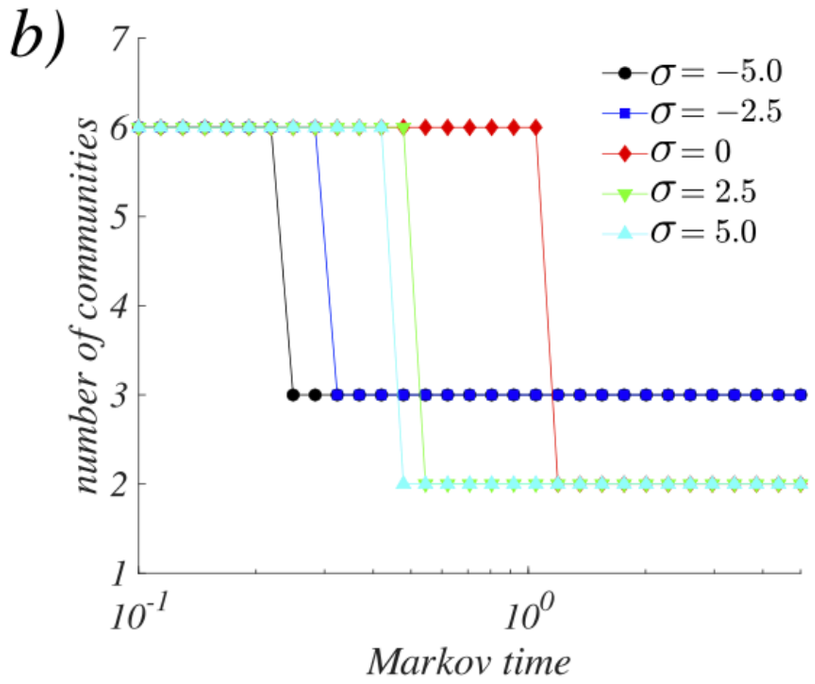
## Abstract

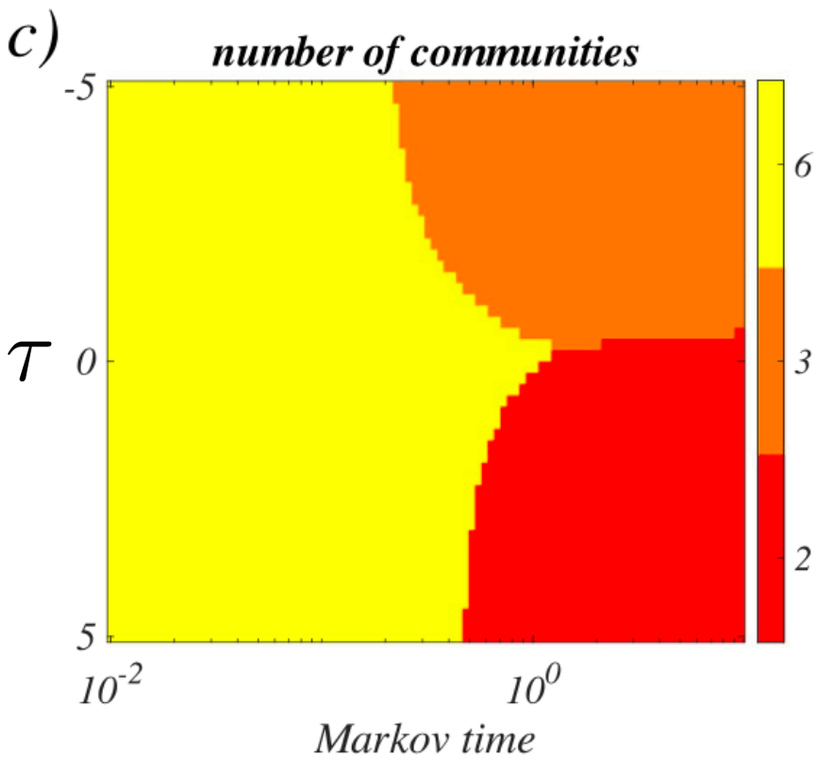
We propose a one-parameter family of random walk processes on hypergraphs, where a parameter biases the dynamics of the walker towards hyperedges of low or high cardinality. We show that for each value of the parameter, the resulting process defines its own hypergraph projection on a weighted network. We then explore the differences between them by considering the community structure associated to each random walk process. To do so, we adapt the Markov stability framework to hypergraphs and test it on artificial and real-world hypergraphs.

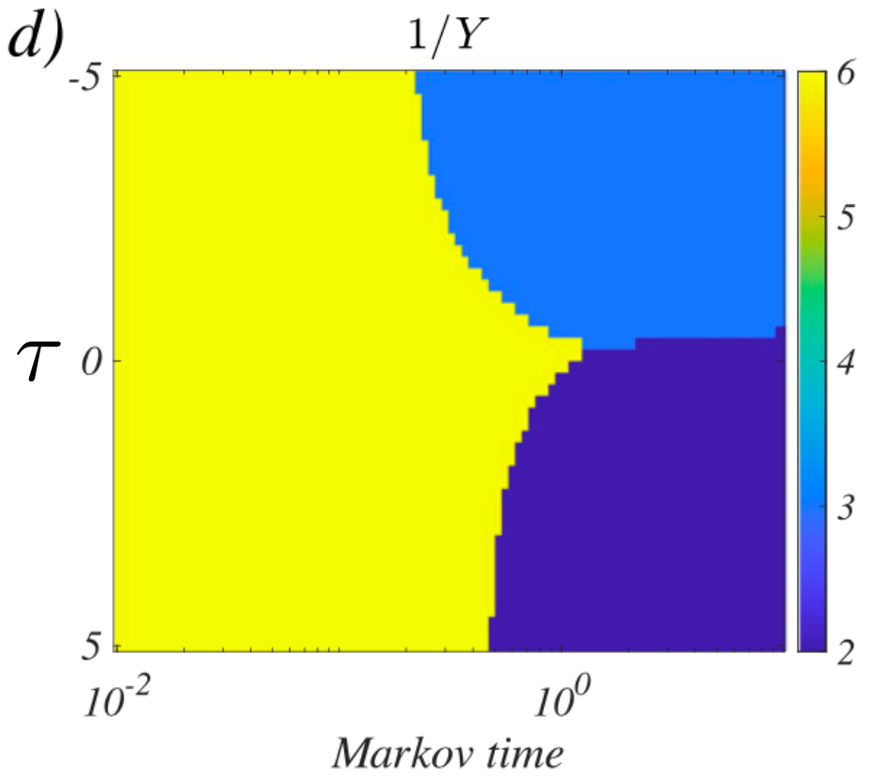
# Community detection

$$k_{ij}^{H} = \sum_{\alpha} (C_{\alpha\alpha} - 1)^{\tau} e_{i\alpha} e_{j\alpha}$$




$$Y = \sum_{i=1}^{M} \frac{S_i^2}{N^2},$$

# Thank you

Timoteo Carletti

Random walks on Hypergraphs

# Any questions?

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