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Abstract

A class of algorithms for optimization in the presence of noise is presented, that does not require the evaluation of the objective function. This class generalizes the well-known Adagrad method. The complexity of this class is then analyzed as a function of its parameters, and it is shown that some methods of the class enjoy a better asymptotic convergence rate than previously known. A new class of algorithms is then derived with similar characteristics. Initial numerical experiments suggest that it may have some merits in practice.

Keywords: First-order methods, objective-function-free optimization, noisy gradients, Adagrad, convergence bounds, evaluation complexity.

1 Introduction

Minimization algorithms which can handle noisy evaluations of the objective function and/or gradients have generated a significant amount of research in the last years [3, 4, 5, 7, 8, 9, 10, 15, 16, 19, 20, 21, 22, 23, 24]. Interestingly, a number of these contributions [3, 4, 5, 7, 8, 19] indicate that, when the (noisy) objective function is evaluated, its accuracy is significantly more critical to ensure convergence than that of the computed (noisy) derivatives. This may be the reason why methods where the objective function is *not* evaluated, such as Adagrad [10], RMSProp [21], Adam [15] or AMSGrad [20], have become very popular in the context of finite-sum minimization, where noise in the evaluation arises from sampling among a very large number of terms. That such methods can be provably convergent to first-order stationary points is quite remarkable. Moreover, several authors have been able to prove global convergence rates, including the recent contributions by [9], where an improved (compared to earlier analysis) such rate was proved for the Adagrad algorithm, and by [23] where the analysis was refined to take “sparsity” of the gradient sequence and optimal[1] learning rates into account and to cover AMSgrad and RMSProp.

The present paper builds on the results of [22, 9] and achieves several goals.

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[1]A priori unknown.

1. The global rate of convergence result of [9] is extended to a parametric class of methods comprising the Adagrad algorithm.

2. An improved asymptotic rate is also derived for these methods under an additional conditional variance condition, indicating that the results of [9] cannot be sharp if this condition holds. This new complexity bound is independent from gradient "sparsity" and allows an essentially arbitrary choice of the learning rate. It therefore provides an alternative to that of [23].

3. A new class of methods is then proposed, whose global rate of convergence is shown to be very close to that of methods using (exact) function evaluations.

The presentation is organized as follows. The new class of first-order algorithms is introduced in Section 2, of which two subclasses (one of them containing the Adagrad method) are defined and analyzed (complexity-wise) in Sections 3 and 4, respectively. A numerical illustration in the finite-sum minimization context is presented in Section 5 and some conclusions outlined in Section 6.

2 A class of first-order methods for minimizing noisy functions

We are interested in (approximately) solving the problem

$$\min_{x \in \mathbb{R}^n} F(x) \tag{1}$$

where F is a function from $\mathbb{R}^n$ to $\mathbb{R}$ contaminated by noise. Moreover, we assume that evaluating F at any given x to sufficient accuracy is either impossible or too costly. Evaluating a noisy gradient is however possible... and our only source of information about the problem. While access to F or its exact gradient is impossible, we nevertheless make the following assumptions.

Assumption 2.1. The objective function $F(x)$ is continuously differentiable.

Assumption 2.2. Its exact gradient $G(x) \stackrel{\text{def}}{=} \nabla_x^1 f(x)$ is Lipschitz continuous with Lipschitz constant L, that is

$$\|G(x) - G(y)\| \leq L\|x - y\|$$

for all $x, y \in \mathbb{R}^n$.

Assumption 2.3. There exists a constant F_{low} such that, for all x, $F(x) \geq F_{\text{low}}$.

A standard consequence of Assumption 2.2 is that, for, any $x, s \in \mathbb{R}^n$,

$$F(x+s) \leq F(x) + G(x)^T s + \frac{L}{2}\|s\|^2 \tag{2}$$

(see Lemma 2.1 in [2], for instance).
We now present a first-order *adaptively scaled gradient* algorithm (ASGRADn), where, at iteration k, a *noisy* gradient $g_k = g(x_k)$ is evaluated and a step s_k defined that decreases the associated local linear model and whose size is determined by componentwise "scaling factors" $w_{i,k}$ to be chosen at each iteration. Our algorithm, which can be viewed as an Adagrad method with more general scaling factors, is formally described as follows.

Algorithm 2.1: The ASGRADn Class of Algorithms

Step 0: Initialization. x_0 and a constant $\gamma_{\rm low} \in (0,1]$ are given. Set $k = 0$.

Step 1: Step computation. Evaluate g_k and set

$$s_k = \gamma_k s_k^L, \tag{3}$$

with

$$s_{i,k}^L = -\frac{g_{i,k}}{w_{i,k}} \tag{4}$$

for a stepsize $\gamma_k \in [\gamma_{\rm low}, 1]$ and positive scaling factors $w_{i,k}$.

Step 2: New iterate. Define

$$x_{k+1} = x_k + s_k, \tag{5}$$

increment k by one and return to Step 1.

We stress that g_k (as evaluated in Step 1) is a noisy random gradient evaluation. The ASGRADn algorithms therefore generate a stochastic process

$$\{x_k, g_k, \gamma_k, s_k^L, s_k\}$$

on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The associated expectation operator will be denoted by $\mathbb{E}[\cdot]$ and $\mathbb{E}_k[\cdot]$ will stand for the conditional expectation knowing $\{g_0, \dots, g_{k-1}\}$.

We will, in what follows, assume that the noisy gradient g_k is a bounded non-biased estimator of the true gradient, that is

Assumption 2.4. We have that, for all $k \geq 0$, $\mathbb{E}_k[g_k] = G(x_k)$. Moreover, there exists a constant $\kappa_g \geq 1$ such that $\|g_k\|_\infty \leq \kappa_g$ for all $k \geq 0$ and all realizations of the algorithm.

Note that this assumption immediately implies that

$$\|G(x_k)\|_\infty \leq \kappa_g \tag{6}$$

for all $k \geq 0$.

The reader has undoubtly noted that we have not been very specific regarding how the scaling factors $w_{i,k}$ are selected, and a whole range of options is possible. We may therefore view ASGRADn as a parametric *class of algorithms* covering many possible such choices. The rest of this paper is devoted to the analysis of two specific subclasses of interest.

3 An Adagrad-inspired subclass of ASGRADn

In the first considered ASGRADn subclass, the scaling factors are inspired by the definition of the Adagrad algorithm [10]. More specifically, we make the following additional assumptions.

Assumption 3.1. For each $i \in \{1, \dots, n\}$, there exist a constant $\varsigma_i > 0$ and a random variable v_i such that $v_i \geq \varsigma_i$ and $w_{i,k} = (v_{i,k})^\mu$ for some $\mu \in (0,1)$ and all $k \geq 0$. In addition,

$$|\mathbb{E}_k[v_{i,k}] - v_{i,k}| \leq \kappa_v (\mathbb{E}_k\left[g_{i,k}^2\right] + g_{i,k}^2) \tag{7}$$

for some $\kappa_v > 0$ and all $k \geq 0$.

Assumption 3.2. For every realization of the algorithm, we have that that $g_{i,k}^2 \leq v_{i,k}$ for all $i \in \{1, \dots, n\}$ and all $k \geq 0$.

We immediately note that Assumption 3.1 implies that

$$v_{i,k} \geq \min_{i \in \{1,\dots,n\}} \varsigma_i \stackrel{\text{def}}{=} \varsigma_{\min} \tag{8}$$

and Assumption 3.2 ensures that

$$\mathbb{E}_k\left[g_{i,k}^2\right] \leq \mathbb{E}_k[v_{i,k}]\,. \tag{9}$$

The first step in our analysis is to derive a parametric bound on the decrease in the exact linear model of F caused by the step s_k, using a technique inspired by [22] and [9].

Lemma 3.3. *Let s_j^L be the step produced by the* ASGRADn *algorithm at the j-th iteration. Suppose also that Assumptions 2.4, 3.1 and 3.2 hold. Let G_j be the true gradient of F at x_j. Then, for all $i \in \{1, \dots, n\}$,*

$$\mathbb{E}_j\left[\gamma_j G_{i,j} s_{i,j}^L\right] \leq -(1-\frac{\mu}{2})\frac{\gamma_{\text{low}} G_{i,j}^2}{(\mathbb{E}_j[v_{i,j}])^\mu} + 2\kappa_\Delta \mathbb{E}_j\left[\frac{g_{i,j}^2}{w_{i,j}^2}\right], \tag{10}$$

where

$$\kappa_\Delta \stackrel{\text{def}}{=} \frac{\mu\kappa_v^2}{\gamma_{\text{low}}}\left[\kappa_g^{2\mu} + \frac{\kappa_g^2}{\varsigma_{\min}^{1-\mu}} + \frac{\kappa_g^{4-2\mu}}{\varsigma_{\min}^{2-2\mu}} + \kappa_g^{2-2\mu}\kappa_\mu\right] \tag{11}$$

with

$$\kappa_\mu \stackrel{\text{def}}{=} \frac{1}{\varsigma_{\min}^{1-2\mu}}\mathbf{1}_{\mu<\frac{1}{2}} + \kappa_g^{4\mu-2}\mathbf{1}_{\mu\geq\frac{1}{2}},$$

where $\mathbf{1}_e$ stands for the indicator function of the event e.

Proof. See Appendix.

This lemma essentially implies that s^L provides a descent direction on the true F as long as the square of the true gradient's norm remains large compared with the stepsizes. We also need another result inspired by [22, 9], whose utility will be to bound the last term on the right-hand side of (10).

Lemma 3.4. *Let $\{a_k\}_{k\geq 0}$ be a non-negative sequence, $\alpha > 0$ and define, for each $k \geq 0$, $b_k = \sum_{j=0}^k a_j$. Then if $\alpha \neq 1$,*

$$\sum_{j=0}^k \frac{a_j}{(\varsigma + b_j)^\alpha} \leq \frac{1}{(1-\alpha)}((\varsigma + b_k)^{1-\alpha} - \varsigma^{1-\alpha}). \tag{12}$$

Otherwise (i.e. if $\alpha = 1$) (see Lemma 5.2 in [9]),

$$\sum_{j=0}^k \frac{a_j}{\varsigma + b_j} \leq \log\left(\frac{\varsigma + b_k}{\varsigma}\right). \tag{13}$$

Proof. See Appendix. Note that (13) is the limit of (12) when α tends to one.

Using both Lemmas 3.3 and 3.4, we are now in position to deduce a first result on the global convergence rate of the ASGRADn subclass using specific "Adagrad-like" scaling factors satisfying Assumptions 3.1 and 3.2.

Theorem 3.5. *Suppose that Assumptions 2.1–2.4 hold and that the* ASGRADn *algorithm is applied to problem (1). Suppose also that, for some $\varsigma \in (0, \kappa_g]$, some $\mu \in (0,1)$, all $k \geq 0$ and all $i \in \{1, \dots, n\}$,*

$$w_{i,k} = \left(\varsigma + \sum_{\ell=0}^{k} g_{i,\ell}^2 \right)^{\mu}. \tag{14}$$

Then the following bounds hold for κ_Δ given by (11) and

$$\kappa_\square \stackrel{\text{def}}{=} \frac{\kappa_g^{2\mu}(4\kappa_\Delta + L)}{(1 - \frac{\mu}{2})\gamma_{\text{low}}}. \tag{15}$$

(i) If $\mu \in (0, \frac{1}{2})$, then

$$\begin{aligned}
&\mathbb{E}\left[\operatorname*{average}_{j \in \{0,\dots,k\}} \|G_j\|^2 \right] \\
&\leq \frac{2\kappa_g^{2\mu}}{(1 - \frac{\mu}{2})\gamma_{\text{low}}(k+1)^{1-\mu}} \Big[F(x_0) - F_{\text{low}} \Big] \\
&\quad + \frac{n\kappa_\square}{1 - 2\mu} \frac{(\varsigma + \kappa_g^2(k+1))^{1-2\mu} - \varsigma^{1-2\mu}}{(k+1)^{1-\mu}}.
\end{aligned} \tag{16}$$

(ii) If $\mu = \frac{1}{2}$, then

$$\begin{aligned}
&\mathbb{E}\left[\operatorname*{average}_{j \in \{0,\dots,k\}} \|G_j\|^2 \right] \\
&\leq \frac{8\kappa_g}{3\gamma_{\text{low}}\sqrt{(k+1)}} \Big[F(x_0) - F_{\text{low}} \Big] \\
&\quad + n\kappa_\square \frac{\log\left(1 + (k+1)\frac{\kappa_g^2}{\varsigma} \right)}{\sqrt{(k+1)}}.
\end{aligned} \tag{17}$$

(iii) If $\mu \in (\frac{1}{2}, 1)$, then

$$\begin{aligned}
&\mathbb{E}\left[\operatorname*{average}_{j \in \{0,\dots,k\}} \|G_j\|^2 \right] \\
&\leq \frac{2\kappa_g^{2\mu}}{(1 - \frac{\mu}{2})\gamma_{\text{low}}(k+1)^{1-\mu}} \Big[F(x_0) - F_{\text{low}} \Big] \\
&\quad + \frac{n\kappa_\square}{2\mu - 1} \frac{\varsigma^{1-2\mu} - (\varsigma + \kappa_g^2(k+1))^{1-2\mu}}{(k+1)^{1-\mu}}.
\end{aligned} \tag{18}$$

Proof. See Appendix.

Note that the last fractions in the last terms of (16) and (18) have been written in a form stressing the continuity with (17), but could obviously be bounded above by the simpler

$$\frac{(\varsigma + \kappa_g^2)^{1-2\mu}}{(k+1)^{\mu}} \quad \text{and} \quad \frac{\varsigma^{1-2\mu}}{(k+1)^{1-\mu}}$$

respectively.

Theorem 3.5 suggests a few comments. The first is that (16), (17) and (18) guarantee the convergence of the ASGRADn algorithm to first-order critical points, because their right-hand sides all tend to zero when k tends to infinity. The rate at which this convergence occurs differs however for the three cases, depending on the parameter μ. If constants are lumped into a generic $\mathcal{O}(\cdot)$ notation, we obtain, using Jensen's inequality, that

$$\mathbb{E}\left[\operatorname*{average}_{j\in\{0,\ldots,k\}} \|G_j\|\right] \le \begin{cases} \mathcal{O}\left(\dfrac{1}{(k+1)^{\frac{1}{2}\mu}}\right) & (\mu \in (0,\tfrac{1}{2})),\\ \mathcal{O}\left(\dfrac{\log(k+1)}{(k+1)^{\frac{1}{4}}}\right) & (\mu = \tfrac{1}{2}),\\ \mathcal{O}\left(\dfrac{1}{(k+1)^{\frac{1}{2}(1-\mu)}}\right) & (\mu \in (\tfrac{1}{2},1)).\end{cases}$$

Examining these "k-order" bounds indicates that the best bound is that corresponding to $\mu = \frac{1}{2}$. This is nothing but the standard Adagrad algorithm.

One may then ask whether the bounds given by Theorem 3.5 are sharp. We now show that is is not the case under an additional conditional variance condition on the gradient estimator. Note that both Assumptions 3.1–3.2 automatically hold for (14) with $v_{i,j} = \varsigma + \sum_{\ell=0}^{j} g_{i,\ell}^2$ (for convenience, set $v_{i,-1} = \varsigma$).

Theorem 3.6. *Suppose that Assumptions 2.1–2.4 hold and that the* ASGRADn *algorithm is applied to problem (1) with the scaling factors defined by (14) for some $\varsigma \in (0, \kappa_g]$, $\mu \in (0,1)$, all $i \in \{1,\ldots,n\}$ and all $k \ge 0$. Suppose also that, for all $i \in \{1,\ldots,n\}$ and all $k \ge 0$*

$$\operatorname{Var}_k\left[g_{i,k}\right] = \mathbb{E}_k\left[g_{i,k}^2 - G_{i,k}^2\right] \le \kappa_{\rm var} G_{i,k}^2 \tag{19}$$

for some $\kappa_{\rm var} \ge 0$. Then, for any $\theta \in (0, (1-\frac{1}{2}\mu)\gamma_{\rm low})$, there exists a finite $j_\theta \ge 0$ such that

$$\begin{aligned}\mathbb{E}\left[\operatorname*{average}_{j\in\{j_\theta+1,\ldots,k\}} \|G_j\|^2\right] &\le \kappa_*(\theta)\frac{(k+2)^\mu}{k-j_\theta} + \frac{n\kappa_\triangledown}{k-j_\theta}\\ &\le \frac{j_\theta+3}{(k+2)}\left(\frac{\kappa_*(\theta)}{(k+2)^{-\mu}} + n\kappa_\triangledown\right),\end{aligned} \tag{20}$$

where

$$\kappa_*(\theta) \stackrel{\rm def}{=} \frac{\kappa_g^{2\mu}}{\theta}\left\{F(x_0) - F_{\rm low} + n\,\kappa_\diamond\right\} \tag{21}$$

with

$$\begin{aligned}\kappa_\triangledown &\stackrel{\rm def}{=} \left(\frac{\tilde\kappa}{\gamma_{\rm low}(1-\frac{1}{2}\mu)-\theta}\right)^{\frac{1}{\mu}},\\ \kappa_\diamond &\stackrel{\rm def}{=} \frac{\tilde\kappa}{\varsigma^{2\mu}}\left[(j_\theta+1)\kappa_g^2 + \kappa_\triangledown\right],\\ \tilde\kappa &\stackrel{\rm def}{=} \left(2\kappa_\Delta + \frac{L}{2}\right)(1+\kappa_{\rm var})\left(1+\frac{\kappa_g^2}{\varsigma}\right)^\mu,\end{aligned} \tag{22}$$

and κ_Δ defined by (11).

Proof. We have verified (at the beginning of the proof of Theorem 3.5) that the proposed scaling factors verify Assumptions 3.1 and 3.2. We invoke (2) to deduce that

$$\begin{aligned} F(x_{j+1}) &\leq F(x_j) + \gamma_j G_j^\intercal s_j^L + \frac{L}{2}\gamma_j^2 \|s_j^L\|^2 \\ &\leq F(x_j) + \gamma_j G_j^\intercal s_j^L + \frac{L}{2}\|s_j^L\|^2, \end{aligned}$$

and, taking the conditional expectation and using Lemma 3.3, that

$$\begin{aligned} &\mathbb{E}_j[F(x_{j+1})] \\ &\quad \leq F(x_j) + \sum_{i=1}^n \mathbb{E}_j\left[\gamma_j G_{i,j} s_{i,j}^L\right] + \frac{L}{2}\mathbb{E}_j\left[\|s_j^L\|^2\right], \\ &\quad \leq F(x_j) - \sum_{i=1}^n (1-\frac{\mu}{2})\gamma_{\text{low}} \frac{G_{i,j}^2}{(\mathbb{E}_j[v_{i,j}])^\mu} \\ &\qquad + \left(2\kappa_\Delta + \frac{L}{2}\right)\mathbb{E}_j\left[\frac{g_{i,j}^2}{v_{i,j}^{2\mu}}\right]. \end{aligned}$$

Observe now that, for each $i \in \{1, \dots, n\}$, $v_{i,j-1} + \kappa_g^2 \geq v_{i,j} \geq v_{i,j-1}$ and that $v_{i,j-1}$ is measurable knowing $g_0, \dots, g_{j-1}$. Hence,

$$\begin{aligned} &\mathbb{E}_j[F(x_{j+1})] \\ &\leq F(x_j) - \sum_{i=1}^n \left[1 - \frac{\mu}{2}\right] \frac{\gamma_{\text{low}} G_{i,j}^2}{(\mathbb{E}_j[v_{i,j}])^\mu} + \left[2\kappa_\Delta + \frac{L}{2}\right]\mathbb{E}_j\left[\frac{g_{i,j}^2}{v_{i,j-1}^{2\mu}}\right] \\ &\leq F(x_j) - \sum_{i=1}^n \left[1 - \frac{\mu}{2}\right] \frac{\gamma_{\text{low}} G_{i,j}^2}{(\mathbb{E}_j[v_{i,j}])^\mu} + \left[2\kappa_\Delta + \frac{L}{2}\right]\frac{\mathbb{E}_j\left[g_{i,j}^2\right]}{v_{i,j-1}^{2\mu}}. \end{aligned}$$

We now use (19) to deduce that

$$\begin{aligned} \mathbb{E}_j[F(x_{j+1})] \leq F(x_j) &- \sum_{i=1}^n \left[1 - \frac{\mu}{2}\right] \frac{\gamma_{\text{low}} G_{i,j}^2}{(\mathbb{E}_j[v_{i,j}])^\mu} \\ &+ \left[2\kappa_\Delta + \frac{L}{2}\right](1 + \kappa_{\text{var}})\frac{G_{i,j}^2}{v_{i,j-1}^{2\mu}}. \end{aligned}$$

But

$$\frac{\mathbb{E}_j[v_{i,j}]}{v_{i,j-1}} \leq 1 + \frac{\kappa_g^2}{v_{i,j-1}} \leq 1 + \frac{\kappa_g^2}{\varsigma}$$

and thus

$$\frac{G_{i,j}^2}{v_{i,j-1}^{2\mu}} = \frac{(\mathbb{E}_j[v_{i,j}])^\mu}{(v_{i,j-1}\mathbb{E}_j[v_{i,j}])^\mu}\frac{G_{i,j}^2}{v_{i,j-1}^\mu} \leq \frac{(1 + \frac{\kappa_g^2}{\varsigma})^\mu}{(\mathbb{E}_j[v_{i,j}])^\mu}\frac{G_{i,j}^2}{v_{i,j-1}^\mu},$$

so that

$$\begin{aligned} &\mathbb{E}_j[F(x_{j+1})] \\ &\leq F(x_j) - \sum_{i=1}^n \left[1 - \frac{\mu}{2}\right] \frac{\gamma_{\text{low}} G_{i,j}^2}{(\mathbb{E}_j[v_{i,j}])^\mu} + \frac{\tilde{\kappa}}{(\mathbb{E}_j[v_{i,j}])^\mu}\frac{G_{i,j}^2}{v_{i,j-1}^\mu} \end{aligned}$$

where $\tilde{\kappa}$ is defined by (22). Summing over iterations in $j \in \{0, \ldots, k\}$ in this last inequality yields that

$$\sum_{j=0}^{k} \mathbb{E}_j[F(x_{j+1})] \leq \sum_{j=0}^{k} F(x_j) + \sum_{j=0}^{k}\sum_{i=1}^{n} \frac{G_{i,j}^2}{(\mathbb{E}_j[v_{i,j}])^\mu}\left[-\gamma_{\rm low}\left(1-\frac{\mu}{2}\right)+\frac{\tilde{\kappa}}{v_{i,j-1}^{\mu}}\right]. \tag{23}$$

Now select an arbitrary $\theta \in (0, \gamma_{\rm low}(1-\frac{1}{2}\mu))$ and let $\mathcal{I}$ be the (possibly empty) subset of $\{1, \ldots, n\}$ such that, for all $j \geq 0$ and all $i \in \mathcal{I}$,

$$-\gamma_{\rm low}\left(1-\frac{\mu}{2}\right)+\frac{\tilde{\kappa}}{v_{i,j-1}^{\mu}} \geq -\theta.$$

This last inequality implies that, for all $j \geq 0$ and all $i \in \mathcal{I}$,

$$\mathbb{E}[v_{i,j-1}] \leq \left(\frac{\tilde{\kappa}}{\gamma_{\rm low}(1-\frac{1}{2}\mu)-\theta}\right)^{\frac{1}{\mu}},$$

and hence, using Jensen's inequality, that, for all $k \geq 0$ and all $i \in \mathcal{I}$,

$$\begin{aligned}\mathbb{E}\left[\sum_{j=0}^{k} G_{i,j}^2\right] &= \mathbb{E}\left[\sum_{j=0}^{k} \mathbb{E}_j[g_{i,j}]^2\right] = \sum_{j=0}^{k} \mathbb{E}[g_{i,j}]^2 \\ &\leq \varsigma + \sum_{j=0}^{k} \mathbb{E}\left[g_{i,j}^2\right] = \mathbb{E}[v_{i,k}] \\ &\leq \left(\frac{\tilde{\kappa}}{\gamma_{\rm low}(1-\frac{1}{2}\mu)-\theta}\right)^{\frac{1}{\mu}}.\end{aligned} \tag{24}$$

Consider now $i \in \mathcal{J} \stackrel{\rm def}{=} \{1, \ldots, n\} \setminus \mathcal{I}$ (assuming therefore that $\mathcal{J} \neq \emptyset$). For such an i, there must exist a $j_i(\theta)$ sufficiently large such that

$$-\gamma_{\rm low}\left(1-\frac{\mu}{2}\right)+\frac{\tilde{\kappa}}{v_{i,j-1}^{\mu}} \leq -\theta,$$

for $j = j_i(\theta)+1$ and hence, since $v_{i,j}$ is an increasing function of j, for all $j \geq j_i(\theta)+1$. If we now set

$$j_\theta = \begin{cases} \max_{i\in\mathcal{J}} j_i(\theta) & \text{if } \mathcal{J} \neq \emptyset \\ -1 & \text{otherwise.}\end{cases} \tag{25}$$

we then verify, using Assumption 2.4, that

$$\sum_{j=0}^{j_\theta}\sum_{i\in\mathcal{J}} \frac{G_{i,j}^2}{(\mathbb{E}_j[v_{i,j}])^\mu}\left(-(1-\frac{\mu}{2})\gamma_{\rm low}+\tilde{\kappa}\frac{1}{v_{i,j-1}^{\mu}}\right) \leq n\tilde{\kappa}(j_\theta+1)\frac{\kappa_g^2}{\varsigma^{2\mu}}. \tag{26}$$

Returning to inequality (23) and using (26) for the $\mathcal{J}$ terms and the fact that $\varsigma \leq v_{i,j} \leq (k+2)\kappa_g^2$, we deduce that

$$
\begin{aligned}
&\sum_{j=0}^{k} \mathbb{E}_j[F(x_{j+1})] \\
&\quad \leq \sum_{j=0}^{k} F(x_j) - \sum_{j=j_\theta+1}^{k} \sum_{i\in\mathcal{J}} \frac{G_{i,j}^2}{\mathbb{E}_j[v_{i,j}]^\mu}\theta \\
&\qquad\quad + n\tilde{\kappa}(j_\theta+1)\frac{\kappa_g^2}{\varsigma^{2\mu}} + \sum_{j=0}^{k}\sum_{i\in\mathcal{I}} \frac{\tilde{\kappa} G_{i,j}^2}{\varsigma^{2\mu}} \\
&\quad \leq \sum_{j=0}^{k} F(x_j) - \sum_{j=j_\theta+1}^{k} \sum_{i\in\mathcal{J}} \frac{\theta G_{i,j}^2}{\kappa_g^{2\mu}(k+2)^\mu} \\
&\qquad\quad + \frac{\tilde{\kappa}}{\varsigma^{2\mu}}\left[n(j_\theta+1)\kappa_g^2 + \sum_{j=0}^{k}\sum_{i\in\mathcal{I}} G_{i,j}^2\right].
\end{aligned}
$$

We now take the full expectation and use (24) for each $i \in \mathcal{I}$ to obtain that

$$
\begin{aligned}
\mathbb{E}[F(x_{k+1})] \leq F(x_0) &- \frac{\theta}{\kappa_g^{2\mu}(k+2)^\mu}\mathbb{E}\left[\sum_{j=j_\theta+1}^{k}\sum_{i\in\mathcal{J}} G_{i,j}^2\right] \\
&+ \frac{n\tilde{\kappa}}{\varsigma^{2\mu}}\left[(j_\theta+1)\kappa_g^2 + \left(\frac{\tilde{\kappa}}{\gamma_{\text{low}}(1-\frac{1}{2}\mu)-\theta}\right)^{\frac{1}{\mu}}\right].
\end{aligned}
$$

Rearranging the terms of the last inequality and using the fact that $F(x_{k+1}) \geq F_{\text{low}}$ by Assumption 2.3 and (24) for the $\mathcal{I}$ terms gives that

$$
\begin{aligned}
\mathbb{E}\left[\sum_{j=j_\theta+1}^{k} \|G_j\|^2\right] &= \mathbb{E}\left[\sum_{j=j_\theta+1}^{k}\sum_{i\in\mathcal{J}} G_{i,j}^2 + \sum_{j=j_\theta+1}^{k}\sum_{i\in\mathcal{I}} G_{i,j}^2\right] \\
&\leq \frac{\kappa_g^{2\mu}(k+2)^\mu}{\theta}\Bigg\{F(x_0) - F_{\text{low}} + \frac{n\tilde{\kappa}}{\varsigma^{2\mu}}\Bigg[(j_\theta+1)\kappa_g^2 \\
&\quad + \left[\frac{\tilde{\kappa}}{\gamma_{\text{low}}(1-\frac{1}{2}\mu)-\theta}\right]^{\frac{1}{\mu}}\Bigg]\Bigg\} + n\left(\frac{\tilde{\kappa}}{\gamma_{\text{low}}(1-\frac{1}{2}\mu)-\theta}\right)^{\frac{1}{\mu}}.
\end{aligned}
$$

Dividing by $(k - j_\theta)$ gives (20)–(22). To conclude our proof, we need to examine the situation where $\mathcal{J} = \emptyset$ and (24) holds for all $i \in \mathcal{I} = \{1, \ldots, n\}$. Therefore

$$
(k+1)\mathbb{E}\left[\operatorname*{average}_{j\in\{0,\ldots,k\}} \|G_j\|^2\right] \leq n\left(\frac{\tilde{\kappa}}{\gamma_{\text{low}}(1-\frac{1}{2}\mu)-\theta}\right)^{\frac{1}{\mu}}. \tag{27}
$$

We may then ignore this situation in our worst-case analysis since this last bound is clearly better than (20).

It is important to note that, at variance with Theorem 3.5 which states *global*[2] convergence rates, Theorem 3.6 only gives *asymptotic*[3] rates. Indeed, our proof does not give an explicit expression of the index j_θ as a function of problem-dependent quantities only.

Our result complements that stated in Corollary 7 of [23] in that it now allows a very flexible choice of the learning rate γ_k (i.e. $\gamma_k \in [\gamma_{\rm low}, 1]$) while γ_k has to be chosen as a specific function of *a priori* unknown constants[4] in this reference, if the best achievable rate of convergence is to be achieved. However, our result does so at the price of the variance condition (19) and does not take any sparsity of the gradient sequence into account.

Note that expressions (21) and (20) in Theorem 3.6 have an explicit linear dependence on the problem dimension n (as in [23]), but caution should be exercized in interpreting this observation since (potentially severe) dependence on dimension may lurk in the Lipschitz constant L of Assumption 2.2.

Interestingly, Theorem 3.6 also raises the possibility (which we dismissed in the worst-case but could well positively influence the practical convergence behaviour) that $\mathcal{I} = \{1, \ldots, n\}$ and (27) holds, which is then significantly better than both (16)–(18) and (20).

Even if the favourable situation we just discussed does not occur, (20) shows that (16)–(18) are not sharp whenever the conditional variance condition (19) holds. Indeed, for k sufficiently large, (20) indicates that a bound of the form

$$\mathbb{E}\left[\underset{j\in\{j_\theta+1,\ldots,k\}}{\text{average}} \|G_j\|\right] = \mathcal{O}\left(\frac{1}{(k+1)^{\frac{1}{2}(1-\mu)}}\right), \tag{28}$$

Observe that this order is very close, for small μ, to that achieved by standard methods using function evaluations to enforce descent (such as steepest descent [17] or first-order adaptive regularization methods [18]) in the noiseless case.

4 A "divergent series" subclass of ASGRADn

The key to Theorem 3.6 and its improved convergence rate is the existence of the (implicit) j_θ index. One might then wonder if a subclass of ASGRADn exists where such an index can be explicitly computed and a similar convergence rate achievable. This is the object of our next theorems.

Theorem 4.1. *Suppose that Assumptions 2.1–2.4 hold and that the* ASGRADn *algorithm is applied to problem (1) with its scaling factors being defined, for some $\nu \in (0, 1)$ and $\mu \in [\nu, \max(1, 2\nu))$, all $i \in \{1, \ldots, n\}$ and all $k \geq 0$ by*

$$\rho_{i,k}(k+1)^\nu \leq w_{i,k} \leq \xi_{i,k}(k+1)^\mu, \tag{29}$$

where $\rho_{i,k}$, $w_{i,k}$ and $\xi_{i,k}$ are random variables depending on iterations $\{0, \ldots, k\}$ such that $\varsigma \leq \rho_{i,k}$ and $\xi_{i,k} \leq \kappa_\xi$ for some constants $0 < \varsigma \leq \kappa_\xi$, for all $i \in \{1, \ldots, n\}$ and all $k \geq 0$.

[2] That is valid for each $k \geq 0$.
[3] That is valid for k sufficiently large.
[4] Such as κ_g, L and the index of the final iteration.

Then, for $\nu \neq \frac{1}{2}$,

$$\mathbb{E}\left[\operatorname*{average}_{j\in\{0,\ldots,k\}} \|G_j\|^2\right] \leq \frac{\kappa_\xi(F(x_0)-F_{\rm low})}{\gamma_{\rm low}(k+1)^{1-\mu}} + \frac{n\,L\kappa_\xi\kappa_g^2}{2(1-2\nu)\varsigma^2\gamma_{\rm low}}\left[\frac{1}{(k+1)^{2\nu-\mu}} - \frac{2\nu}{(k+1)^{1-\mu}}\right], \tag{30}$$

while, if $\nu = \frac{1}{2}$,

$$\mathbb{E}\left[\operatorname*{average}_{j\in\{0,\ldots,k\}} \|G_j\|^2\right] \leq \frac{\kappa_\xi(F(x_0)-F_{\rm low})}{\gamma_{\rm low}(k+1)^{1-\mu}} + \frac{n\,L\kappa_\xi\kappa_g^2}{2\varsigma^2\gamma_{\rm low}}\left(\frac{1+\log(k+1)}{(k+1)^{1-\mu}}\right), \tag{31}$$

Proof. See Appendix.

The choice (29) is of course reminiscent, in a smooth but stochastic and nonconvex setting, of the "divergent stepsize" subgradient method for non-smooth convex optimization (see [1] and the many references therein), for which a $\mathcal{O}(1/\sqrt{k})$ global rate of convergence is known (Theorems 8.13 and 8.30 in this last reference).

The bounds given by Theorem 4.1 are qualitatively similar to those of Theorem 3.5, but, as in Theorem 3.6 for this case, they may be improved if we strengthen our assumptions, this time explicitly rather than implicitly.

Theorem 4.2. *Suppose that Assumptions 2.1–2.4 hold and that the* ASGRADn *algorithm is applied to problem (1) with its scaling factors being defined as in Theorem 4.1 with $\nu \in (0,1)$ and $\mu \in [\nu, \max(\frac{4}{3}\nu, 1))$. Suppose additionally that the conditional variance condition (19) holds for all $i \in \{1,\ldots,n\}$ and all $k \geq 0$. Then, for any $\theta \in (0, \frac{\gamma_{\rm low}}{\kappa_\xi}))$,*

$$\mathbb{E}\left[\operatorname*{average}_{j\in\{j_\theta+1,\ldots,k\}} \|G_j\|^2\right] \leq \kappa_\#(\theta)\frac{(k+1)^\mu}{k-j_\theta} \leq \frac{\kappa_\#(\theta)(j_\theta+2)}{(k+1)^{1-\mu}}, \tag{32}$$

where

$$\kappa_\#(\theta) \stackrel{\rm def}{=} \frac{1}{\theta}\left(F(x_0) - F_{\rm low} + \frac{n\,\kappa_g^2 L\kappa_\xi^2\,(1+\kappa_{\rm var})}{2^{1-\mu}\varsigma^4} j_\theta\right), \tag{33}$$

and

$$j_\theta \stackrel{\rm def}{=} \left\lfloor\left(\frac{L\kappa_\xi^3(1+\kappa_{\rm var})}{2^{1-\mu}\varsigma^4(\gamma_{\rm low}-\theta\kappa_\xi)}\right)^{\frac{1}{4\nu-3\mu}}\right\rfloor + 1. \tag{34}$$

Proof. See Appendix.

The k-order of convergence of $\mathbb{E}\Big[\text{average}_{j\in\{j_\theta+1,\ldots,k\}} \|G_j\|\Big]$ implied by (32) is therefore

$$\mathcal{O}\left(\frac{1}{(k+1)^{\frac{1}{2}(1-\mu)}}\right)$$

which is identical to (28), except that now (34) now gives an explicit formula for j_θ, thereby quantifying what is meant by "k sufficienlty large" used to derive (28).

It is of course possible to combine the ideas of the two ASGRADn subclasses considered so far, for instance by defining both $\xi_{i,k}$ and $\rho_{i,k}$ in (29) to be equal to $\varsigma + \sum_{j=0}^k g_{i,j}^2/(k+1)$, without altering the results of Theorem 4.2.

5 Numerical illustration

We now provide some numerical illustration of the algorithmic variants discussed in the previous sections. We trained a simple convolutional network of [11] (denoted in the paper as cifar10-nv) and a small resnet18 model [12] on the CIFAR-10 image classification dataset[5]. For these experiments, we used haiku [13] and optax [14] two JAX [6] based libraries on a workstation with four GTX 1080TI. We now compare the numerical performance of (14) for various μ values in $(0.1, 0.5, 0.9)$ and of two scaling factor choices satisfying (29). In order to define these, first set

$$\xi_k = \max(\varsigma, \xi_{k-1}, |g_k|),\ (k \geq 0), \tag{35}$$

with $\xi_{-1} = 0$ and the max being understood componentwise. They are then defined by

$$w_{i,k} = \xi_k (k+1)^\nu \tag{36}$$

and

$$w_{i,k} = \max(\varsigma, \frac{1}{k+1} \sum_{j=0}^{k} |g_{i,k}|)(k+1)^\nu. \tag{37}$$

These two scalings, denoted respectively by *maxgi* and *avrgi*, verify (29) with $\varsigma = \varsigma$ and $\kappa_\xi = \kappa_g$. In (36) and (37), we chose $\nu = 0.1$ and $\varsigma = 0.01$. The corresponding variants are called *maxgi* and *avrgi*, respectively. Note that the scaling factors are increasing in *maxgi* but no necessarily so in *avrgi*.
For all experiments, we also chose a fixed[6] learning rate policy with $\gamma_k = \gamma = 5.\{10^{-4}, 10^{-5}\}$ for all $k \geq 0$. We used the same random initialization for all scaling choices and followed the data-augmentation procedure of [11], both for training and testing. We trained the models for a total of 100000 steps with a batchsize of 128 using the mean-cross entropy loss function. We report the training and test accuracies (the latter on a sample of size 128 from the test dataset) every 500 steps.

The results of these experiments (averaged over three random runs) are presented in Figures 1–4. In each figure, the top panel shows the evolution (as a function of the number of steps) of the training accuracy, and the bottom panel that of the test accuracy. The average values are shown as thick lines and the shaded areas of corresponding colour give the 67% confidence intervals.

These simple numerical illustrations are obviously not meant to replace significant numerical testing, but, albeit caution must be exercised not to extrapolate from limited data, they still suggest a few tentative comments.

- The relative behaviour of the tested variants does not differ significantly between the two tested network architectures, even if the test accuracy is (as expected) slighly lower for the resnet18 case.
- For fixed learning rates, the methods *maxgi* and *avrgi* of the second ASGRADn subclass (introduced in Section 4) seem to produce relatively good results on our example, both in training and testing, often outperforming the Adagrad-like variants of the first subclass (of Section 3).
- Among Adagrad-like variants, those with a larger μ handle smaller learning rates better on these examples, a behaviour admittedly not predicted by our theory.

[5] https://www.cs.toronto.edu/ kriz/cifar.html

[6] Our choice of a fixed learning rate policy is meant to focus on the intrinsic properties of each scaling factor option.

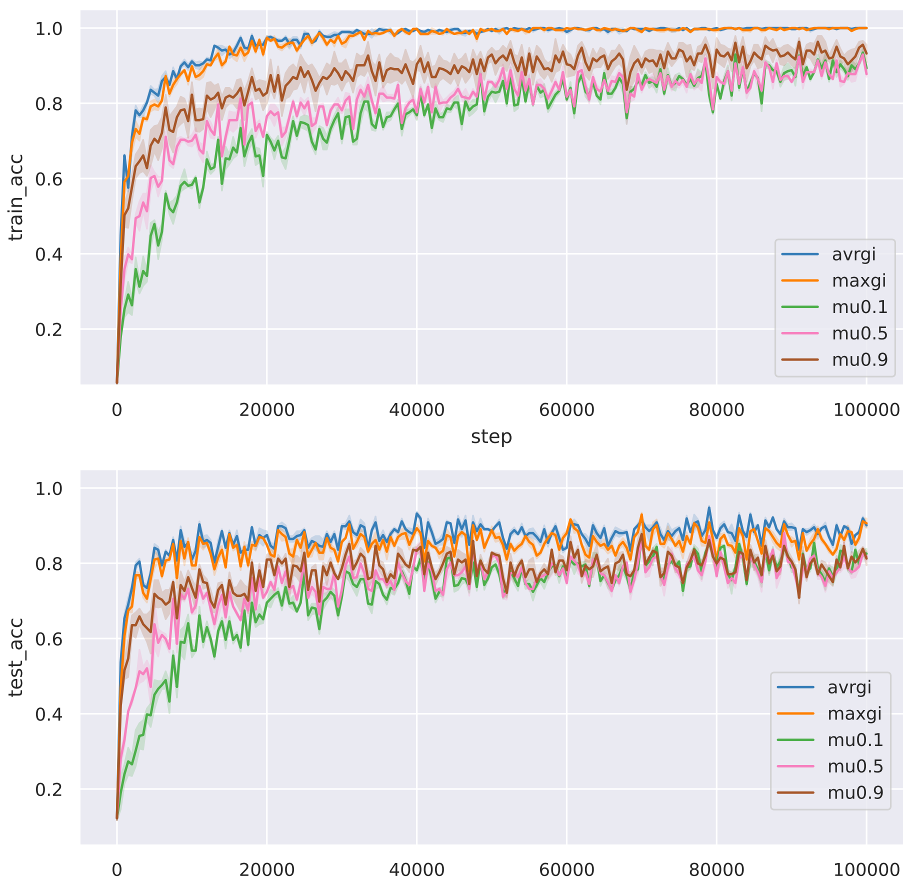

Figure 1: Training (top) and test (bottom) accuracies for the Adagrad-like ($\mu \in (0.1, 0.5, 0.9)$), *maxgi* and *avrgi* variants with $\gamma = 5.10^{-4}$ on the cifar10-nv architecture

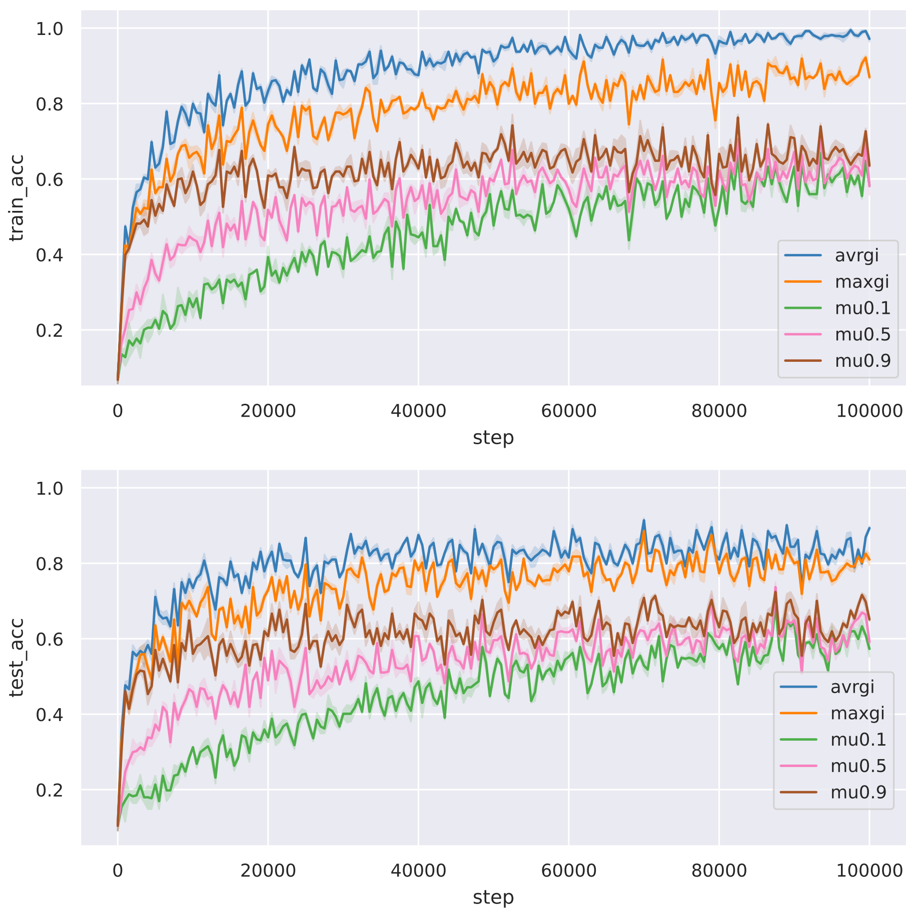

Figure 2: Training (top) and test (bottom) accuracy for the Adagrad-like ($\mu \in (0.1, 0.5, 0.9)$), *maxgi* and *avrgi* variants with $\gamma = 5.10^{-5}$ on the cifar10-nv architecture

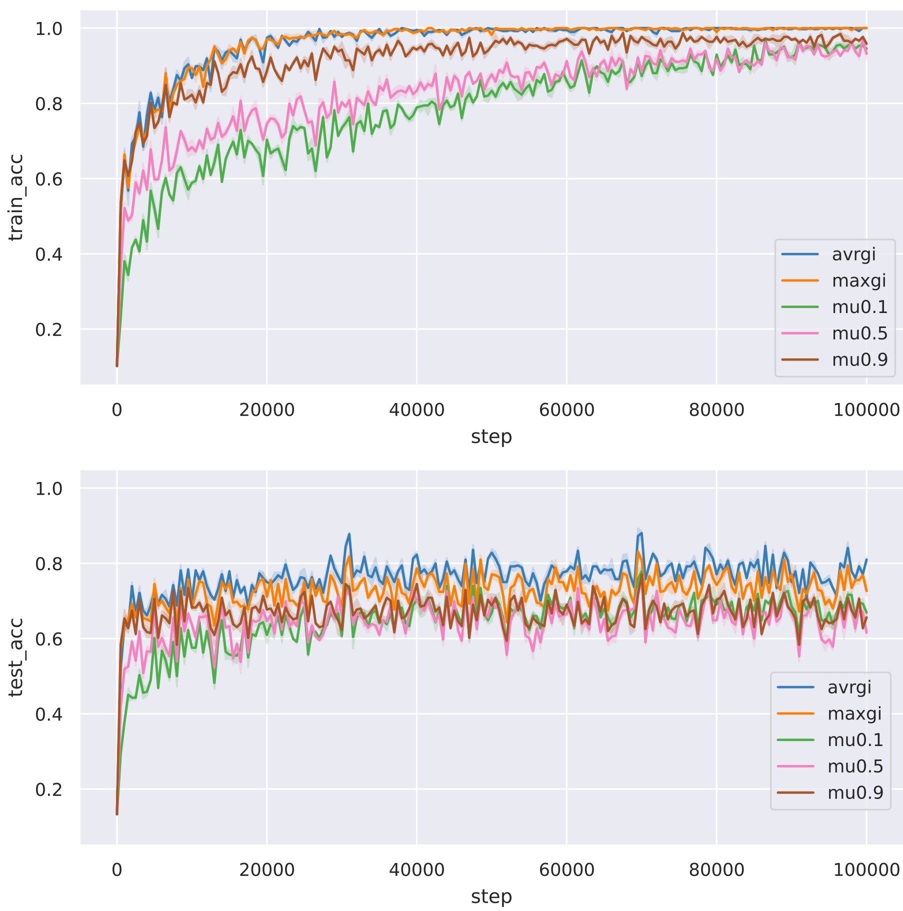

Figure 3: Training (top) and test (bottom) accuracies for the Adagrad-like ($\mu \in (0.1, 0.5, 0.9)$), *maxgi* and *avrgi* variants with $\gamma = 5.10^{-4}$ on the resnet18 architecture

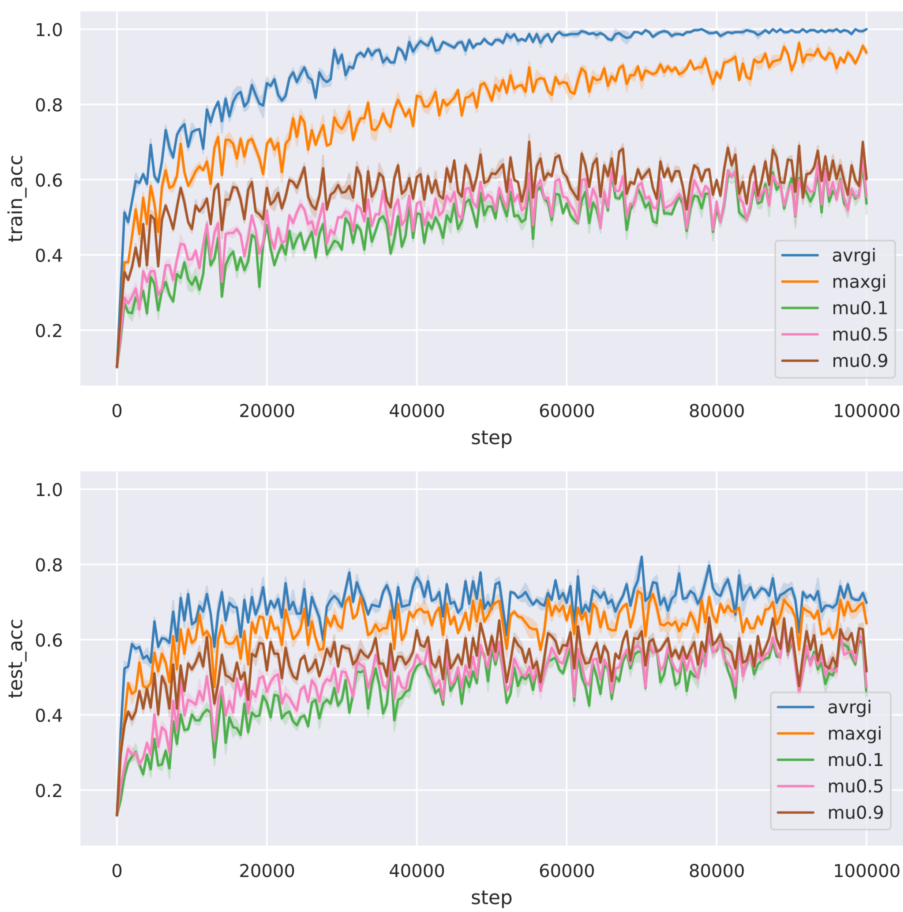

Figure 4: Training (top) and test (bottom) accuracy for the Adagrad-like ($\mu \in (0.1, 0.5, 0.9)$), *maxgi* and *avrgi* variants with $\gamma = 5.10^{-5}$ on the resnet18 architecture

- The comparison of Figures 1 and 3 with Figures 2 and 4 unsurprisingly shows that, albeit our theory does not depend on the choice of γ_k, the practical convergence behaviour may be affected by this choice (and other factors such as the batchsize).

6 Conclusions

We have extended the Adagrad algorithm to a parametric class of minimization methods and derived complexity upper bounds for two subclasses of interest, the first containing the standard Adagrad. These bounds give the best complexity to values of the class parameters corresponding, in the first subclass, to Adagrad. We have also shown these bounds can be improved for both subclasses under an additional variance condition, in which case the parameter choice yielding the best bounds no longer corresponds to Adagrad. This improvement is asymptotic and implicit for the first class and explicit for the second. However, our numerical illustrations of the discussed methods on simple examples indicate that methods of the second class have merits, but also that, at least in our examples, there remains some distance from the above theory to real behaviour. This may possibly be because the complexity bounds may not be sharp, but also, fortunately, because the worst-case happens very rarely in practice.

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A first technical lemma

Lemma .1. *Let* $\mu \in (0, 1]$. *Let* $x, y \in \mathbb{R}^+ \setminus \{0\}$. *Then*

$$\frac{|x^\mu - y^\mu|}{x^\mu y^\mu} \leq \mu \frac{|x - y|}{x y^\mu} + \mu \frac{|x - y|}{x^\mu y}. \tag{38}$$

Proof. Let us first consider the case $x \geq y$. Remembering that $u^\mu \leq 1 + \mu(u - 1)$ for $u > 0$ and taking $u = \frac{x}{y}$, we successively derive that

$$\begin{aligned}
\frac{x^\mu}{y^\mu} &\leq 1 + \mu \left(\frac{x}{y} - 1 \right), \\
x^\mu - y^\mu &\leq \mu \left(\frac{x y^\mu}{y} - y^\mu \right), \\
x^\mu - y^\mu &\leq \mu y^{\mu - 1}(x - y), \\
\frac{x^\mu - y^\mu}{x^\mu y^\mu} &\leq \mu \frac{x - y}{x^\mu y}.
\end{aligned} \tag{39}$$

Hence the inequality (38) is valid when $x \geq y$. For the symmetric case ($y \geq x$), we similarly obtain that

$$\frac{y^\mu - x^\mu}{x^\mu y^\mu} \leq \mu \frac{y - x}{y^\mu x}. \tag{40}$$

Combining (39) and (40) yields the desired result.

Proof of Lemma 3.3

Let us consider an iteration index $j \geq 0$ and a component index $i \in \{1, \ldots, n\}$. We first use the definition of s^L in (4) and the fact that $w_{i,j} = v_{i,j}^\mu$ (Assumption 3.1) to obtain that

$$\mathbb{E}_j \left[\gamma_j G_{i,j} s_{i,j}^L \right] = -\mathbb{E}_j \left[\gamma_j \frac{G_{i,j} g_{i,j}}{v_{i,j}^\mu} \right] = -\mathbb{E}_j \left[\gamma_j \frac{G_{i,j} g_{i,j}}{\mathbb{E}_j[v_{i,j}]^\mu} \right] + \mathbb{E}_j \left[\gamma_j G_{i,j} g_{i,j} \left(\frac{1}{\mathbb{E}_j[v_{i,j}]^\mu} - \frac{1}{v_{i,j}^\mu} \right) \right]. \tag{41}$$

If $G_{i,j} g_{i,j} \geq 0$, then

$$-\frac{\gamma_j G_{i,j} g_{i,j}}{\mathbb{E}_j[v_{i,j}]^\mu} \leq -\gamma_{\text{low}} \frac{G_{i,j} g_{i,j}}{\mathbb{E}_j[v_{i,j}]^\mu}$$

and thus

$$\mathbb{E}_j \left[-\frac{\gamma_j G_{i,j} g_{i,j}}{\mathbb{E}_j[v_{i,j}]^\mu} \right] \leq -\gamma_{\text{low}} \frac{G_{i,j}}{\mathbb{E}_j[v_{i,j}]^\mu} \mathbb{E}_j[g_{i,j}] = -\gamma_{\text{low}} \frac{G_{i,j}^2}{\mathbb{E}_j[v_{i,j}]^\mu}. \tag{42}$$

Otherwise, if $G_{i,j} g_{i,j} < 0$, we may deduce that

$$-\frac{\gamma_j G_{i,j} g_{i,j}}{\mathbb{E}_j[v_{i,j}]^\mu} \leq -\frac{G_{i,j} g_{i,j}}{\mathbb{E}_j[v_{i,j}]^\mu},$$

implying that

$$\mathbb{E}_j \left[-\frac{\gamma_j G_{i,j} g_{i,j}}{\mathbb{E}_j[v_{i,j}]^\mu} \right] \leq -\frac{G_{i,j}}{\mathbb{E}_j[v_{i,j}]^\mu} \mathbb{E}_j[g_{i,j}] = -\frac{G_{i,j}^2}{\mathbb{E}_j[v_{i,j}]^\mu} \leq -\gamma_{\text{low}} \frac{G_{i,j}^2}{\mathbb{E}_j[v_{i,j}]^\mu}. \tag{43}$$

Combining (41), (42) and (43) gives that

$$\mathbb{E}_j\left[\gamma_j G_{i,j} s_{i,j}^L\right] \leq -\gamma_{\text{low}} \frac{G_{i,j}^2}{\mathbb{E}_j[v_{i,j}]^\mu} + \mathbb{E}_j\left[\underbrace{\gamma_j G_{i,j} g_{i,j} \frac{v_{i,j}^\mu - \mathbb{E}_j[v_{i,j}]^\mu}{v_{i,j}^\mu \mathbb{E}_j[v_{i,j}]^\mu}}_{A}\right]. \tag{44}$$

We now derive an upper bound on the absolute value of the A term by successively using Lemma (.1), Assumption 3.1 and the bound $\gamma_j \leq 1$ to obtain that

$$\begin{aligned}
|A| &= |\gamma_j G_{i,j} g_{i,j}| \frac{|v_{i,j}^\mu - \mathbb{E}_j[v_{i,j}]^\mu|}{v_{i,j}^\mu \mathbb{E}_j[v_{i,j}]^\mu} \\
&\leq \mu |\gamma_j G_{i,j} g_{i,j}| \frac{|v_{i,j} - \mathbb{E}_j[v_{i,j}]|}{v_{i,j}^\mu \mathbb{E}_j[v_{i,j}]} + \mu |\gamma_j G_{i,j} g_{i,j}| \frac{|v_{i,j} - \mathbb{E}_j[v_{i,j}]|}{v_{i,j} \mathbb{E}_j[v_{i,j}]^\mu} \\
&\leq \mu \underbrace{|G_{i,j} g_{i,j}| \kappa_v \frac{\mathbb{E}_j\left[g_{i,j}^2\right]}{v_{i,j}^\mu \mathbb{E}_j[v_{i,j}]}}_{B} + \mu \underbrace{|G_{i,j} g_{i,j}| \kappa_v \frac{g_{i,j}^2}{v_{i,j}^\mu \mathbb{E}_j[v_{i,j}]}}_{C} \\
&\quad + \mu \underbrace{|G_{i,j} g_{i,j}| \kappa_v \frac{\mathbb{E}_j\left[g_{i,j}^2\right]}{v_{i,j} \mathbb{E}_j[v_{i,j}]^\mu}}_{D} + \mu \underbrace{|G_{i,j} g_{i,j}| \kappa_v \frac{g_{i,j}^2}{v_{i,j} \mathbb{E}_j[v_{i,j}]^\mu}}_{E}.
\end{aligned}$$

We now use Young's inequality with $p = q = 2$, that is

$$\forall \lambda > 0, x,\, y \in \mathbb{R}^+,\, xy \leq \frac{\lambda}{2} x^2 + \frac{y^2}{2\lambda}, \tag{45}$$

to successively handle the 4 terms of the last bound.

- For the first term B, we choose

$$x = \frac{|G_{i,j}|}{\mathbb{E}_j[v_{i,j}]^\mu}, \quad \lambda = \frac{\gamma_{\text{low}} \mathbb{E}_j[v_{i,j}]^\mu}{4} \text{ and } y = \kappa_v |g_{i,j}| \frac{\mathbb{E}_j\left[g_{i,j}^2\right]}{v_{i,j}^\mu \mathbb{E}_j[v_{i,j}]^{1-\mu}}.$$

Using (45), Assumptions 2.4, 3.2 and (9), we obtain that

$$\begin{aligned}
B &\leq \gamma_{\text{low}} \frac{G_{i,j}^2}{8\mathbb{E}_j[v_{i,j}]^\mu} + 2 \frac{\kappa_v^2}{\gamma_{\text{low}}} \frac{g_{i,j}^2}{v_{i,j}^{2\mu}} \frac{\mathbb{E}_j\left[g_{i,j}^2\right]^2}{\mathbb{E}_j[v_{i,j}]^{2-\mu}}, \\
&\leq \gamma_{\text{low}} \frac{G_{i,j}^2}{8\mathbb{E}_j[v_{i,j}]^\mu} + 2 \frac{\kappa_v^2}{\gamma_{\text{low}}} \mathbb{E}_j\left[g_{i,j}^2\right]^\mu \frac{g_{i,j}^2}{v_{i,j}^{2\mu}} \\
&\leq \gamma_{\text{low}} \frac{G_{i,j}^2}{8\mathbb{E}_j[v_{i,j}]^\mu} + 2 \frac{\kappa_v^2}{\gamma_{\text{low}}} \kappa_g^{2\mu} \frac{g_{i,j}^2}{v_{i,j}^{2\mu}}.
\end{aligned}$$

Taking now the expectation over $\mathbb{E}_j[.]$ yields that

$$\mathbb{E}_j[B] \leq \gamma_{\text{low}} \frac{G_{i,j}^2}{8\mathbb{E}_j[v_{i,j}]^\mu} + 2 \frac{\kappa_v^2}{\gamma_{\text{low}}} \kappa_g^{2\mu} \mathbb{E}_j\left[\frac{g_{i,j}^2}{w_{i,j}^2}\right]. \tag{46}$$

• Now consider the C term. In this case, we choose

$$x = \frac{|G_{i,j} g_{i,j}|}{\mathbb{E}_j[v_{i,j}]^{\mu}}, \quad \lambda = \gamma_{\rm low} \frac{\mathbb{E}_j[v_{i,j}]^{\mu}}{4\mathbb{E}_j\left[g_{i,j}^2\right]} \text{ and } y = \kappa_v \frac{g_{i,j}^2}{v_{i,j}^{\mu}\mathbb{E}_j[v_{i,j}]^{1-\mu}}$$

to deduce from (45) that

$$\begin{aligned} C &\leq \gamma_{\rm low} \frac{G_{i,j}^2}{8\mathbb{E}_j[v_{i,j}]^{\mu}} \frac{g_{i,j}^2}{\mathbb{E}_j\left[g_{i,j}^2\right]} + 2\frac{\kappa_v^2}{\gamma_{\rm low}} \frac{g_{i,j}^4}{v_{i,j}^{2\mu}} \frac{\mathbb{E}_j\left[g_{i,j}^2\right]}{\mathbb{E}_j[v_{i,j}]^{2-\mu}} \\ &\leq \gamma_{\rm low} \frac{G_{i,j}^2}{8\mathbb{E}_j[v_{i,j}]^{\mu}} \frac{g_{i,j}^2}{\mathbb{E}_j\left[g_{i,j}^2\right]} + 2\frac{\kappa_v^2}{\gamma_{\rm low}} \kappa_g^2 \frac{g_{i,j}^2}{v_{i,j}^{2\mu}} \frac{1}{\mathbb{E}_j[v_{i,j}]^{1-\mu}} \\ &\leq \gamma_{\rm low} \frac{G_{i,j}^2}{8\mathbb{E}_j[v_{i,j}]^{\mu}} \frac{g_{i,j}^2}{\mathbb{E}_j\left[g_{i,j}^2\right]} + 2\frac{\kappa_v^2}{\gamma_{\rm low}} \frac{\kappa_g^2}{\varsigma_{\rm min}^{1-\mu}} \frac{g_{i,j}^2}{v_{i,j}^{2\mu}}, \end{aligned}$$

where we successively used the facts that $\mathbb{E}_j\left[g_{i,j}^2\right] \leq \mathbb{E}_j[v_{i,j}]$ (because of (9)), $g_{i,j}^2 \leq \kappa_g^2$ (because of Assumption 2.4) and $\mathbb{E}_j[v_{i,j}]^{1-\mu} \geq \varsigma_{\rm min}^{1-\mu}$ (because of (8)). Taking the expectation over $\mathbb{E}_j[.]$ then gives that

$$\mathbb{E}_j[C] \leq \gamma_{\rm low} \frac{G_{i,j}^2}{8\mathbb{E}_j[v_{i,j}]^{\mu}} + 2\frac{\kappa_v^2}{\gamma_{\rm low}} \frac{\kappa_g^2}{\varsigma_{\rm min}^{1-\mu}} \mathbb{E}_j\left[\frac{g_{i,j}^2}{w_{i,j}^2}\right]. \tag{47}$$

(Note that we can divide by $\mathbb{E}_j\left[g_{i,j}^2\right]$ above, as it suffice to notice that $\mathbb{E}_j\left[g_{i,j}^2\right] = 0$ implies $g_{i,j}^2 = 0$. C would then be equal to zero and (47) would still be verified.)

• Let us now handle the D term. Choosing

$$x = \frac{|G_{i,j}|}{\mathbb{E}_j[v_{i,j}]^{\mu}}, \quad \lambda = \gamma_{\rm low} \frac{\mathbb{E}_j[v_{i,j}]^{\mu}}{4} \text{ and } y = \kappa_v |g_{i,j}| \frac{\mathbb{E}_j\left[g_{i,j}^2\right]}{v_{i,j}},$$

we now deduce from (45) that

$$\begin{aligned} D &\leq \gamma_{\rm low} \frac{G_{i,j}^2}{8\mathbb{E}_j[v_{i,j}]^{\mu}} + 2\frac{\kappa_v^2}{\gamma_{\rm low}} \frac{g_{i,j}^2 \mathbb{E}_j\left[g_{i,j}^2\right]^2}{\mathbb{E}_j[v_{i,j}]^{\mu}\, v_{i,j}^2}, \\ &\leq \gamma_{\rm low} \frac{G_{i,j}^2}{8\mathbb{E}_j[v_{i,j}]^{\mu}} + 2\frac{\kappa_v^2}{\gamma_{\rm low}} \frac{g_{i,j}^2}{v_{i,j}^{2\mu}} \frac{1}{v_{i,j}^{2-2\mu}} \frac{\mathbb{E}_j\left[g_{i,j}^2\right]^2}{\mathbb{E}_j[v_{i,j}]^{\mu}} \\ &\leq \gamma_{\rm low} \frac{G_{i,j}^2}{8\mathbb{E}_j[v_{i,j}]^{\mu}} + 2\frac{\kappa_v^2}{\gamma_{\rm low}} \frac{g_{i,j}^2}{v_{i,j}^{2\mu}} \frac{1}{v_{i,j}^{2-2\mu}} \mathbb{E}_j\left[g_{i,j}^2\right]^{2-\mu} \\ &\leq \gamma_{\rm low} \frac{G_{i,j}^2}{8\mathbb{E}_j[v_{i,j}]^{\mu}} + 2\frac{\kappa_v^2}{\gamma_{\rm low}} \frac{\kappa_g^{4-2\mu}}{\varsigma_{\rm min}^{2-2\mu}} \frac{g_{i,j}^2}{v_{i,j}^{2\mu}}, \end{aligned}$$

where, as for the C term, we used the facts that $\mathbb{E}_j\left[g_{i,j}^2\right]^{\mu} \leq \mathbb{E}_j[v_{i,j}]^{\mu}$, $g_{i,j}^2 \leq \kappa_g^2$ and $v_{i,j}^{2-2\mu} \geq \varsigma_{\rm min}^{2-2\mu}$. Taking the expectation $\mathbb{E}_j[.]$ yields, in this case, that

$$\mathbb{E}_j[D] \leq \gamma_{\rm low} \frac{G_{i,j}^2}{8\mathbb{E}_j[v_{i,j}]^{\mu}} + 2\frac{\kappa_v^2}{\gamma_{\rm low}} \frac{\kappa_g^{4-2\mu}}{\varsigma_{\rm min}^{2-2\mu}} \mathbb{E}_j\left[\frac{g_{i,j}^2}{w_{i,j}^2}\right]. \tag{48}$$

- Finally consider the E term. Choosing

$$x = \frac{|G_{i,j} g_{i,j}|}{\mathbb{E}_j[v_{i,j}]^\mu}, \quad \lambda = \gamma_{\text{low}} \frac{\mathbb{E}_j[v_{i,j}]^\mu}{4\mathbb{E}_j\left[g_{i,j}^2\right]} \text{ and } y = \kappa_v \frac{g_{i,j}^2}{v_{i,j}}$$

in (45) then gives that

$$\begin{aligned}
\mathbb{E}_j[E] &\le \gamma_{\text{low}} \frac{G_{i,j}^2}{8\mathbb{E}_j[v_{i,j}]^\mu} \frac{g_{i,j}^2}{\mathbb{E}_j\left[g_{i,j}^2\right]} + 2\frac{\kappa_v^2}{\gamma_{\text{low}}} \frac{g_{i,j}^4 \mathbb{E}_j\left[g_{i,j}^2\right]}{\mathbb{E}_j[v_{i,j}]^\mu \, v_{i,j}^2} \\
&\le \gamma_{\text{low}} \frac{G_{i,j}^2}{8\mathbb{E}_j[v_{i,j}]^\mu} \frac{g_{i,j}^2}{\mathbb{E}_j\left[g_{i,j}^2\right]} \\
&\qquad + 2\frac{\kappa_v^2}{\gamma_{\text{low}}} \mathbb{E}_j\left[g_{i,j}^2\right]^{1-\mu} \frac{g_{i,j}^2}{v_{i,j}^{2\mu}} \left(\frac{1}{v_{i,j}^{1-2\mu}} \frac{g_{i,j}^2}{v_{i,j}} \mathbf{1}_{\mu<\frac{1}{2}} + \frac{|g_{i,j}^{4-4\mu}|}{v_{i,j}^{2-2\mu}} |g_{i,j}^{4\mu-2}| \mathbf{1}_{\mu\ge\frac{1}{2}} \right) \\
&\le \gamma_{\text{low}} \frac{G_{i,j}^2}{8\mathbb{E}_j[v_{i,j}]^\mu} \frac{g_{i,j}^2}{\mathbb{E}_j\left[g_{i,j}^2\right]} + 2\frac{\kappa_v^2}{\gamma_{\text{low}}} \kappa_g^{2-2\mu} \frac{g_{i,j}^2}{v_{i,j}^{2\mu}} \left(\frac{1}{\varsigma_{\min}^{1-2\mu}} \mathbf{1}_{\mu<\frac{1}{2}} + \kappa_g^{4\mu-2} \mathbf{1}_{\mu\ge\frac{1}{2}} \right),
\end{aligned}$$

where we once more used the facts that $\mathbb{E}_j\left[g_{i,j}^2\right]^\mu \le \mathbb{E}_j[v_{i,j}]^\mu$ and $|g_{i,j}| \le \kappa_g$, in turn implying that

$$g_{i,j}^2 \le v_{i,j} \text{ and } v_{i,j} \ge \varsigma_{\min} \quad \text{if} \quad \mu < \tfrac{1}{2}$$

and

$$|g_{i,j}^{4-4\mu}| \le v_{i,j}^{2-2\mu} \text{ and } |g_{i,j}^{4\mu-2}| \le \kappa_g^{4\mu-2} \quad \text{if} \quad \mu \ge \tfrac{1}{2}.$$

Taking the expectation $\mathbb{E}_j[.]$, we deduce that

$$\mathbb{E}_j[E] \le \gamma_{\text{low}} \frac{G_{i,j}^2}{8\mathbb{E}_j[v_{i,j}]^\mu} + \frac{\kappa_v^2}{\gamma_{\text{low}}} \kappa_g^{2-2\mu} \left(\frac{1}{\varsigma_{\min}^{1-2\mu}} \mathbf{1}_{\mu<\frac{1}{2}} + \kappa_g^{4\mu-2} \mathbf{1}_{\mu\ge\frac{1}{2}} \right) \mathbb{E}_j\left[\frac{g_{i,j}^2}{w_{i,j}^2}\right]. \tag{49}$$

Summing now (46), (47), (48) and (49) and substituting the obtained upper-bound of A in (44), we finally obtain (10) with (11).

Proof of Lemma 3.4

Consider first the case where $\alpha \ne 1$ and note that $\frac{1}{(1-\alpha)} x^{1-\alpha}$ is then a non-decreasing and concave function on $(0, +\infty)$. Setting $b_{-1} = 0$ and using these properties, we obtain that, for $j \ge 0$,

$$\begin{aligned}
\frac{a_j}{(\varsigma + b_j)^\alpha} &\le \frac{1}{1-\alpha} \left((\varsigma + b_j)^{1-\alpha} - (\varsigma + b_j - a_j)^{1-\alpha} \right) \\
&\le \frac{1}{1-\alpha} \left((\varsigma + b_j)^{1-\alpha} - (\varsigma + b_{j-1})^{1-\alpha} \right).
\end{aligned}$$

We then obtain (12) by summing this inequality for $j \in \{0, \ldots, k\}$.

Suppose now that $\alpha = 1$, We then use the concavity and non-decreasing character of the logarithm to derive that

$$\frac{a_j}{(\varsigma + b_j)^\alpha} = \frac{a_j}{(\varsigma + b_j)} \leq \log(\varsigma + b_j) - \log(\varsigma + b_j - a_j) \leq \log(\varsigma + b_j) - \log(\varsigma + b_{j-1}).$$

The inequality (13) then again follows by summing for $j \in \{0, \dots, k\}$.

Proof of Theorem 3.5

It is clear from (14) that $w_{i,k} \geq \varsigma^\mu$. Moreover, if we define $v_{i,k} \stackrel{\text{def}}{=} \varsigma + \sum_{\ell=0}^k g_{i,\ell}^2$, then we have that $w_{i,k} = v_{i,k}^\mu$, $v_{i,k} \geq g_{i,k}^2$ and

$$|\mathbb{E}_k[v_{i,k}] - v_{i,k}| = |\mathbb{E}_k\left[g_{i,k}^2\right] - g_{i,k}^2| \leq \mathbb{E}_k\left[g_{i,k}^2\right] + g_{i,k}^2.$$

Thus the proposed scaling factors verify Assumptions 3.1 and 3.2 with $\kappa_v = 1$.
Using (2), we derive that

$$F(x_{j+1}) \leq F(x_j) + \gamma_j G_j^T s_j^L + \frac{L}{2}\gamma_j^2 \|s_j^L\|^2 \leq F(x_j) + \gamma_j G_j^T s_j^L + \frac{L}{2}\|s_j^L\|^2.$$

Taking the conditional expectation, using Lemma 3.3, the fact that $v_{i,j} \leq (k+2)\kappa_g^2$ (because we assumed that $\varsigma \leq \kappa_g$), (4), we deduce that, for $j \in \{0, \dots, k\}$,

$$\begin{aligned}
\mathbb{E}_j[F(x_{j+1})] &\leq F(x_j) + \sum_{i=1}^n \mathbb{E}_j\left[\gamma_j G_{i,j} s_{i,j}^L\right] + \frac{L}{2}\mathbb{E}_j\left[\|s_j^L\|^2\right], \\
&\leq F(x_j) - \sum_{i=1}^n (1 - \frac{\mu}{2})\gamma_{\text{low}} \frac{G_{i,j}^2}{(\mathbb{E}_j[v_{i,j}])^\mu} + 2\kappa_\Delta \mathbb{E}_j\left[\frac{g_{i,j}^2}{w_{i,j}^2}\right] + \frac{L}{2}\mathbb{E}_j\left[\|s_j^L\|^2\right], \\
&\leq F(x_j) - (1 - \frac{\mu}{2})\gamma_{\text{low}} \frac{\|G_j\|^2}{\kappa_g^{2\mu}(k+2)^\mu} + \left(\frac{L}{2} + 2\kappa_\Delta\right)\mathbb{E}_j\left[\|s_j^L\|^2\right].
\end{aligned}$$

We may now sum the previous inequality for $j \in \{0, \dots, k\}$ and take the full expectation to derive that

$$\begin{aligned}
\mathbb{E}[F(x_{k+1})] &\leq F(x_0) - (1 - \frac{\mu}{2})\frac{\gamma_{\text{low}}}{\kappa_g^{2\mu}(k+2)^\mu} \sum_{j=0}^k \mathbb{E}\left[\|G_j\|^2\right] + \left(\frac{L}{2} + 2\kappa_\Delta\right)\sum_{j=0}^k \mathbb{E}\left[\|s_j^L\|^2\right] \\
&\leq F(x_0) - (1 - \frac{\mu}{2})\frac{\gamma_{\text{low}}}{\kappa_g^{2\mu}(k+2)^\mu} \sum_{j=0}^k \mathbb{E}\left[\|G_j\|^2\right] + \left(\frac{L}{2} + 2\kappa_\Delta\right)\sum_{i=1}^n \sum_{j=0}^k \mathbb{E}\left[(s_{i,j}^L)^2\right].
\end{aligned} \tag{50}$$

Using now Lemma 3.4 with $\alpha = 2\mu$ for each $s_{i,j}^L$, (4), (14) and Assumption 2.4, we derive that, for $\mu \in (0, \frac{1}{2})$,

$$\begin{aligned}\sum_{j=0}^{k} (s_{i,j}^L)^2 &= \sum_{j=0}^{k} \frac{g_{i,j}^2}{(\varsigma + \sum_{j=0}^{k} g_{i,j}^2)^{2\mu}} \\ &\leq \frac{1}{1-2\mu} \left[\left(\varsigma + \sum_{j=0}^{k} g_{i,j}^2 \right)^{1-2\mu} - \varsigma^{1-2\mu} \right] \\ &\leq \frac{1}{1-2\mu} \left[\left(\varsigma + (k+1)\kappa_g^2 \right)^{1-2\mu} - \varsigma^{1-2\mu} \right].\end{aligned}$$

Plugging this inequality in (50) and using Assumption 2.3, we obtain that

$$\begin{aligned}F_{\rm low} \leq \mathbb{E}[F(x_{k+1})] \leq F(x_0) - (1 - \frac{\mu}{2}) \frac{\gamma_{\rm low}}{\kappa_g^{2\mu}(k+2)^\mu} \sum_{j=0}^{k} \mathbb{E}\big[\|G_j\|^2\big] \\ + \frac{n}{1-2\mu} \left(\frac{L}{2} + 2\kappa_\Delta \right) \big[(\varsigma + (k+1)\kappa_g^2)^{1-2\mu} - \varsigma^{1-2\mu} \big]\end{aligned}$$

and thus, since $(k+2)^\mu \leq 2(k+1)^\mu$, that

$$\begin{aligned}(k+1)\mathbb{E}\left[\underset{j \in \{0,\dots,k\}}{\text{average}} \|G_j\|^2 \right] &\leq \sum_{j=0}^{k} \mathbb{E}\big[\|G_j\|^2\big] &\quad (51)\\ &\leq \frac{2\kappa_g^{2\mu}(F(x_0) - F_{\rm low})}{(1-\frac{\mu}{2})\gamma_{\rm low}(k+1)^{-\mu}} &\quad (52)\\ &\quad + \frac{n\left[(\varsigma + \kappa_g^2(k+1))^{1-2\mu} - \varsigma^{1-2\mu} \right]}{(1-2\mu)(k+1)^{-\mu}} \left(\frac{\kappa_g^{2\mu}\Big(L + 4\kappa_\Delta\Big)}{\gamma_{\rm low}(1-\frac{\mu}{2})} \right),\end{aligned}$$

which is (16).
If $\mu = \frac{1}{2}$, we reuse (50) and Lemma 3.4 for each $s_{i,j}^L$ with $\alpha = 1$, and derive that, in this case,

$$\mathbb{E}[F(x_{k+1})] \leq F(x_0) - \frac{3}{4} \frac{\gamma_{\rm low}}{\sqrt{(k+2)}\kappa_g} \sum_{j=0}^{k} \mathbb{E}\big[\|G_j\|^2\big] + n \left(\frac{L}{2} + 2\kappa_\Delta \right) \log \left(1 + (k+1)\frac{\kappa_g^2}{\varsigma} \right).$$

By a reasoning similar to that leading to (51) we now obtain that

$$\begin{aligned}(k+1)\mathbb{E}\left[\underset{j \in \{0,\dots,k\}}{\text{average}} \|G_j\|^2 \right] &\leq \sum_{j=0}^{k} \mathbb{E}\big[\|G_j\|^2\big] \\ &\leq \left(\frac{4}{3} \right) \frac{2\kappa_g(F(x_0) - F_{\rm low})\sqrt{(k+1)}}{\gamma_{\rm low}} \\ &\quad + \left(\frac{4n}{3} \right) \frac{\kappa_g}{\gamma_{\rm low}} (L + 4\kappa_\Delta) \log \Big(1 + (k+1)\frac{\kappa_g^2}{\varsigma} \Big) \sqrt{(k+1)}.\end{aligned}$$

Rearranging the terms yields (17).

Finally, if $\mu \in (\frac{1}{2}, 1)$, we again reuse (50) and Lemma 3.4 for each $s_{i,j}^L$ with $\alpha = 2\mu > 1$, and deduce that

$$\mathbb{E}[F(x_{k+1})] \le F(x_0) - (1 - \frac{\mu}{2})\frac{\gamma_{\text{low}}}{(k+2)^{\mu}\kappa_g^{2\mu}} \sum_{j=0}^{k} \mathbb{E}\left[\|G_j\|^2\right] + \left(\frac{L}{2} + 2\kappa_\Delta\right)\frac{n}{2\mu - 1}\left(\varsigma^{1-2\mu} - (\varsigma + \kappa_g^2(k+1))^{1-2\mu}\right).$$

Following the same argument as above yields that

$$(k+1)\mathbb{E}\left[\underset{j\in\{0,\ldots,k\}}{\text{average}} \|G_j\|^2\right] \le \sum_{j=0}^{k} \mathbb{E}\left[\|G_j\|^2\right] \le \frac{2\kappa_g^{2\mu}(F(x_0) - F_{\text{low}})}{(1-\frac{\mu}{2})\gamma_{\text{low}}(k+1)^{-\mu}} + \frac{n}{2\mu-1}\left(\frac{\kappa_g^{2\mu}\left(L + 4\kappa_\Delta\right)}{\gamma_{\text{low}}(1-\frac{\mu}{2})}\right) \times \frac{\varsigma^{1-2\mu} - (\varsigma + \kappa_g^2(k+1))^{1-2\mu}}{(k+1)^{-\mu}}.$$

Rearranging the terms gives (18).

Proof of Theorem 4.1

By using (2) and the bounds on γ_j, we derive that

$$F(x_{j+1}) \le F(x_j) + \gamma_j G_j^T s_j^L + \frac{L}{2}\gamma_j^2\|s_j^L\|^2 \le F(x_j) + \gamma_j G_j^T s_j^L + \frac{L}{2}\|s_j^L\|^2. \tag{53}$$

We now derive an upper bound on the expectation $\mathbb{E}_j\left[\frac{-\gamma_j G_{i,j} g_{i,j}}{w_{i,j}}\right]$ using an argument similar to that used in the beginning of the proof of Lemma 3.3.

Consider first the case where $G_{i,j}g_{i,j} < 0$ then, $\frac{-\gamma_j G_{i,j} g_{i,j}}{w_{i,j}} \le \frac{-G_{i,j} g_{i,j}}{\varsigma(j+1)^{\nu}}$ as $\gamma_j \le 1$ and $\varsigma(j+1)^{\nu} \le w_{i,j}$ (by using (29) and the bounds on $\rho_{i,k}$). Hence,

$$\mathbb{E}_j\left[\frac{-\gamma_j G_{i,j} g_{i,j}}{w_{i,j}}\right] \le -\frac{G_{i,j}^2}{\varsigma(j+1)^{\nu}}. \tag{54}$$

Otherwise (i.e. if $G_{i,j}g_{i,j} \ge 0$), then, $\frac{-\gamma_j G_{i,j} g_{i,j}}{w_{i,j}} \le \frac{-\gamma_{\text{low}} G_{i,j} g_{i,j}}{\kappa_\xi (j+1)^{\mu}}$ because $\gamma_{\text{low}} \le \gamma_j$ and $w_{i,j} \le \kappa_\xi (j+1)^{\mu}$ (by using (29) and the bounds on $\xi_{i,k}$). As a consequence, we have that

$$\mathbb{E}_j\left[\frac{-\gamma_j G_{i,j} g_{i,j}}{w_{i,j}}\right] \le -\frac{\gamma_{\text{low}} G_{i,j}^2}{\kappa_\xi (j+1)^{\mu}}, \tag{55}$$

Defining now $\kappa_\gamma \stackrel{\text{def}}{=} \frac{\gamma_{\text{low}}}{\kappa_\xi}$, noting that $\kappa_\gamma \le \frac{1}{\varsigma}$ and combining (54) and (55) yields that

$$\mathbb{E}_j\left[\frac{-\gamma_j G_{i,j} g_{i,j}}{w_{i,j}}\right] \le -\frac{\kappa_\gamma G_{i,j}^2}{(j+1)^{\mu}},$$

so that, taking the conditional expectation of (53), the last inequality and using Assumption 2.4,

$$\begin{aligned}
\mathbb{E}_j[F(x_{j+1})] &\le F(x_j) + \sum_{i=1}^n \mathbb{E}_j\left[\gamma_j G_{i,j} s_{i,j}^L\right] + \frac{L}{2}\mathbb{E}_j\left[\|s_j^L\|^2\right], \\
&\le F(x_j) + \sum_{i=1}^n \mathbb{E}_j\left[\gamma_j G_{i,j}\frac{g_{i,j}}{w_{i,j}}\right] + \frac{L}{2}\mathbb{E}_j\left[\frac{g_{i,j}^2}{w_{i,j}^2}\right], \\
&\le F(x_j) - \sum_{i=1}^n \frac{\kappa_\gamma G_{i,j}^2}{(j+1)^\mu} + \frac{L}{2}\mathbb{E}_j\left[\frac{g_{i,j}^2}{w_{i,j}^2}\right], \qquad (56) \\
&\le F(x_j) - \sum_{i=1}^n \frac{\kappa_\gamma G_{i,j}^2}{(j+1)^\mu} + \frac{L\kappa_g^2}{2}\mathbb{E}_j\left[\frac{1}{w_{i,j}^2}\right].
\end{aligned}$$

Using now (29) and the bounds on $\rho_{i,k}$ yields that

$$\mathbb{E}_j[F(x_{j+1})] \le F(x_j) - \sum_{i=1}^n \kappa_\gamma \frac{G_{i,j}^2}{(j+1)^\mu} + \frac{nL\kappa_g^2}{2\varsigma^2(j+1)^{2\nu}}.$$

Summing over all iterations from 0 to k and taking the full expectation gives that

$$\mathbb{E}[F(x_{k+1})] \le F(x_0) - \kappa_\gamma \sum_{j=0}^k \sum_{i=1}^n \frac{\mathbb{E}\left[G_{i,j}^2\right]}{(j+1)^\mu} + \frac{nL\kappa_g^2}{2\varsigma^2}\sum_{j=0}^k \frac{1}{(j+1)^{2\nu}}.$$

If we now define

$$\phi_\nu(x) \stackrel{\text{def}}{=} \begin{cases} \dfrac{(x+1)^{1-2\nu}-1}{1-2\nu} & \text{if} \quad \nu \neq \frac{1}{2} \\ \log(x+1) & \text{otherwise,} \end{cases}$$

we may bound the last inequality, using a simple sum-integral comparison and Assumption 2.3 to obtain that

$$\sum_{j=0}^k \sum_{i=1}^n \mathbb{E}\left[G_{i,j}^2\right] \le \frac{(k+1)^\mu (F(x_0) - F_{\text{low}})}{\kappa_\gamma} + \frac{nL\kappa_g^2 (k+1)^\mu}{2\varsigma^2 \kappa_\gamma}(1+\phi_\nu(k)).$$

Substituting κ_γ by its value gives then,

$$\mathbb{E}\left[\operatorname*{average}_{j\in\{0,\dots,k\}} \|G_j\|^2\right] \le \frac{\kappa_\xi (F(x_0)-F_{\text{low}})}{\gamma_{\text{low}}(k+1)^{1-\mu}} + \frac{nL\kappa_\xi\kappa_g^2}{2\varsigma^2\gamma_{\text{low}}}\left(\frac{1+\phi_\nu(k)}{(k+1)^{1-\mu}}\right).$$

This gives (31) when $\nu = \frac{1}{2}$. Otherwise, (30) follows by using the fact that

$$1 + \phi_\nu(k) = \frac{1}{1-2\nu}\left[\frac{1}{(k+1)^{2\nu-1}} - 2\nu\right].$$

Proof of Theorem 4.2

Proof. To simplify notation, set, for the course of this proof, $w_{i,-1} = \varsigma$, $i \in \{1, \dots, n\}$, $\frac{0}{0} = 1$ and (as in the previous theorem) $\kappa_\gamma = \gamma_{\rm low}/\kappa_\xi$. As in the proof of Theorem 4.1, we derive (see (56)) that

$$\begin{aligned}
\mathbb{E}_j[F(x_{j+1})] &\leq F(x_j) - \sum_{i=1}^n \frac{\kappa_\gamma G_{i,j}^2}{(j+1)^\mu} + \frac{L}{2}\mathbb{E}_j\left[\frac{g_{i,j}^2}{w_{i,j}^2}\right] \\
&\leq F(x_j) - \sum_{i=1}^n \frac{\kappa_\gamma G_{i,j}^2}{(j+1)^\mu} + \frac{L}{2}\mathbb{E}_j\left[\frac{g_{i,j}^2}{w_{i,j-1}^2}\left(\frac{w_{i,j-1}}{w_{i,j}}\right)^2\right] \\
&\leq F(x_j) - \sum_{i=1}^n \frac{\kappa_\gamma G_{i,j}^2}{(j+1)^\mu} + \frac{L\kappa_\xi^2 j^{2\mu}}{2\varsigma^2 j^{2\nu}}\mathbb{E}_j\left[\frac{g_{i,j}^2}{w_{i,j-1}^2}\right] \\
&\leq F(x_j) - \sum_{i=1}^n \frac{\kappa_\gamma G_{i,j}^2}{(j+1)^\mu} + \frac{L\kappa_\xi^2 j^{2\mu}}{2\varsigma^2 j^{2\nu} w_{i,j-1}^2}(1+\kappa_{\rm var})G_{i,j}^2,
\end{aligned}$$

where we have used the fact that $\left(\frac{w_{i,j-1}}{w_{i,j}}\right)^2 \leq \frac{\kappa_\xi^2 j^{2\mu}}{\varsigma^2 j^{2\nu}}$ (because of (29)), the measurability of $w_{i,j-1}$ w.r.t the past and (19) to deduce the last inequality. Using now the bound $\frac{(j+1)^\mu}{w_{i,j-1}} \leq \frac{2^\mu j^\mu}{\varsigma j^\nu}$ and summing over the iterations for $j \in \{0, \dots, k\}$ then yields that

$$\sum_{j=0}^k \mathbb{E}_j[F(x_{j+1})] \leq \sum_{j=0}^k F(x_j) + \sum_{j=0}^k \sum_{i=1}^n \frac{G_{i,j}^2}{(j+1)^\mu}\left(-\kappa_\gamma + \frac{\widehat{\kappa}\, j^{3\mu}}{w_{i,j-1}\, j^{3\nu}}\right), \tag{57}$$

with $\widehat{\kappa} = \frac{L\kappa_\xi^2}{2^{1-\mu}\varsigma^3}(1+\kappa_{\rm var})$. Note now that the definition of j_θ in (34), the fact that $4\nu > 3\mu$ and that $w_{i,j-1} \geq \varsigma\, j^\nu$ together imply that

$$\left(-\kappa_\gamma + \frac{\widehat{\kappa}\, j^{3\mu}}{w_{i,j-1}\, j^{3\nu}}\right) \leq -\theta, \tag{58}$$

for $j \geq j_\theta$. Hence, from (57),

$$\sum_{j=0}^k \mathbb{E}_j[F(x_{j+1})] \leq \sum_{j=0}^k F(x_j) - \theta \sum_{j=j_\theta}^k \sum_{i=1}^n \frac{G_{i,j}^2}{(j+1)^\mu} + \sum_{j=0}^{j_\theta-1} \sum_{i=1}^n \frac{G_{i,j}^2}{(j+1)^\mu}\left(-\kappa_\gamma + \frac{\widehat{\kappa}\, j^{3\mu}}{w_{i,j-1}\, j^{3\nu}}\right), \tag{59}$$

while the last term of the previous equation is bounded by

$$\sum_{j=0}^{j_\theta-1} \sum_{i=1}^n \frac{G_{i,j}^2}{(j+1)^\mu}\left(-\kappa_\gamma + \frac{\widehat{\kappa}\, j^{3\mu}}{w_{i,j}\, j^{3\nu}}\right) \leq \sum_{j=0}^{j_\theta-1} \sum_{i=1}^n \widehat{\kappa}\frac{G_{i,j}^2}{\varsigma} \leq \frac{n\kappa_g^2\widehat{\kappa}}{\varsigma} j_\theta, \tag{60}$$

where we used the facts that $\|G\|_\infty \leq \kappa_g$ (because of (6)), $w_{i,j} \geq \varsigma$ (because of (29)) and $3\nu > \frac{9}{4}\mu > 2\mu$ (because of the bounds $\nu \leq \mu < \frac{4}{3}\nu$). Injecting (60) in (59), we deduce that

$$\theta \sum_{j=j_\theta}^k \sum_{i=1}^n \frac{G_{i,j}^2}{(j+1)^\mu} \leq \sum_{j=0}^k F(x_j) - \sum_{j=0}^k \mathbb{E}_j[F(x_{j+1})] + \frac{n\kappa_g^2\widehat{\kappa}}{\varsigma} j_\theta.$$

Taking the full expectation then gives that

$$(k - j_\theta)\mathbb{E}\left[\underset{j\in\{j_\theta+1,\ldots,k\}}{\text{average}} \|G_j\|^2\right] \leq \mathbb{E}\left[\sum_{j=j_\theta}^{k}\sum_{i=1}^{n} G_{i,j}^2\right] \leq \frac{(k+1)^\mu}{\theta}\left[F(x_0) - F_{\text{low}} + \frac{n\kappa_g^2\widehat{\kappa}}{\varsigma} j_\theta\right]. \tag{61}$$

which gives the desired result.