Timoteo Carletti

Complex Networks & Dynamical Systems

naxys
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Complex Networks & Dynamical Systems

and beyond

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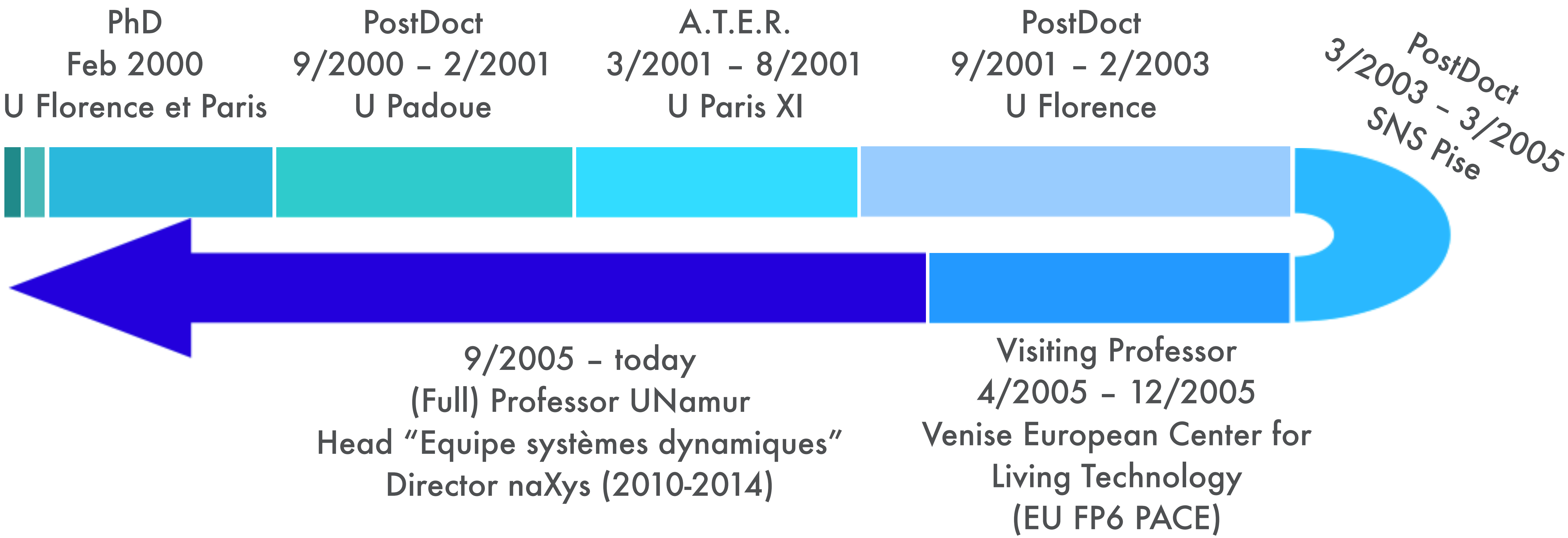
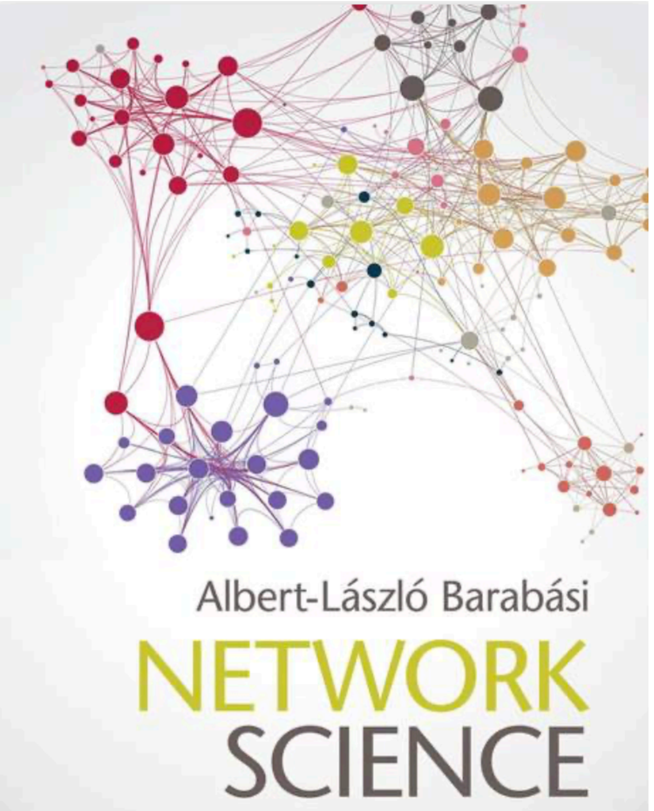
Who am I?


PhD
Feb 2000
U Florence et Paris
PostDoct
9/2000 – 2/2001
U Padoue
A.T.E.R.
3/2001 – 8/2001
U Paris XI
PostDoct
9/2001 – 2/2003
U Florence
PostDoct
3/2003 – 3/2005
SNS Pise
Visiting Professor
4/2005 – 12/2005
Venise European Center for
Living Technology
(EU FP6 PACE)
9/2005 – today
(Full) Professor UNamur
Head "Equipe systèmes dynamiques"
Director naXys (2010-2014)

Motivation : We live in an interconnected world …



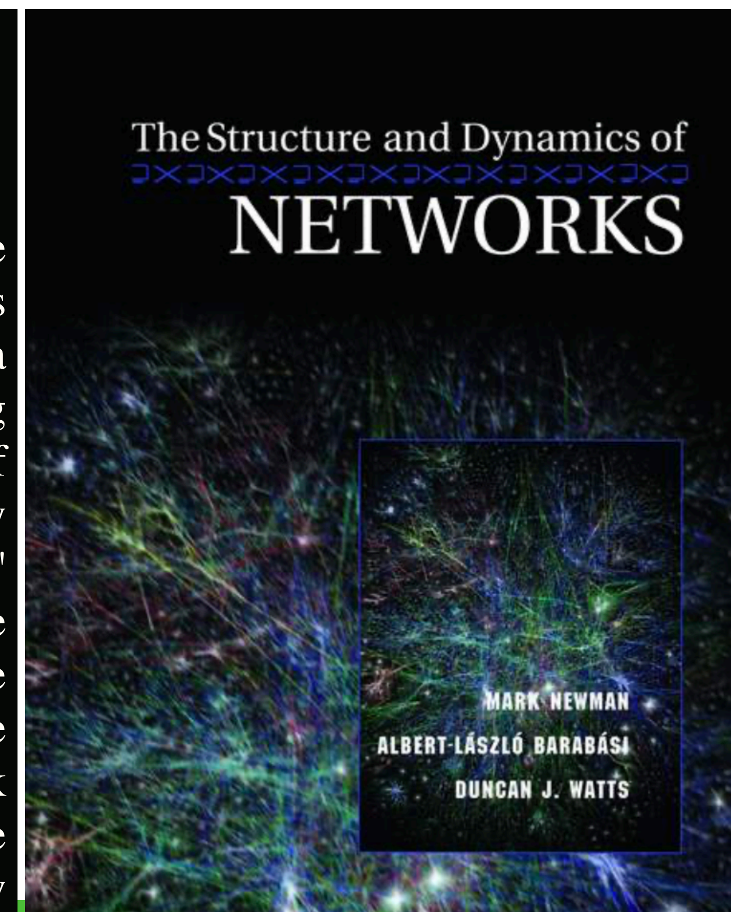
Network Science A.-L. Barabási

Networks are everywhere, from the Internet, to social networks, and the genetic networks that determine our biological existence. Illustrated throughout in full colour, this pioneering textbook, spanning a wide range of topics from physics to computer science, engineering, economics and the social sciences, introduces network science to an interdisciplinary audience. From the origins of the six degrees of separation to explaining why networks are robust to random failures, the author explores how viruses like Ebola and H1N1 spread, and why it is that our friends have more friends than we do. Using numerous real-world examples, this innovatively designed text includes clear delineation between undergraduate and graduate level material. The mathematical formulas and derivations are included within Advanced Topics sections, enabling use at a range of levels. Extensive online resources, including films and software for network analysis, make this a multifaceted companion for anyone with an interest in network science.

The Structure and Dynamics of Networks

A.-L. Barabási, M. Newman, D.J.Watts

From the Internet to networks of friendship, disease transmission, and even terrorism, the concept-and the reality-of networks has come to pervade modern society. But what exactly is a network? What different types of networks are there? Why are they interesting, and what can they tell us? In recent years, scientists from a range of fields-including mathematics, physics, computer science, sociology, and biology-have been pursuing these questions and building a new "science of networks." This book brings together for the first time a set of seminal articles representing research from across these disciplines. It is an ideal sourcebook for the key research in this fast-growing field. The book is organized into four sections, each preceded by an editors' introduction summarizing its contents and general theme. The first section sets the stage by discussing some of the historical antecedents of contemporary research in the area. From there the book moves to the empirical side of the science of networks before turning to the foundational modeling ideas that have been the focus of much subsequent activity. The book closes by taking the reader to the cutting edge of network science--the relationship between network structure and system dynamics. From network robustness to the spread of disease, this section offers a potpourri of topics on this rapidly expanding frontier of the new science.



… where "basic" units interact each others

Support for the spread of

- Information
- Opinions
- Likes
- Viruses
- …

… where "basic" units interact each others

At larger scale

support for the spread of goods

Support for the spread of

- Information
- Opinions
- Likes
- Viruses
- …

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… where "basic" units interact each others

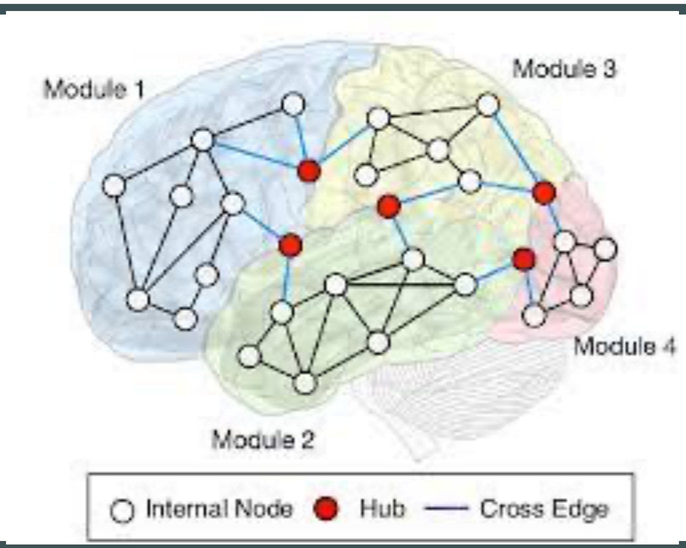
At larger scale

support for the spread of goods

Support for the spread of

- Information
- Opinions
- Likes
- Viruses
- …

At smaller scale



support for the spread of signals (memory, actions, thoughts, …)

Research question: understand the system behaviour

Reductionist approach
(e.g., grand unified theory)

vs.

Holistic approach
(e.g., complex systems)

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Outlook of the lectures

- Some basic facts about dynamical systems theory ;
- A short introduction about network theory ;
- Networked dynamical systems: main questions and results ;
- Beyond network theory.

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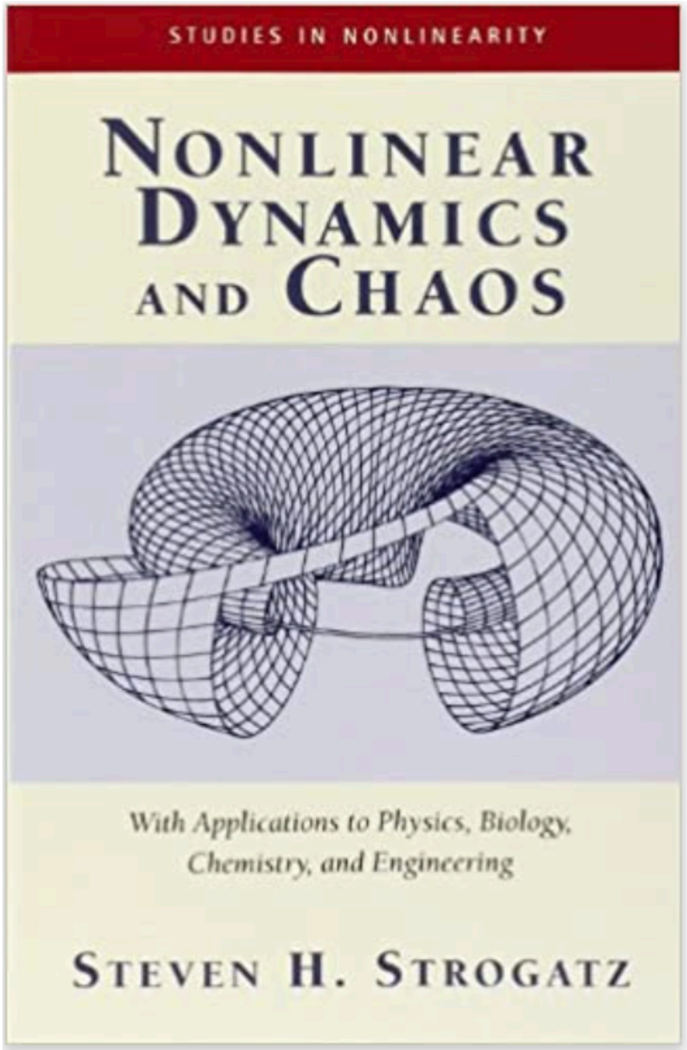
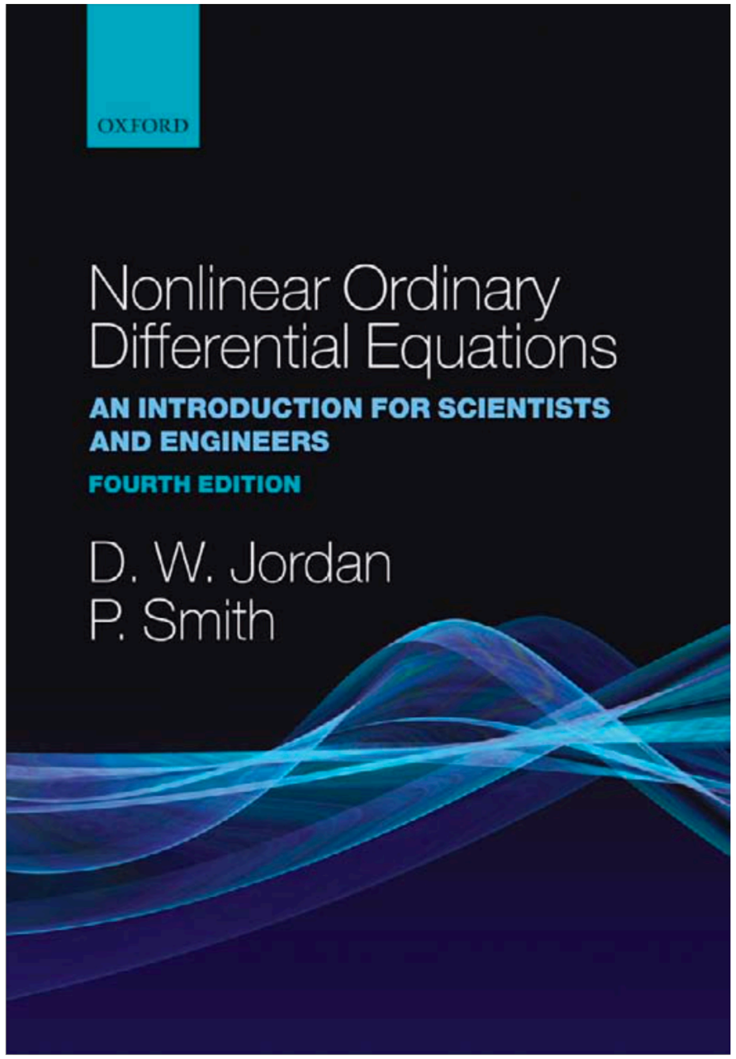
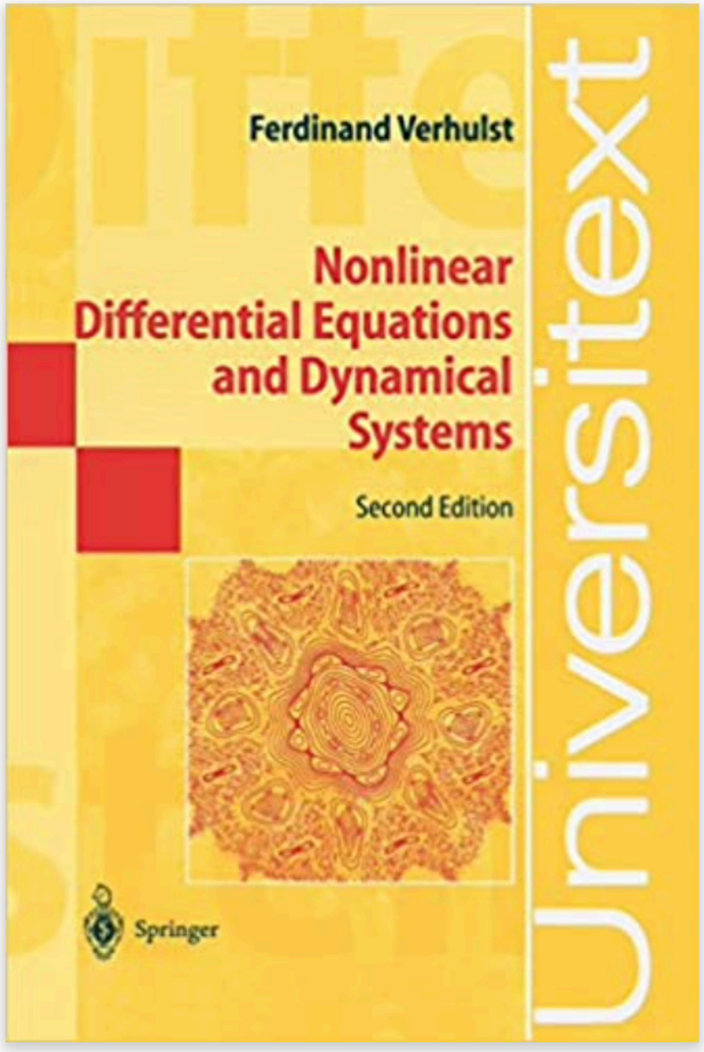
Dynamical Systems







Dynamical Systems

Continuous time $t \in \mathbb{R}$

$\vec{x} = (x_1, \dots, x_n)^\top \in \mathcal{A} \subset \mathbb{R}^n$ phase space (state space)

$\vec{f} : \mathcal{A} \to \mathbb{R}^n$

Time evolution $\dot{\vec{x}} := \dfrac{d\vec{x}}{dt} = \vec{f}(\vec{x})$

$$\left.\begin{array}{l}\begin{cases} \dot{x}_1 = \frac{dx_1}{dt} = f_1(x_1, \dots, x_n) \\ \quad\vdots \\ \dot{x}_n = \frac{dx_n}{dt} = f_n(x_1, \dots, x_n) \end{cases} \\ \vec{x}(t_0) = \vec{x}_0 \quad \text{initial condition} \end{array}\right\} \text{Cauchy problem}$$

Continuous time $t \in \mathbb{R}$

$\vec{f} : \mathcal{A} \to \mathbb{R}^n$

Sufficiently regular to ensure existence and uniqueness of the solution $\vec{\varphi}(t; \vec{x}_0)$ of the Cauchy problem

f Lipschitz is enough [Picard theorem]

Dynamical Systems

Continuous time $t \in \mathbb{R}$

$\vec{f} : \mathcal{A} \to \mathbb{R}^n$ Sufficiently regular to ensure existence and uniqueness of the solution $\vec{\varphi}(t; \vec{x}_0)$ of the Cauchy problem

Linear case if "only monomials in x_i are present"

$\vec{f}(\vec{x}) = \mathbf{A}\vec{x}$ $\mathbf{A} \in \mathbb{R}^{n\times n}$ $\vec{\varphi}(t; \vec{x}_0) = e^{\mathbf{A}(t-t_0)}\vec{x}_0$

$$e^{\mathbf{A}t} := \sum_{n\geq 0} \frac{t^n}{n!}\mathbf{A}^n$$

Non-Linear case, all the remaining ones

Dynamical Systems

Continuous time $t \in \mathbb{R}$

$\vec{f} : \mathcal{A} \to \mathbb{R}^n$ Sufficiently regular to ensure existence and uniqueness of the solution $\vec{\varphi}(t;\vec{x}_0)$ of the Cauchy problem

Linear case if "only monomials in x_i are present"

$$\vec{f}(\vec{x}) = \mathbf{A}\vec{x} \qquad \mathbf{A} \in \mathbb{R}^{n\times n} \qquad \vec{\varphi}(t;\vec{x}_0) = e^{\mathbf{A}(t-t_0)}\vec{x}_0$$

Non-Linear case, all the remaining ones

Autonomous vs non-autonomous

$$\vec{f}(\vec{x}) \qquad \vec{f}(\vec{x}, t)$$

explicit dependence on time

Dynamical Systems

Discrete time $n \in \mathbb{N}$

$\vec{x} = (x_1, \dots, x_n)^\top \in \mathcal{A} \subset \mathbb{R}^n$ phase space (state space)

$\vec{f} : \mathcal{A} \to \mathbb{R}^n$

Time evolution

$$\vec{x}_{n+1} = \vec{f}(\vec{x}_n)$$

$\vec{x}_0$ initial condition

Dynamical Systems

Discrete time $n \in \mathbb{N}$

$\vec{x} = (x_1, \dots, x_n)^\top \in \mathcal{A} \subset \mathbb{R}^n$ phase space (state space)

$\vec{f} : \mathcal{A} \to \mathbb{R}^n$

Time evolution

$$\vec{x}_{n+1} = \vec{f}(\vec{x}_n)$$

$\vec{x}_0$ initial condition

IT CAN BE STUDIED
BUT WE WILL NOT DO IT

Equilibrium point :

$$\vec{x}^* \in \mathcal{A} \ : \ \vec{f}(\vec{x}^*) = 0 \Rightarrow \vec{x}(t) = \vec{x}^* \quad \forall t$$

Dynamical Systems

Equilibrium point :

$$\vec{x}^* \in \mathcal{A} \ : \ \vec{f}(\vec{x}^*) = 0 \Rightarrow \vec{x}(t) = \vec{x}^* \quad \forall t$$

Stability of the equilibrium point :

$\vec{x}^*$ is (locally) <u>stable</u> (or <u>Lyapunov stable</u>) if

$$\forall \epsilon > 0 \, \exists \delta(\epsilon) > 0 \ : \forall \vec{x}_0 : |\vec{x}_0 - \vec{x}^*| < \delta$$

$$\Rightarrow \ |\vec{\varphi}(t; \vec{x}_0) - \vec{x}^*| < \epsilon \, \forall t \geq 0$$

Equilibrium point :

$$\vec{x}^* \in \mathcal{A} \ : \ \vec{f}(\vec{x}^*) = 0 \Rightarrow \vec{x}(t) = \vec{x}^* \quad \forall t$$

Stability of the equilibrium point :

$\vec{x}^*$ is <u>asymptotically</u> (locally) <u>stable</u> if

$$\exists \delta(\vec{x}^*) > 0 \ : \forall \vec{x}_0 : |\vec{x}_0 - \vec{x}^*| < \delta \quad \Rightarrow \quad \vec{\varphi}(t; \vec{x}_0) \underset{t \to +\infty}{\longrightarrow} \vec{x}^*$$

Linear stability analysis of the equilibrium point :

$$\vec{x}^* \in \mathcal{A} \; : \; \vec{f}(\vec{x}^*) = 0 \Rightarrow \vec{x}(t) = \vec{x}^* \quad \forall t$$

Let $\mathbf{J}(\vec{x}_0) = \frac{\partial \vec{f}}{\partial \vec{x}}(\vec{x}_0)$ be the Jacobian matrix, i.e.,

$$J_{ij}(\vec{x}_0) = \frac{\partial f_i}{\partial x_j}(\vec{x}_0)$$

Linear stability analysis of the equilibrium point :

$$\vec{x}^* \in \mathcal{A} \ : \ \vec{f}(\vec{x}^*) = 0 \Rightarrow \vec{x}(t) = \vec{x}^* \quad \forall t$$

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$$J_{ij}(\vec{x}_0) = \frac{\partial f_i}{\partial x_j}(\vec{x}_0)$$

Let $\lambda_1, \dots, \lambda_n$ be the eigenvalues of $\mathbf{J}(\vec{x}_0)$ then

if $\Re\lambda_i < 0 \ \forall i = 1, \dots, n$ then $\vec{x}^*$ is asymptotically stable

The case of multiple eigenvalues can be handled as well

Attention to non-normal matrices

A matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$ is non-normal iff $\mathbf{A}\mathbf{A}^\top \neq \mathbf{A}^\top\mathbf{A}$

This implies that $\mathbf{A}$ cannot be diagonalised with orthogonal vectors

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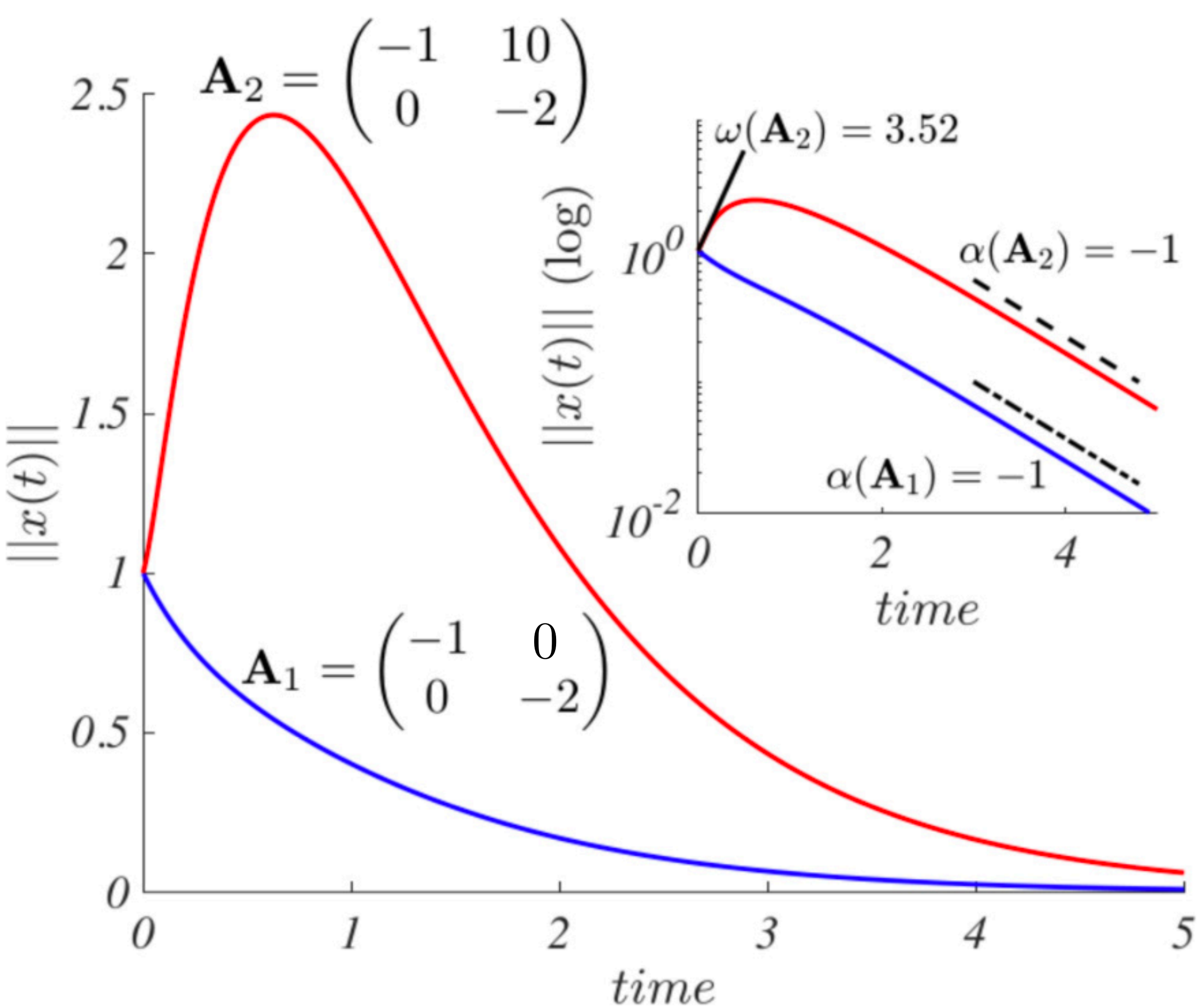
Dynamical Systems

Attention to <u>non-normal</u> matrices

A matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$ is <u>non-normal</u> iff $\mathbf{A}\mathbf{A}^\top \neq \mathbf{A}^\top\mathbf{A}$

This implies that $\mathbf{A}$ cannot be diagonalised with orthogonal vectors

$$\frac{d\vec{x}}{dt} = \mathbf{A}\vec{x} \qquad ||\vec{x}||^2 = \vec{x}^\top \cdot \vec{x}$$



$$\alpha(\mathbf{A}) = \sup \Re\sigma(\mathbf{A})$$

spectral abscissa

$$\omega(\mathbf{A}) = \sup \sigma\left(\frac{\mathbf{A}+\mathbf{A}^\top}{2}\right)$$

numerical abscissa (reactivity)

Dynamical Systems

Solution

$$\vec{x}_0 \in \mathcal{A} \ : \ \frac{d\vec{\psi}(t;\vec{x}_0)}{dt} = \vec{f}(\vec{\psi}(t;\vec{x}_0)) \quad \forall t$$

Stability of a solution :

A solution $\vec{\psi}(t;\vec{x}_0)$ is said <u>orbitally stable</u> (or <u>Poincaré stable</u>)

$$\forall \epsilon > 0 \, \exists \delta(\epsilon) > 0 \ : \forall \vec{x}_0 : |\vec{x}_0 - \vec{x}_0'| < \delta$$

$$\Rightarrow \ |\vec{\varphi}(t;\vec{x}_0') - \vec{\psi}(t;\vec{x}_0)| < \epsilon \, \forall t \geq 0$$

T - Periodic solution

$$\vec{x}_0 \in \mathcal{A} \; : \; \frac{d\vec{\psi}(t;\vec{x}_0)}{dt} = \vec{f}(\vec{\psi}(t;\vec{x}_0)) \quad \forall t$$

$$\vec{\psi}(t;\vec{x}_0) = \vec{\psi}(t+T;\vec{x}_0)\,\forall t$$

T - Periodic solution

$$\vec{x}_0 \in \mathcal{A} \ : \ \frac{d\vec{\psi}(t;\vec{x}_0)}{dt} = \vec{f}(\vec{\psi}(t;\vec{x}_0)) \quad \forall t$$

$$\vec{\psi}(t;\vec{x}_0) = \vec{\psi}(t+T;\vec{x}_0)\,\forall t$$

Poincaré map

Let V be a (n-1)-dimensional manifold transverse to the flow of the ODE st $\vec{x}_0 \in V$, then $\vec{\psi}(T;\vec{x}_0) = \vec{x}_0 \in V$. Let $\vec{x}' \in V$ close (enough) to $\vec{x}_0$ then let $t' > 0$ the smallest time st $P(\vec{x}') = \vec{x}'' = \vec{\varphi}(t';\vec{x}') \in V$

Stability of T - Periodic solution

$$\forall \epsilon > 0 \, \exists \delta(\epsilon) > 0 \ : \forall \vec{x}' \ : |\vec{x}_0 - \vec{x}'| < \delta$$

$$\Rightarrow \ |P^{\circ n}(\vec{x}') - \vec{x}_0| < \epsilon \, \forall n \geq 0$$

Note : $P(\vec{x}_0) = \vec{x}_0$

Asymptotically stability

$$|P^{\circ n}(\vec{x}') - \vec{x}_0| \underset{n \to +\infty}{\to} 0$$

Stability of T - Periodic solution with the Floquet theory

$$\frac{d\vec{x}}{dt} = \mathbf{A}(t)\vec{x} \qquad \mathbf{A}(t) = \mathbf{A}(t+T) \quad \forall t$$

Then there exist $\mathbf{B} \in \mathbb{R}^{n \times n}$ and $\mathbf{S}(t) \in \mathbb{R}^{n \times n}$ T-periodic and invertible

such that $$\vec{x}(t) = \mathbf{S}(t)e^{\mathbf{B}t}\vec{x}_0$$

Moreover $$\frac{d\mathbf{S}}{dt} = \mathbf{AS} - \mathbf{SB}$$

Let $\mathbf{C} = e^{\mathbf{B}T}$ be the monodromy matrix. The eigenvalues ρ_j of $\mathbf{C}$ are called characteristic multipliers and are related to those of $\mathbf{B}$, μ_j characteristic exponents $\rho_j = e^{\mu_j T}$

[Andronov-Witte Theorem] If $\Re\mu_j < 0 \quad \forall j = 2, \dots, n$ then the periodic orbit is stable.

Let $\mathbf{C} = e^{\mathbf{B}T}$ be the monodromy matrix. The eigenvalues ρ_j of $\mathbf{C}$ are called characteristic multipliers and are related to those of $\mathbf{B}$, μ_j characteristic exponents $\rho_j = e^{\mu_j T}$

[Andronov-Witte Theorem] If $\Re\mu_j < 0 \quad \forall j = 2, \dots, n$ then the periodic orbit is stable.

Note 1: $\mu_1 = 0$

Note 2: there are explicit general ways to compute $\mathbf{S}(t)$ or μ_j

Note 3: $$\rho_1 \dots \rho_n = \exp\left(\int_0^T \mathrm{tr}\mathbf{A}(t)\, dt\right)$$

Limit cycles : isolated periodic solutions

To find limit cycles is a difficult task

(Second part of) the 16th Hilbert problem :

Determine an upper bound to the number of limit cycles in a polynomial planar EDO, as function of the degree and/or the coefficients.

This number is finite (Yulii Ilyashenko and Jean Écalle, 1991-1992)

To find limit cycles is a difficult task

In the plane one can use the Poincaré - Bendixon theorem.

In general the Brower fixed point theorem can be used.

van der Pol oscillator

$$\frac{d^2x}{dt^2} - \mu(1-x^2)\frac{dx}{dt} + x = 0$$

State dependent non linear damping

Small parameter case

$$|\mu| < 2$$

van der Pol oscillator

$$\frac{d^2x}{dt^2} - \mu(1-x^2)\frac{dx}{dt} + x = 0$$

Liénard coordinates

$$\dot{x} = \mu\left(x - \frac{x^3}{3}\right) - y$$
$$\dot{y} = x$$

van der Pol oscillator

$$\dot{x} = \mu\left(x - \frac{x^3}{3}\right) - y$$

$$\dot{y} = x$$

$(0,0)$ is the unique equilibrium

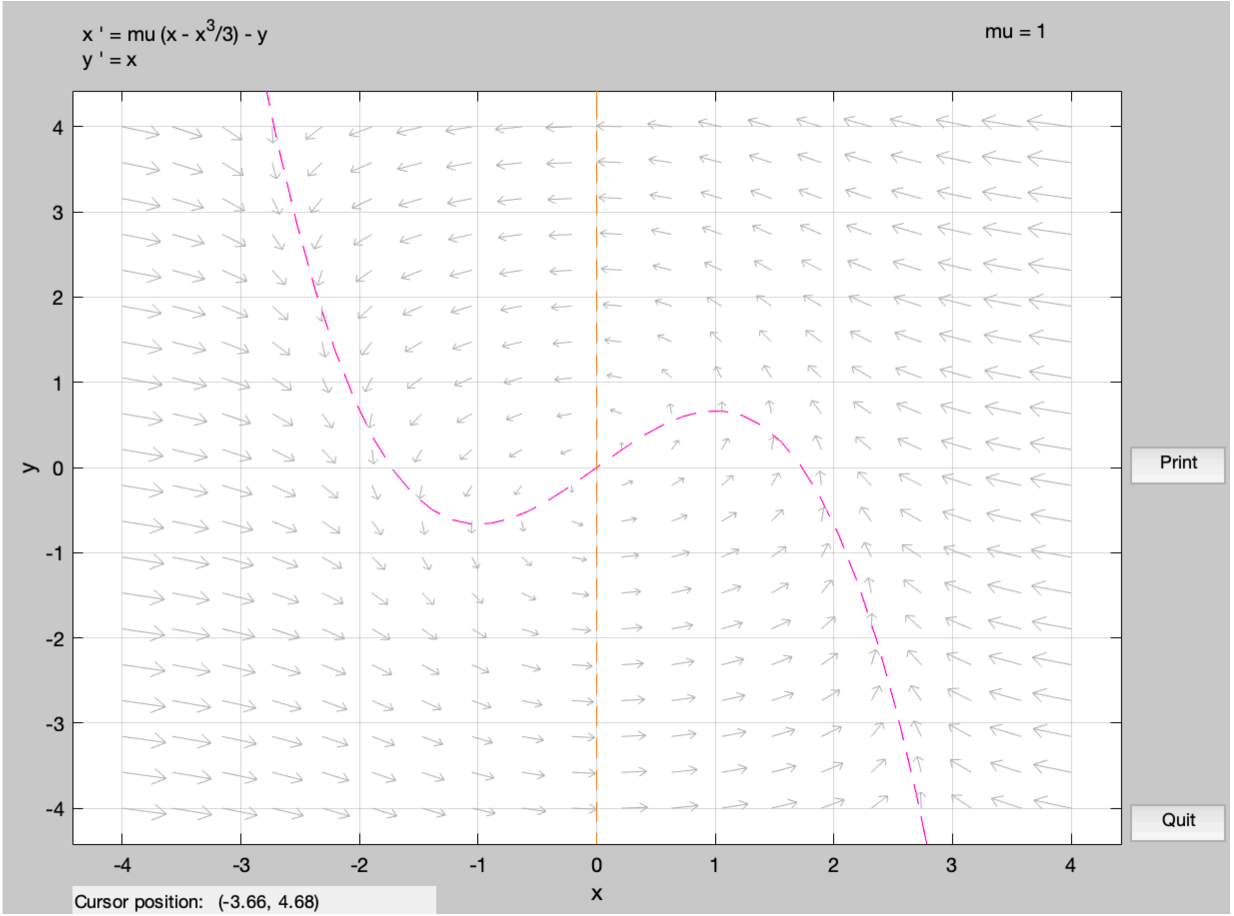
The Jacobian matrix is $\mathbf{J} = \begin{pmatrix} \mu & -1 \\ 1 & 0 \end{pmatrix}$

whose eigenvalues are

$$\lambda = \frac{\mu}{2} \pm i\sqrt{1 - \left(\frac{\mu}{2}\right)^2}$$

0 - isocline

$$f(x) = \mu\left(x - \frac{x^3}{3}\right)$$



van der Pol oscillator

$$\dot{x} = \mu\left(x - \frac{x^3}{3}\right) - y$$

$$\dot{y} = x$$

$$\mu < 0 \Rightarrow \Re\lambda = \frac{\mu}{2} < 0$$

Stable focus

0 - isocline

$$f(x) = \mu\left(x - \frac{x^3}{3}\right)$$

van der Pol oscillator

$$\dot{x} = \mu\left(x - \frac{x^3}{3}\right) - y$$
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Unstable focus

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Unstable focus

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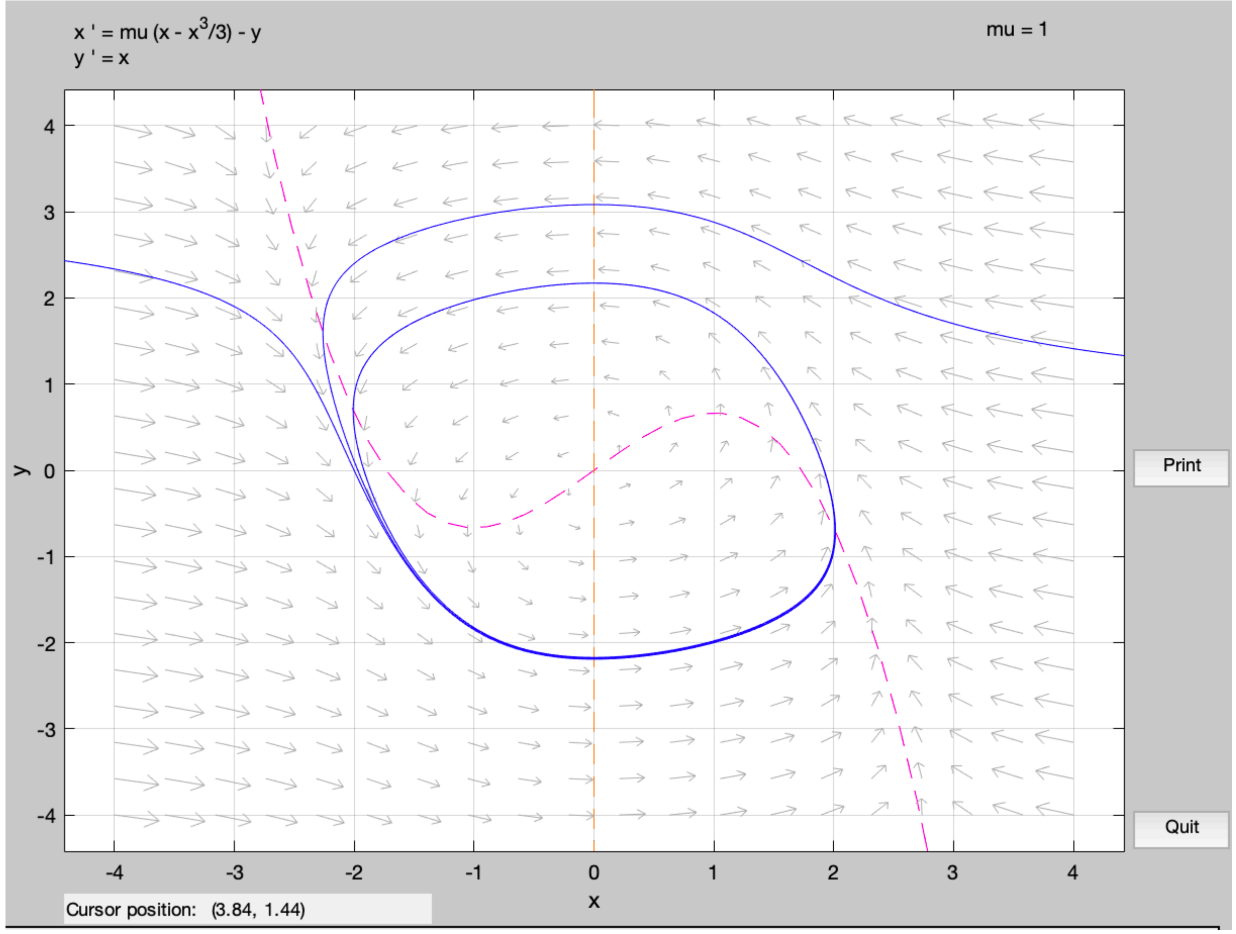


van der Pol oscillator

$$\dot{x} = \mu\left(x - \frac{x^3}{3}\right) - y$$

$$\dot{y} = x$$

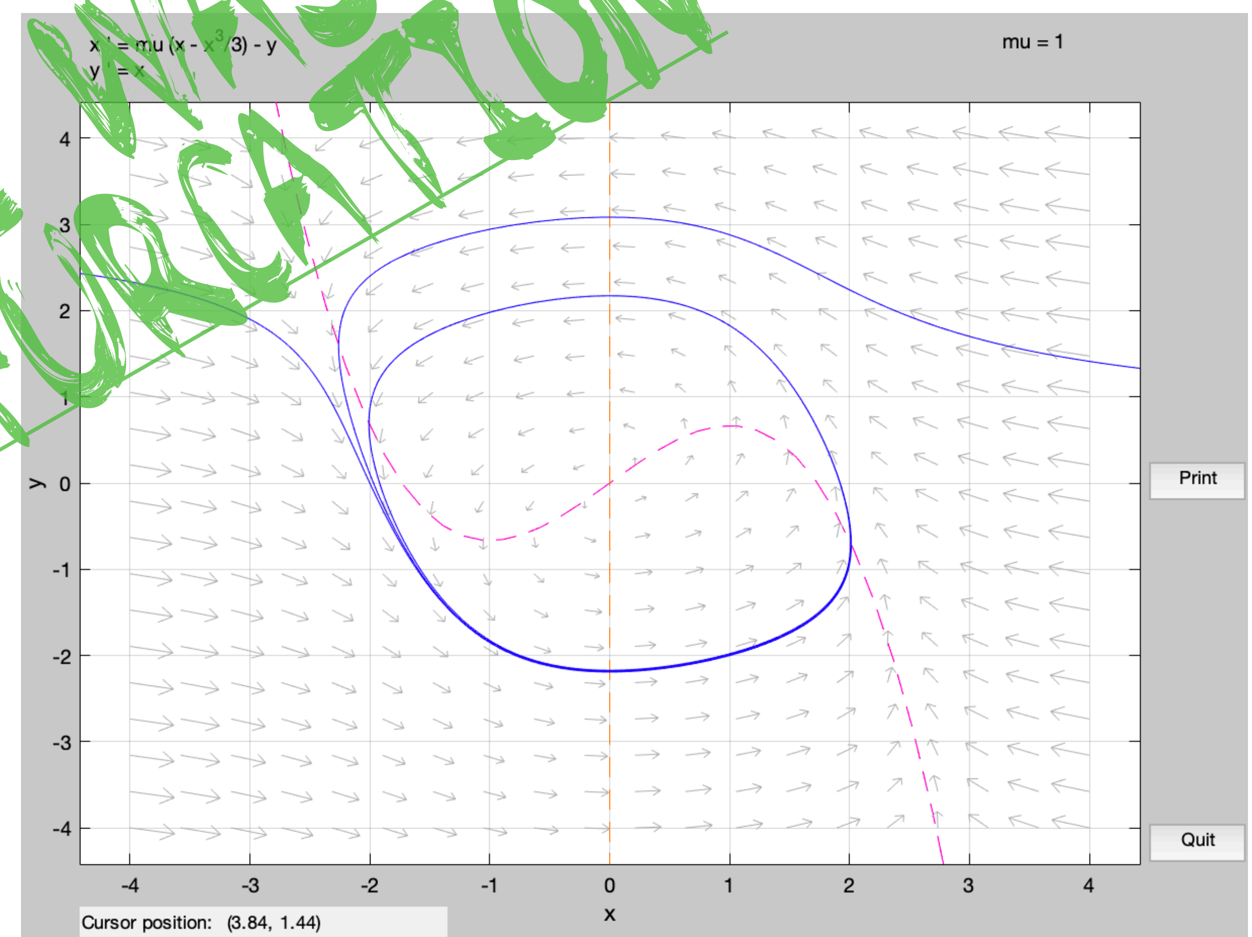
$$\mu > 0 \Rightarrow \Re\lambda = \frac{\mu}{2} > 0$$

Unstable focus

0 - isocline

$$f(x) = \mu\left(x - \frac{x^3}{3}\right)$$

A LIMIT CYCLE WAS BORN FROM A BIFURCATION



Stuart - Landau oscillator

$$\frac{dz}{dt} = z(a + ib - |z|^2)$$

$z = x + iy \in \mathbb{C}$ $a \in \mathbb{R}$ $b \in \mathbb{R}_+$

complex amplitude

Stuart - Landau oscillator

$$\frac{dz}{dt} = z(a + ib - |z|^2) \qquad z = x + iy \in \mathbb{C} \quad a \in \mathbb{R} \quad b \in \mathbb{R}_+$$

complex amplitude

Real variables

$$\begin{cases} \dot{x} &= ax - by - x(x^2 + y^2) \\ \dot{y} &= bx + ay - y(x^2 + y^2) \end{cases}$$

$(x, y) = (0, 0)$ Equilibrium

Stable if $a < 0$, unstable if $a > 0$

Stuart - Landau oscillator

$$\frac{dz}{dt} = z(a + ib - |z|^2) \qquad z = x + iy \in \mathbb{C} \quad a \in \mathbb{R} \quad b \in \mathbb{R}_+$$

complex amplitude

Polar coordinates $z(t) = \rho(t)e^{i\theta(t)}$

$$\begin{cases} \dot{\rho} & = \rho(a - \rho^2) \\ \dot{y} & = b \end{cases}$$

A limit cycle emerges once a passes from negative to positive values

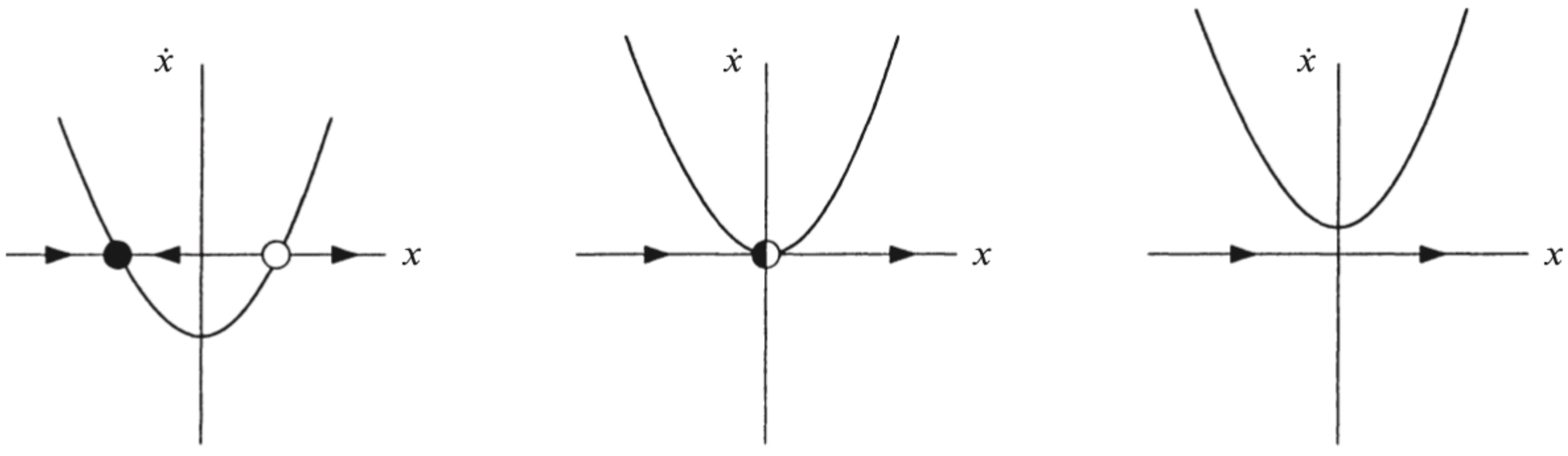
Bifurcation.

The qualitative behaviour of the system suddenly changes once a parameter reaches a critical value.

Saddle - node Bifurcation.

By varying a parameter a saddle equilibrium and a stable node equilibrium merge and disappear.

$$\dot{x} = r + x^2$$

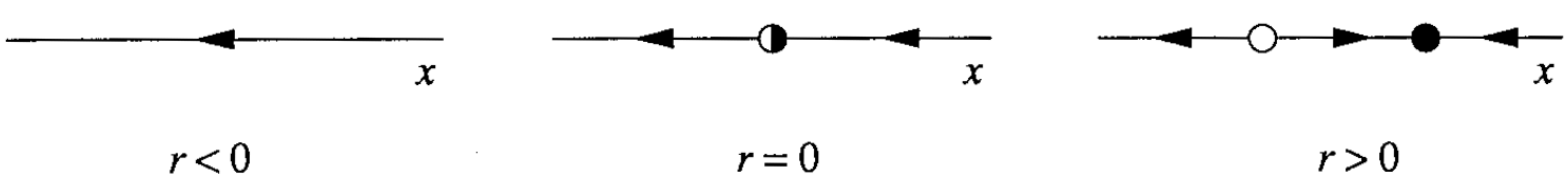


(a) $r<0$ (b) $r=0$ (c) $r>0$

Saddle - node (blue sky) Bifurcation.

By varying a parameter a saddle equilibrium and a node equilibrium appear "out of the clear blue sky"

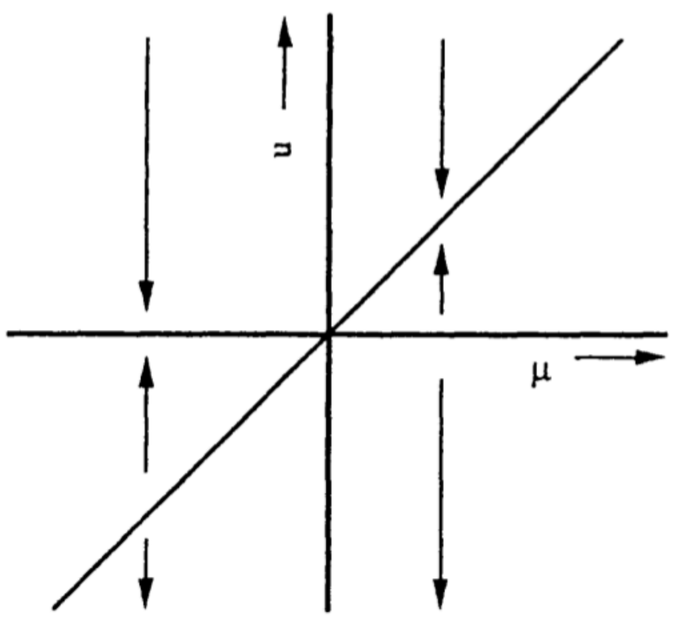
$$\dot{x} = r - x^2$$

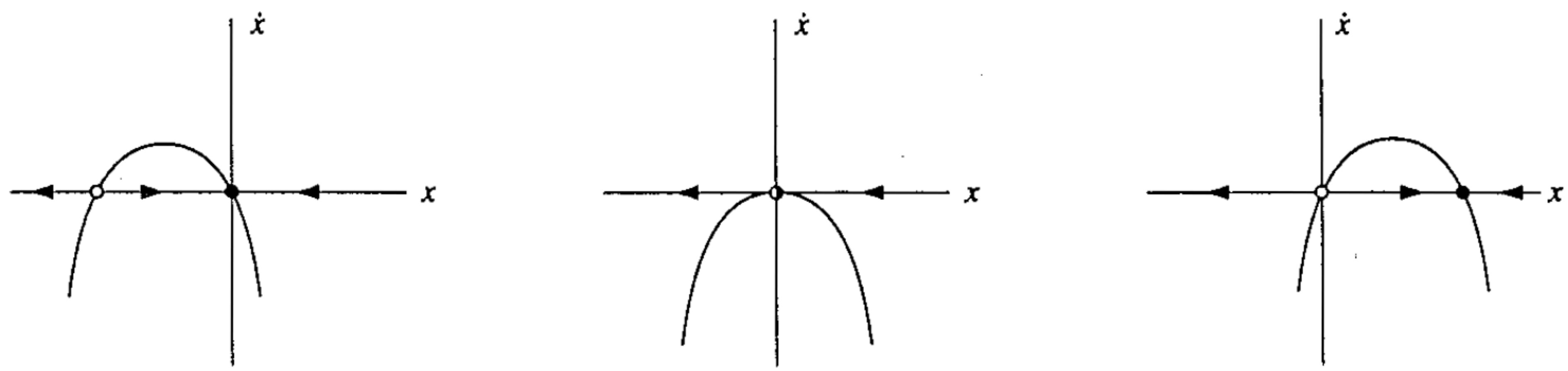


Transcritical Bifurcation.

The equilibrium (here x=0) always exits but it changes its character by varying a parameter.

$$\dot{x} = rx - x^2$$



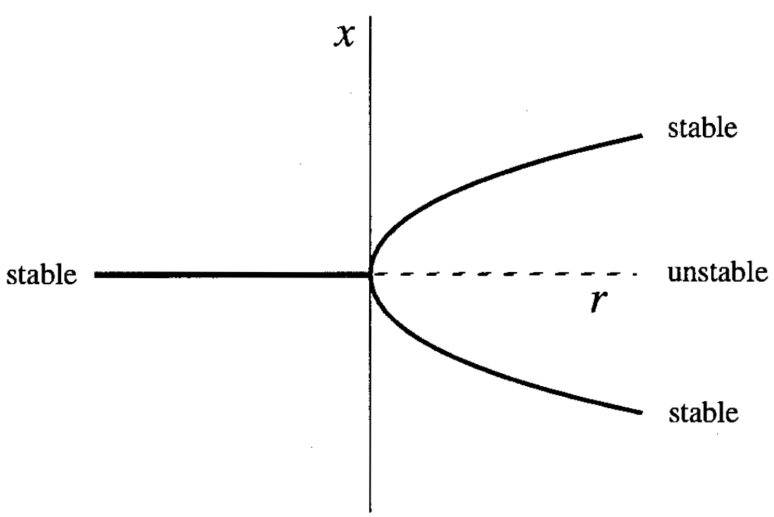


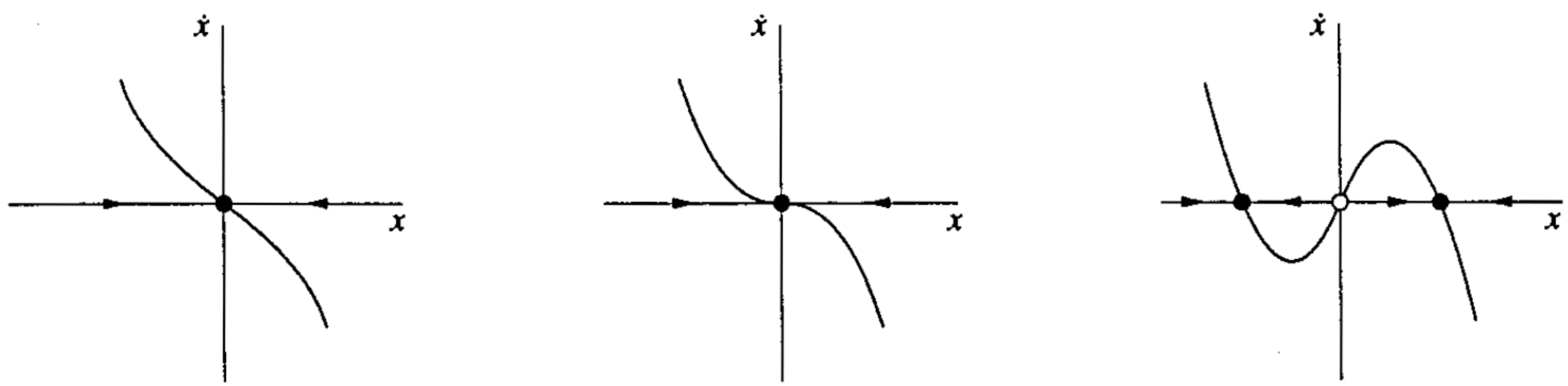
(a) $r<0$ (b) $r=0$ (c) $r>0$

Supercritical Pitchfork Bifurcation.

By varying a parameter a stable equilibrium (here x=0) becomes unstable and two new stable equilibria emerge.

$$\dot{x} = rx - x^3$$



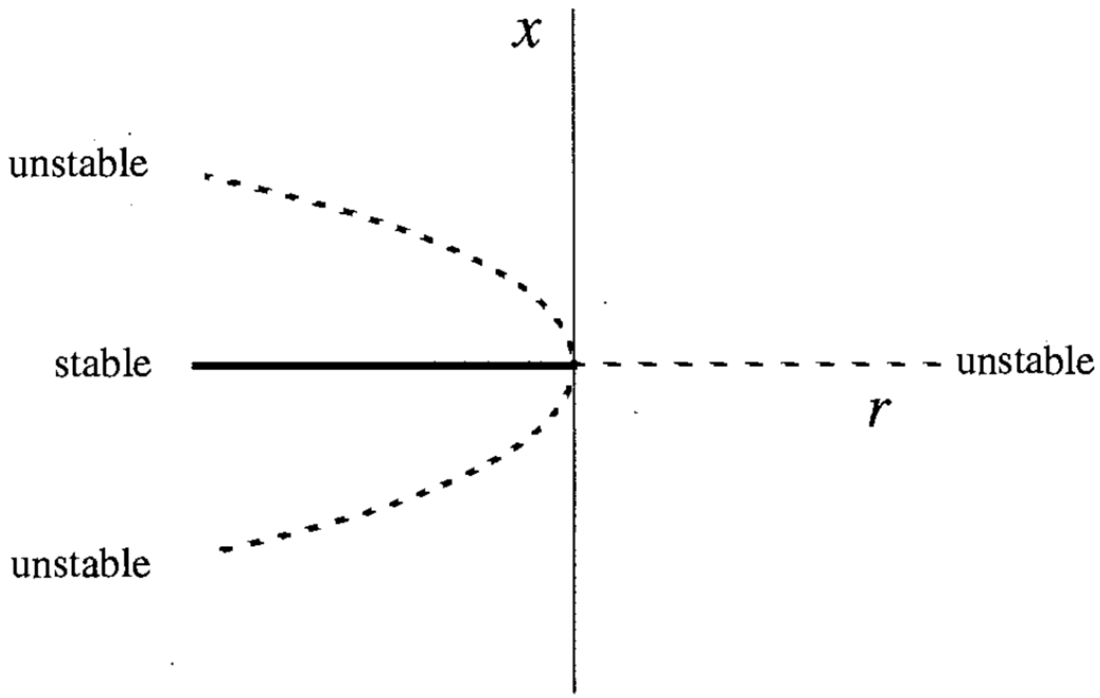


(a) $r < 0$ (b) $r = 0$ (c) $r > 0$

Subcritical Pitchfork Bifurcation.

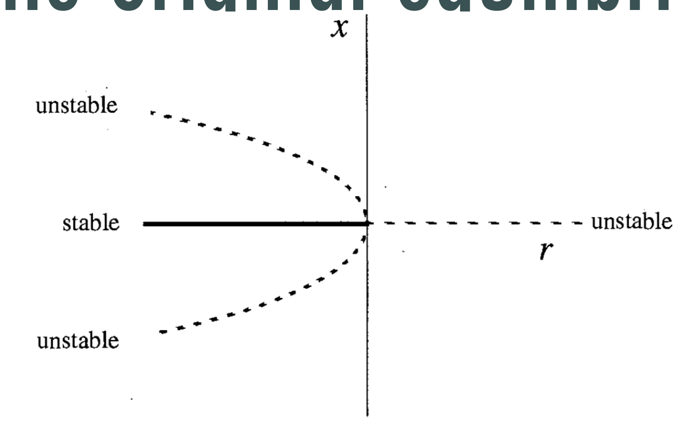
By varying a parameter a stable equilibrium (here x=0) and two unstable ones, merge and make the original equilibrium to be unstable

$$\dot{x} = rx + x^3$$



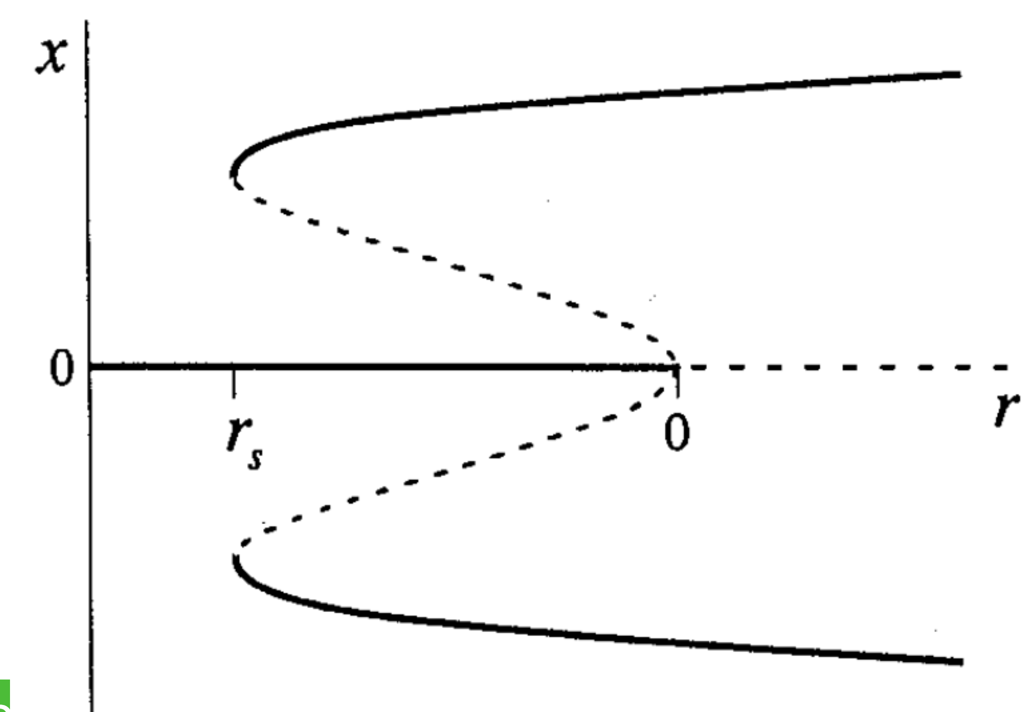
Subcritical Pitchfork Bifurcation.

By varying a parameter a stable equilibrium (here x=0) and two unstable ones, merge and make the original equilibrium to be unstable



$$\dot{x} = rx + x^3$$

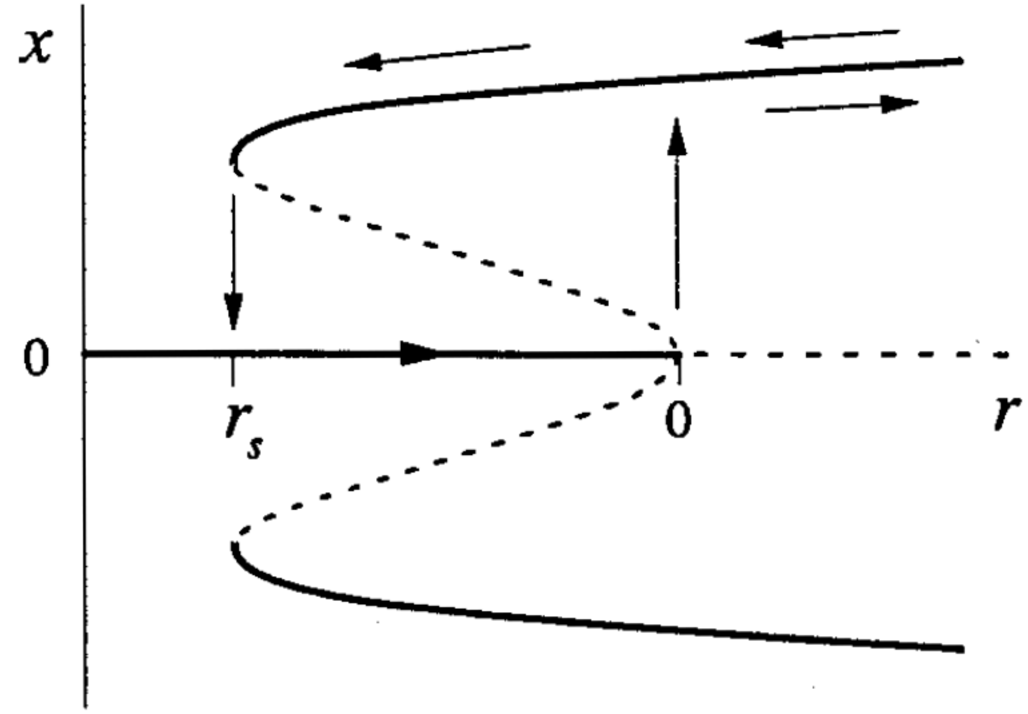
To avoid escaping orbits, one should usually introduce high order stabilising terms



$$\dot{x} = rx + x^3 - x^5$$

Subcritical Pitchfork Bifurcation and Hysteresis

$$\dot{x} = rx + x^3 - x^5$$



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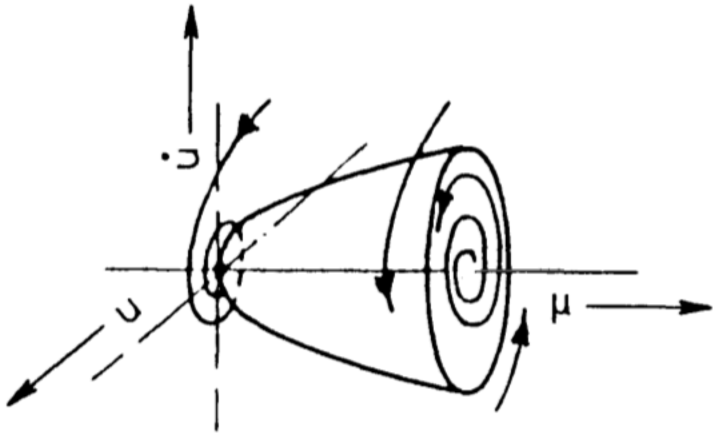
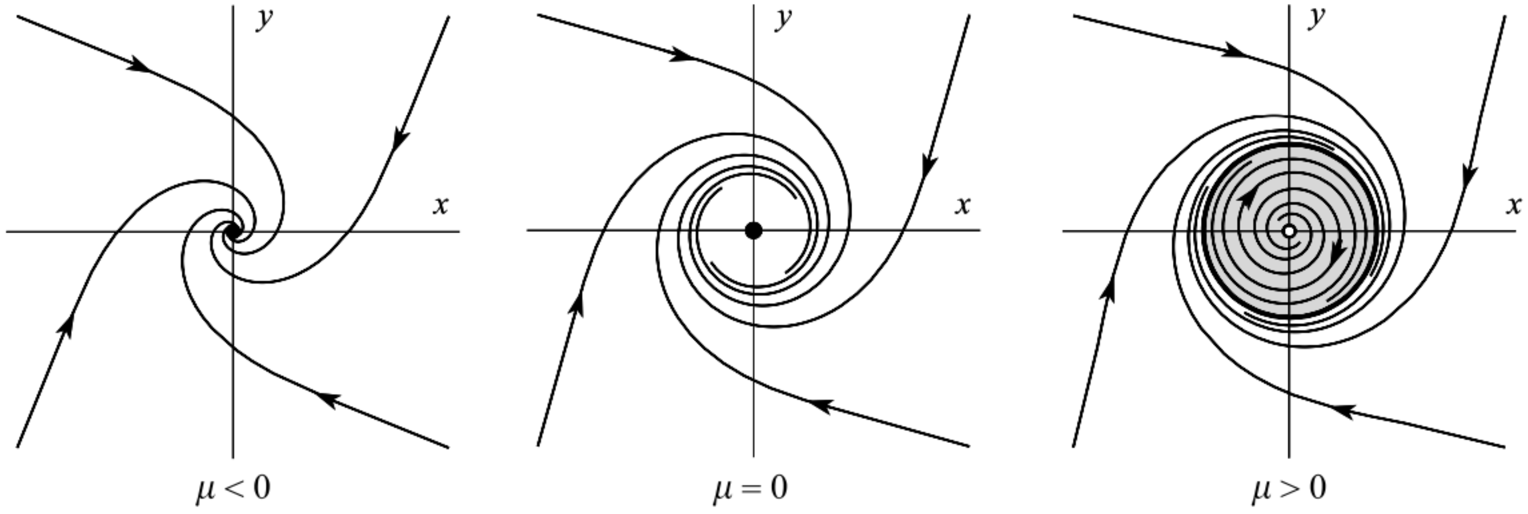
Hopf Bifurcation.

By varying a parameter, a stable equilibrium becomes unstable and a limit cycle emerges

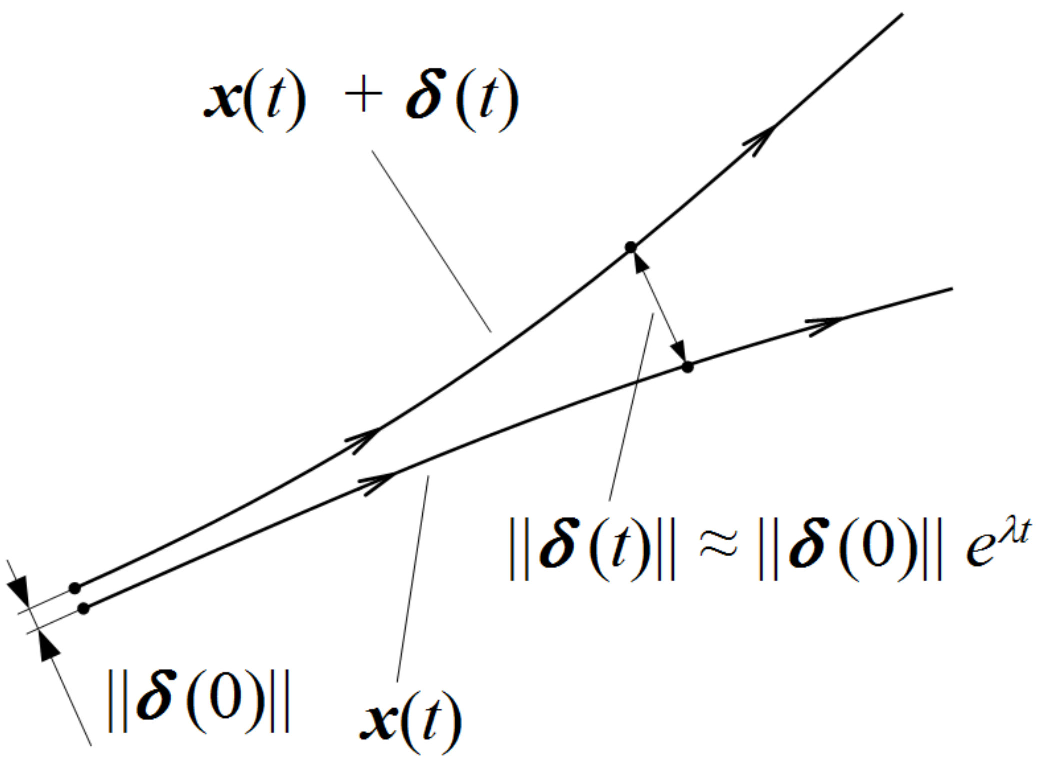
$$\begin{cases} \dot{r} & = r(\mu - r^2) \\ \dot{\theta} & = -1 \end{cases}$$





Chaotic behaviour.

Roughly speaking, high sensitivity to initial conditions, i.e., nearby orbits diverge each others.



$$\lambda = \lim_{t\to\infty} \lim_{||\delta(0)||\to 0} \frac{1}{t} \log \frac{||\delta(t)||}{||\delta(0)||}$$

Maximal Lyapunov exponent

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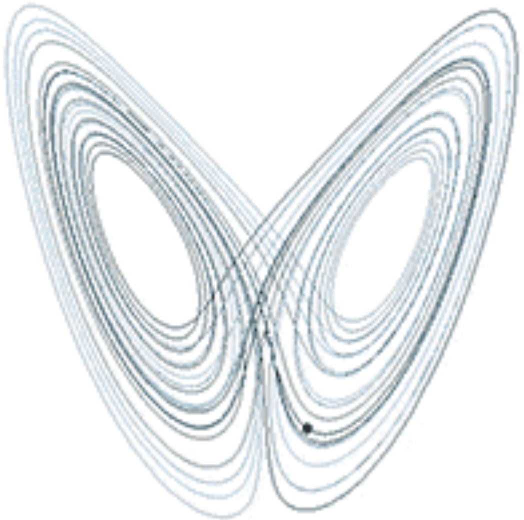
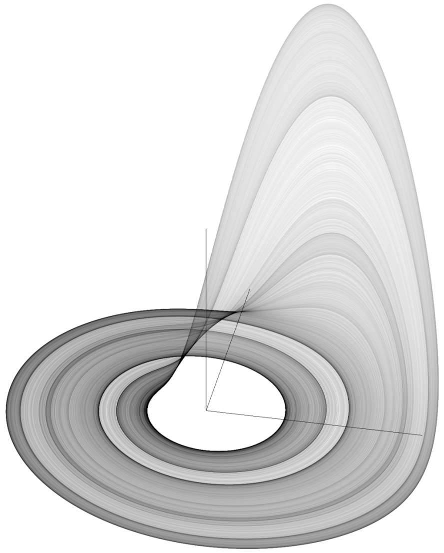
Chaotic behaviour. Two "main" examples

Lorenz system



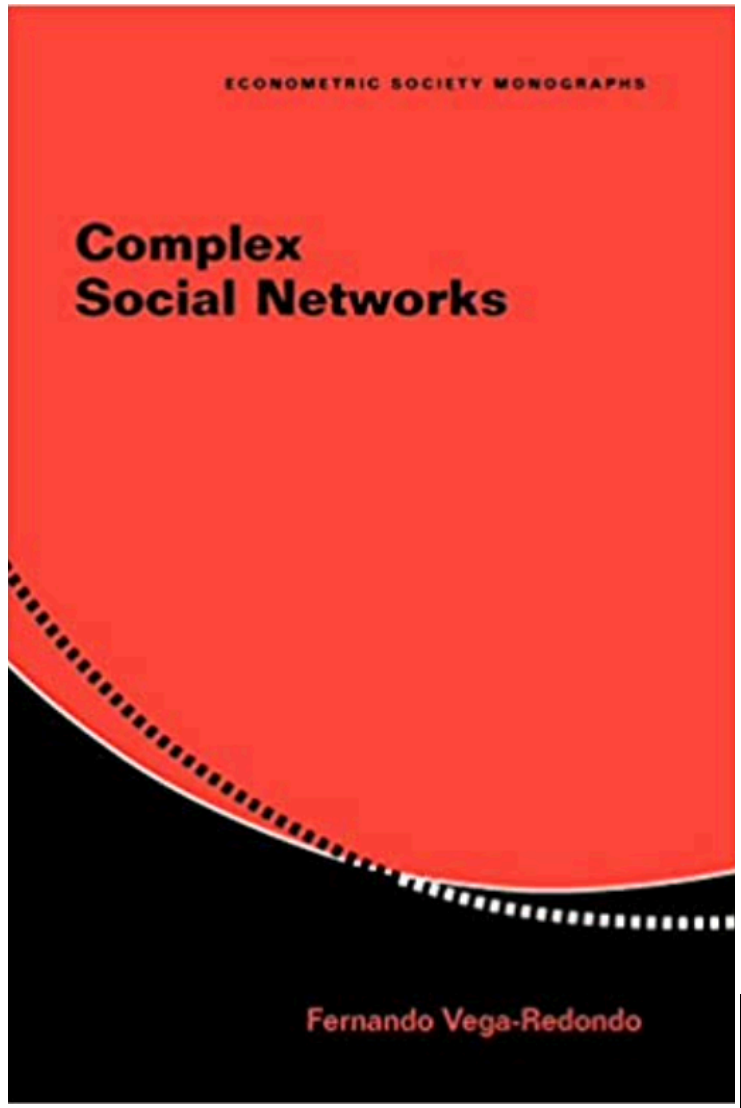
$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = \sigma(y - x), \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = x(\rho - z) - y, \\ \dfrac{\mathrm{d}z}{\mathrm{d}t} = xy - \beta z. \end{cases}$$

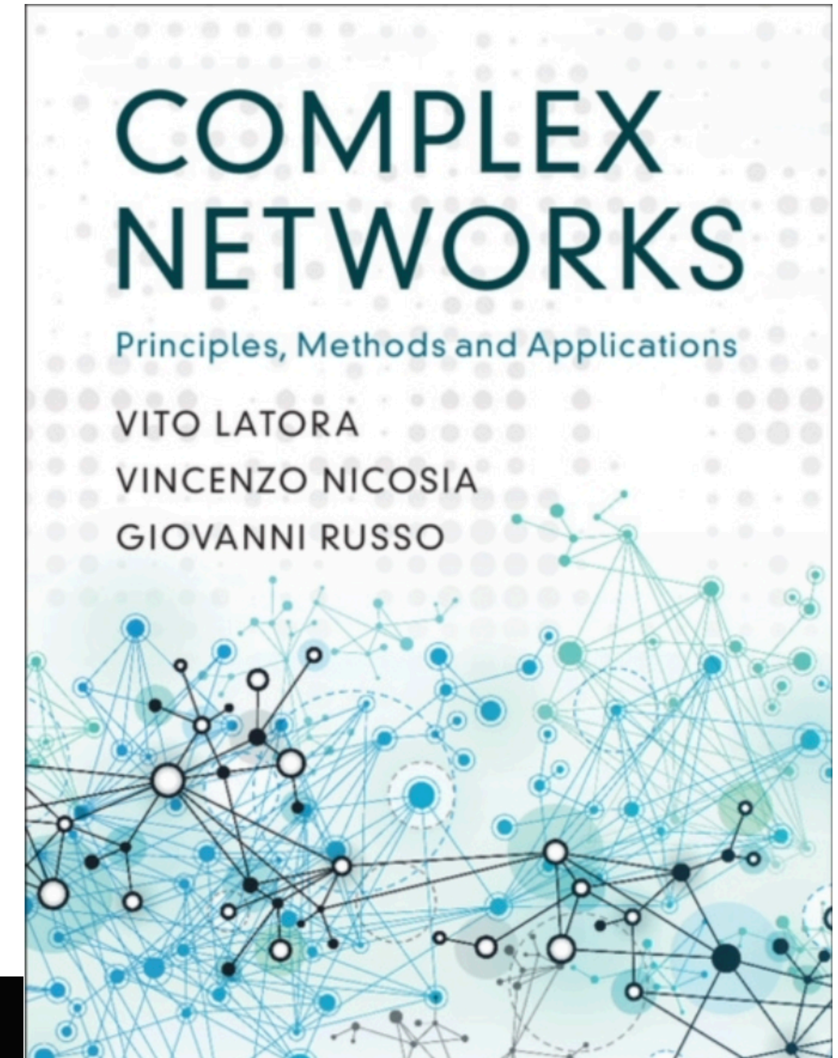
Rössler system

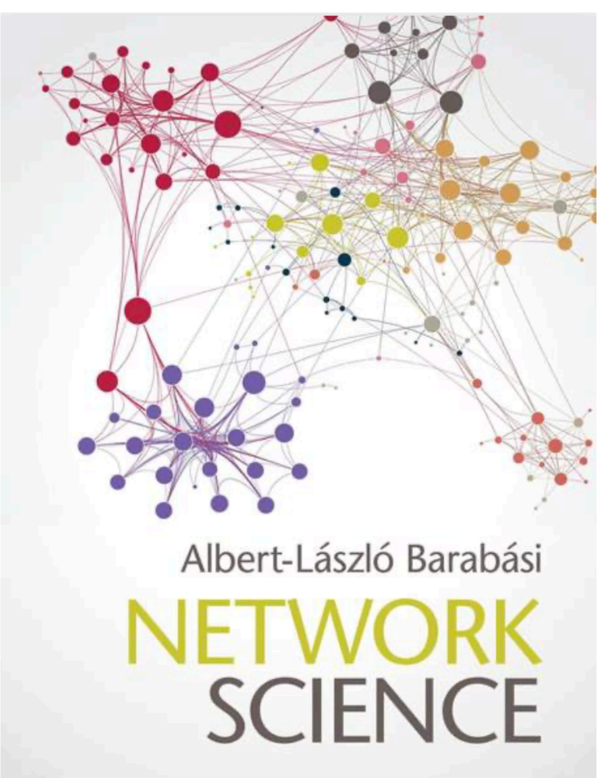


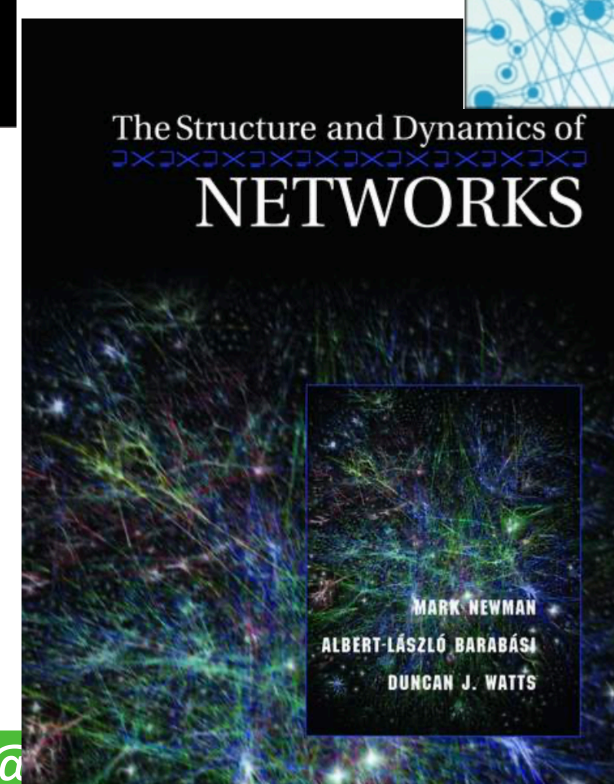
$$\begin{cases} \dfrac{dx}{dt} = -y - z \\ \dfrac{dy}{dt} = x + ay \\ \dfrac{dz}{dt} = b + z(x - c) \end{cases}$$

Network theory









Network theory

Non physical networks :

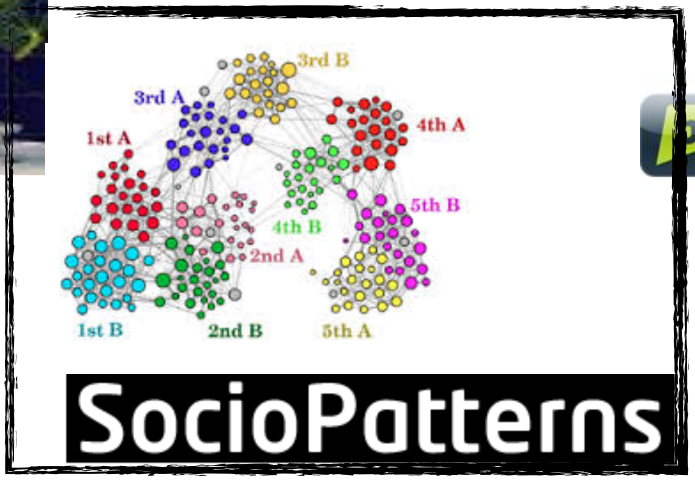
- Friendships
- www
- email / texto / call
- ...

Physical networks :

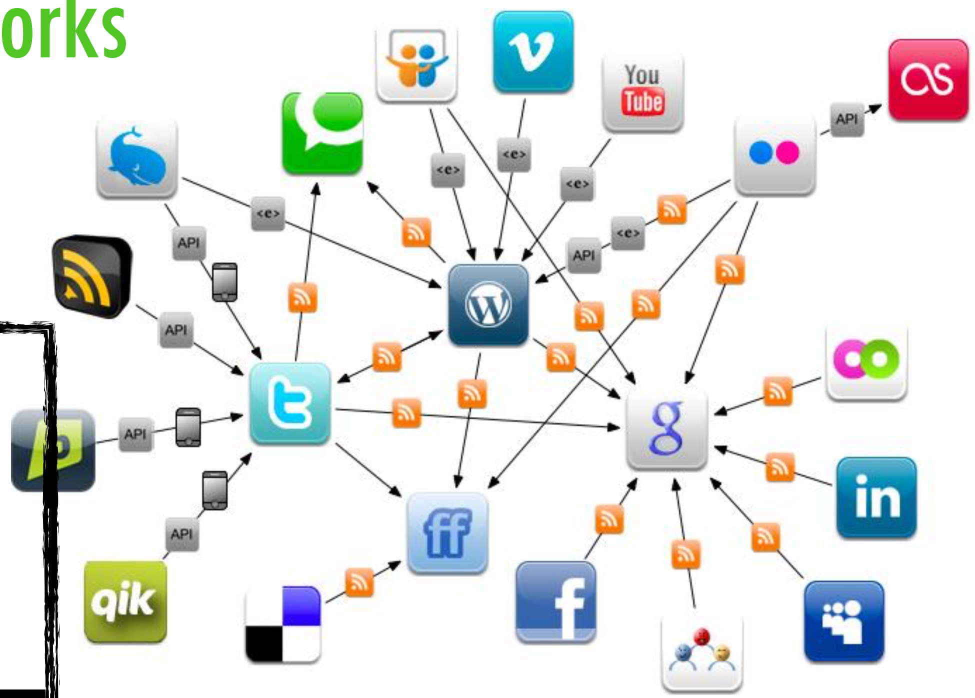
- Power plant
- internet
- road / train / flight
- ...

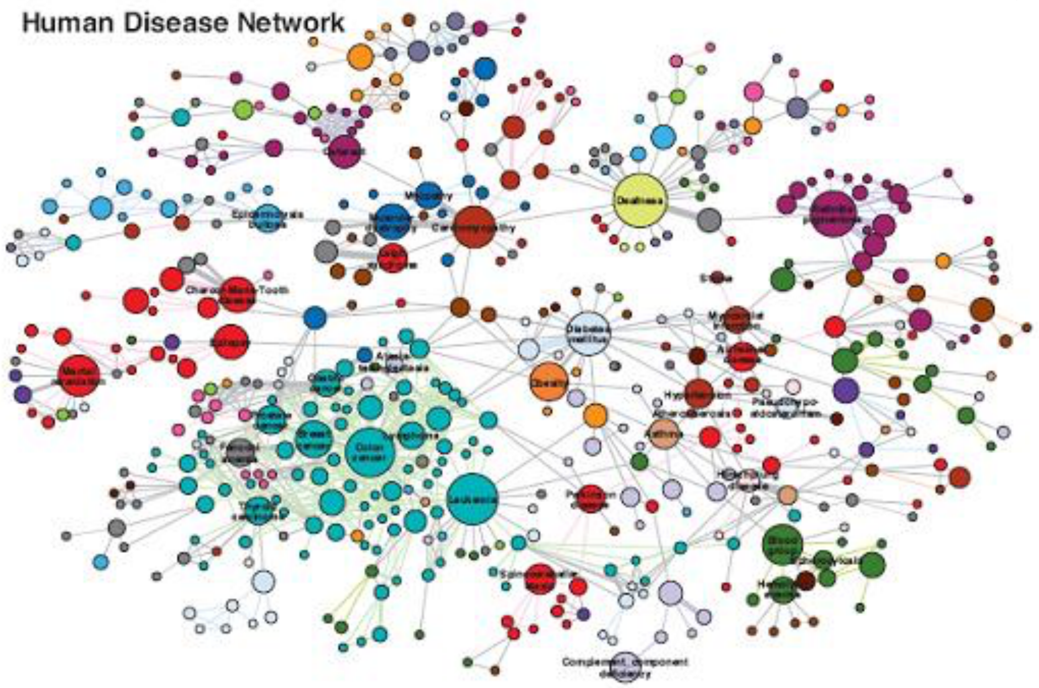
Networks are everywhere

world flights map

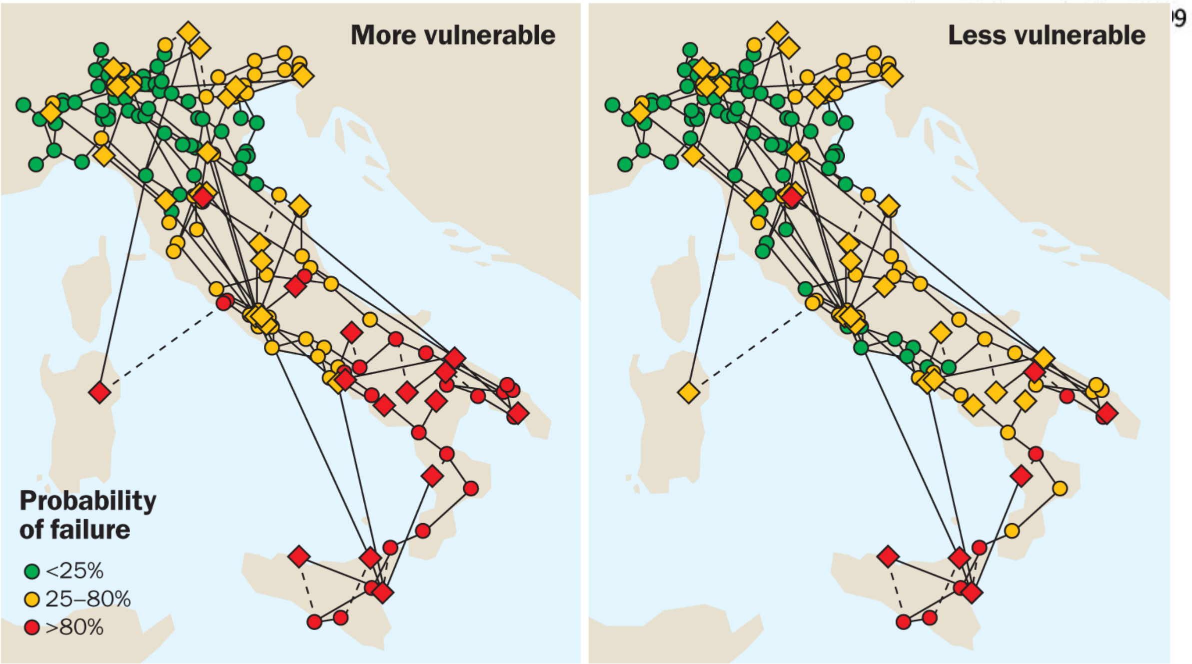


social networks





proteins networks



technological networks

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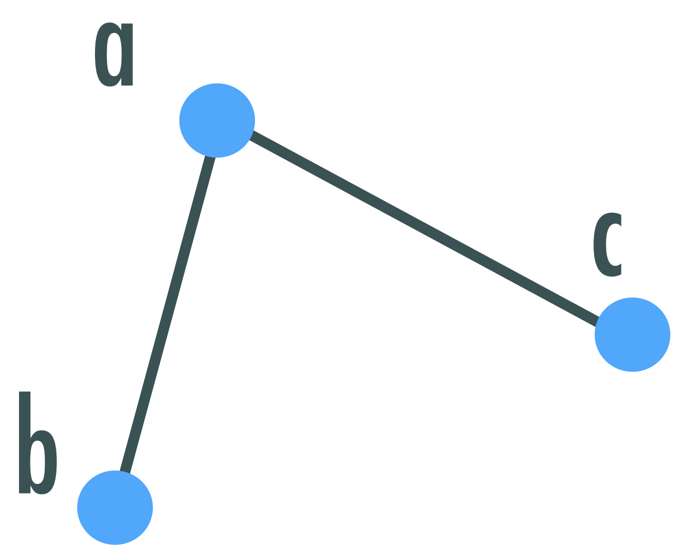
Network theory

Network = finite set of nodes pairwise connected, i.e., there is a link (edge) among the two nodes if there is some interaction among them



a, b and c are human beings

a and b are friends, they can exchange ideas

a and c are friends, they can exchange ideas

Network theory

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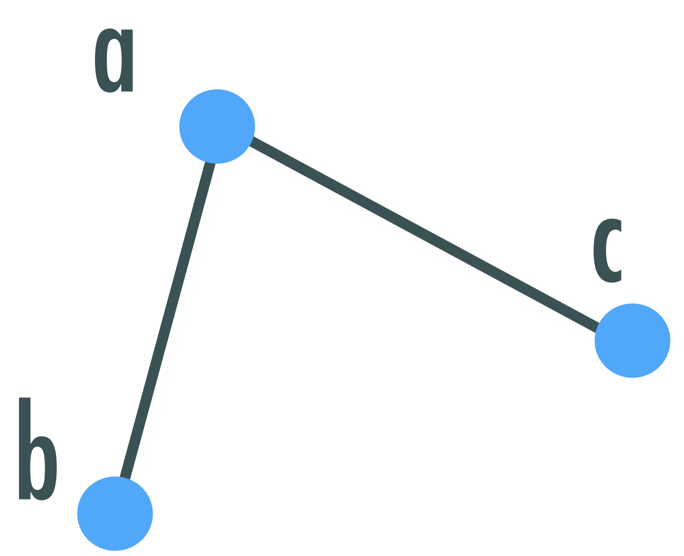

a, b and c are web pages

a and b are linked, they share hyperlinks

a and c are linked, they share hyperlinks

Network theory

A network can be encoded by the adjacency matrix $\mathbf{A} \in \{0,1\}^{n\times n}$

$$A_{ij} = 1 \quad \text{iff} \quad i \text{ and } j \text{ are linked}$$

The degree of a node is the number of its neighbours

$$k_i = \sum_{j=1}^{n} A_{ij}$$

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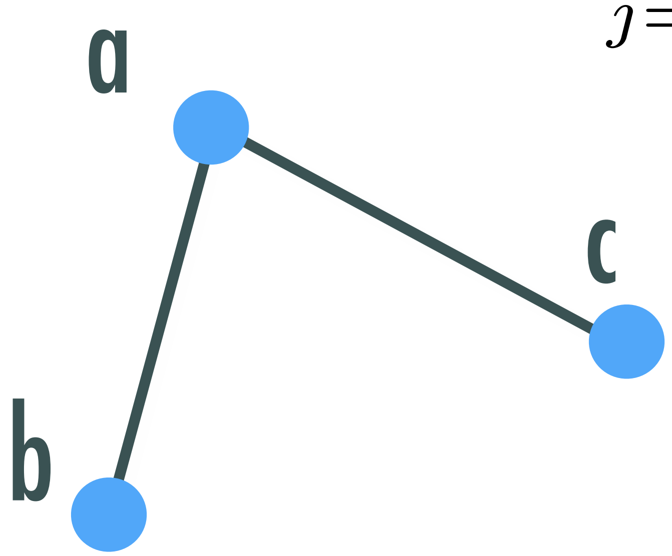

$$A_{ij} = \begin{array}{c} \\ \\ \end{array} \begin{array}{c} \begin{array}{ccc} \text{a} & \text{b} & \text{c} \end{array} \\ \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \end{array} \qquad k_a = 2\,, k_b = 1\,, k_c = 1$$

Network theory

If all the links are reciprocal ones, then we have an undirected network

$$A_{ij} = A_{ji} \quad \forall i, j$$

Otherwise, we have an directed network

$$A_{ij} = 1 \quad \text{iff} \quad j \to i$$

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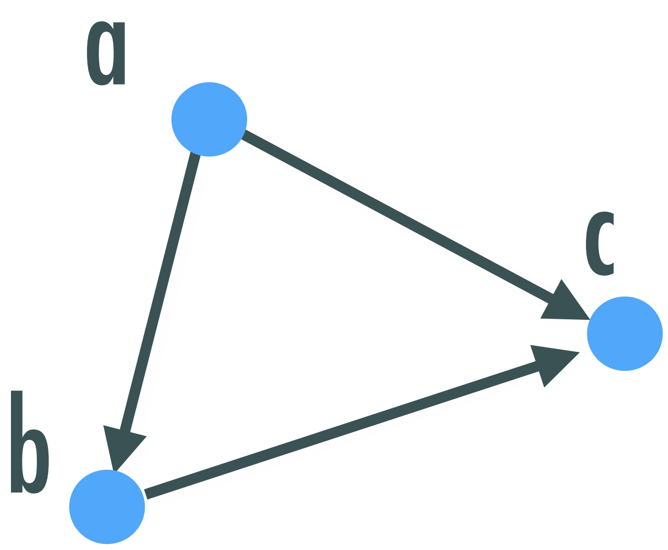

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The out-degree of a node is the number of exiting links

$$k_i^{out} = \sum_{j=1}^{n} A_{ji}$$

The in-degree of a node is the number of entering links

$$k_i^{in} = \sum_{j=1}^{n} A_{ij}$$

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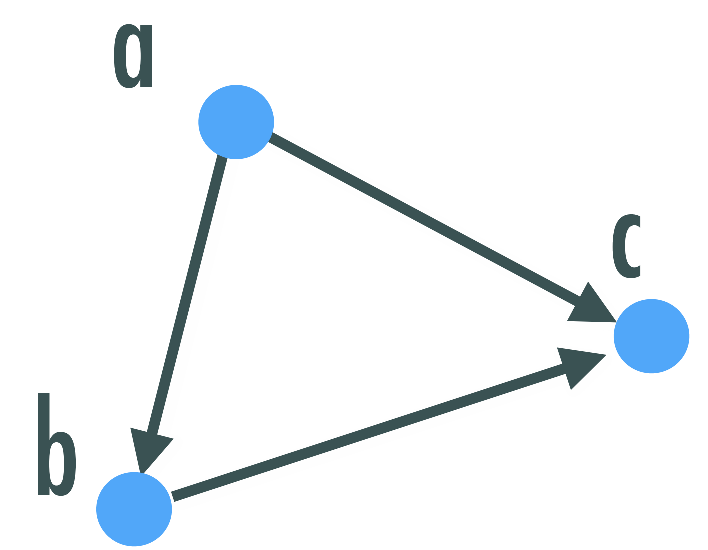

$$k_a^{out} = 2\,, k_b^{out} = 1\,, k_c^{out} = 0$$

$$k_a^{in} = 0\,, k_b^{in} = 1\,, k_c^{in} = 2$$

Links can be weighted $\mathbf{A} \in \mathbb{R}_{+}^{n\times n}$

$$A_{ij} = s \quad \text{iff} \quad i \text{ and } j \text{ have a connection whose weight is } s$$

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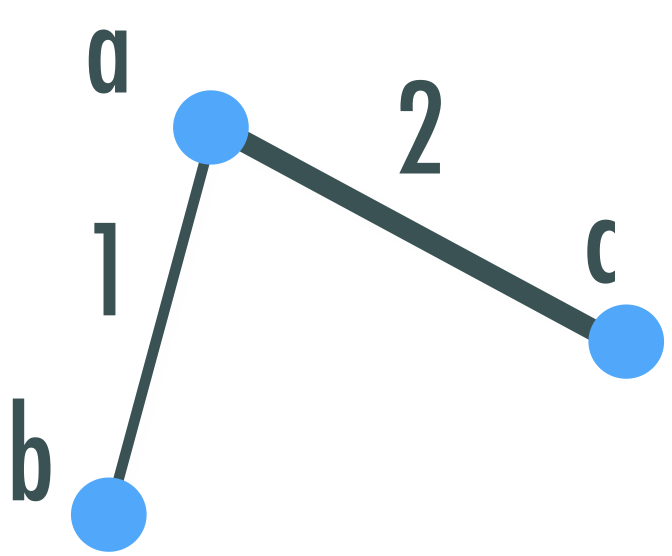
$$A_{ij} = s \quad \text{iff} \quad i \text{ and } j \text{ have a connection whose weight is } s$$

The <u>degree</u> is replaced by the notion of <u>strength</u>

$$s_i = \sum_j A_{ij}$$



$$A_{ij} = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}$$

$$s_a = 3\,,\, s_b = 1\,,\, s_c = 2$$

Network theory

A network can be also encoded by the incidence matrix $\mathbf{M} \in \{0,1\}^{m \times n}$

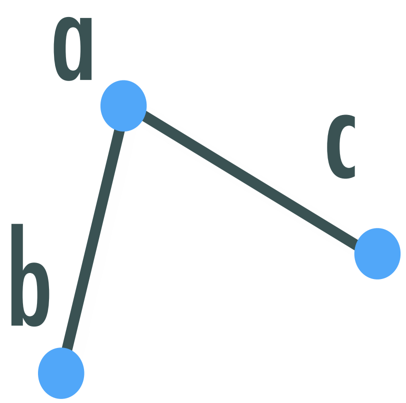
m number of links, n number of nodes

Let $e = (i, j)$ be link in the network, then

$$M_{e,i} = 1\,, M_{e,j} = -1 \qquad\qquad M_{e,k} = 0 \quad \forall k \neq i, j$$

Network theory

$$e = (i, j) \qquad M_{e,i} = 1\,, M_{e,j} = -1 \qquad M_{e,k} = 0 \quad \forall k \neq i, j$$



$$A_{ij} = \begin{array}{c} \begin{array}{ccc} \text{a} & \text{b} & \text{c} \end{array} \\ \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \end{array} \qquad M_{ij} = \begin{array}{cc} \begin{array}{ccc} \text{a} & \text{b} & \text{c} \end{array} & \\ \begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix} & \begin{array}{c} \text{(a,b)} \\ \text{(a,c)} \end{array} \end{array}$$

$$-\mathbf{M}^\top \mathbf{M} = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix}$$

$$-\mathbf{M}^\top \mathbf{M} = \mathbf{A} - \mathrm{diag}(k_1, k_2, k_3)$$

Network theory

Models of networks.

Erdős–Rényi (random network) $G(n,p)$

Given n nodes, consider all pairs and with probability $p \in (0,1)$ link them.

1) The average number of links is $\binom{n}{2}p$

Erdős, P.; Rényi, A. (1959). "On Random Graphs", *Publicationes Mathematicae*, **6**, 290–297

Network theory

Models of networks.

Erdős–Rényi (random network) $G(n,p)$

Given n nodes, consider all pairs and with probability $p \in (0,1)$ link them.

1) The average number of links is $\binom{n}{2}p$

2) The probability to have a node with degree k is

$$P(k) = \binom{n-1}{2} p^k (1-p)^{n-1-k}$$

3) if $n \to \infty$ and $np = const$ then

$$P(k) \sim \frac{(np)^k e^{-np}}{k!}$$

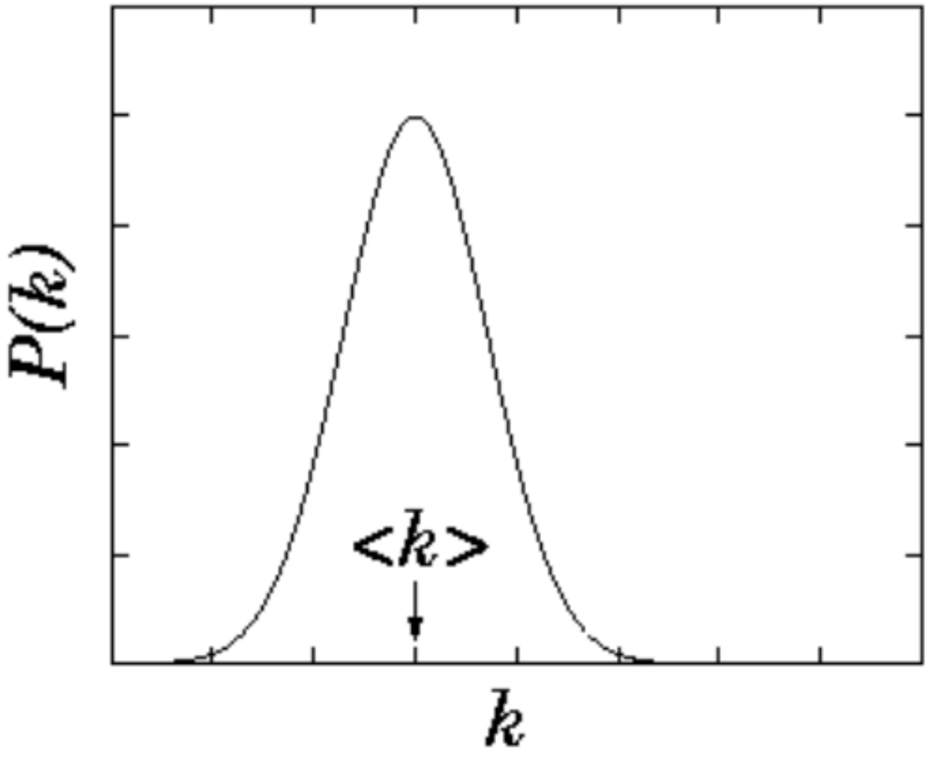
Models of networks.

Erdős–Rényi (random network) $G(n, p)$

Given n nodes, consider all pairs and with probability $p \in (0,1)$ link them.

$$n \to \infty \qquad np = const \qquad P(k) \sim \frac{(np)^k e^{-np}}{k!}$$

$$\langle k \rangle = \sum_k k P(k)$$



Models of networks.

Watts–Strogatz (random network with small world property)

ℓ_{ij} Distance among two nodes = number of "hops" needed to connect them

Average shortest path : $$\ell_G = \frac{1}{n(n-1)}\sum_{i\neq j}\ell_{ij}$$

Complete network $\ell_G = 1$

d-dimensional lattice (n nodes) $\ell_G \sim n^{1/d}$

Erdős–Rényi (n,p) $$\ell_G \sim \frac{\log n}{\log(np)}$$

Models of networks.

Watts–Strogatz (random network with small world property)

Let y_i be the actual number of links between the neighbours of node i

the (local) clustering coefficient is

$$C_i = \frac{y_i}{k_i(k_i-1)/2}$$

Network theory

Models of networks.

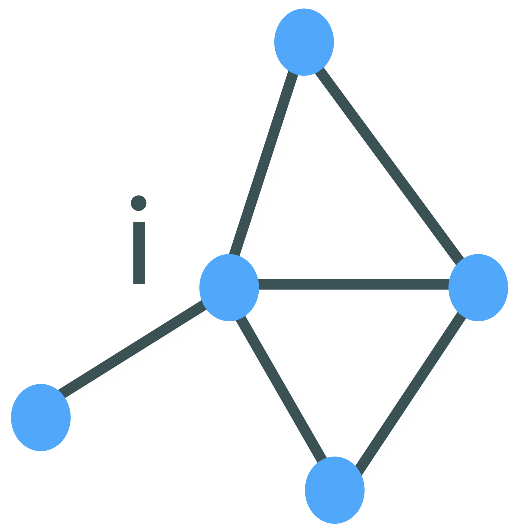
Watts–Strogatz (random network with small world property)

$$C_i = \frac{y_i}{k_i(k_i-1)/2}$$

$$C_i = \frac{2}{4\times 3/2} = \frac{1}{3}$$

$$k_i = 4$$

$$y_i = 2$$

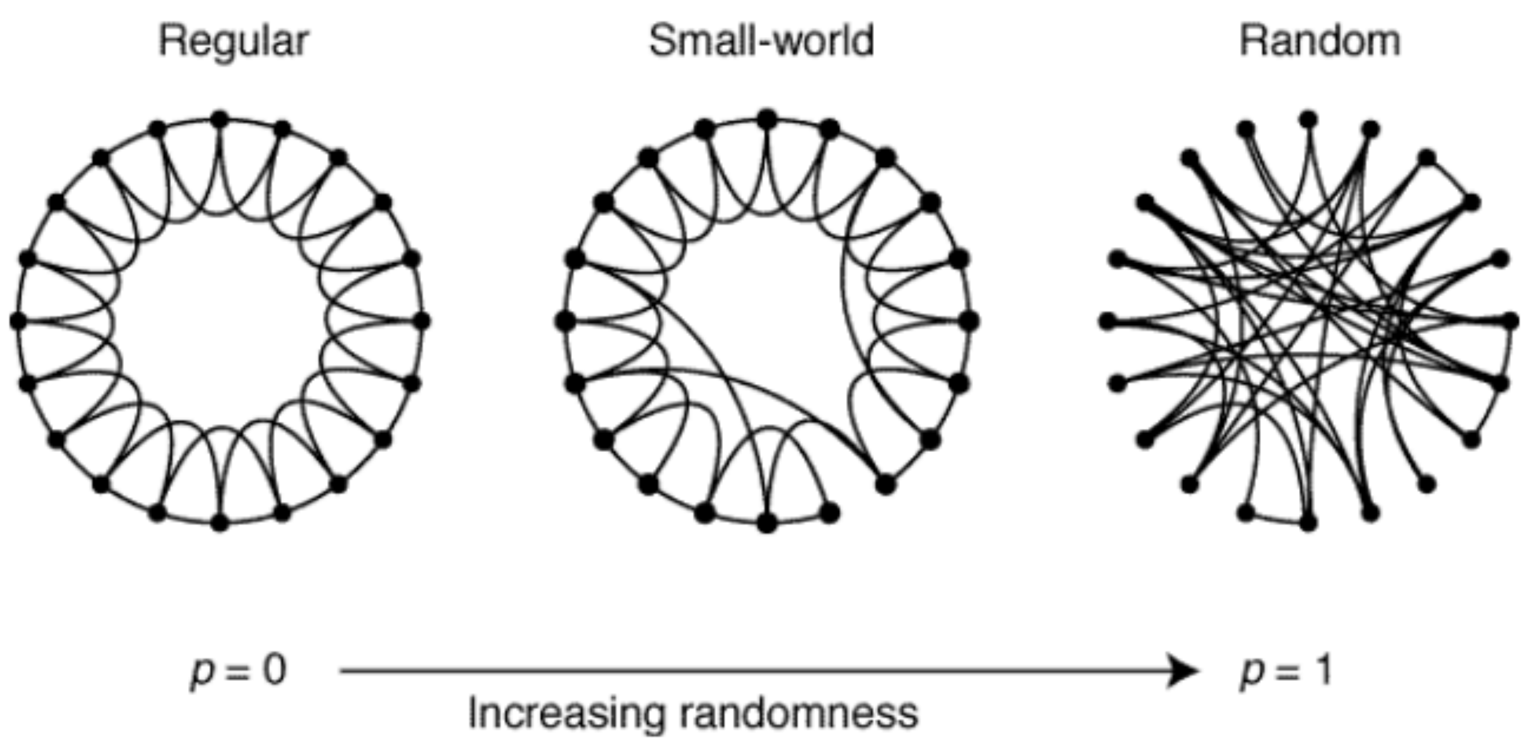


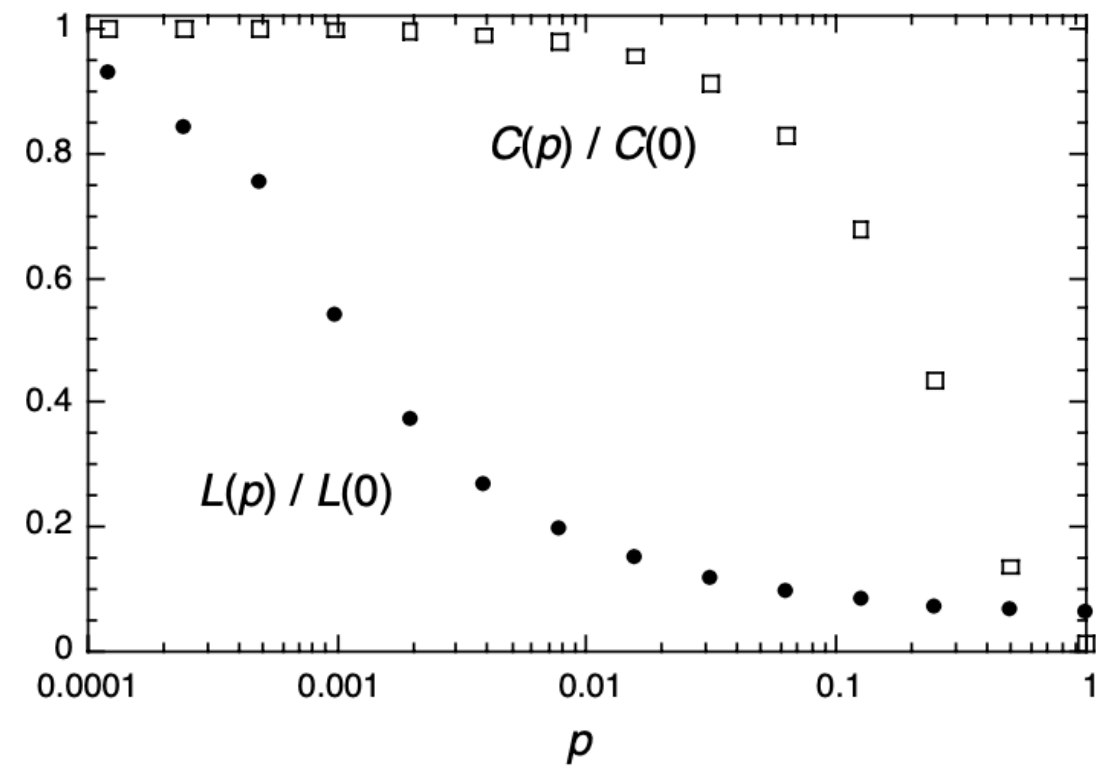
That is, the number of triangles among all the possible ones

the clustering coefficient is $C_G = \frac{1}{n}\sum_i C_i$

Models of networks.

Watts–Strogatz (random network with small world property)



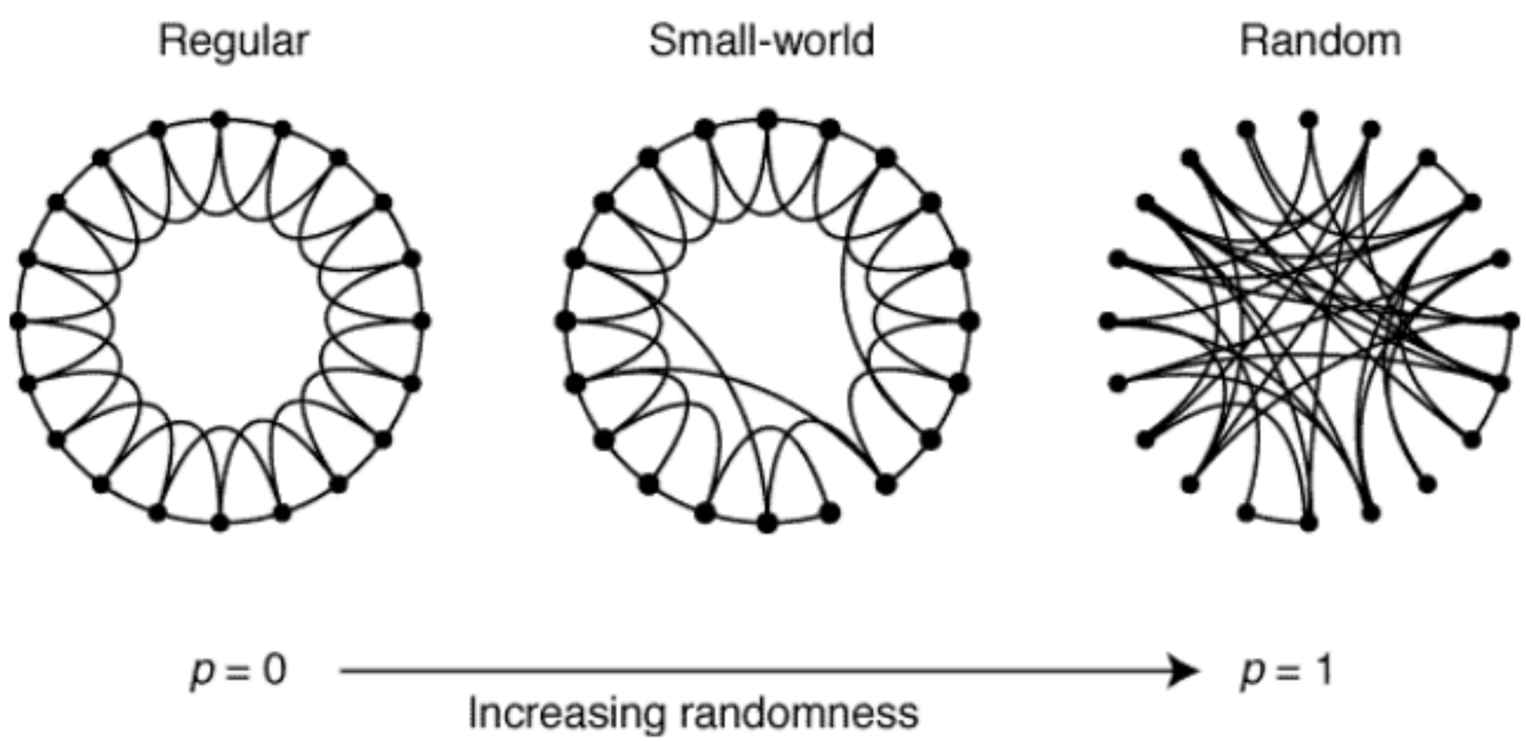


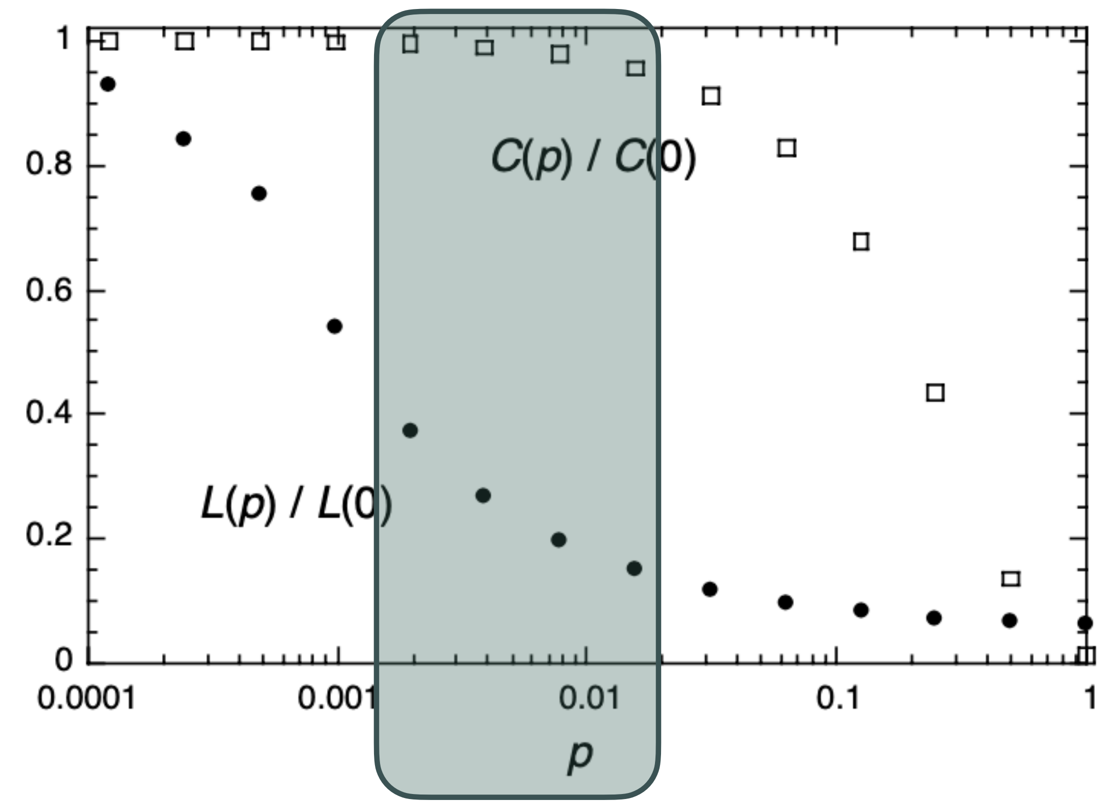
Watts, D. J.; Strogatz, S. H. (1998). "Collective dynamics of 'small-world' networks", *Nature*, **393** (6684): 440–442.

Network theory

Models of networks.

Watts–Strogatz (random network with small world property)

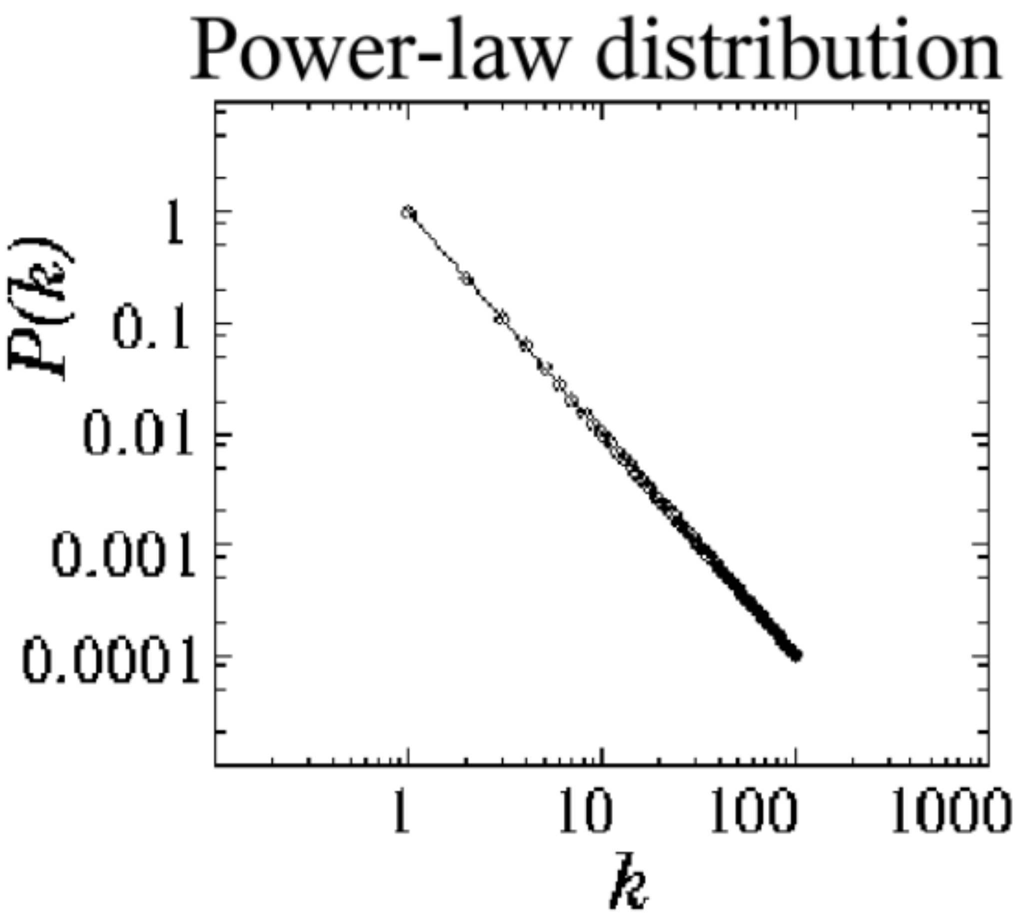




Small distance but large clustering (as in lattices)

Models of networks.

Barabási - Albert (random network with scale free property)



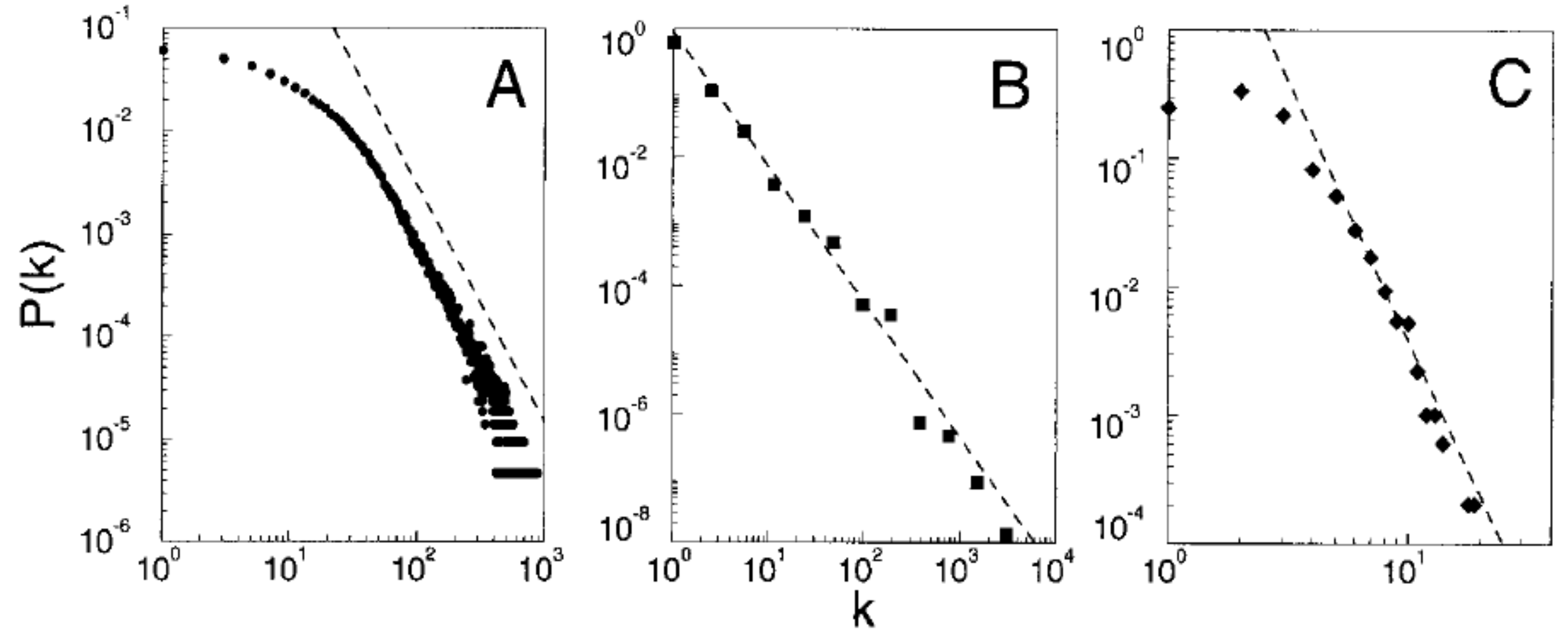


Fig. 1. The distribution function of connectivities for various large networks. (**A**) Actor collaboration graph with $N = 212{,}250$ vertices and average connectivity $\langle k \rangle = 28.78$. (**B**) WWW, $N = 325{,}729$, $\langle k \rangle = 5.46$ (6). (**C**) Power grid data, $N = 4941$, $\langle k \rangle = 2.67$. The dashed lines have slopes (A) $\gamma_{actor} = 2.3$, (B) $\gamma_{www} = 2.1$ and (C) $\gamma_{power} = 4$.

Models of networks.

Barabási - Albert (random network with scale free property)

1) Start with m nodes connected among them

2) At each time step add 1 new node with a new link to existing nodes

3) Preferential attachement : the new node will select existing nodes according to their degree

$$P(\text{new node} \to i) = \frac{k_i}{\sum_j k_j}$$

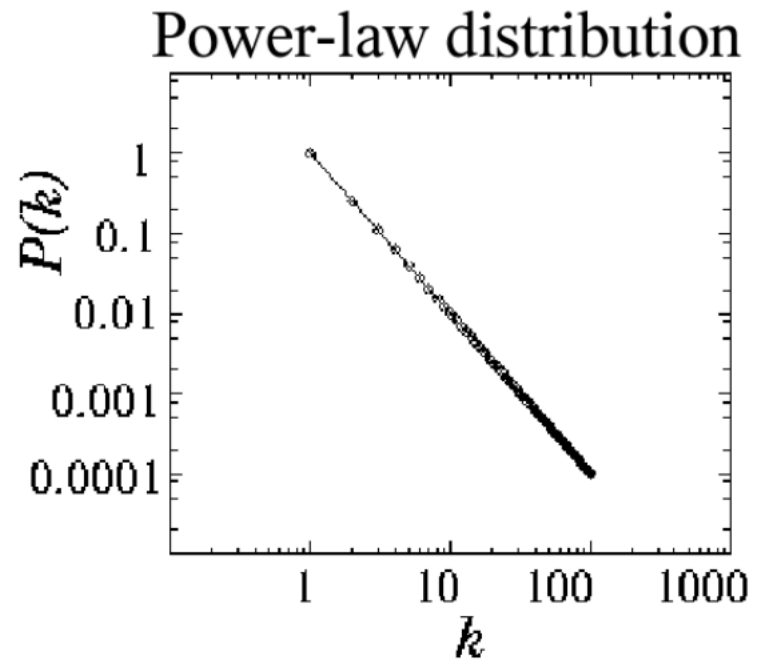
Barabási, A.; Albert, R. (1999), "Emergence of scaling in random networks", *Science*, **286** (5439): 509–512.

Models of networks.

Barabási - Albert (random network with scale free property)

In the limit of large networks one has

$$P(k) \sim \frac{1}{k^3}$$



Network theory

Models of networks.

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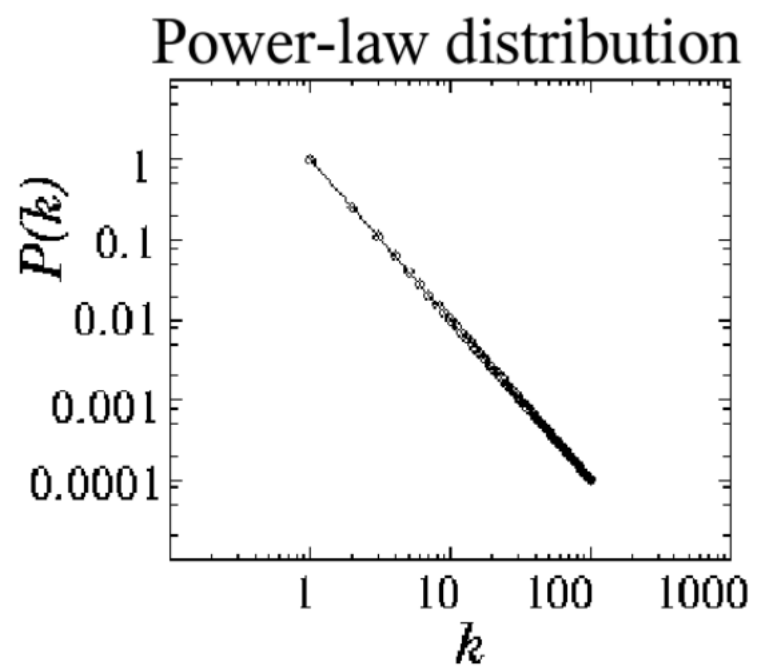


A network for which $P(k) \sim \dfrac{1}{k^\gamma}$ is called scale free

Models of networks.

Scale free network $$P(k) \sim \frac{1}{k^\gamma}$$

If $2 < \gamma \leq 3$ then $\langle k \rangle < \infty$ and $\langle k^2 \rangle = \infty$

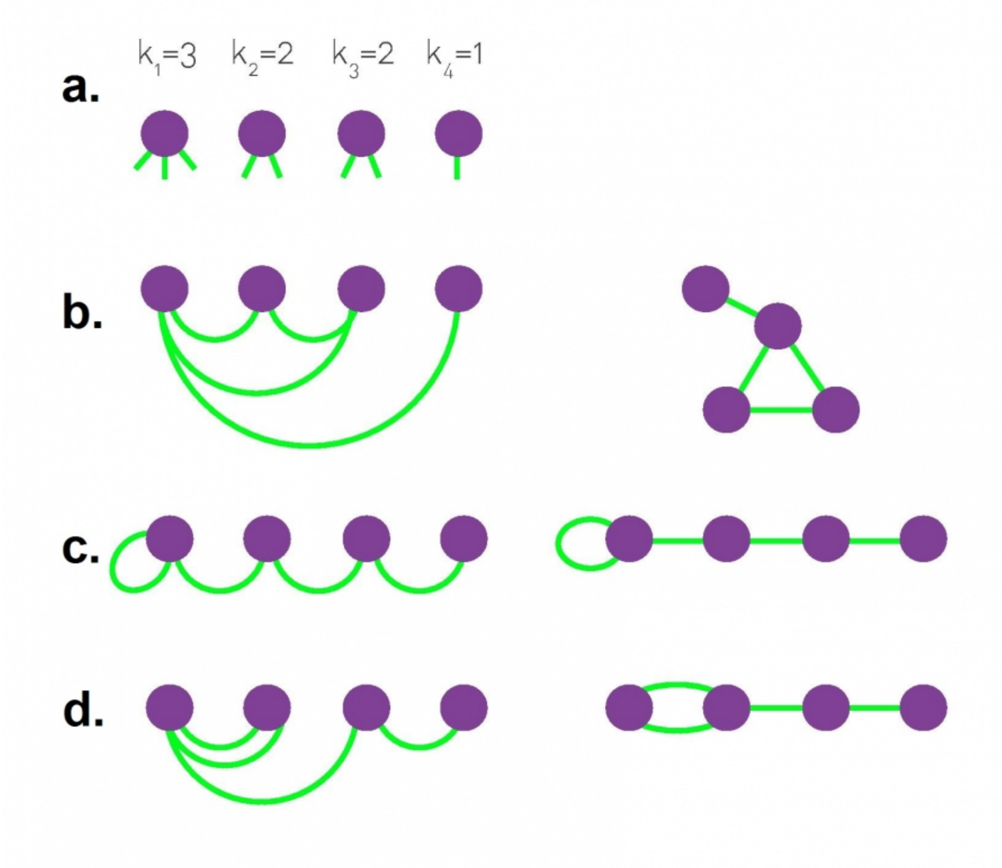
If $\gamma > 3$ then $\langle k \rangle < \infty$ and $\langle k^2 \rangle < \infty$

$$\langle k \rangle = \int_{k_{min}}^{k_{max}} k^{1-\gamma} dk \sim \int_1^\infty k^{1-\gamma} dk$$

Models of networks.

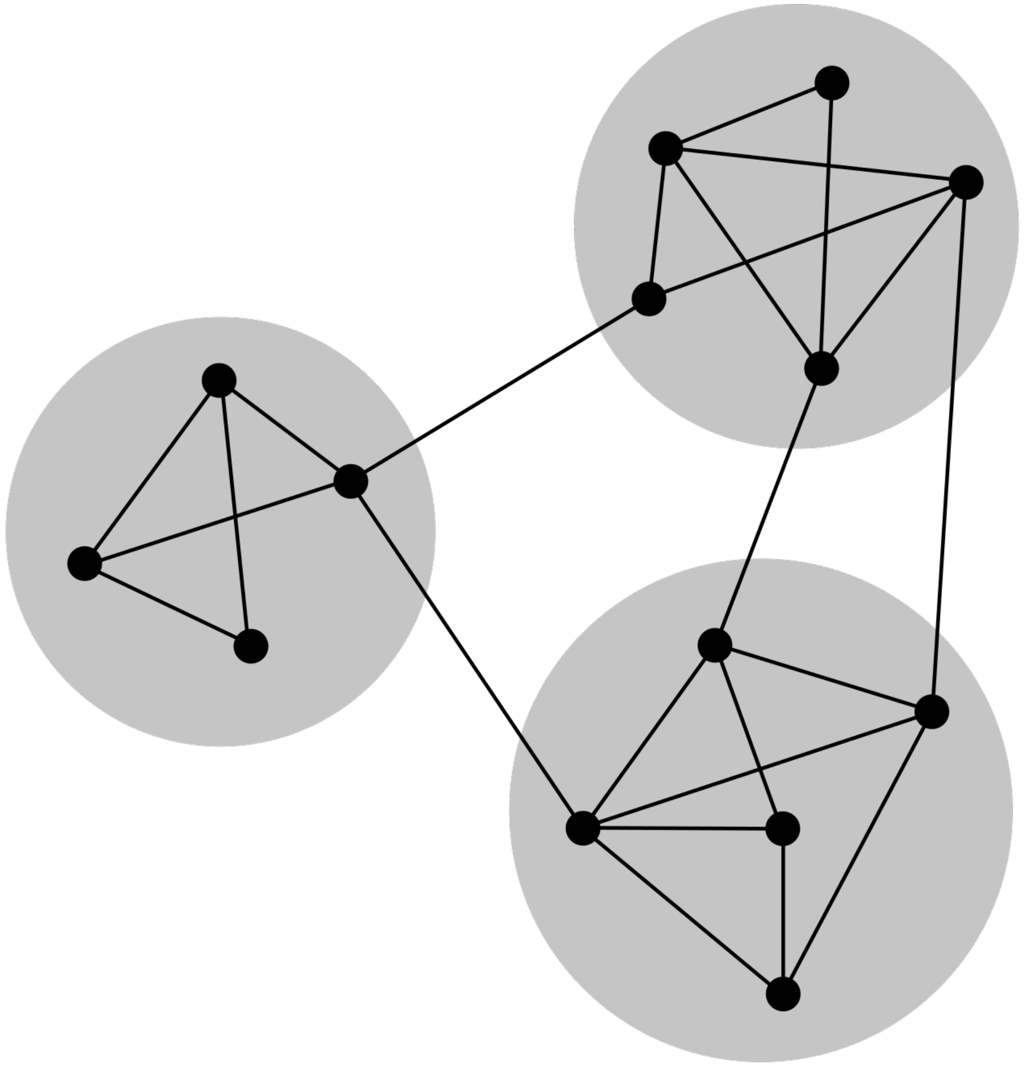
Configuration model

Given the degree sequence $k_1, \ldots, k_n$ reconstruct the network that exhibits such degrees



Community detection

Group of node tightly connected among them and weakly connected with the rest of the network.



Community detection

Group of node tightly connected among them and weakly connected with the rest of the network.

Communities allow to “simplify” the network structure

Communities allow to “better” understand the network dynamics