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Existence of harmonic solutions for some generalisation of the non-autonomous Liénard equations

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Abstract

We study the problem of existence of harmonic solutions for some generalisations of the periodically perturbed Liénard equation, where the damping function depends both on the position and the velocity. In the associated phase-space this corresponds to a term of the form $f(x, y)$ instead of the standard dependence on x alone. We introduce suitable autonomous systems to control the orbits behaviour, allowing thus to construct invariant regions in the extended phase-space and to conclude about the existence of the harmonic solution, by invoking the Brouwer fixed point Theorem applied to the Poincaré map. Applications are given to the case of the p-Laplacian and the prescribed curvature equation.

Keywords: Non-autonomous systems, Generalized Liénard equations, Prescribed curvature operator, Relativistic acceleration, φ-Laplacian, Positively invariant sets, Brouwer fixed point theorem

MSC Classification: 35C25 , 34L30 , 34A26

1 Introduction

Starting from the pioneering work by Liénard [14] in 1928, the quest for the existence of periodic solutions for the following model, nowadays called Liénard equation

$$\ddot{x} + f(x)\dot{x} + g(x) = 0\,, \tag{1}$$

has attracted a lot of attention and there is an enormous amount of works available in the literature. The interested reader can consult for instance [8, 26, 27] and the references quoted therein.

Generalisations, taking into account nonlinear acceleration terms, have been recently studied [19, 22]

$$\frac{d}{dt}\varphi(\dot{x}) + f(x)\dot{x} + g(x) = 0\,, \tag{2}$$

where a nonlinear function φ of the velocity is considered.

In particular assuming $\varphi(\dot{x}) = \dot{x}/\sqrt{1+\dot{x}^2}$ allows to recast the previous model into the *prescribed curvature* equation of Liénard type [20]

$$\frac{d}{dt}\frac{\dot{x}}{\sqrt{1+\dot{x}^2}} + f(x)\dot{x} + g(x) = 0\,. \tag{3}$$

Similarly, one can study the *relativistic* Liénard equation

$$\frac{d}{dt}\frac{\dot{x}}{\sqrt{1-\dot{x}^2}} + f(x)\dot{x} + g(x) = 0\,, \tag{4}$$

by taking $\varphi(\dot{x}) = \dot{x}/\sqrt{1-\dot{x}^2}$. The latter corresponding to a relativistic acceleration term, hence the name.

Motivated by the recent work [5], the results hereby presented are a natural development of the former one. In particular we will extend previous existence results to the non-autonomous case and allow for a more general velocity term depending on both x and $\dot{x}$. More precisely we will introduce a *T-periodic continuous function*, $e(t)$, a function $\hat{f}(x,\dot{x})$, and thus consider the *φ-Laplacian non-autonomous* equation of Liénard type

$$\frac{d}{dt}\varphi(\dot{x}) + \hat{f}(x,\dot{x})\dot{x} + g(x) = e(t)\,, \tag{5}$$

where $\varphi : (-\rho,\rho) \to (-\omega,\omega)$, is an odd increasing homeomorphism, with $0 < \rho, \omega \leq +\infty$. Notice that, at a first glance, it would be more natural to assume that φ is a diffeomorphism. However, by suitably considering the solution as an orbit of an equivalent planar system (see (6)), one can see that the above assumption about φ is sufficient. We also observe that (3) is a particular case of (5) with $\rho = +\infty$, $\omega = 1$ and $\varphi(s) = s/\sqrt{1+s^2}$, while (4) corresponds to (5) with $\rho = 1$, $\omega = +\infty$ and $\varphi(s) = s/\sqrt{1-s^2}$. In particular, due to the presence of the periodic forcing term $e(t)$, we will be interested in studying the existence of *harmonic solutions*, namely periodic solutions with the same period T of the forcing term.

Models close to these analyzed here, have been recently investigated by several authors, using either the Leray-Schauder-type methods initiated in [1], or variational methods initiated in [2], or symplectic methods initiated in [10]. We refer the interested reader to the surveys [17, 18] for broader discussion about these methods. We hereby will adopt a completely different approach; we will employ standard phase-space analysis coupled with comparison methods to construct invariant regions allowing us to apply the Brouwer fixed point Theorem [9] to the T-time Poincaré map. Indeed Eq. (5) is equivalent to the time dependent system defined on the phase-space

$$\begin{cases} \dot{x} = \varphi^{-1}(y) \\ \dot{y} = -f(x,y)\varphi^{-1}(y) - g(x) + e(t)\,, \end{cases} \tag{6}$$

with $f(x,y) = \hat{f}(x,\varphi^{-1}(y))$. Hence, defining $E_+ = \max_t e(t)$ and $E_- = \min_t e(t)$, one can compare the previous system (6) with the autonomous ones

$$\begin{cases} \dot{x} &= \varphi^{-1}(y) \\ \dot{y} &= -f(x,y)\varphi^{-1}(y) - g(x) + E_+\,, \end{cases} \tag{7}$$

and

$$\begin{cases} \dot{x} &= \varphi^{-1}(y) \\ \dot{y} &= -f(x,y)\varphi^{-1}(y) - g(x) + E_-\,. \end{cases} \tag{8}$$

Under suitable assumptions (see Lemma 1 and Lemma 4) one can control the behaviour of the orbits of system (6) using the solutions of systems (7) and (8) and eventually construct a bounded region of the phase-space, homeomorphic to a disk, that is mapped onto itself by the (time T) Poincaré map of the system (6). Being the flow of the latter, a continuous function of the initial conditions, one can apply the Brouwer Theorem and prove the existence of a fixed point for the T-Poincaré map, that is a T-periodic solution of the non-autonomous system (6) (see Theorem 1).

Let us observe that the case of the prescribed curvature operator corresponds to

$$\varphi^{-1}(y) = \frac{y}{\sqrt{1-y^2}}\,,$$

while that of the relativistic acceleration is given by

$$\varphi^{-1}(y) = \frac{y}{\sqrt{1+y^2}}\,.$$

In general, we have that

$$\varphi^{-1} : (-\omega,\omega) \to (-\rho,\rho) \subset \mathbb{R},$$

is an odd increasing homemorphism with $0 < \omega \leq +\infty$. We do not assume that φ^{-1} is bounded, even if actually this case may occur in the case of the relativistic acceleration.

The paper is organized as follows. We present the main definitions and hypotheses and then we prove our main result, Theorem 1, for equation (5). Subsequently, we present some examples and applications for different choices of the operator φ, including the above mentioned prescribed curvature case, the relativistic acceleration and the p-Laplacians as well. We will also briefly discuss the presence of more complex solutions of recurrence type.

2 Definitions and results

Throughout the paper, we assume the functions $f(x,y)$ and $g(x)$ to satisfy the standard hypotheses to ensure the existence and uniqueness of the associated Cauchy problem, and the forcing function $e(t)$ to be continuous and T-periodic for some $T > 0$. Moreover we assume a suitable sign condition for the function $g(x)$, namely $g(x) > E_+$ for large enough $x > 0$ and $g(x) < E_-$ for small enough $x < 0$, where E_+ and E_- denote the bounds on $e(t)$ previously defined.

Let us start by considering the case of the non-autonomous φ-Laplacian Liénard equation given by Eq. (6) in phase-space. To prove the existence of a periodic solution for the latter, we will first introduce two autonomous systems, obtained by replacing the forcing term $e(t)$ with its maximum and minimum. Then the solutions of each systems can be studied and their behaviour quantitatively determined using ideas taken from [5].

It will be convenient to introduce the auxiliary function

$$h(y) := \frac{y}{\varphi^{-1}(y)},$$

which corresponds to $\sqrt{1-y^2}$ for the prescribed curvature case and to $\sqrt{1+y^2}$ for the relativistic one. Notice that h is an even and positive function, being φ^{-1} odd.

Before to prove our main result, let us present some preliminary Lemmas. We start with the following Lemma which is similar to the one discussed in [5] and based on previous results [3, 4, 24, 25]. This result has a general validity as it does not require $\omega = +\infty$.

Lemma 1. *Let us assume there exist $\alpha < 0 < \beta$ and two smooth functions $\phi(x)$ and $\psi(x)$ such that*

$$0 < \phi(x) < \omega \quad \forall x \leq \alpha \ \textit{and} \ -\omega < \psi(x) < 0 \quad \forall x \geq \beta \,.$$

Assume moreover

$$f(x,\phi(x))\phi(x) > -\phi'(x)\phi(x) - (g(x) - E_+)\, h(\phi(x)) \quad \forall x \leq \alpha \,, \tag{9}$$

and

$$f(x,\psi(x))\psi(x) < -\psi'(x)\psi(x) - (g(x) - E_-)\,h(\psi(x)) \quad \forall x \geq \beta\,. \tag{10}$$

Then the orbits of system (6) enter the regions bounded by $y = \phi(x)$ for $x \leq \alpha$ and $y = \psi(x)$ for $x \geq \beta$ (see Fig. 1).

Proof Let us consider in detail the case involving $\phi(x)$; being the one for $\psi(x)$ similar, the associated proof will be omitted. The slope of system (6) is given by

$$\frac{dy}{dx}(x) = -f(x,y) - \frac{g(x)}{\varphi^{-1}(y)} + \frac{e(t)}{\varphi^{-1}(y)}\,, \tag{11}$$

and thus evaluated on the graph of the function $\phi(x)$ we obtain:

$$\begin{aligned} \frac{dy}{dx}(x)\Big|_{y=\phi(x)} &= -f(x,\phi(x)) - \frac{g(x)-e(t)}{\varphi^{-1}(\phi(x))} \\ &= \frac{1}{\phi(x)}\left[-f(x,\phi(x))\phi(x) - (g(x) - e(t))\,h(\phi(x))\right]\,, \end{aligned} \tag{12}$$

where we have used the position that $h(y) = y/\varphi^{-1}(y)$.

Recalling the assumption (9) and the positiveness of $\phi(x)$ for $x \leq \alpha$ we get:

$$\frac{dy}{dx}(x)\Big|_{y=\phi(x)} < \phi'(x) + (e(t) - E_+)\,\frac{1}{\varphi^{-1}(\phi(x))} \quad \forall x \leq \alpha\,,$$

recalling the definition of E_+ we can finally conclude

$$\frac{dy}{dx}(x)\Big|_{y=\phi(x)} < \phi'(x) \quad \forall x \leq \alpha\,.$$

Namely, any orbits starting on $y = \phi(x)$ will enter the region bounded by such curve if $x \leq \alpha$ (see Fig. 1). □

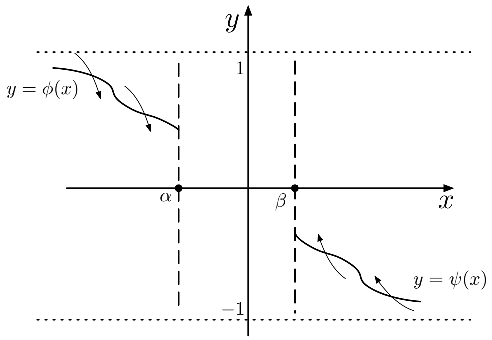

Fig. 1 The curves $y = \phi(x)$ and $y = \psi(x)$ and their relation with the trajectories of the system (6). In this illustrative example, we have assumed that $\omega = 1$, corresponding to the case of the prescribed curvature operator.

We are now treating the cases which allow us to rule out the presence of vertical asymptotes for the orbits.

Lemma 2. *Let us assume $\omega = +\infty$ and that there exist a positive continuous function $T(x)$ and three positive constants L, D and M such that*

$$|f(x,y)| \leq LT(x)|y| \quad \forall x \in [-M,M] \text{ and } |y| \geq D\,, \tag{13}$$

where $M \geq \max\{-\alpha,\beta\}$, for α and β being the constants introduced in Lemma 1. Then any orbit of system (6) starting on the lines $x = \pm M$ cannot escape to infinity (i.e. no vertical asymptote is allowed).

Proof One can easily get the following bound on the slope of system (6)

$$\left|\frac{dy}{dx}(x)\right| \leq |f(x,y)| + (|g(x)| + E_+)\,\frac{1}{\varphi^{-1}(|y|)} \leq LT(x)|y| + (|g(x)| + E_+)\,\frac{1}{\varphi^{-1}(D)}\,,$$

hence the orbit can be continued in the future starting from any $x_0 = -M$ and $y_0 \geq D$ up to $x = M$. □

Remark 1. *We observe that if $\omega < +\infty$, we need to use a slightly different lemma in our setting in which one must also control the growth of the orbit $y(x)$, that is the latter should remain inside the strip $\mathbb{R} \times (-\omega,\omega)$. This constraint could be obtained by multiplying $f(x,y)$ by a positive small parameter, as in [20]. We hereby propose an alternative way relying on the values of $f(x,y)$ close to the upper/lower boundary of the strip.*

Lemma 3. *Let us assume $\omega < +\infty$ and that there exist three positive constants M, δ_ω and $\delta_{-\omega}$ such that for all $x \in [-M,M]$ the following limits do exist (uniformly with respect to $x \in [-M,M]$)*

$$\lim_{y\to\omega^-} f(x,y) = \delta_\omega \text{ and } \lim_{y\to-\omega^+} f(x,y) = \delta_{-\omega}\,. \tag{14}$$

Then any orbit of system (6) starting on the lines $x = \pm M$ cannot escape from the strip $\mathbb{R} \times (-\omega,\omega)$.

Proof The time evolution of the vertical component, y, of system (6) is given by

$$\frac{dy}{dt}(x) = -f(x,y)\varphi^{-1}(y) - g(x) + e(t)\,; \tag{15}$$

Let $x \in [-M,M]$ and consider values of y close but smaller than ω, hence the term $-f(x,y)\varphi^{-1}(y)$ is negative because of the assumption (14) and very large in absolute value. In conclusion the whole right hand side of Eq. (15) is strictly negative and $y(t)$ should be a decreasing function, preventing thus it from reaching the value ω.

A similar analysis can be done considering y values close, but larger, than $-\omega$. In this case, the term $-f(x,y)\varphi^{-1}(y)$ is positive and large and thus $y(t)$ is an increasing function of t. □

The last step is to show that orbits "turn clockwise", that is they cannot escape "horizontally". This goal is achieved using the following Lemmas 4 and 5 aimed at comparing the slope of system (6) with the one for Eq. (7) in the region $y > 0$, and the one for Eq. (8) in the complementary region $y < 0$.

Lemma 4. *Let the slope of system* (6) *given for* $y \neq 0$ *by:*

$$\frac{dy}{dx}(x) = -f(x,y) - (g(x) - e(t))\,\frac{1}{\varphi^{-1}(y)}\,.$$

Similarly the slopes of systems (7) *and* (8) *are obtained as:*

$$\frac{dy_+}{dx}(x) = -f(x,y) - (g(x) - E_+)\,\frac{1}{\varphi^{-1}(y)}\,,$$

and

$$\frac{dy_-}{dx}(x) = -f(x,y) - (g(x) - E_-)\,\frac{1}{\varphi^{-1}(y)}\,,$$

where we recall that $E_+ = \max e(t)$ *and* $E_- = \min e(t)$.
Then

$$\frac{dy}{dx}(x) \leq \frac{dy_+}{dx}(x) \quad \forall y > 0\,, \tag{16}$$

and

$$\frac{dy}{dx}(x) \leq \frac{dy_-}{dx}(x) \quad \forall y < 0\,. \tag{17}$$

Proof The proof of the claim is straightforward by comparing the definitions of the slopes and the bounds $E_- \leq e(t) \leq E_+$. □

The next result allows to show the aimed behaviour for the solutions of the autonomous systems (7) and (8). Hence orbits of (6) should “follow” the same guidelines because of the previous Lemma.

Lemma 5. *Assume* $\omega = +\infty$. *Let us assume* $G(x)$ *to be unbounded and to grow faster than* E_+x. *Let moreover* $f(x,y) > 0$ *for* $x > b$, *for some* $0 < b \leq \beta$, *then any trajectory of* (7) *starting on the vertical line* (b, y_b), $y_b > 0$, *will first intersect the positive* x*-axis and then again the vertical line* $x = d$ *at some* $y_{b'} < 0$.

Proof Let us define the energy function for the autonomous Duffing type system associated to Eq. (7)

$$\begin{cases} \dot{x} &= \varphi^{-1}(y) \\ \dot{y} &= -g(x) + E_+\,, \end{cases} \tag{18}$$

namely

$$H_+(x,y) = \Phi_*(y) + G(x) - E_+x\,, \qquad \text{with } \Phi_*(y) := \int_0^y \phi^{-1}(s)\,ds. \tag{19}$$

Because of the assumption on $G(x)$, its level sets are closed if $x > 0$ is large enough. Moreover, a direct computation allows to get

$$\begin{aligned} \tfrac{dH_+}{dt}(x,y) \;&=\; \varphi^{-1}(y)\left(-f(x,y)\varphi^{-1}(y) - (g(x) - E_+)\right) \\ &\quad + (g(x) - E_+)\,\varphi^{-1}(y) = -f(x,y)\left[\varphi^{-1}(y)\right]^2\,. \end{aligned} \tag{20}$$

The assumption on the sign of $f(x,y)$ for large x permits to conclude that orbits of (7) starting on the vertical line (b, y_b), $y_b > 0$, will enter the level set of $H_+(x,y)$ which is a closed curve as already noticed, and thus they must reach the positive x-axis somewhere. From there, they are again driven by the level set of $H_+(x,y)$ and thus have to reach the vertical line $x = d$ at some negative y. □

Let us observe that a similar result holds true for solutions of (8) with initial conditions on the vertical line (a, y_a), $y_a < 0$, provided $f(x, y) > 0$ for $x < a$, for some $\alpha \leq a < 0$. In this case one should use the energy function

$$H_-(x, y) = \Phi_*(y) + G(x) - E_- x$$

and require again that $G(x)$ grows faster than $E_- x$ for negative x.

Corollary 1. *Under the assumptions of Lemmas 4 and 5, any solution of (6) starting on the vertical line (b, y_b), $y_b > 0$, will first intersect the positive x-axis and then again the vertical line $x = b$ at some $y'_b < 0$.*

The proof is a straightforward application of the comparison method. Indeed from Lemma 4 we know that solutions of (6) should remain "inside" the solutions of (7), and the latter have the required property, namely starting from (b, y_b), $y_b > 0$, they will arrive again at $x = b$ for some $y'_b < 0$, after passing through $y = 0$. By comparison this behaviour holds true also for orbits of (6).

We can now prove our main result.

Theorem 1. *Let us assume $\omega = +\infty$. Consider system (6) with φ^{-1} defined on $\mathbb{R}$. Let us assume the above introduced regularities conditions on the functions $f(x, y)$ and $g(x)$, and let $e(t)$ be a continuous and T-periodic function for some $T > 0$. Assume the existence of $\alpha < 0 < \beta$ such that $f(x, y) > 0$ for $(x, y) \in (\beta, +\infty) \times \mathbb{R}_+$ and for $(x, y) \in (-\infty, \alpha) \times \mathbb{R}_-$, and moreover $g(x) > E_+$ for $x > \beta$ and $g(x) < E_-$ for $x < \alpha$, where E_+ and E_- denote the bounds on $e(t)$ previously defined. Assume the existence of two smooth functions $0 < \phi(x)$ and $\psi(x) < 0$ in the range of φ^{-1} such that:*

$$f(x, \phi(x))\phi(x) > -\phi'(x)\phi(x) - (g(x) - E_+)\, h(\phi(x)) \quad \forall x \leq \alpha < 0\,, \tag{21}$$

and

$$f(x, \psi(x))\psi(x) < -\psi'(x)\psi(x) - (g(x) - E_-)\, h(\psi(x)) \quad \forall x \geq \beta > 0\,, \tag{22}$$

Moreover, let there exists a positive function $T(x)$ such that

$$|f(x, y)| \leq LT(x)|y| \quad \forall x \in [\alpha, \beta] \text{ and } |y| \geq D > 0\,, \tag{23}$$

and finally that $G(x) - E_\pm x \to \infty$ for $x \to \pm\infty$. Then the system (6) exhibits a least a T-periodic solution.

Proof In comparison with the above quoted recent work [5], now we are dealing with a non-autonomous system, and therefore we cannot use standard tools such as the Poincaré-Bendixon Theorem. Instead we will build our proof on the continuity of the solutions of (6) with respect to initial conditions and on the construction of a region homeomorphic to a disk and invariant for the period-T Poincaré map. In such a way we can apply the Brouwer fixed point Theorem to ensure the existence of an initial condition which will result to be a fixed point for the Poincaré map and thus a periodic solution of the system (6).

The reader can consult panel a) of Fig. 2 to follow the progress of the proof. Let us consider a point $A = (x_A, y_A)$, $x_A < -M$ and $y_A = \phi(x_A) > 0$, in virtue of Lemma 1 the trajectory originating from this point it is bounded away from the x-axis for negative t and it enters the graph of the function $\phi(x)$; therefore it will

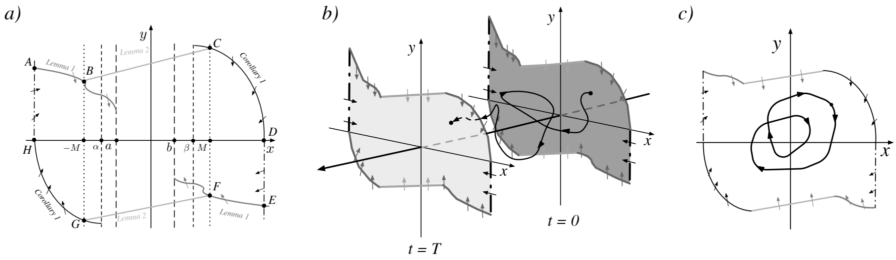

Fig. 2 **Phase-space and extended phase-space**. Panel a): Scheme of the proof of the Theorem 1. Panel b): Invariant domain for the Brouwer Theorem and the T-periodic orbit, at $t=0$ (dark grey) and after one period, $t=T$, (light grey). Panel c): schematic representation of the harmonic orbit.

intercept the vertical line $x = x_B$ where $x_B = \min\{\alpha, M\}$ (for a sake of definitiveness we assumed $x_B = -M$ in the panel a) of Fig. 2). By Lemma 2 (or Lemma 3) the forward trajectory will reach the line $x = M$ at some $y_C > 0$ and from this point on, by Corollary 1, the trajectory will be guided by the level sets of the energy function H_+, thus it will reach the x-axis at some point $x < X_D$ where $x_D > \max\{M, \beta\}$ is sufficiently large. The solutions cannot escape through the vertical segment DE because $\dot{x}|_{0<y<y_E} < 0$.

From this point on, we can use similar arguments, lemmas and corollary as the ones invoked in the construction the first part of the orbit, to prove that the trajectory cannot escape from the graph of the function $y = \psi(x)$, nor to have a vertical asymptote and thus it must reach the vertical line $x = -M$ at some $y_G < 0$. From there, the solution is guided by the level sets of H_- and then by the vertical segment HA to go inward.

The piecewise regular path ABCDEFGHA is the boundary of a domain, which is by construction invariant for the flow and homeomorphic to a disk. This completes the proof by applying the Brouwer Theorem (see panels b) and c) of Fig. 2 for a schematic representation). □

Remark 2. *When $\omega < +\infty$, Theorem 1 holds true provided that one can control the growth of the comparison orbits, as discussed in Remark 1. In the case of the prescribed curvature operator, in order to achieve the result, condition* (23) *is replaced multiplying $f(x,y)$ by a positive small parameter, as in [20]. Alternatively, we can use condition* (14) *and apply Lemma 3.*

Remark 3. *A crucial observation is the following. At first we note that the constants α and β provide a couple of lower and upper solutions. Hence, in the special case in which $\omega = +\infty$ and φ^{-1} is bounded, as, for instance, in the case of the relativistic curvature equation, then a classical result of Bereanu and Mawhin [1, Corollary 5], guarantees the existence of a T-periodic solution if f, g, e are continuous and*

$$g(x) > E_+ \quad \textit{for} \quad x > \beta \quad \textit{and} \quad g(x) < E_- \quad \textit{for} \quad x < \alpha.$$

This holds true because a direct computation shows that in our case the relation (30) *in [1] is verified and hence Corollary 5 of the above mentioned paper holds. Therefore the result in [1] is more general, as there are no extra conditions on $f(x,y)$, which are required. However, if φ^{-1} is not bounded, the result in [1] cannot be applied and it is well known that the above sign condition on $g(x)$ is not enough to guarantee the* *existence of T-periodic solutions. Moreover, with our approach, we also provide some information on the dynamics of the solutions, namely that they are bounded in the future time and we can also locate the periodic solutions in a trapping region. In this light, we will consider also in the applications the relativistic case.*

3 Applications and examples

In this Section we present some applications of the results in Section 2. For sake of simplicity, we will present some examples in which Lemma 1 holds with ϕ and ψ constant functions. Indeed, this was the first case considered in [25] and further extended to the case $\phi(x) = -g(x)$ and $\psi(x) = g(x)$ in [24] and for more general situations in [3, 4]. Under this choice, conditions (9) and (10) can be equivalently written as

$$\exists\, K \in (0,\omega): \quad f(x,K)\varphi^{-1}(K) > -(g(x) - E_+), \forall\, x \leq \alpha$$

and

$$\exists\, L \in (0,\omega): \quad f(x,-L)\varphi^{-1}(-L) < -(g(x) - E_-), \forall\, x \geq \beta,$$

respectively. This simplify the computations, because conditions (9) and (10) can be verified more easily. Alternatively, one can use as ϕ and ψ suitable branches of the 0-isocline for the autonomous system, because now it suffices to have that the slope dy/dx of the non-autonomous system is negative on such branches. Figure 3 below illustrates this situation.

At first we concentrate on two examples, that is the relativistic case and the prescribed curvature one. As already mentioned, the existence result for the relativistic case is well known, but the example shows as the additional assumptions give further information on the dynamical behavior of the solutions, in view of the explicit construction of the invariant region, which is provided also in all other cases.

The first example is a straightforward generalization of the one presented in Section 4.1 of [5], to which we refer for more details, to the non-autonomous case. We thus consider the system

$$\begin{cases} \dot{x} &= \frac{y}{\sqrt{1+y^2}} \\ \dot{y} &= -f(x,y)\frac{y}{\sqrt{1+y^2}} - g(x) + e(t)\,, \end{cases} \tag{24}$$

with $f(x,y) = |y|\cos^2(y)(x^2-1)$, $g(x) = x$ and we set the periodic forcing equals to $e(t) = A\sin(2\pi t)$. In [5] it has been shown that one can use the 0-isocline for the function $\phi(x)$ and $\psi(x)$; in the present framework this accounts to use the 0-isoclines of autonomous systems obtained replacing $e(t)$ in Eq. (24) with its maximum or minimum, $E_\pm = \pm A$.

As a second case, we exhibit an example belonging to the class of prescribed curvature equation which can be equivalently represented as the system

$$\begin{cases} \dot{x} = \frac{y}{\sqrt{1-y^2}} \\ \dot{y} = -f(x,y)\frac{y}{\sqrt{1-y^2}} - g(x) + e(t)\,, \end{cases} \tag{25}$$

in the phase-plane. We follow Section 4.3 of [5], to which we refer for more details, with the addition of a periodic term $e(t) = A\sin(2\pi t)$. More precisely we set $f(x,y) = \lambda|y|\cos^2(3\pi y)(x^2-1)$ and $g(x) = \mu x e^{-|x|}$. Let us observe that the (small) parameter λ has been introduced to constraint the orbits into the strip $|y| < 1$. Again the application relies [5] on the use of the 0-isocline for the function $\phi(x)$ and $\psi(x)$.

Using the MATLAB software [29] we numerically solved system (25) and the results reported in the right panel of Fig. 3 show the existence of a unique periodic solution.

We performed a numerical integration of (24) using the MATLAB software [29] and the results show the existence of a unique periodic solution (see Fig. 3 left panel). Observe that we are not able to prove the uniqueness of such periodic solution, and at this stage this remains a numerical observation.

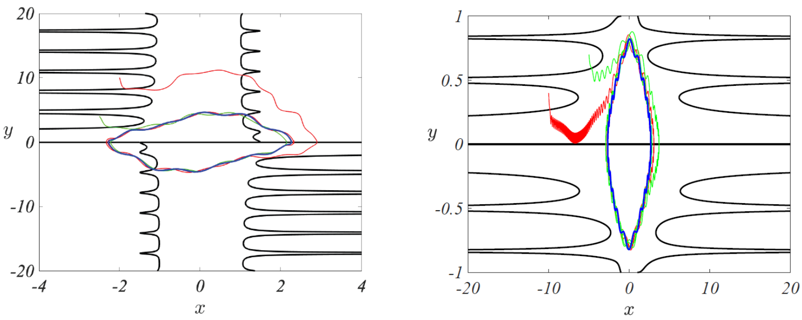

Fig. 3 **Two numerical examples**. In the left panel we show the phase-portrait of the system (24) with $f(x,y) = |y|\cos^2 y(x^2-1)$, $g(x) = x$ and $e(t) = A\sin(2\pi t)$, with $A = 1.5$ while in the right panel we consider the system (25) with $f(x,y) = \lambda|y|\cos^2(3\pi y)(x^2-1)$, $g(x) = \mu x e^{-|x|}$ and $e(t) = A\sin(2\pi t)$, with $\lambda = 0.05$, $\mu = 0.5$ and $A = 0.5$. In both cases the black curves denote the different branches of the 0-isocline, the blue curve is the numerically computed periodic orbit, while the red and green curves represent generic orbits that accumulate on the stable periodic solution.

As a third class of examples, we exhibit some different situations arising from the study of a second-order generalized Liénard equation involving the p-Laplacian differential operator

$$L_p : x \mapsto \frac{d}{dt}\varphi_p(\dot{x}), \quad \text{with } 1 < p < \infty,$$

where

$$\varphi_p(s) := |s|^{p-2}s, \quad \text{for } s \neq 0 \text{ and } \varphi_p(0) = 0.$$

In this case, the equivalently represented system in the phase-plane becomes

$$\begin{cases} \dot{x} = \varphi_q(y) \\ \dot{y} = -f(x,y)\varphi_q(y) - g(x) + e(t), \end{cases} \tag{26}$$

for

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Notice that, with our convention on the exponents, we have a linear differential operator for $p = 2$ and $\varphi_q(y) = |y|^{q-1}\text{sign}(y)$ is the inverse function of φ_p.

Using the Maple software [28], in order to double check the numerical simulations, we numerically solve system (26) for different values of p and q. The results are reported in Figures 4-5-6.

We observe that, even if our main theorem ensures the existence of a T-periodic solutions, we cannot guarantee, in general, its uniqueness or its stability and the presence of other stable/attracting trajectories cannot be ruled out. This is confirmed by

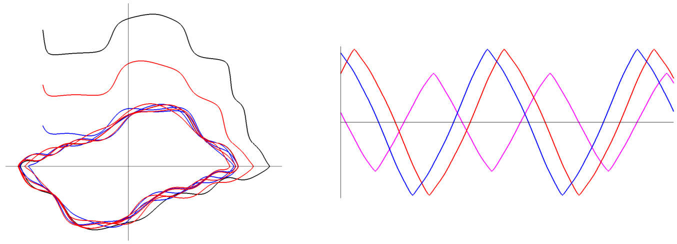

Fig. 4 Case of ϕ_p with $p = 5, q = 5/4, q-1 = 1/4$. In the left panel we show the phase-portrait of the system (26) for with $f(x,y) = |y|\cos^2 y(x^2-1)$, $g(x) = x$ and $e(t) = A\sin(2\pi t)$, with $A = 1.5$, while in the right panel we consider the solutions $(t, x(t))$ with different initial points. From the left panel, it seems that the numerically computed solutions that we find for the initial values $(x(0), y(0)) = (-2, 3)$, $(x(0), y(0)) = (-2, 6)$ and $(x(0), y(0)) = (-2, 10)$, accumulate to a periodic solution. On the other hand, in the right panel, the graphs of $(t, x(t))$ for interval $t \in [300, 320]$ show the presence of two attracting subharmonic solutions of period $T = 9$ (which are one the shift of the other and correspond to the solutions depicted in the left panel) and another attracting subharmonic solution of period $T = 7$ for the initial point $(x(0), y(0)) = (0.01, 0.01)$.

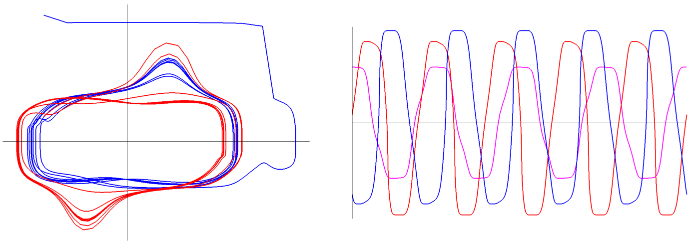

Fig. 5 Case of ϕ_p with $p = 4/3, q = 4, q-1 = 3$. In the left panel we show the phase-portrait of the system (26) for with $f(x,y) = |y|\cos^2 y(x^2-1)$, $g(x) = x$ and $e(t) = A\sin(2\pi t)$, with $A = 1.5$, while in the right panel we consider the solutions $(t, x(t))$ with different initial points. From the left panel, it seems that the numerically computed solutions that we find for the initial values $(x(0), y(0)) = (-2, 1)$ and $(x(0), y(0)) = (-2, 5)$ accumulate to different periodic solutions. On the other hand, in the right panel, the graphs of $(t, x(t))$ for interval $t \in [300, 320]$ show the presence of two different attracting subharmonic solutions of period $T = 4$ (which correspond to the solutions depicted in the left panel) and another attracting subharmonic solution of period $T = 5$ for the initial point $(x(0), y(0)) = (0.01, 0.01)$.

our numerical simulations, where attracting subharmonic solutions (or perhaps, even more complicated orbits for the Poincaré map) are found. This is in complete agreement with the fundamental and pioneering works of Cartwright and Littlewood [6, 7], Levinson [12, 13] and Littewood [15, 16] who analytically predicted these kind of phenomena in the study of the periodically perturbed van der Pol and Liénard equations. Such studies were later expanded by Levi in [11]. As remarked by P.J. Holmes in his review MR617687 of [11], the construction in 1963 by S. Smale [23] of the horseshoe diffeomorphism, was inspired by results in [6] and [13]. For a more recent discussion of the expected dynamics in periodically perturbed dissipative systems, see also [21].

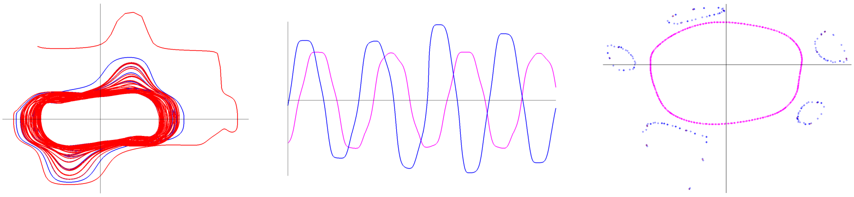

Fig. 6 **Case of ϕ_p with** $p = 3/2, q = 3, q-1 = 2$. In the left panel we show the phase-portrait of the system (26) for with $f(x,y) = |y|\cos^2 y(x^2-1)$, $g(x) = x$ and $e(t) = A\sin(2\pi t)$, with $A = 1.5$, while in the central panel we consider the solutions $(t, x(t))$ with different initial points. From the left panel, it seems that the numerically computed solutions that we find for the initial values $(x(0), y(0)) = (-2, 1)$ and $(x(0), y(0)) = (-2, 5)$ accumulate to the same trajectory of some "recurrent" structure. On the other hand, in the central panel, the graphs of $(t, x(t))$ for interval $t \in [300, 320]$ show the presence of two different attracting solutions which are not periodic. One of these solutions (of larger amplitude) corresponds to the solutions depicted in the left panel (which become practically indistinguishable after some time). Another attracting solution of smaller amplitude emerges for the initial point $(x(0), y(0)) = (0.01, 0.01)$. The nature of these solutions is more evident from the right panel, where we consider the n-th iterates of the Poincaré map for $100 \leq n \leq 800$ for the initial points $(x(0), y(0)) = (-2, 5)$ and $(x(0), y(0)) = (0.01, 0.01)$.

Declarations

The authors declare the absence of any competing interest in the present work.

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