Solvay workshop in memory of Prof. Grégoire Nicolis on "Nonlinear phenomena and complex systems"

Symmetry breaking induced by self-recruitment random walks on regular networks

Jean-François de Kemmeter – 16/06/2021

Joint work with Timoteo Carletti and Malbor Asllani

UNIVERSITÉ DE NAMUR

naxys Namur Institute for Complex Systems

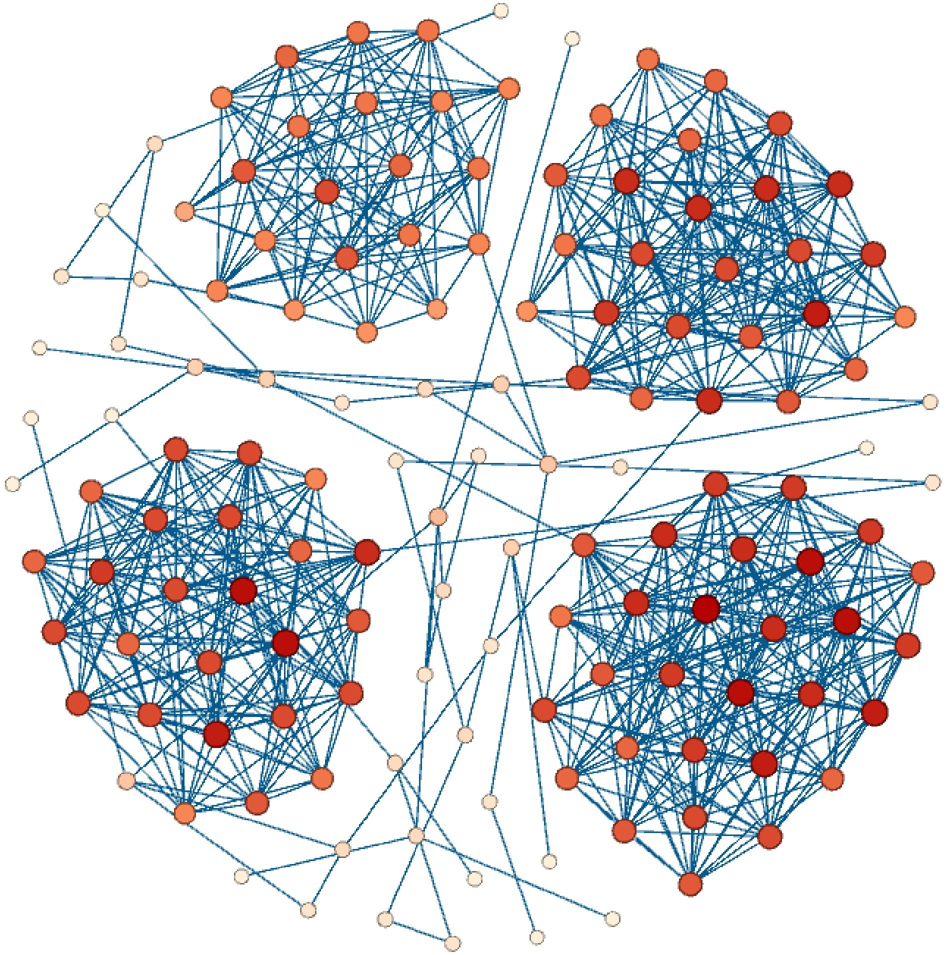
fnrs FREEDOM TO RESEARCH

UCD DUBLIN

Outline

- Linear random walk on top of complex networks
- Self-recruitment random-walk : from a microscopic point of view to the mean-field equations
- Emergence of functional communities
- Symmetry breaking and phase transition on lattices

Linear random walk on complex networks



$\dot{\boldsymbol{\rho}} = \boldsymbol{\rho} L_{RW}$ with $L_{RW} = D^{-1}A - \mathrm{I}$

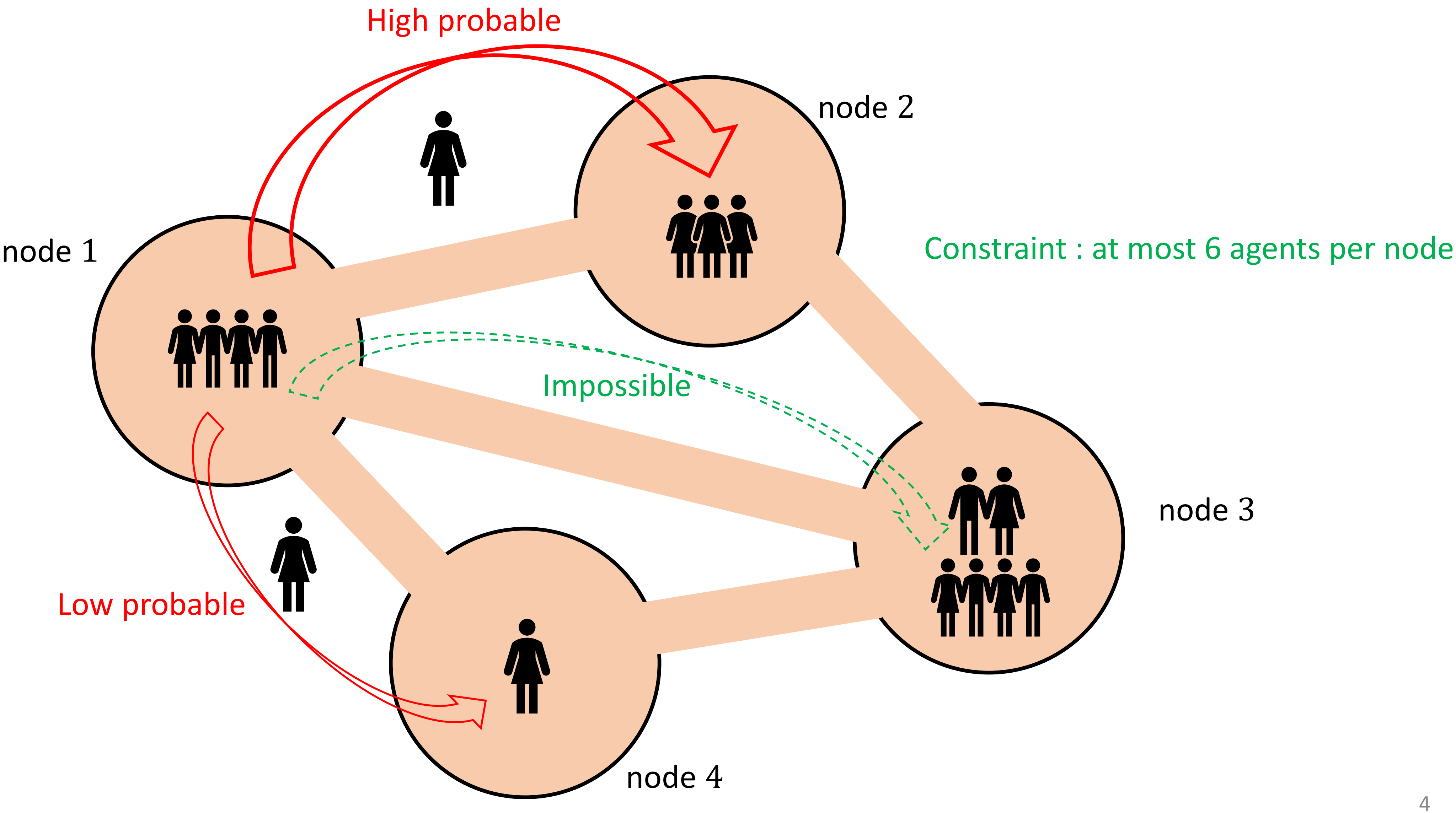
$$A_{ij} = \begin{cases} 1 \text{ if node } i \text{ is connected to node } j \\ 0 \text{ otherwise} \end{cases}$$

Asymptotically : $\rho_i^{\infty} = \frac{k_i}{\langle k \rangle \Omega}$

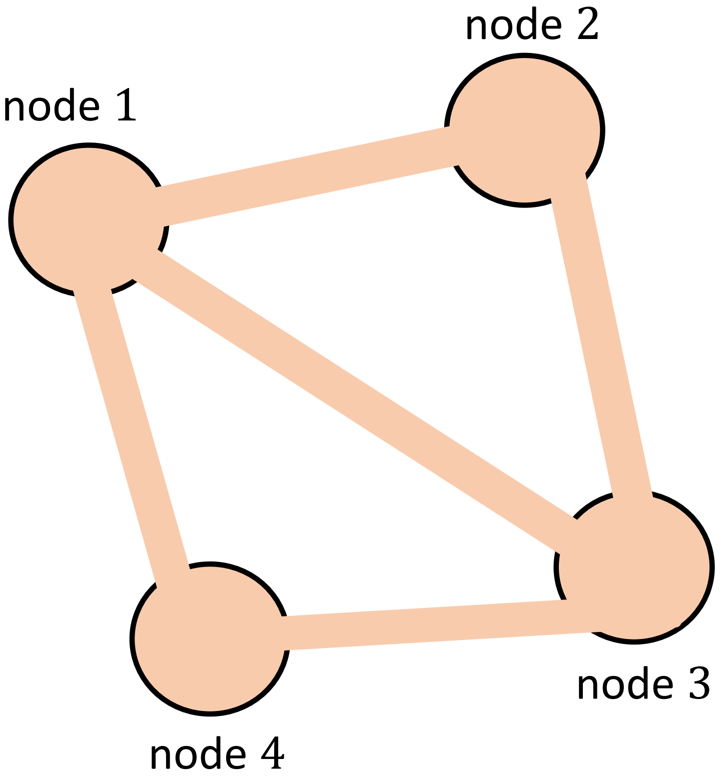
Ω: number of nodes

k_i: degree of node i

Self-recruitment random-walk



Master equation



n_i : number of agents in node i $(i = 1, \cdots, \Omega)$

$0 \leq n_i \leq N$

State of the system at time t

$\boldsymbol{n}(t) = (n_1(t), n_2(t), \cdots, n_\Omega(t))$

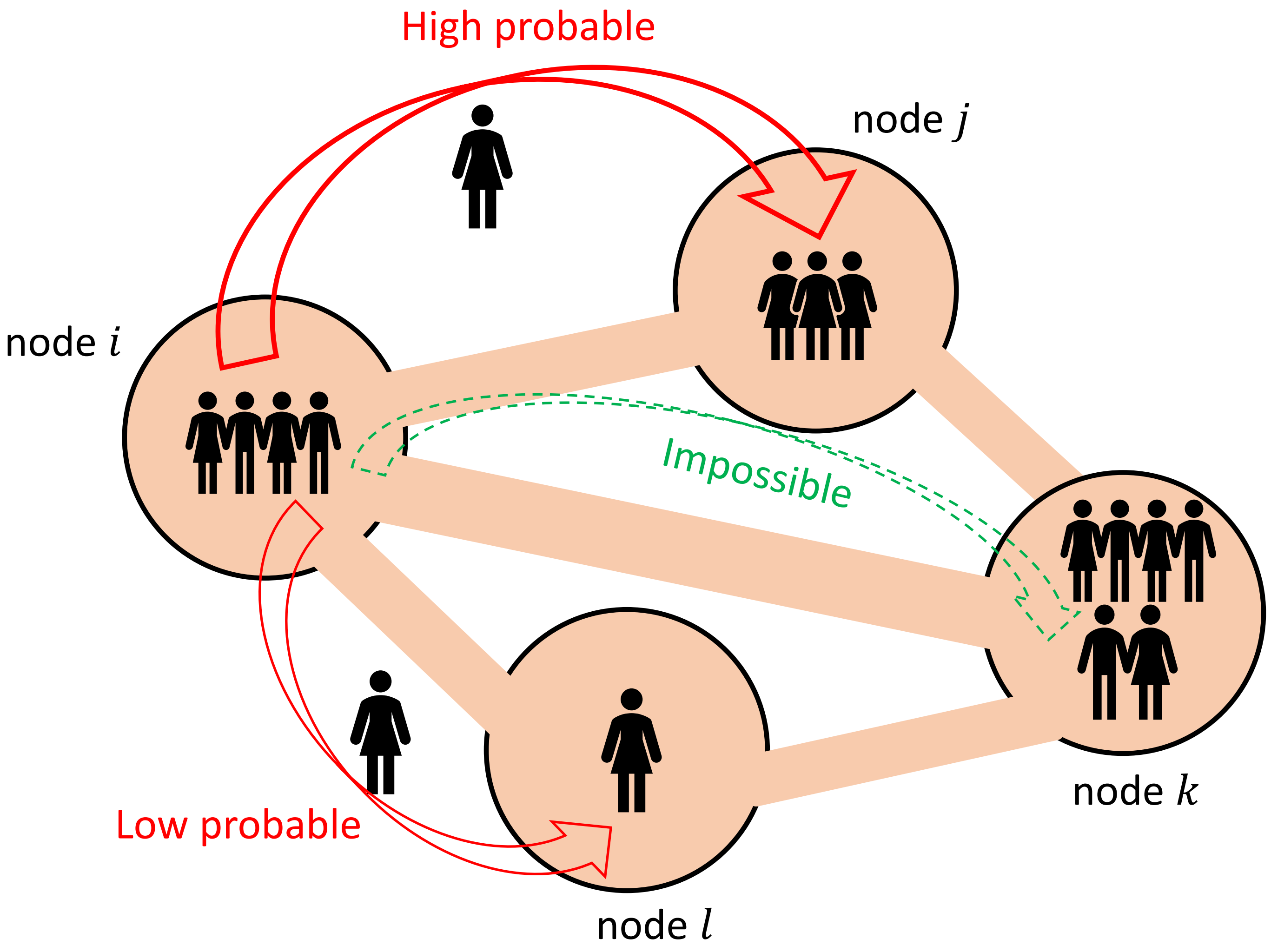
$P(\boldsymbol{n}, t)$: probability of observing state $\boldsymbol{n}$ at time t

Master equation: $$\frac{dP(\boldsymbol{n},t)}{dt} = \sum_{\boldsymbol{n}' \neq \boldsymbol{n}} T(\boldsymbol{n}|\boldsymbol{n}')P(\boldsymbol{n}',t) - T(\boldsymbol{n}'|\boldsymbol{n})\,P(\boldsymbol{n},t)$$

$T(\boldsymbol{n}'|\boldsymbol{n})$: transition probability from state $\boldsymbol{n}$ to state **n’**

[1] Van Kampen, Nicolaas Godfried. *Stochastic processes in physics and chemistry*. Vol. 1. Elsevier, 1992.

Transition matrix



Constraint : at most N agents per node

Willingness of agents to leave a node

$$T(n_i - 1, n_j + 1 | n_i, n_j) = \frac{A_{ij}}{k_i} f\left(\frac{n_i}{N}\right) g\left(\frac{n_j}{N}\right)$$

Ability of agents to settle in a node

$$\begin{cases} f\left(\frac{n_i}{N}\right) = \frac{n_i}{N} \\ g\left(\frac{n_j}{N}\right) = \left(\frac{n_j}{N}\right)^a \left(\frac{N - n_j}{N}\right)^b \end{cases} \qquad a, b > 0$$

[2] Asllani, M., Carletti, T., Di Patti, F., Fanelli, D., & Piazza, F. (2018). Hopping in the crowd to unveil network topology. *Physical review letters*, *120*(15), 158301.

[3] Carletti, T., Asllani, M., Fanelli, D., & Latora, V. (2020). Nonlinear walkers and efficient exploration of congested networks. *Physical Review Research*, *2*(3), 033012.

Mean-field equations

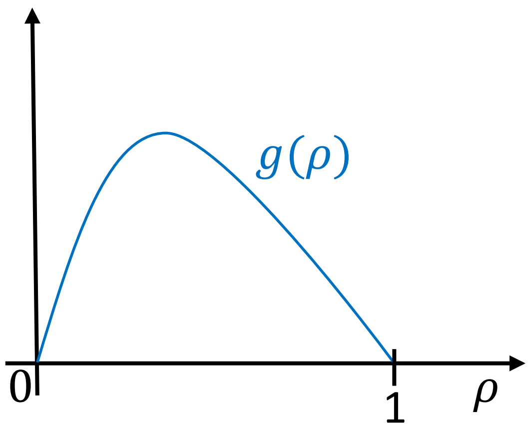
$$\rho_i := \lim_{N\to+\infty} \left\langle \frac{n_i}{N} \right\rangle$$

$$L_{ij} = \frac{A_{ij}}{k_i} - \delta_{ij}$$

$$\Rightarrow \quad \dot{\rho}_i = \sum_{j=1}^{N} \left[-\frac{A_{ij}}{k_i} \rho_i g(\rho_j) + \frac{A_{ji}}{k_j} \rho_j g(\rho_i) \right] = -\sum_{j=1}^{N} L_{ij} \left[\rho_i g(\rho_j) - \frac{k_i}{k_j} \rho_j g(\rho_i) \right]$$

Mass conservation: $\sum_i \dot{\rho}_i = 0$

$$g(\rho_j) = {\rho_j}^a (1 - \rho_j)^b$$



Emergence of empty nodes and metacommunities on heterogeneous networks

$$\dot{\rho}_i = -\sum_{j=1}^{N} L_{ij}\left[\rho_i g(\rho_j) - \frac{k_i}{k_j}\rho_j g(\rho_i)\right]$$

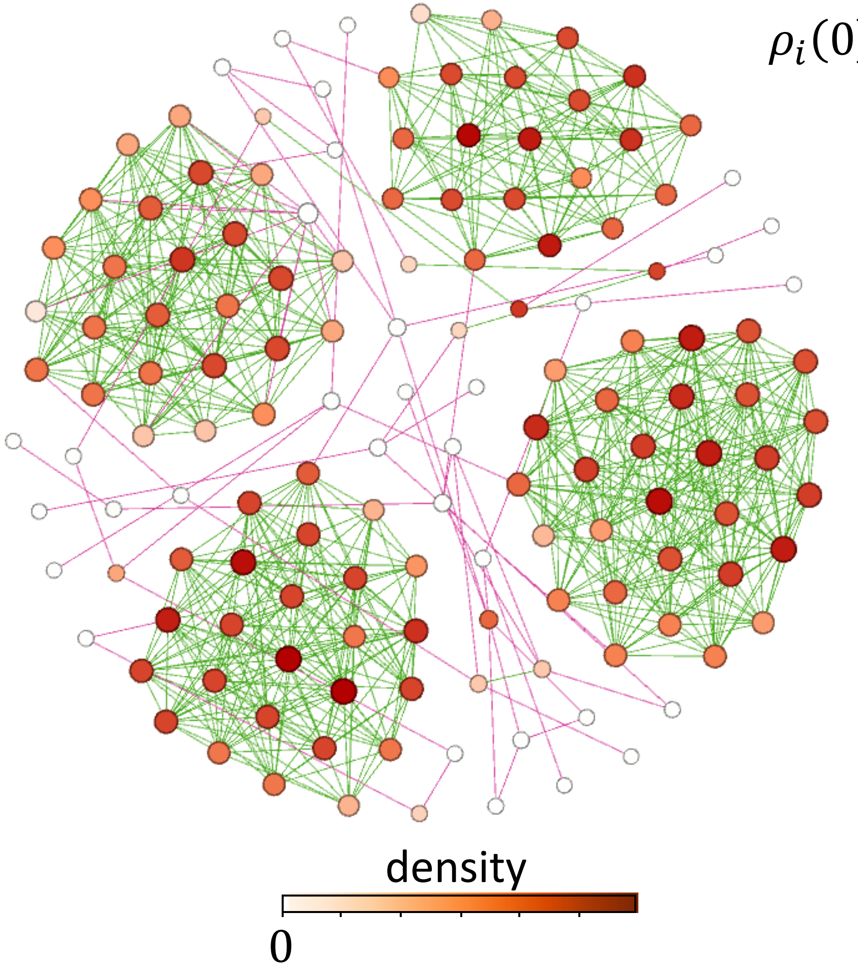
$$g(x) = x(1-x)$$

$$\Rightarrow \quad \dot{\rho}_i = -\rho_i \sum_{j=1}^{N} L_{ij}\rho_j\left[1 - \rho_j - \frac{k_i}{k_j}(1-\rho_i)\right]$$

Stationary solutions for crowded networks

$$\rho_i^{\infty} = 1 - \frac{c}{k_i} \text{ with } c = \frac{1-M/\Omega}{\left\langle \frac{1}{k} \right\rangle}$$

$\rho_i(0) = 0.25 \ \forall i$



Self-recruitment RW on top of lattices

$$\dot{\rho}_i = -\sum_{j=1}^{N} L_{ij}[\rho_i g(\rho_j) - \rho_j g(\rho_i)]$$

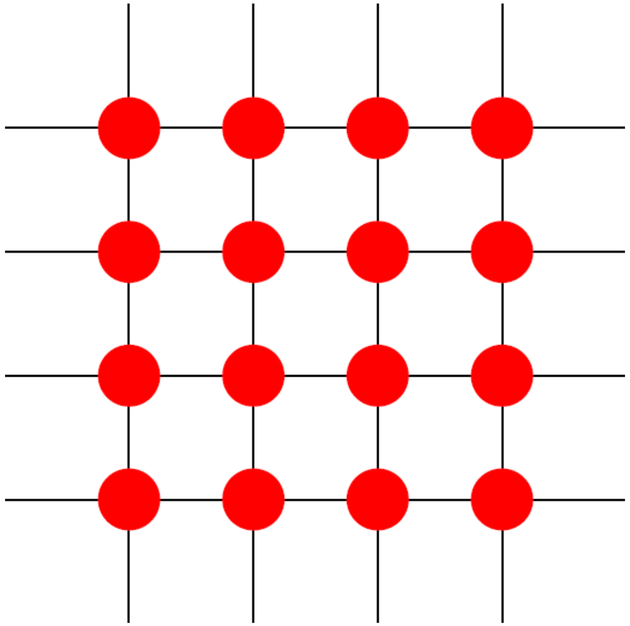
Homogeneous state : $\rho_i = \beta \; \forall i$

Stability of the homogeneous state ?

$$J_{ik} \coloneqq \frac{\partial}{\partial \rho_k}(\dot{\rho}_i)\bigg|_{\rho_i=\beta} = \underbrace{[g(\beta) - \beta g'(\beta)]}_{\xi(\beta)} L_{ik}$$

⇨ Stability condition: $\xi(\beta) < 0$

Square lattice with
periodic boundary conditions



Stability analysis (regular networks)

$$\dot{\rho}_i = -\sum_{j=1}^{N} L_{ij}[\rho_i g(\rho_j) - \rho_j g(\rho_i)]$$

$$g(x) = x^a(1-x)^a$$

Stability condition of the homogeneous state: $\begin{cases} \beta \in]0,1] \text{ if } 0 < a \leq 1 \\ \beta < \frac{a-1}{2a-1} \text{ if } a > 1 \end{cases}$

$a = 2$ ⇨ Homogeneous state unstable if $\beta < \frac{1}{3}$

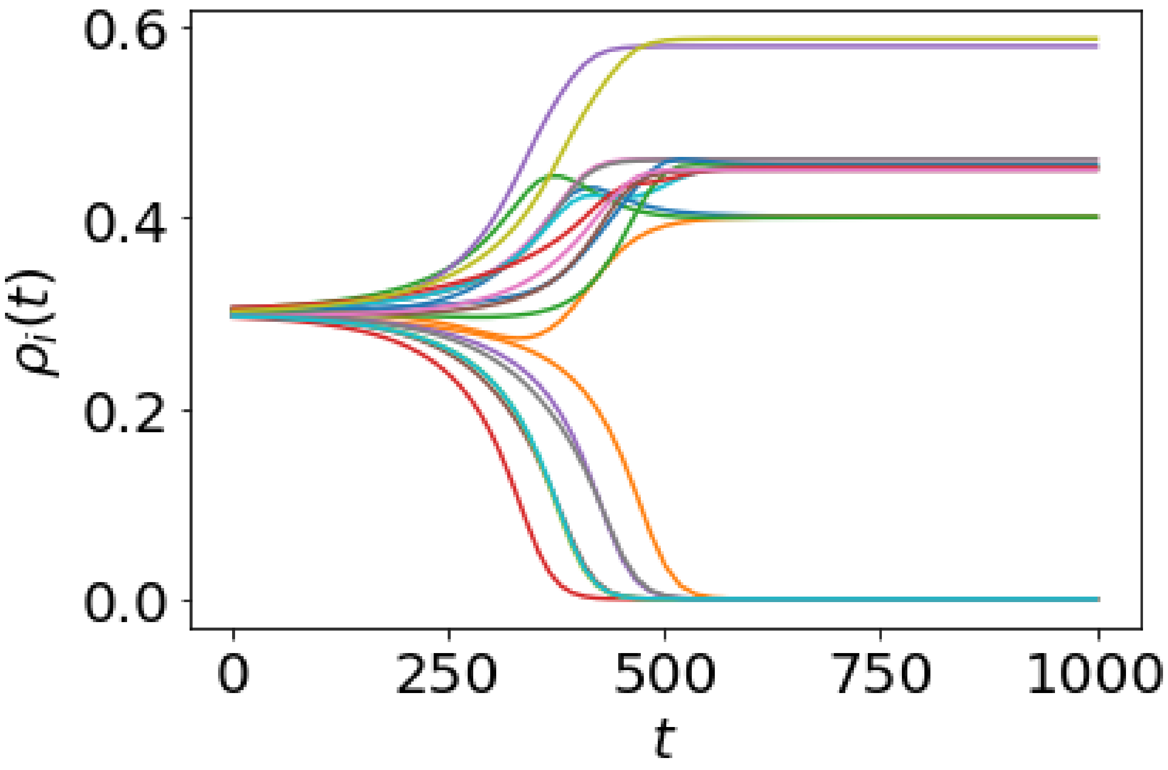
Symmetry breaking

$g(x) = x^2(1-x)^2$

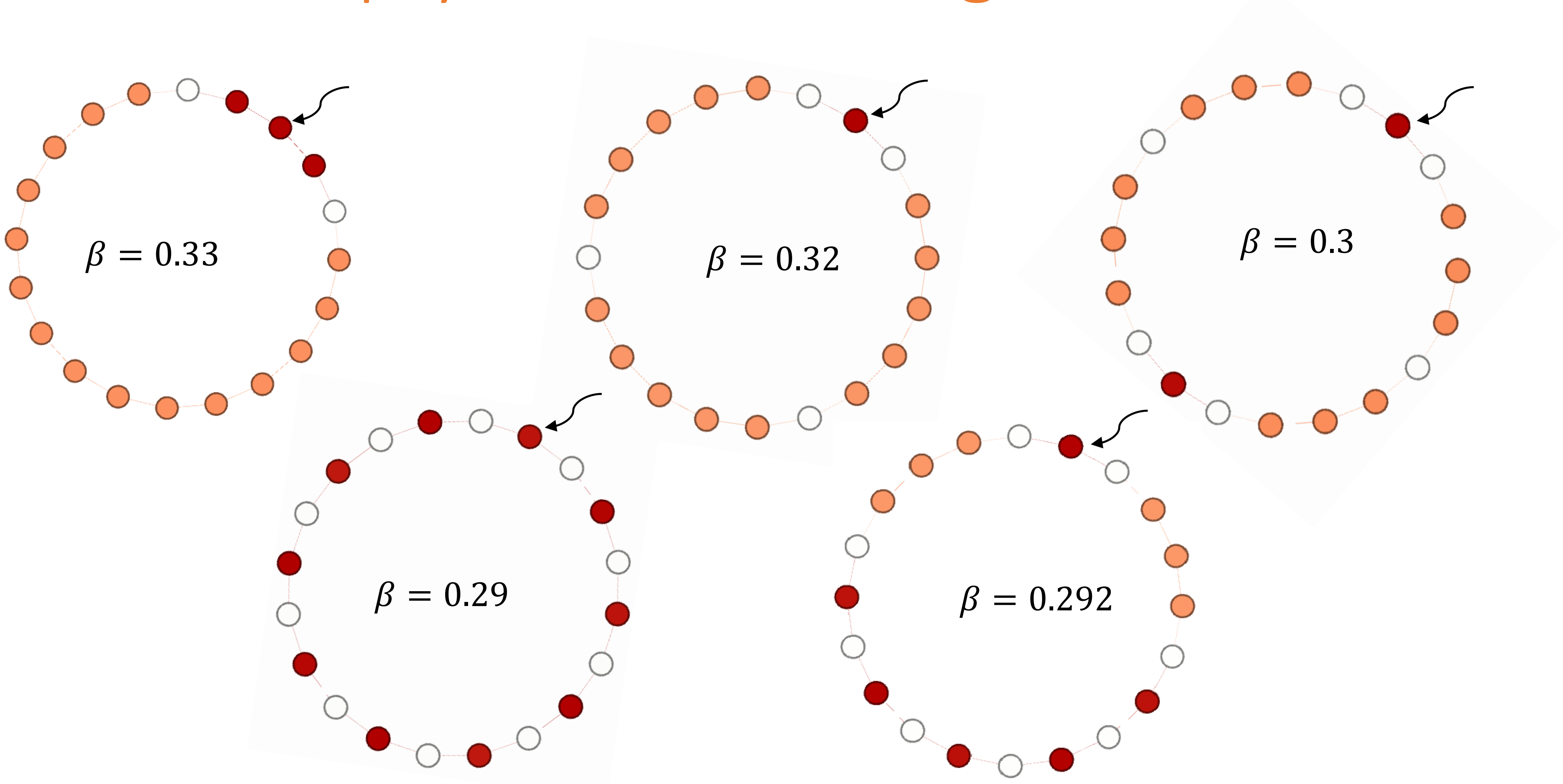
$$\dot{\rho}_i = -\rho_i \sum_{j=1}^{N} L_{ij}\rho_j\left[\rho_i(1-\rho_i)^2 - \rho_j(1-\rho_j)^2\right]$$

$$\rho_i(0) = p(\beta + \sigma\, r_i) \quad \text{with} \quad \begin{cases} \sum_i \rho_i(0) = \Omega\beta \\ \sigma \ll 1 \\ r_i \sim U([0,1]) \end{cases}$$

Ring network ($\beta = 0.3$; $\sigma = 0.01$; $\Omega = 20$)



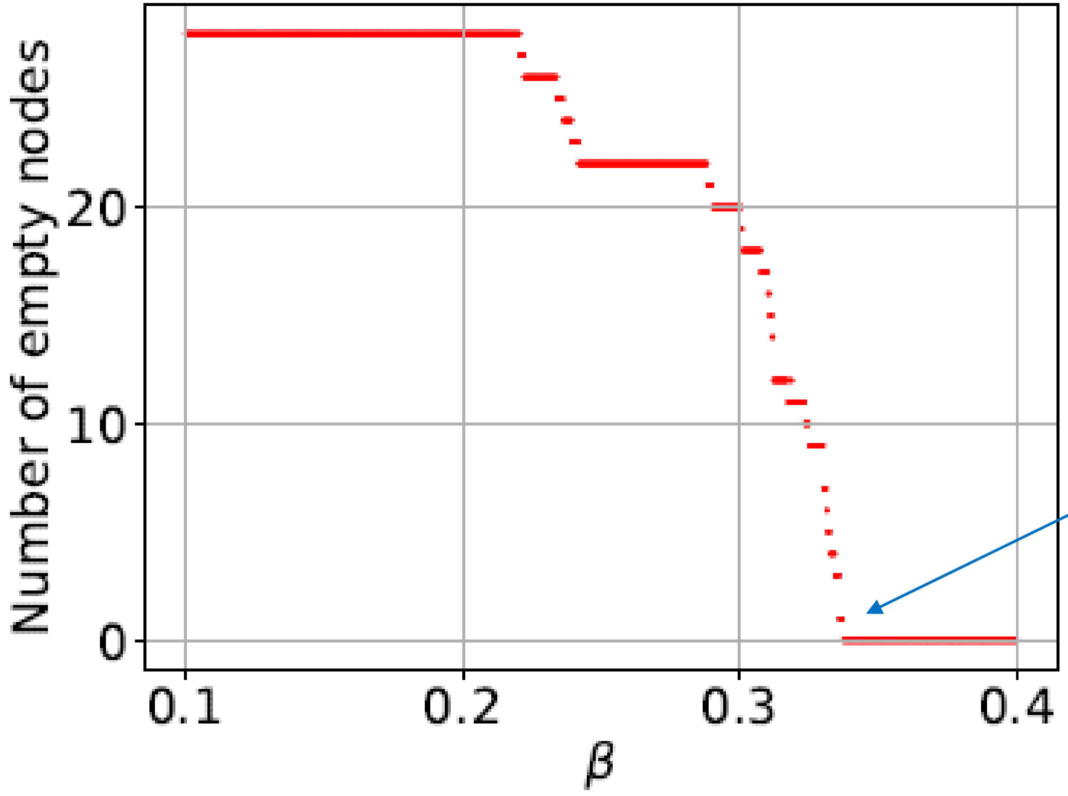
Empty nodes in a ring network



$$\rho_i(0) = \beta + 0.01\delta_{i,1} (i = 1, \cdots, 20)$$

Quantization phenomenon

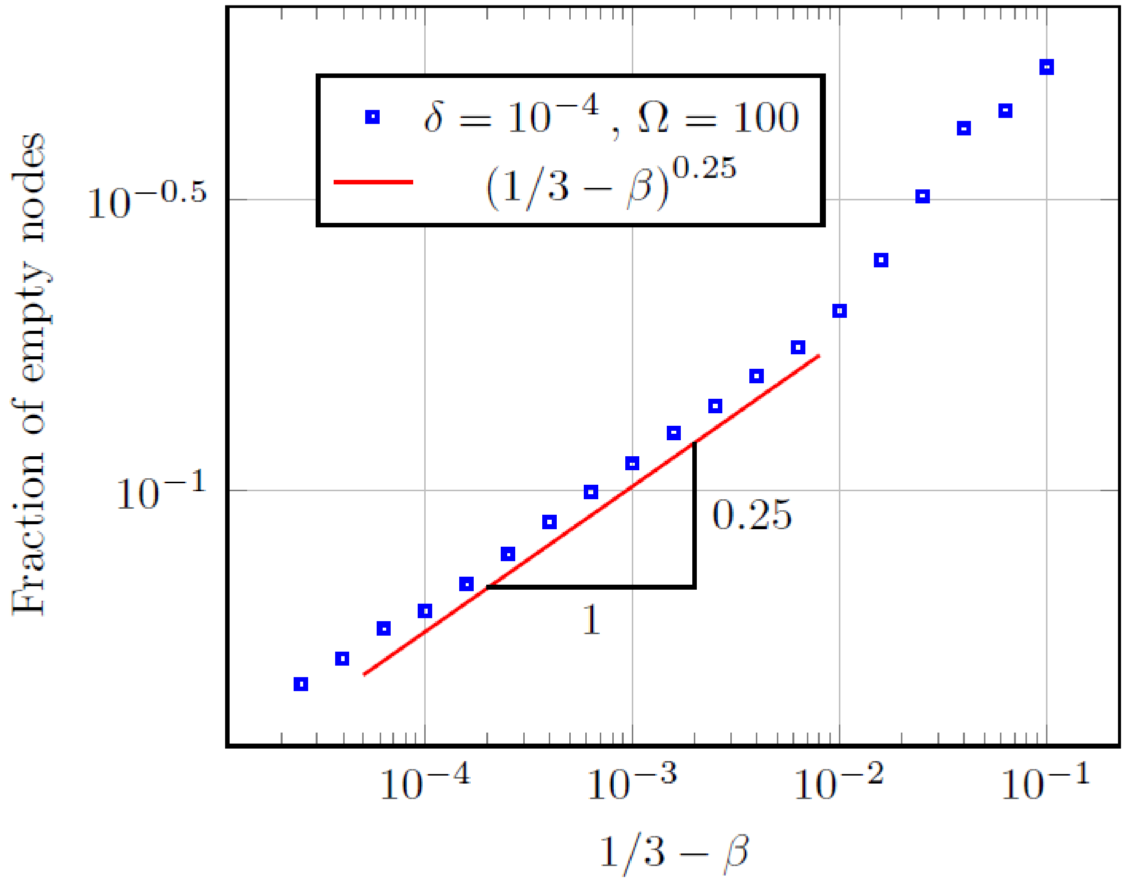
Ring network ($\sigma = 0.01\,;\Omega = 50$)

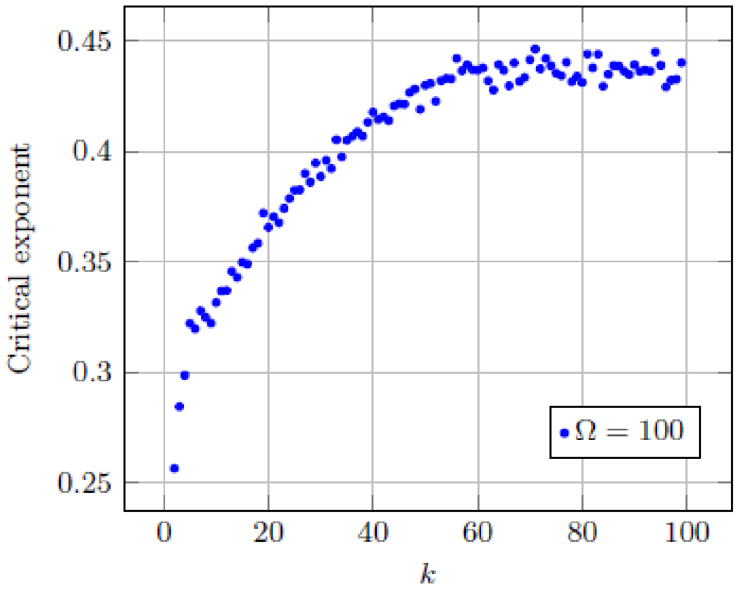


$\rho_i(0) = p(\beta + \sigma\, r_i)$ with $\begin{cases} \sum_i \rho_i(0) = \Omega\beta \\ r_i \sim U([0,1]) \end{cases}$

Phase transition

Second-order phase transition





$$\rho_i(0) = p(\beta + \delta\, r_i) \quad \text{with} \quad \begin{cases} \sum_i \rho_i(0) = \Omega\beta \\ \delta = 10^{-4} \\ r_i \sim U([0,1]) \end{cases}$$

Summary

- We introduced a new individual-based model leading to the emergence of functional communities on networks.

- A symmetry breaking from a homogeneous state to an heterogeneous one can be observed in lattices.

- The model can potentially shed light on the existence of vacant niches in ecology and the formation on urban prairies

Thanks for your attention !