CONFERENCE ON COMPLEX SYSTEMS
PALMA DE MALLORCA
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Nonlinear diffusion in regular graphs

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Joint work with…

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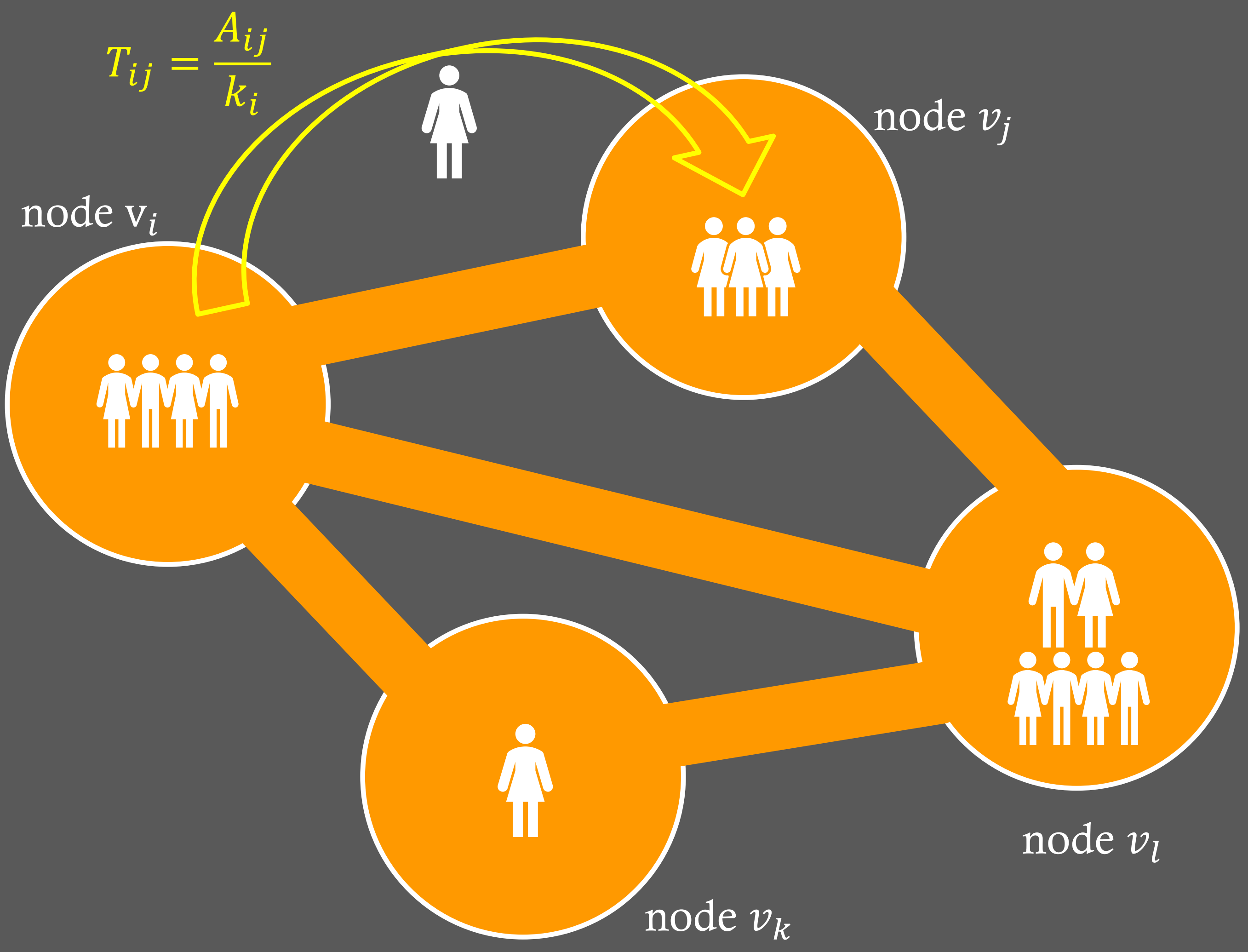
Summary

- Standard random walk on networks
- Random walk with crowding and “social” behaviour
- Emergence of empty nodes and mass segregation:
  - Heterogenous network structure
  - Ring
  - Complete graph

Random walk on networks



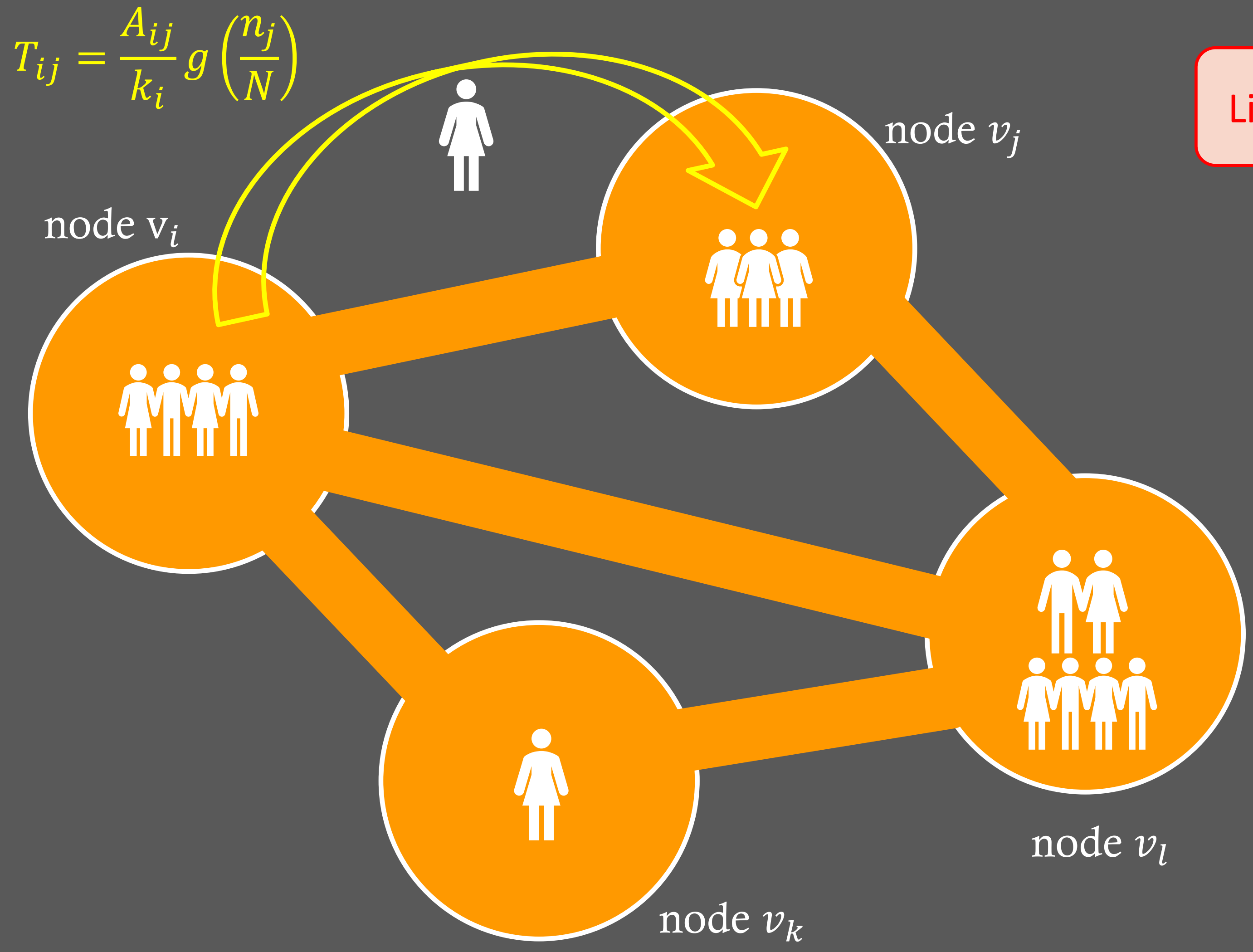
$\rho = [\rho_1, \rho_2, \cdots, \rho_\Omega]^T$

$\dot{\rho} = L\rho$ with $L_{ij} = \frac{A_{ij}}{k_j} - \delta_{ij}$

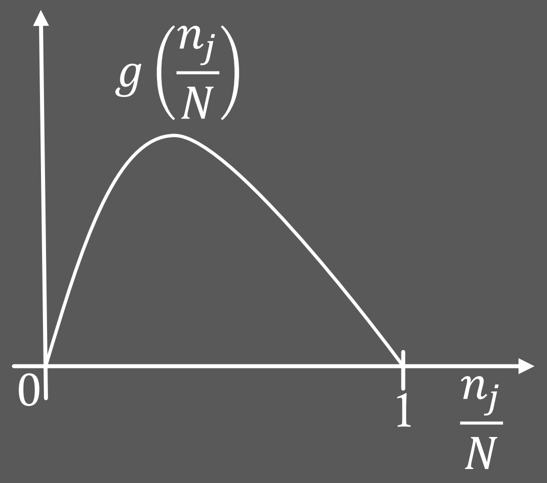
$\rho_i^* \propto k_i$

Random walk on networks with crowding and aggregation

$$T_{ij} = \frac{A_{ij}}{k_i} g\left(\frac{n_j}{N}\right)$$



Limiting carrying capacity: at most N agents per node



$g(0) = 0$: "social" aspect
$g(1) = 0$: crowding

Random walk on networks with crowding and aggregation

$$\rho_i := \lim_{N \to +\infty} \left\langle \frac{n_i}{N} \right\rangle$$

$$\dot{\rho}_i = \sum_{j=1}^{\Omega} \left[-\frac{A_{ij}}{k_i} \rho_i g(\rho_j) + \frac{A_{ji}}{k_j} \rho_j g(\rho_i) \right] = \sum_{j=1}^{\Omega} L_{ij} \left[\rho_j g(\rho_i) - \frac{k_j}{k_i} \rho_i g(\rho_j) \right] \quad \text{with } g(x) = x(1-x)$$

Mass conservation $\sum_i \dot{\rho}_i = 0$

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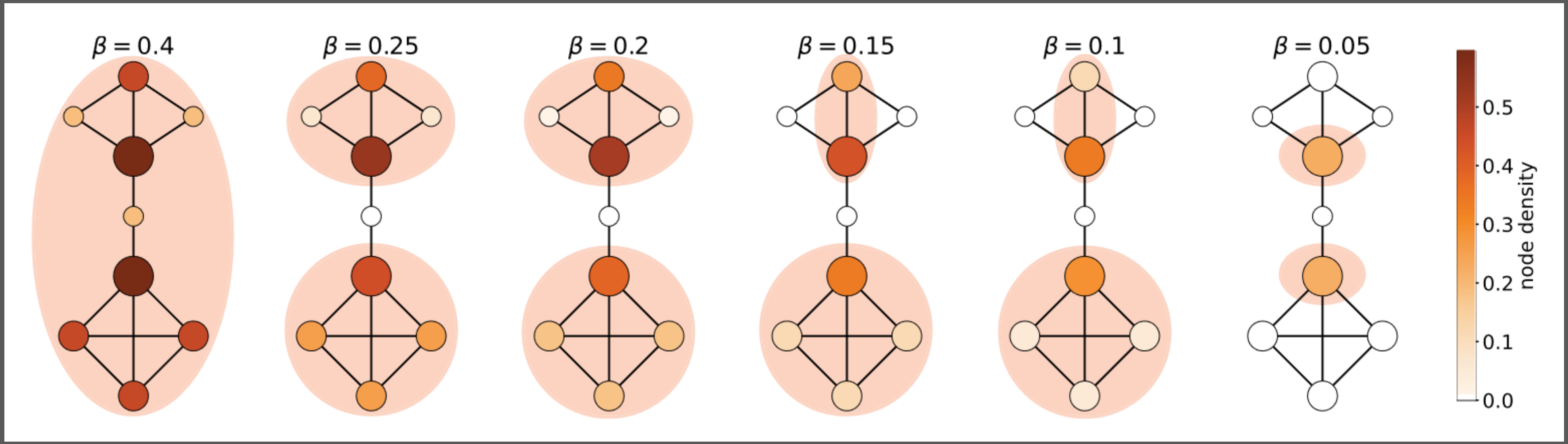
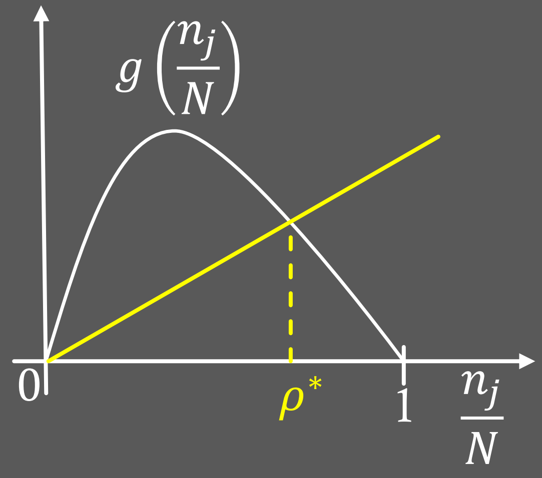
Emergence of empty nodes



Stationary densities: $\rho_j^* g(\rho_i^*) - \frac{k_j}{k_i} \rho_i^* g(\rho_j^*) = 0$

$\Longleftrightarrow \rho_i^* = 0$ or $g(\rho_i^*) = \frac{c_\gamma}{k_i} \rho_i^*$ (γ^{th} (sub)community)



The case of regular networks

$\dot{\rho}_i = \sum_{j=1}^{\Omega} L_{ij}[\rho_j g(\rho_i) - \rho_i g(\rho_j)]$ with $L_{ij} = \frac{A_{ij}}{k} - \delta_{ij}$

Stability of the homogeneous fixed point ?

Let $\rho_i = \beta + \delta_i$ with $\beta \in (0,1)$ and $\delta_i = o(1)$

$\Longrightarrow$ $\dot{\delta} = J\delta$ with $J = \underbrace{[g(\beta) - \beta g'(\beta)]}_{\xi(\beta)} L$

Semi-definite negative matrix

$\Longrightarrow$ Stability condition: $\xi(\beta) > 0$

Ring

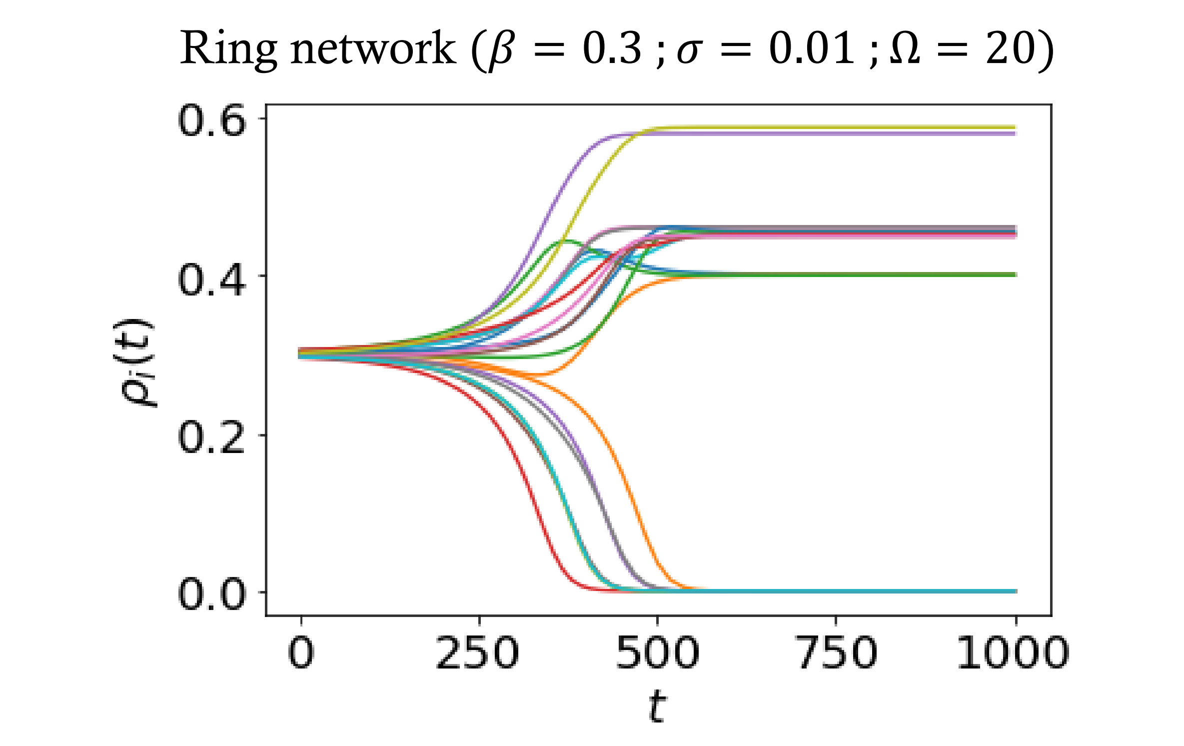
$$g(x) = x^a(1-x)^a \, , a \in \mathbb{R}^+$$

If $a \leq 1$: the homogeneous state is always stable

If $a > 1$: the homogeneous state is stable $\Longleftrightarrow$ $\beta > \dfrac{a-1}{2a-1}$

$a = 2 \implies$ Homogeneous state unstable if $\beta < \frac{1}{3}$

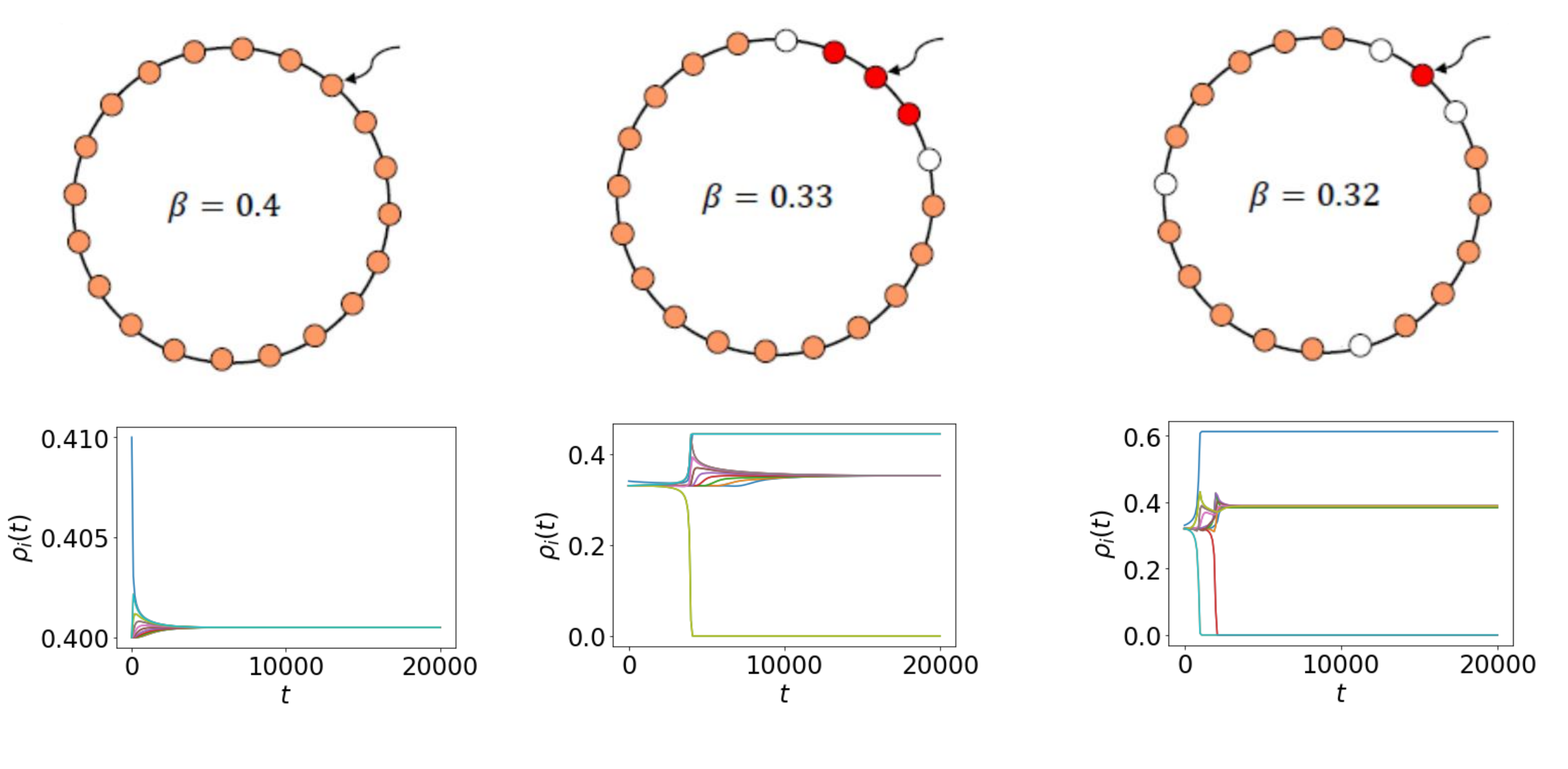
$$\rho_i(0) = p(\beta + \sigma\, r_i) \quad \text{with} \begin{cases} \sum_i \rho_i(0) = \Omega\beta \\ \sigma \ll 1 \\ r_i \sim U([0,1]) \end{cases}$$



Ring

$\rho_i(0) = \beta + 0.01\delta_{i,1}\ (i = 1, \cdots, 20)$ $g(x) = x^2(1-x)^2$

↙ perturbation



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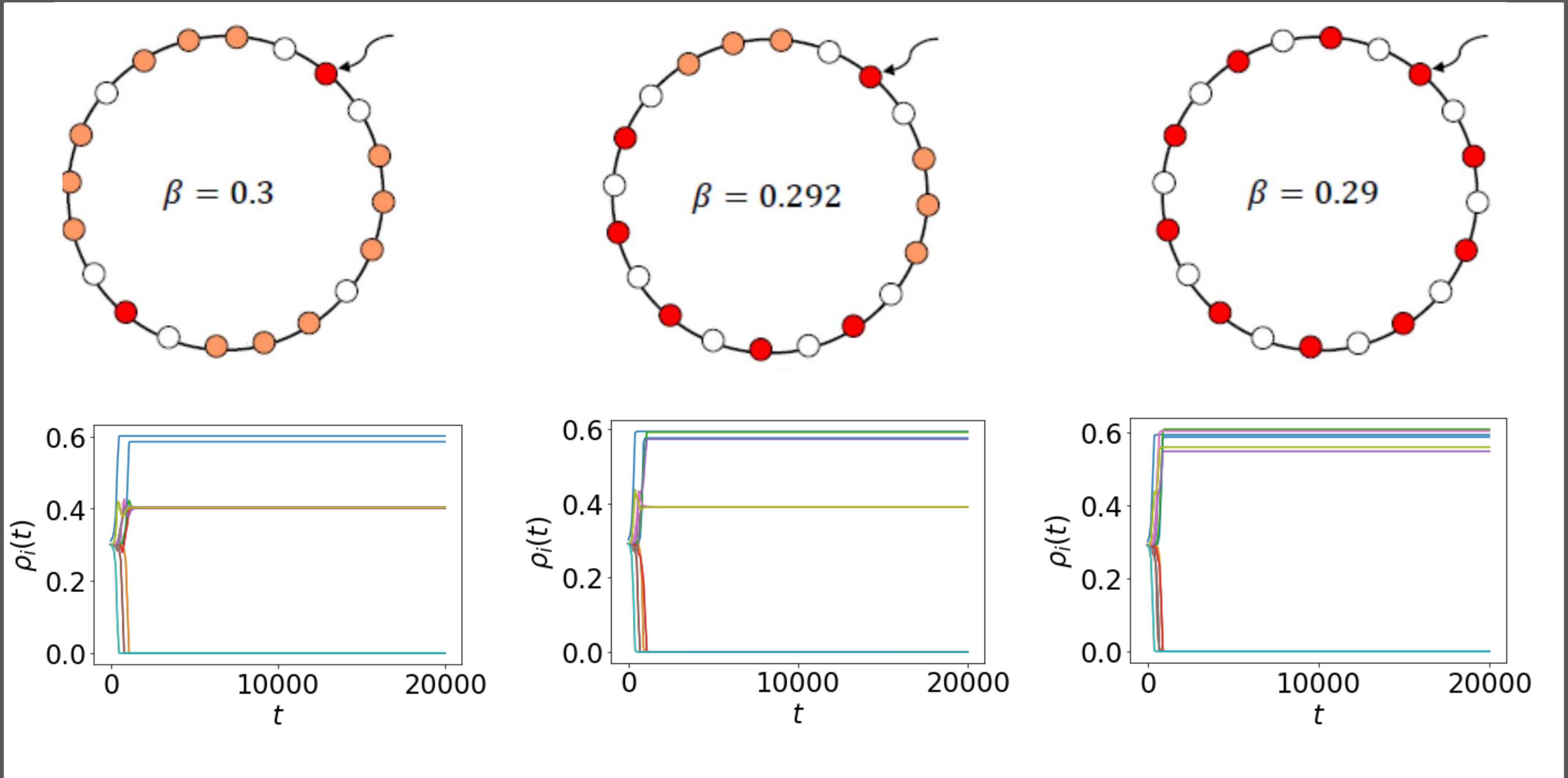
Ring

$\rho_i(0) = \beta + 0.01\delta_{i,1}\ (i = 1, \cdots, 20)$ $\qquad g(x) = x^2(1-x)^2$

perturbation



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Complete graph

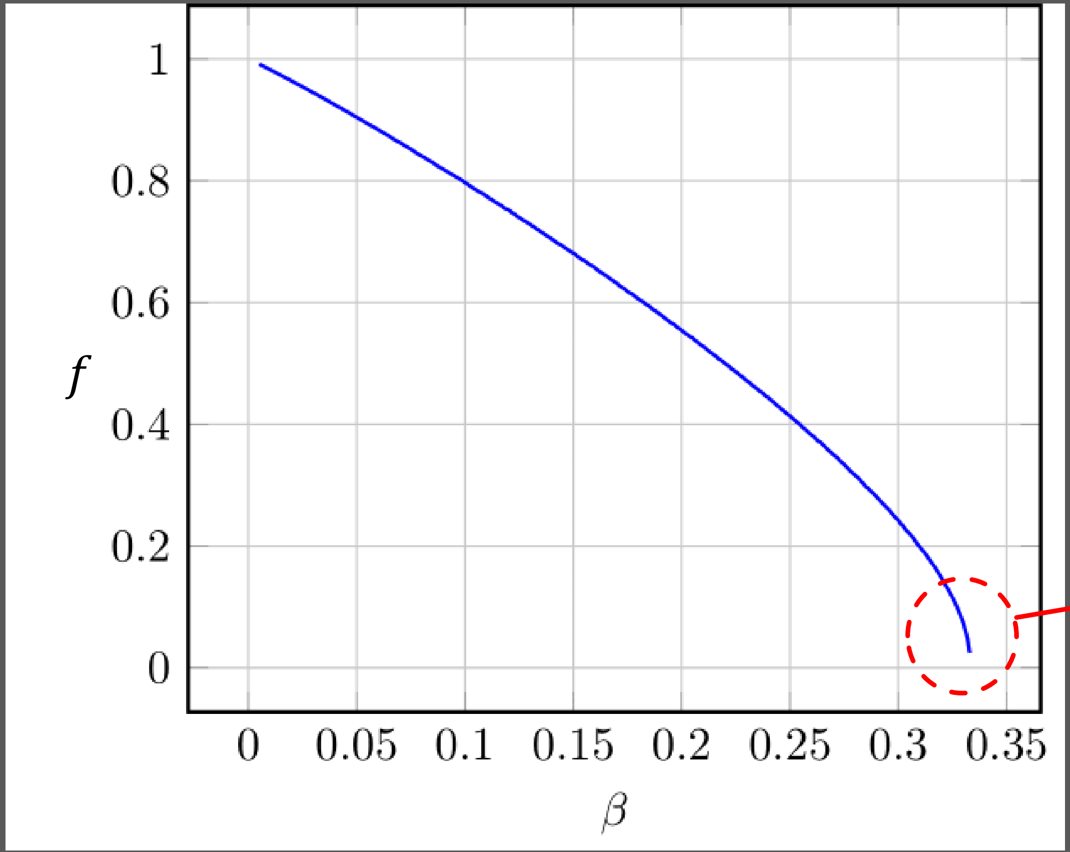
$\dot{\rho}_i = \frac{\beta\Omega}{\Omega-1} g(\rho_i) - \frac{\rho_i}{\Omega-1} \sum_{j=1}^{\Omega} g(\rho_j)$ with $g(x) = x^2(1-x)^2$

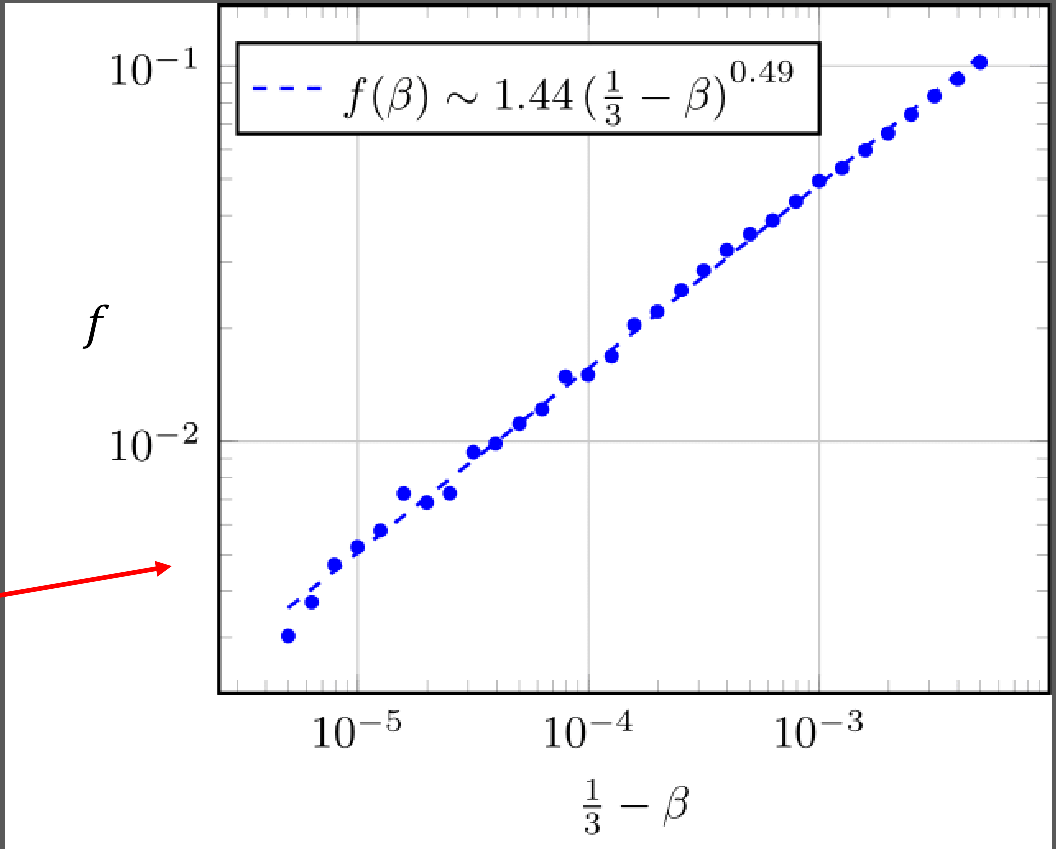
$$\frac{\Omega - 1}{\Omega} \dot{\rho}_i = \beta g(\rho_i) - \frac{\rho_i}{\Omega} \sum_{j=1}^{\Omega} g(\rho_j)$$

$\rho_i(0) < \rho_j(0) \Longrightarrow \rho_i(t) \leq \rho_j(t)\ \forall t \in \mathbb{R}^+$

Indeed if $\rho_i(T) = \rho_j(T)$ for some $T \in \mathbb{R}^+$, then $\dot{\rho}_i(\mathrm{T}) = \dot{\rho}_j(\mathrm{T})$ and thus $\rho_i(t) = \rho_j(t)\ \forall t \geq T$

Complete graph





5000 nodes ; $g(x) = x^2(1-x)^2$; $\rho_i(0) \in \left[\beta - \frac{\delta}{2}, \beta + \frac{\delta}{2}\right]$ with $\delta = 10^{-5}$

From local information to global information

Initial densities: $\rho_1(0) \leq \rho_2(0) \leq \cdots \leq \rho_\Omega(0)$

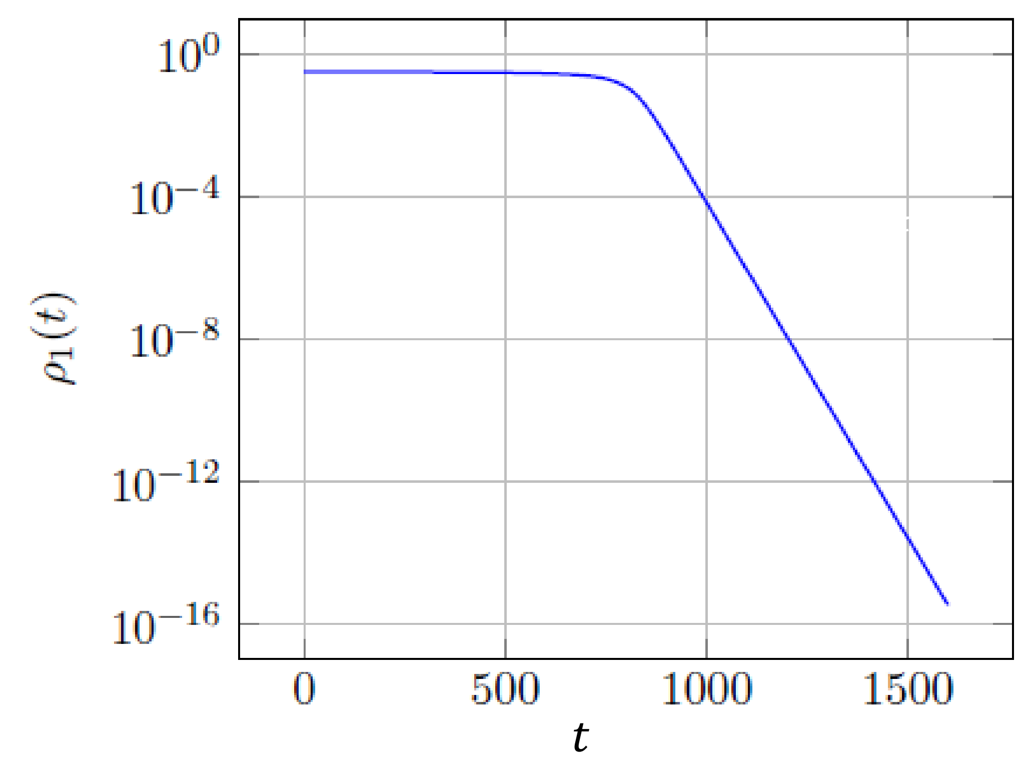
Asymptotically:

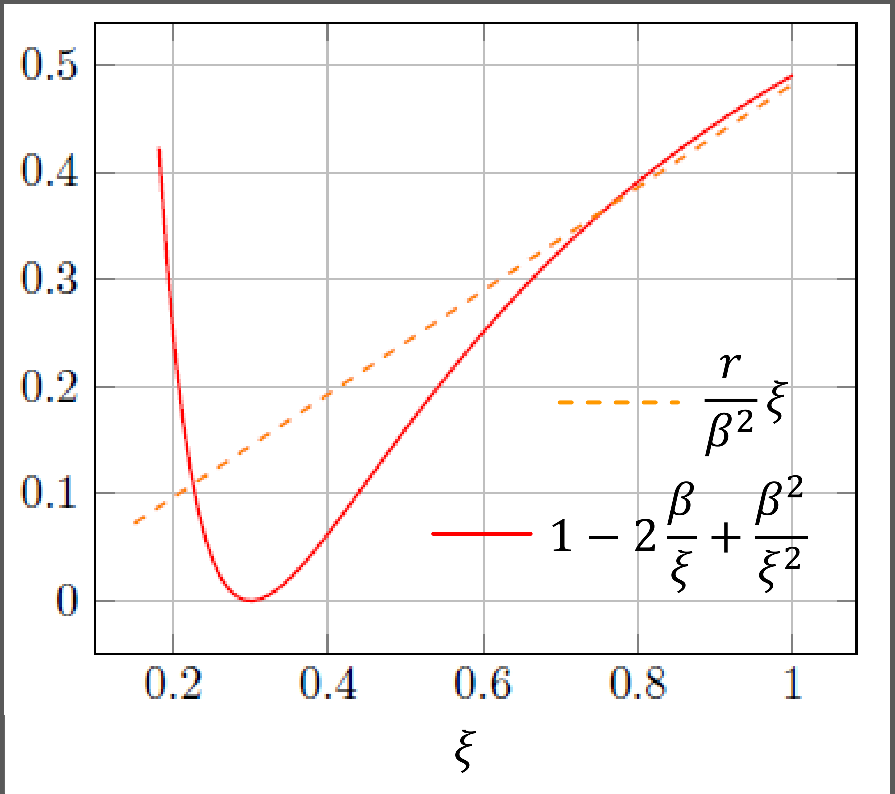
$\rho_i^* = 0$ or $\rho_i^* = \frac{\beta}{1-f}$

$\dot{\rho}_1 \approx -\rho_1(X_2 - 2X_3 + X_4)$ with $X_k = \dfrac{\beta^k}{(1-f)^{k-1}}$

$$r := \lim_{t\to+\infty} -\frac{\dot{\rho}_1}{\rho_1} \Longrightarrow \frac{r}{\beta^2}\xi = 1 - 2\frac{\beta}{\xi} + \frac{\beta^2}{\xi^2} \quad \text{with } \xi = 1 - f$$

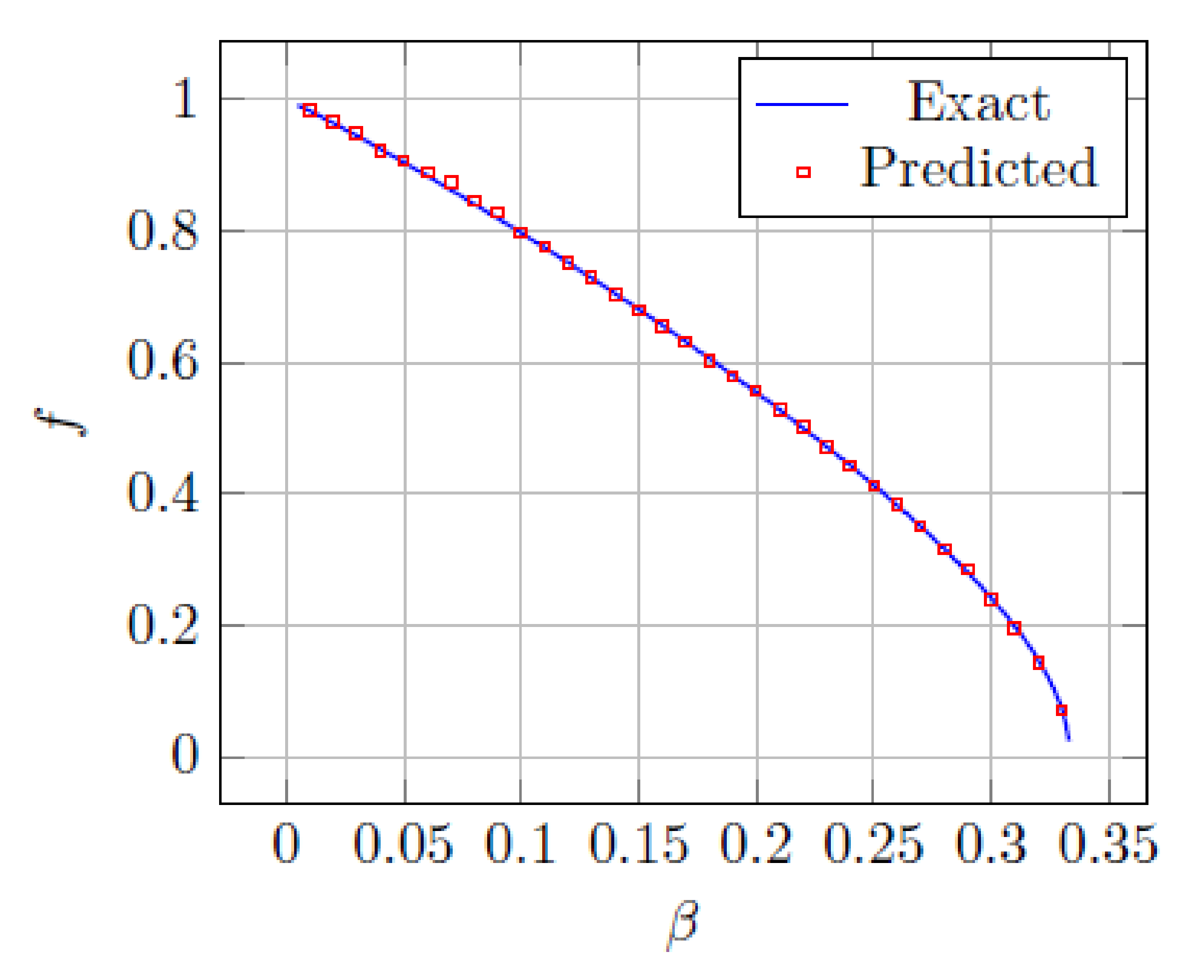
From local information to global information





10 000 nodes ; $\beta = 0.3$; $\xi = 1 - f$

From local information to global information



5000 nodes ; $g(x) = x^2(1-x)^2$; $\rho_i(0) \in \left[\beta - \frac{\delta}{2}, \beta + \frac{\delta}{2}\right]$ with $\delta = 10^{-5}$

Conclusion

- We have introduced a class of non-linear diffusion processes accounting for crowding and “social” behaviour

- In the steady state, empty nodes emerge leading to a segregation of the mass into multiple (sub)communities

- We provided the condition for the homogeneous state to be unstable and investigated how the fraction of empty nodes varies with the average node density

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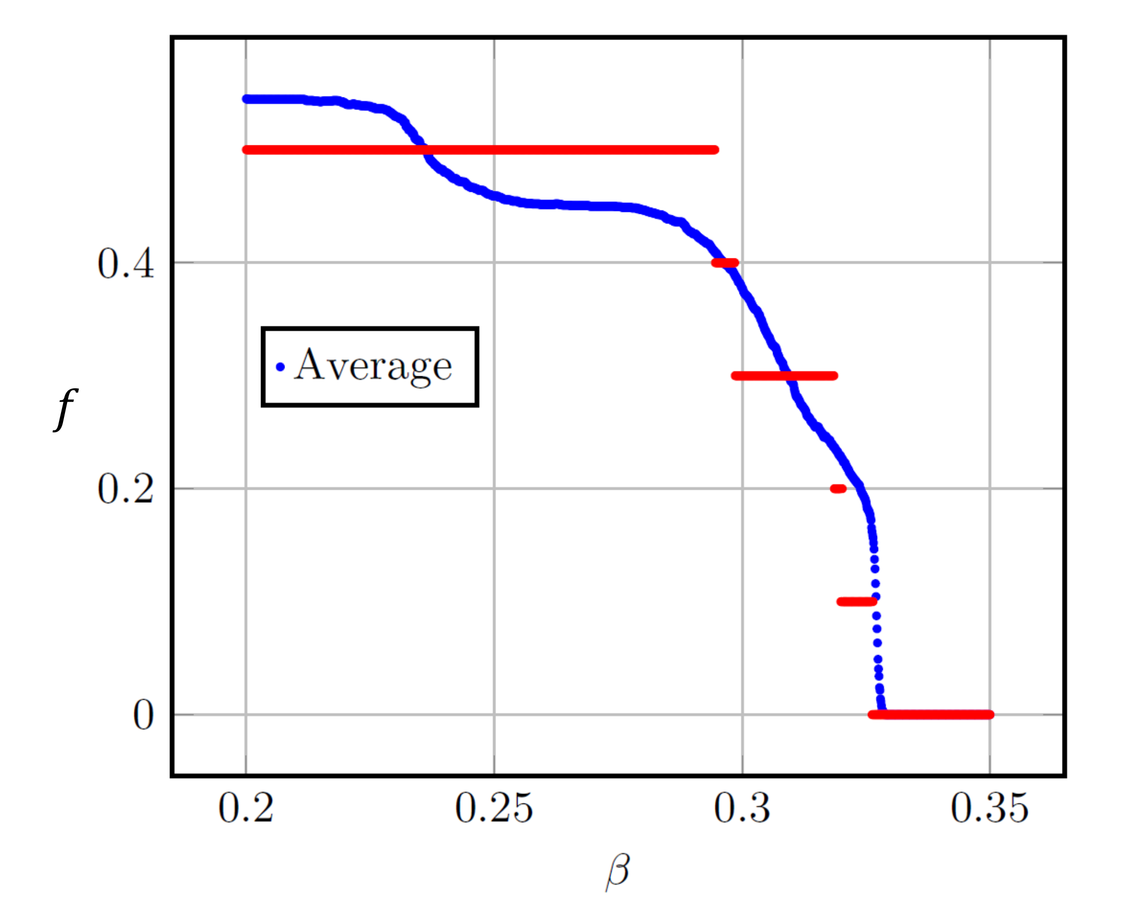
Thank you for your attention !

Appendix: fraction of empty nodes in the ring

$g(x) = x^2(1-x)^2$



$\rho_i(0) = \beta + 0.01\delta_{i,1}\ (i = 1, \cdots, 20)$

$\rho_i(0) = \beta + 10^{-4} r_i\ (i = 1, \cdots, 20)$
r_i ~normal distribution centred at 0 with variance 1