January the 23rd, 2024, ISI Kolkata, India

Timoteo Carletti

A journey in the zoo of Turing patterns

Namur Institute for Complex Systems

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timoteo.carletti@unamur.be

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A personal journey in the zoo of Turing patterns

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timoteo.carletti@unamur.be

Acknowledgements

Master Students

Gwendoline Planchon

Marine Jamoulle

Alice Bellière

PhD Students

Martin Moriamé

Jean-François De Kemmeter

Luca Gallo

Lorenzo Giambagli

Cédric Simal

Sara Nicoletti

Marie Dorchain

PostDocts

Riccardo Muolo

Sarah De Nigris

Giulia Cencetti

Nikos Kouvaris

Thierry-Sainclair Njougouo

Collaborators

Malbor Asllani (Florida State University)

Federico Battiston (Central European University)

Duccio Fanelli (Università di Firenze)

Mattia Frasca (Università di Catania)

Valentina Gambuzza (Università di Catania)

Renaud Lambiotte (University of Oxford)

Vito Latora (Università di Catania & Queen Mary University)

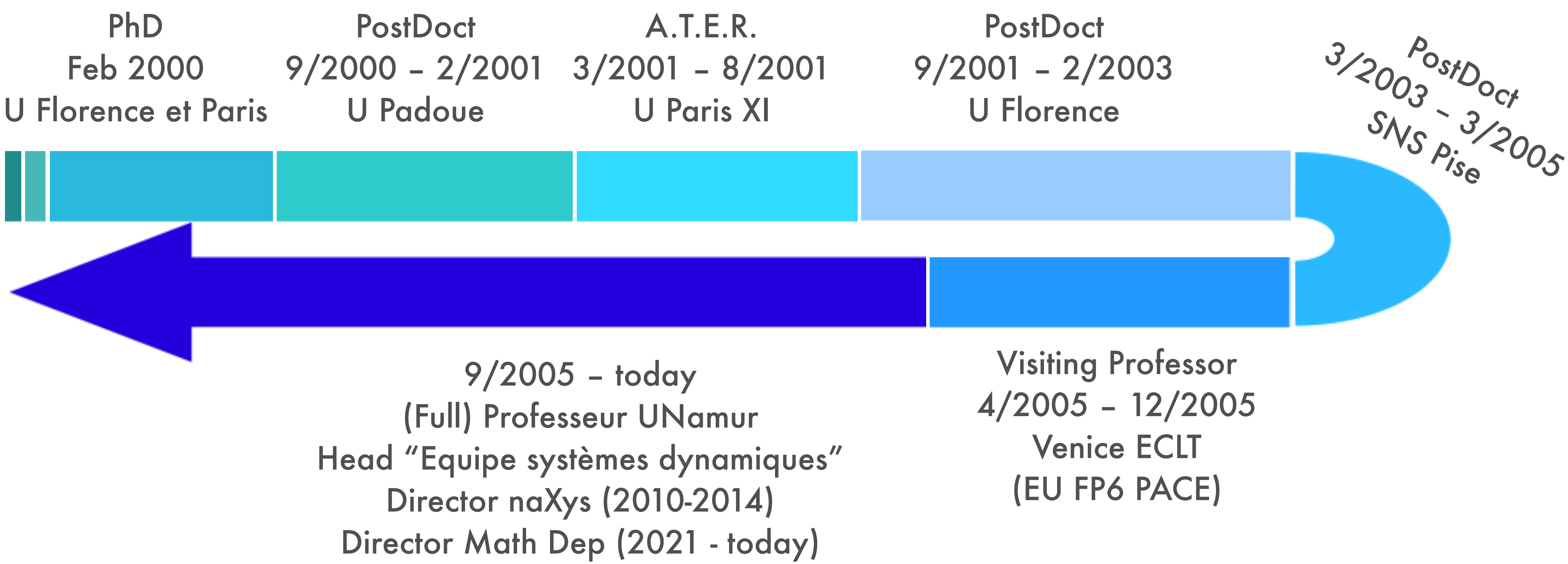
Hiroya Nakao (Tokyo Institute of Technology)

Julien Petit (Ecole Royale Militaire)

Philip Maini (Oxford)

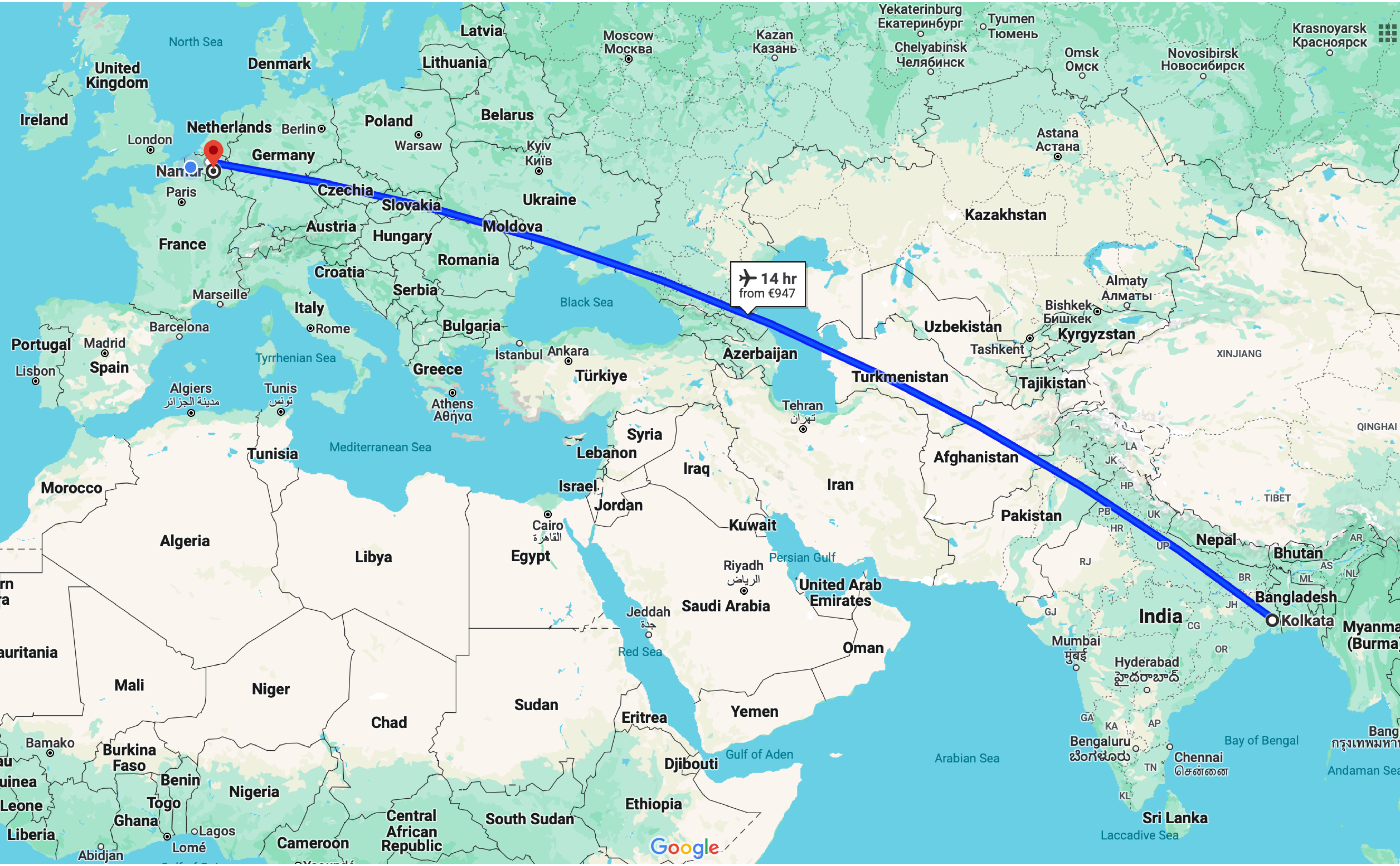
Ginestra Bianconi (Queen Mary University)

Snapshot of my career

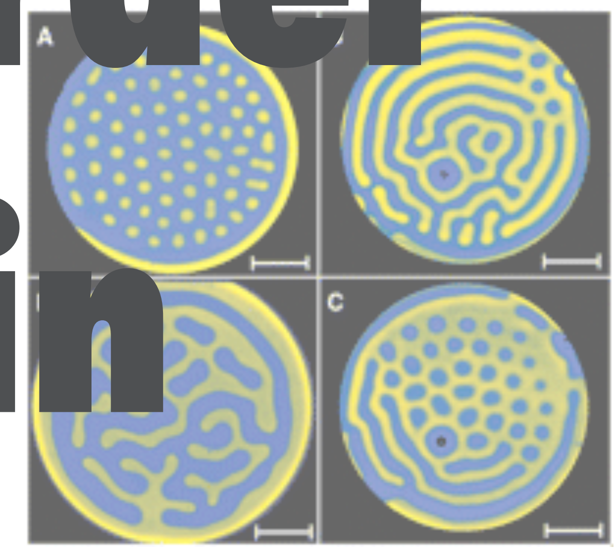


Snapshot of my career

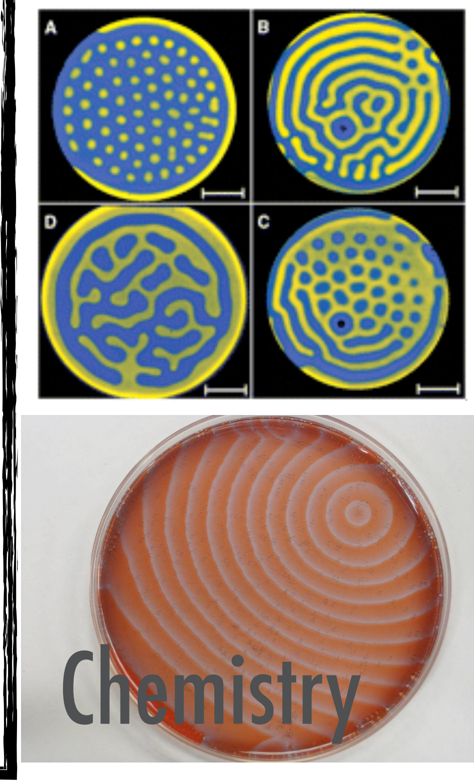
Snapshot of my career

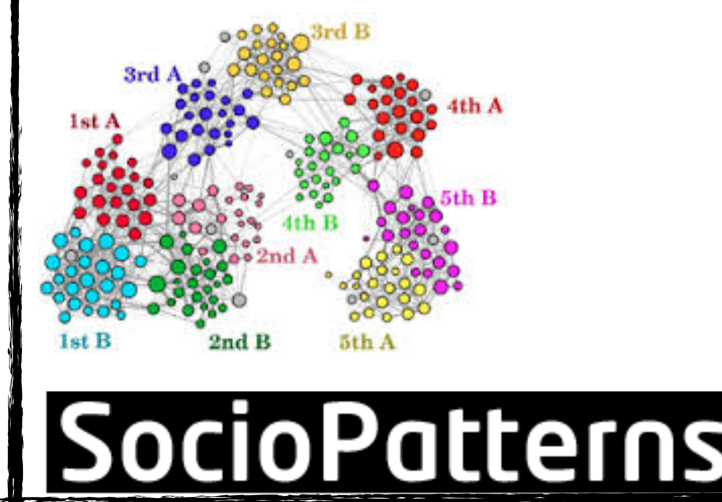


Order from disorder is a leitmotif in Nature



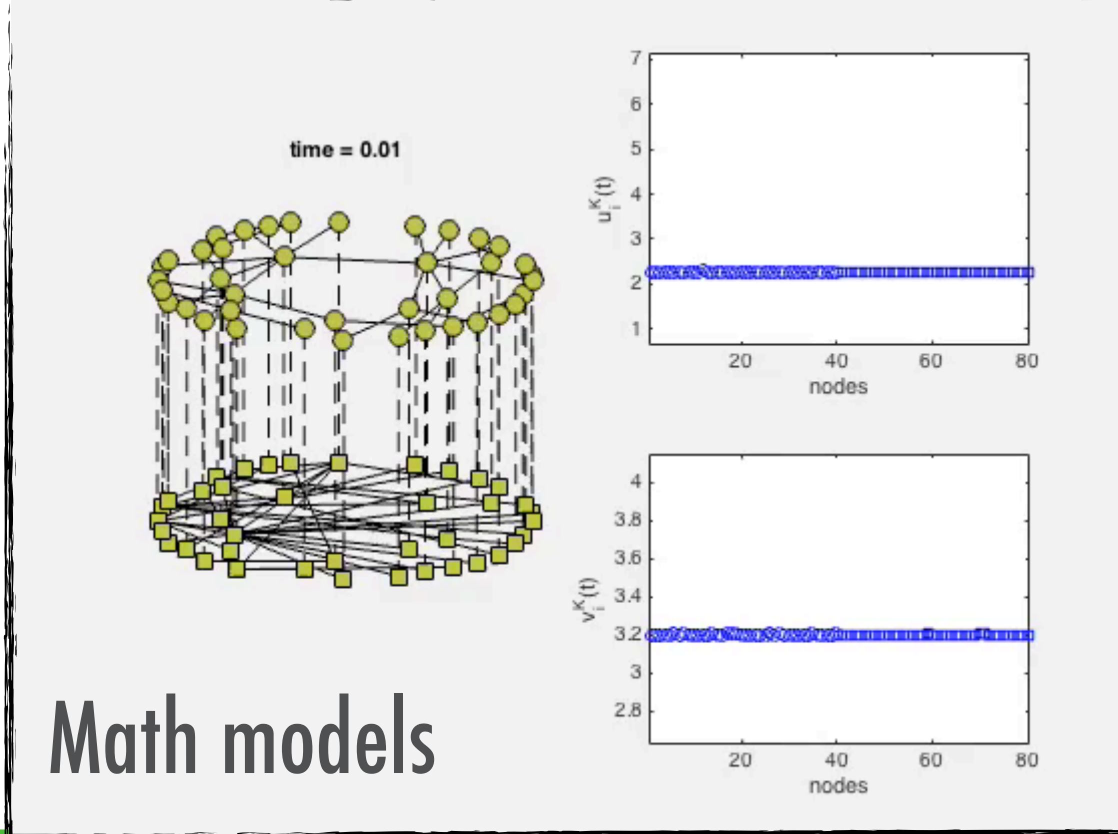
Patterns (motifs) are ubiquitous





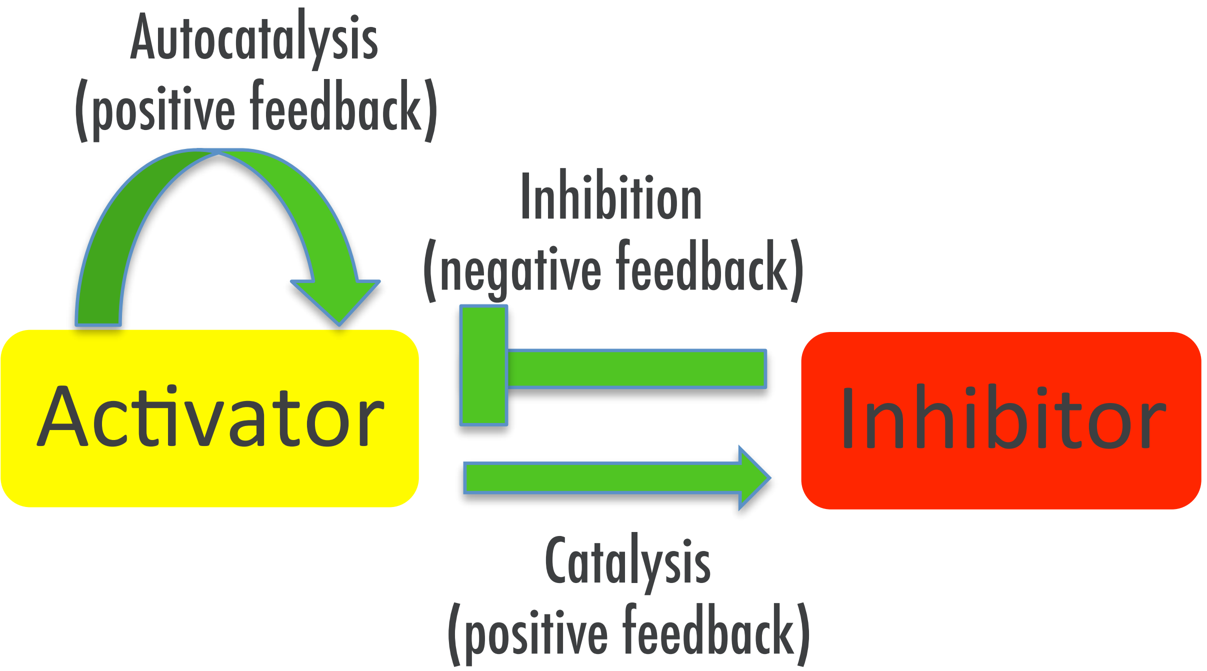
Internet

Twitter



One possible mechanism: Turing instability

Reaction



+

Diffusion

$u(x,y,t)$: Amount of activator at time t and position (x,y)

$v(x,y,t)$: Amount of inhibitor at time t and position (x,y)

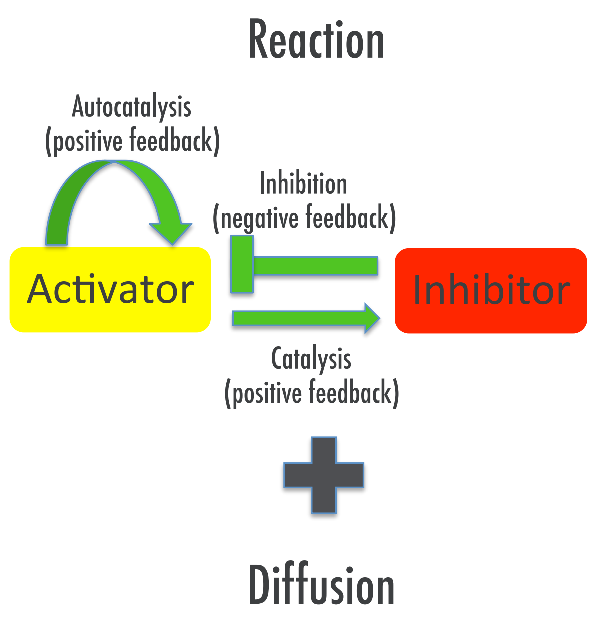
$$\begin{cases} \dfrac{\partial u}{\partial t} = f(u,v) + D_u \nabla^2 u \\ \dfrac{\partial v}{\partial t} = g(u,v) + D_v \nabla^2 v \end{cases}$$

$$(x,y) \in \Omega$$

+ boundary conditions
+ initial condition

A.M.Turing, *The chemical basis of morphogenesis*, Phil. Trans. R Soc London B, **237**, *(1952), pp.37*

One possible mechanism: Turing instability



$u(x,y,t)$: Amount of activator at time t and position (x,y)

$v(x,y,t)$: Amount of inhibitor at time t and position (x,y)

$$\begin{cases} \frac{\partial u}{\partial t} = f(u,v) + D_u \nabla^2 u \\ \frac{\partial v}{\partial t} = g(u,v) + D_v \nabla^2 v \end{cases}$$

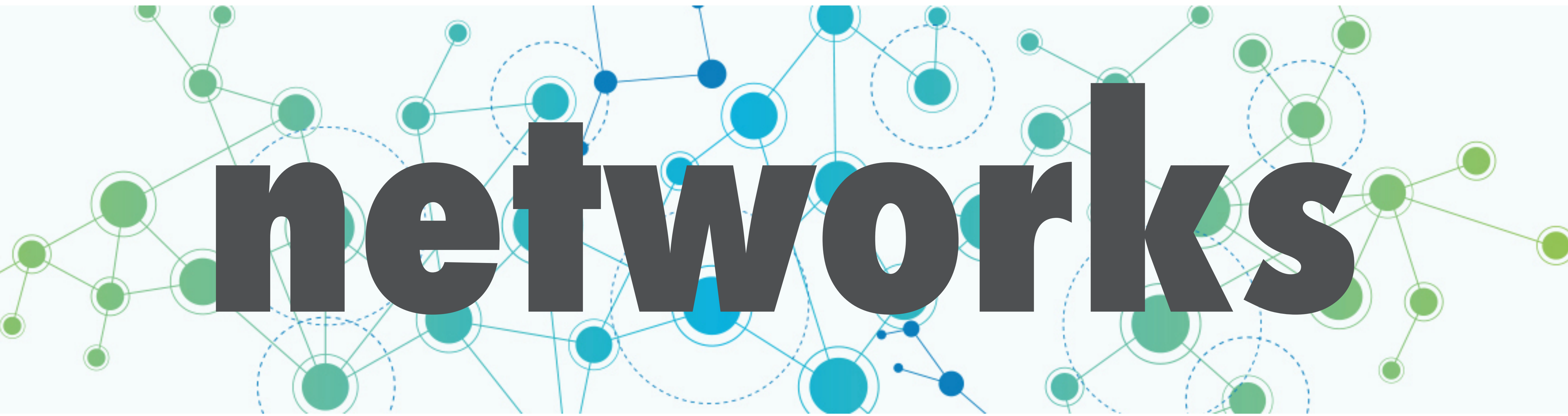
$$(x,y) \in \Omega$$

+ boundary conditions
+ initial condition

Diffusion can drive an instability by perturbing a homogeneous stable (in absence of diffusion) fixed point.

Hence as the perturbation grows, non-linearities enter into the game yielding an asymptotic, spatially inhomogeneous, steady state (stationary pattern) or time varying one (wave like pattern).

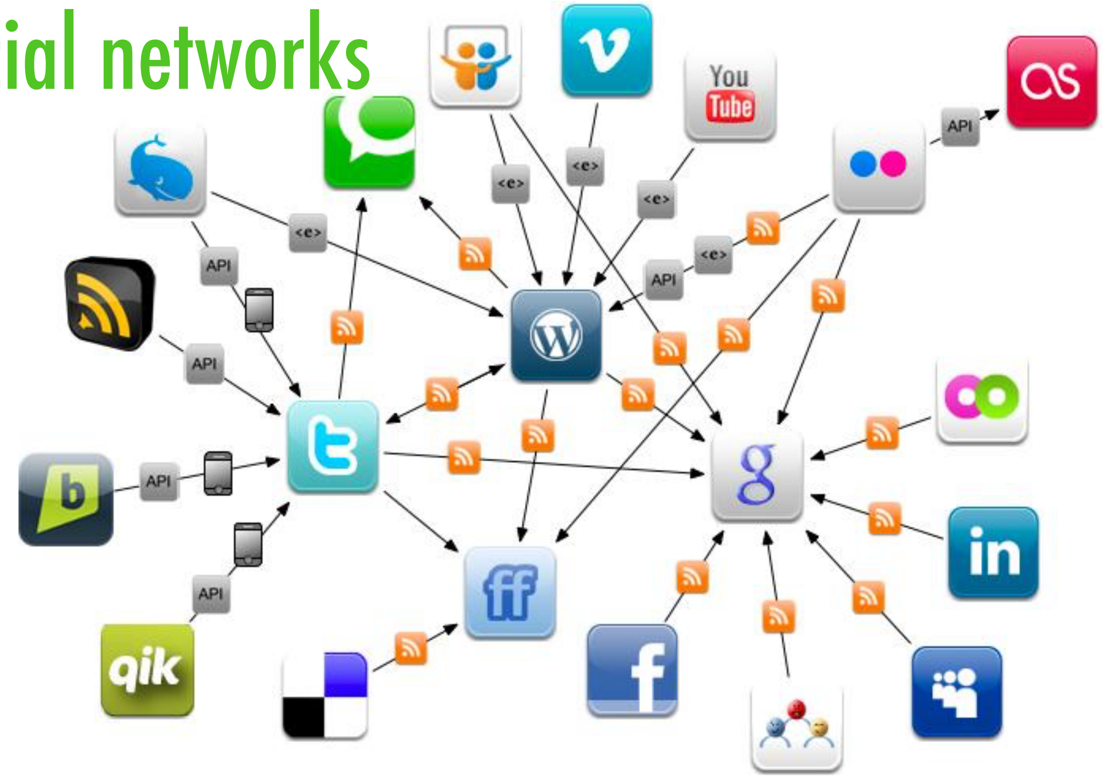
A.M.Turing, *The chemical basis of morphogenesis*, Phil. Trans. R Soc London B, **237**, *(1952), pp.37*

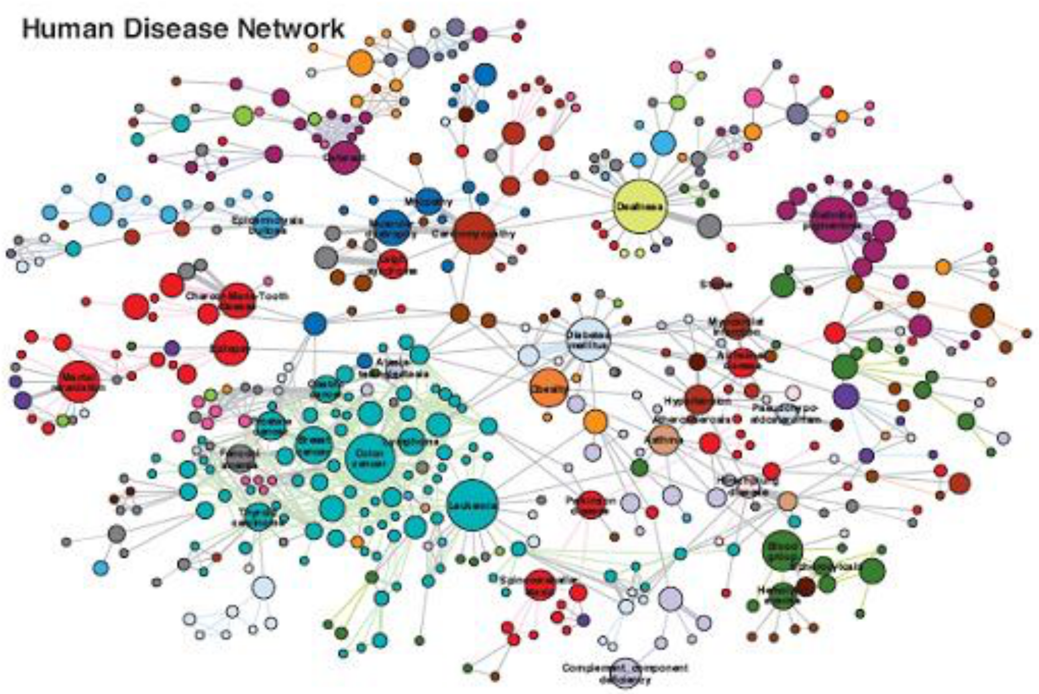

networks

Networks are everywhere

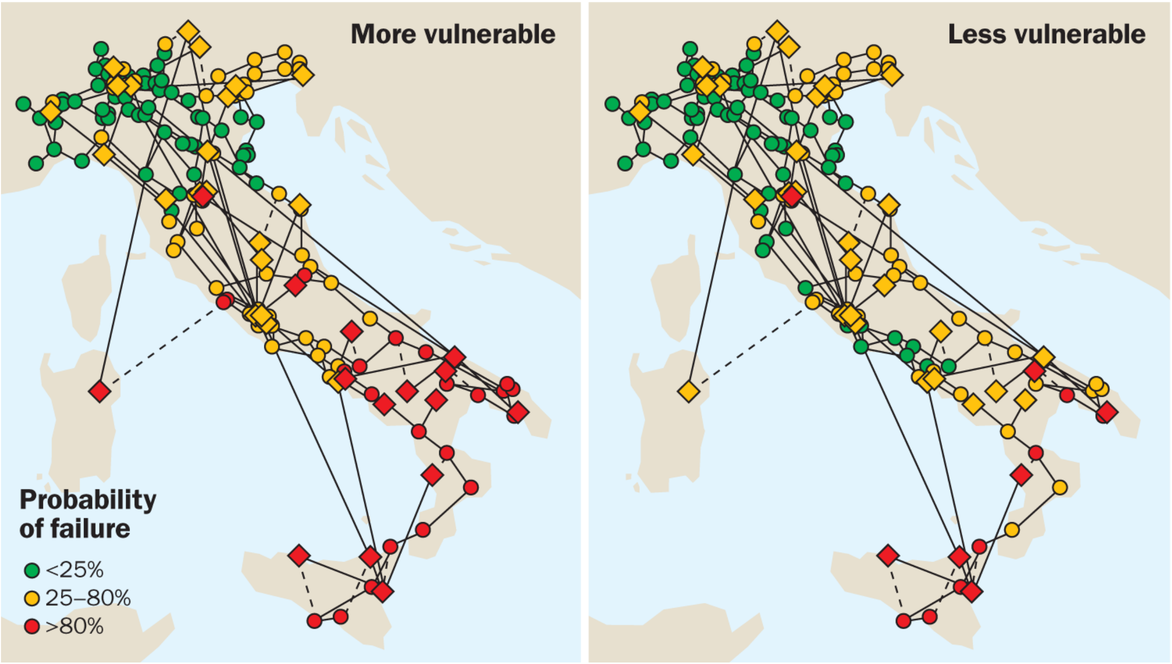
world flights map

social networks





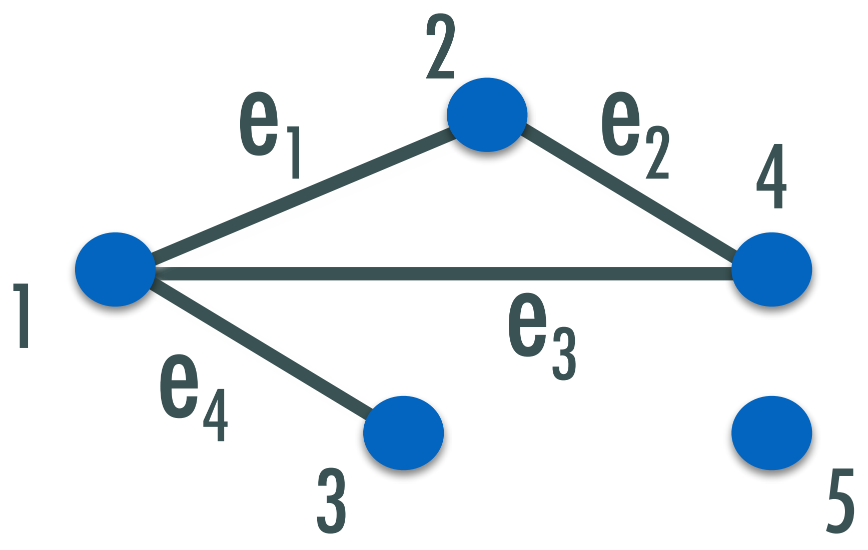
proteins networks



technological networks

(complex) Networks: some definitions

A network is a set of nodes connected by links (edges)



Ex.: 5 nodes and 4 edges (undirected)

Adjacency matrix

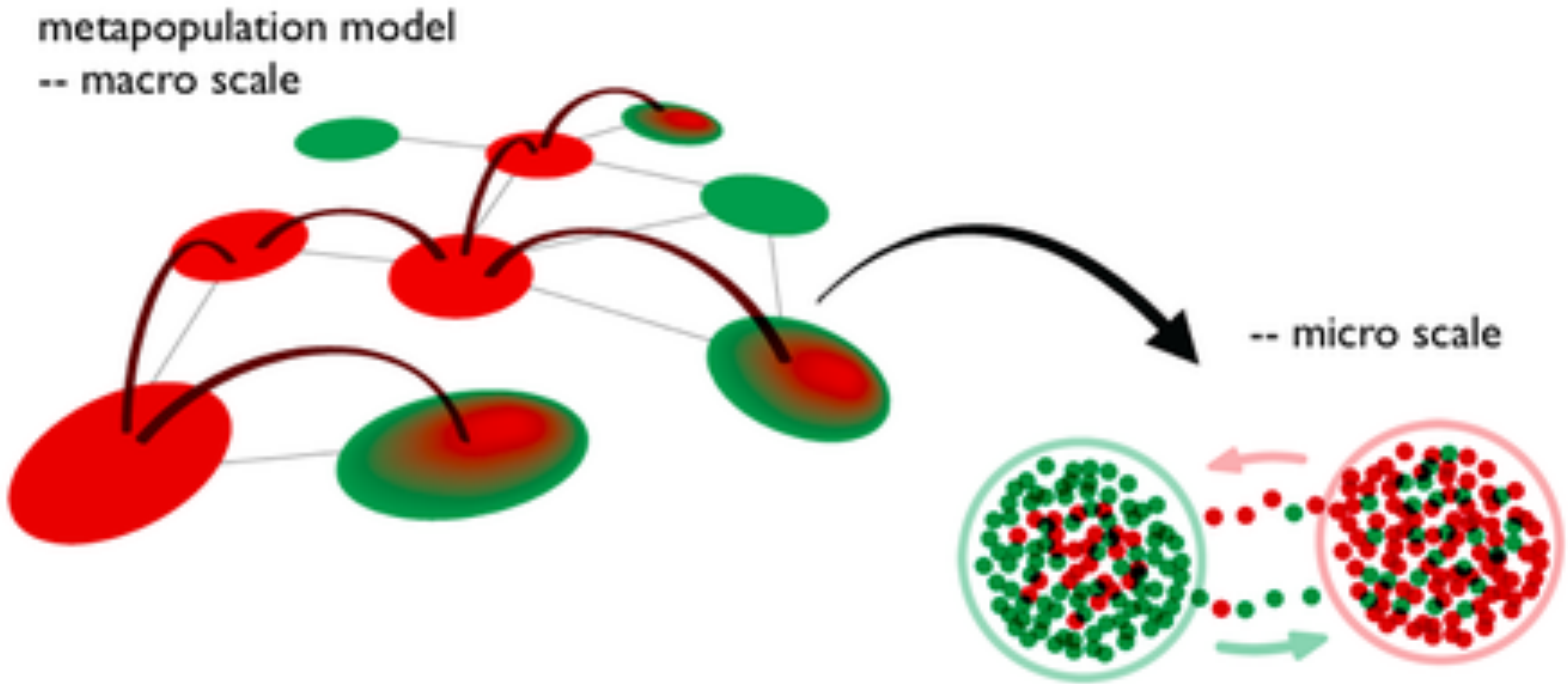
$$A_{ij} = \begin{cases} 1 & \text{if nodes } i \text{ and } j \text{ are linked} \\ 0 & \text{otherwise} \end{cases}$$

The number of links entering (going out) from each node is called in-degree (out-degree)

Ex.: degree node 1 = 3
degree nodes 2 & 4 = 2
degree node 3 = 1
degree node 5 = 0

A network is said to be complex if the degree distribution is not trivial, i.e. not constant (lattice) nor Poissonian (random, Erdős-Rényi)

Reactions occur at each node. Diffusion occurs across edges.



May R., Will a large complex system be stable? Nature, 238, pp. 413, (1972)

Nakao H. and Mikhailov A. S.,
Turing patterns in network-organized activator–inhibitor systems, Nature Physics, 6, pp. 544 (2010)

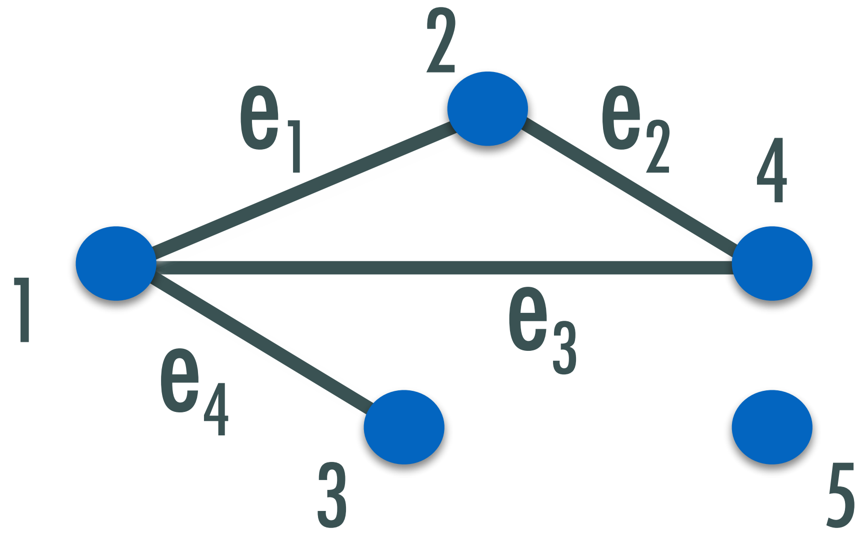
Reaction term:

$$\begin{cases} \dot{u}_i(t) &= f(u_i(t), v_i(t)) \\ \dot{v}_i(t) &= g(u_i(t), v_i(t)) \quad \forall i = 1, \dots, n \text{ and } t > 0. \end{cases}$$

At each node i=1,...,n, "species" u and v react through some non-linear functions f and g depending on the quantities available at node i-th (metapopulation assumption)

Diffusion term:

Diffusive transport of species into a certain node i is given by the sum of incoming fluxes to node i from other connected nodes j, fluxes are proportional to the concentration difference between the nodes (Fick's law).

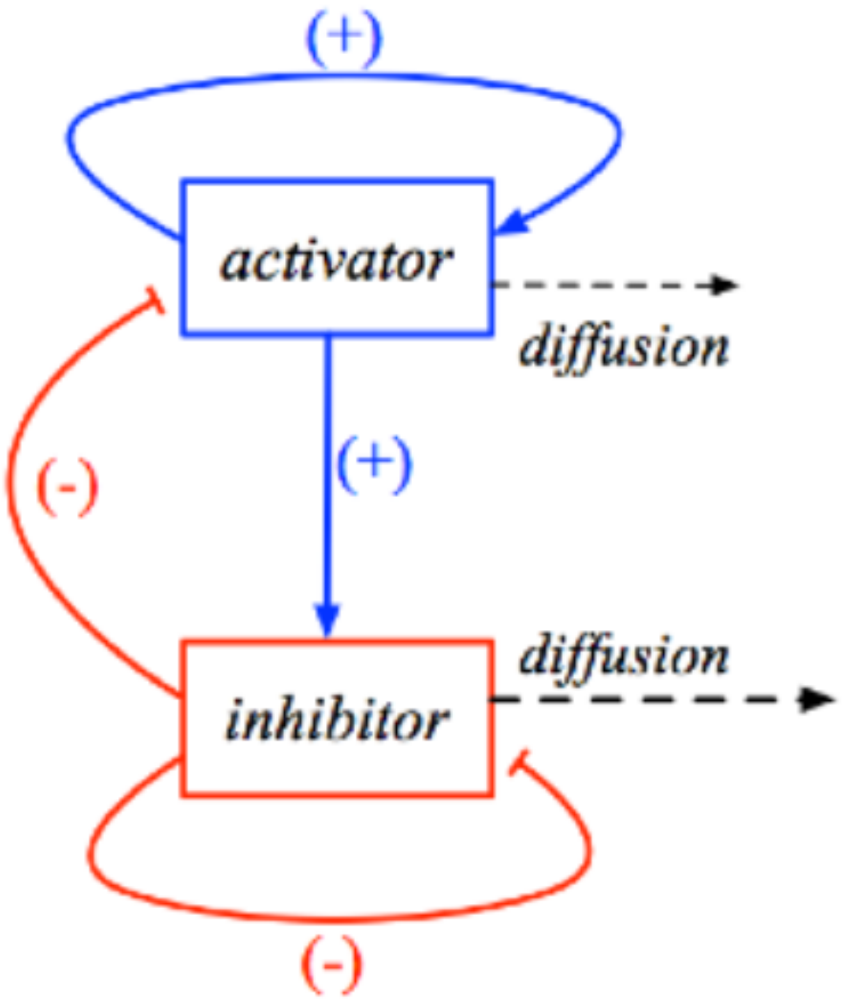


Ex.: consider the amount of u in node 1,
u can enter from 2, 3 and 4
u can leave 1 to go to 2, 3 and 4

$$u_2 + u_3 + u_4 - 3u_1 = \sum_j A_{1j} u_j - k_1 u_1 = \sum_j \left(A_{1j} - \delta_{1j} k_j\right) u_j := \sum_j L_{1j} u_j$$

L is called Laplacian matrix of the network

Turing mechanism: diffusion driven instability



$u_i(t)$ Amount of activator in node i at time t

$v_i(t)$ Amount of inhibitor in node i at time t

$$\begin{cases} \dot{u}_i = f(u_i, v_i) + D_u \sum_j L_{ij} u_j \\ \dot{v}_i = g(u_i, v_i) + D_v \sum_j L_{ij} v_j \, , \end{cases}$$

Local reaction term

Diffusion term (Fick's law)

$L_{ij} = A_{ij} - k_i \delta_{ij}$ Laplace matrix

A_{ij} Adjacency matrix

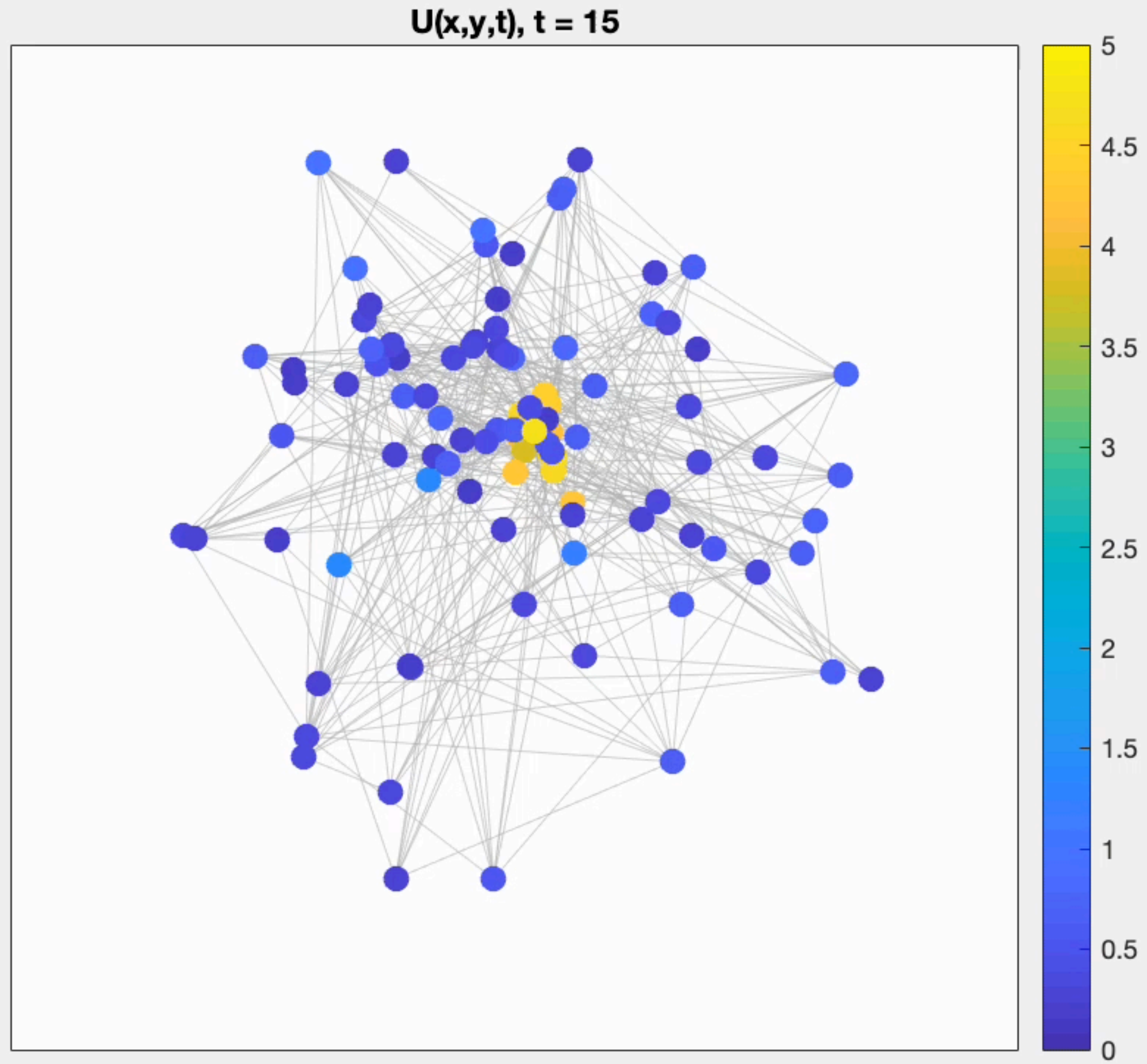
A and L are symmetric

Nakao H. and Mikhailov A. S., Turing patterns in network-organized activator–inhibitor systems, Nature Physics, 6, pp. 544 (2010)

Turing patterns on networks

$u_i(t)$

Patterns:
Sets of nodes whose asymptotic state is far from the homogeneous equilibrium.



Turing mechanism: diffusion driven instability

1) Assume there exists a spatially homogeneous stable solution:

$$u_i = \hat{u} \text{ and } v_i = \hat{v} \quad \forall i$$

2) Linearise around this solution: $u_i = \delta u_i + \hat{u} \text{ and } v_i = \hat{v} + \delta v_i$

$$\begin{pmatrix} \dot{\delta \boldsymbol{u}} \\ \dot{\delta \boldsymbol{v}} \end{pmatrix} = \tilde{\mathcal{J}} \begin{pmatrix} \delta \boldsymbol{u} \\ \delta \boldsymbol{v} \end{pmatrix} \qquad \tilde{\mathcal{J}} = \begin{pmatrix} f_u + D_u \mathbf{L} & f_v \\ g_u & g_v + D_v \mathbf{L} \end{pmatrix}$$

3) Prove that the spatially homogeneous solution:

$$u_i = \hat{u} \text{ and } v_i = \hat{v} \quad \forall i$$

turns out to be unstable once the diffusion is in action

$$D_u > 0 \text{ and } D_v > 0$$

General strategy for the network case

3) Prove that the spatially homogeneous solution:

$$u_i = \hat{u} \text{ and } v_i = \hat{v} \quad \forall i$$

turns out to be unstable once the diffusion is in action

$$D_u > 0 \text{ and } D_v > 0$$

Sketch of the proof

i) Let $L\vec{\phi}^\alpha = \Lambda^\alpha \vec{\phi}^\alpha,\ \alpha = 1, \dots, n$ $\qquad \vec{\phi}^\alpha = (\phi_1^\alpha, \dots, \phi_n^\alpha)$

$$\sum_i \phi_i^\alpha \phi_i^\beta = \delta_{\alpha\beta} \qquad \Lambda^\alpha \leq 0$$

ii) decompose the solution on the eigenbasis and use the ansatz

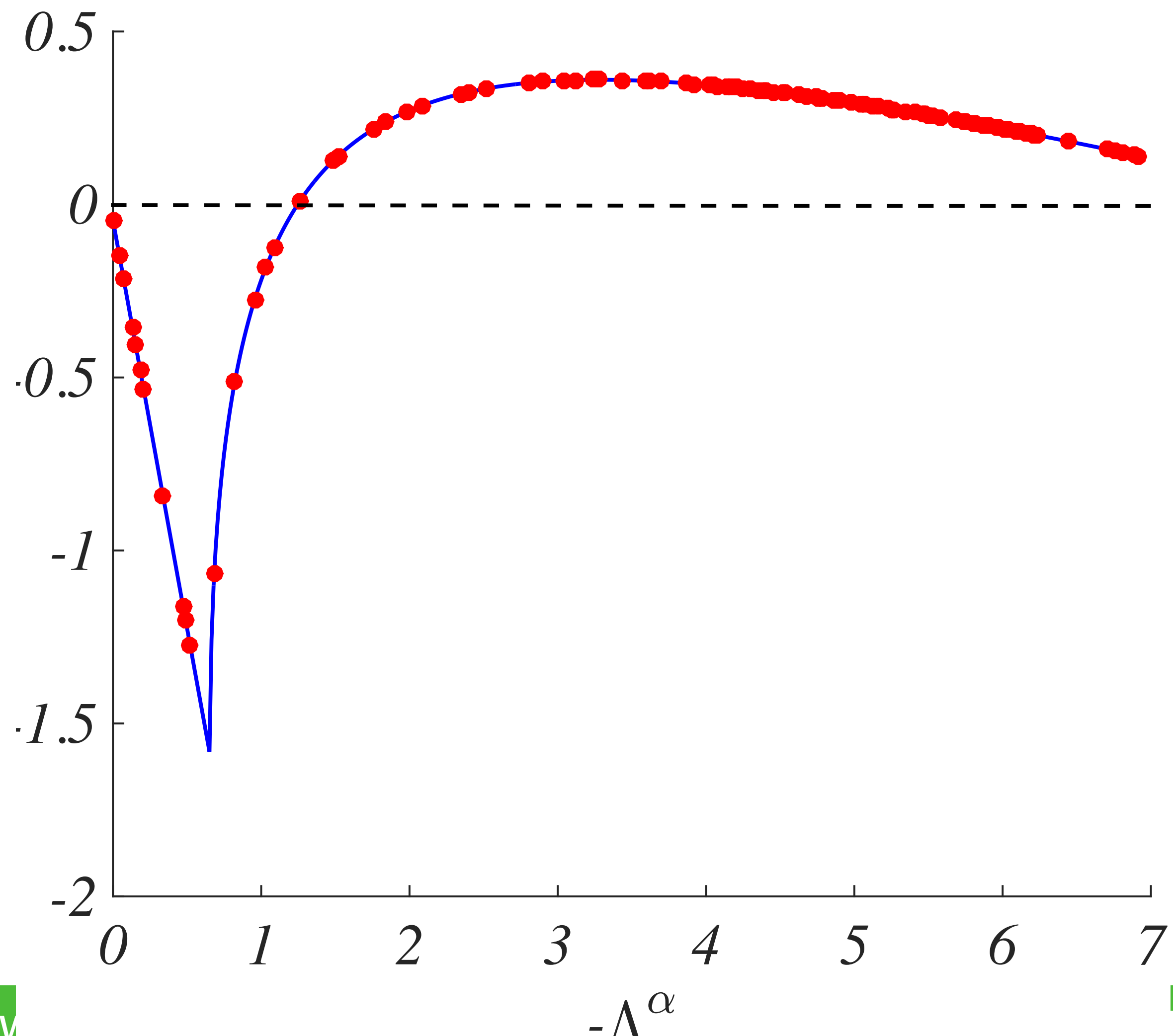
$$\delta u_i(t) = \sum_{\alpha=1}^{n} c_\alpha \phi_i^\alpha e^{\lambda_\alpha t}$$

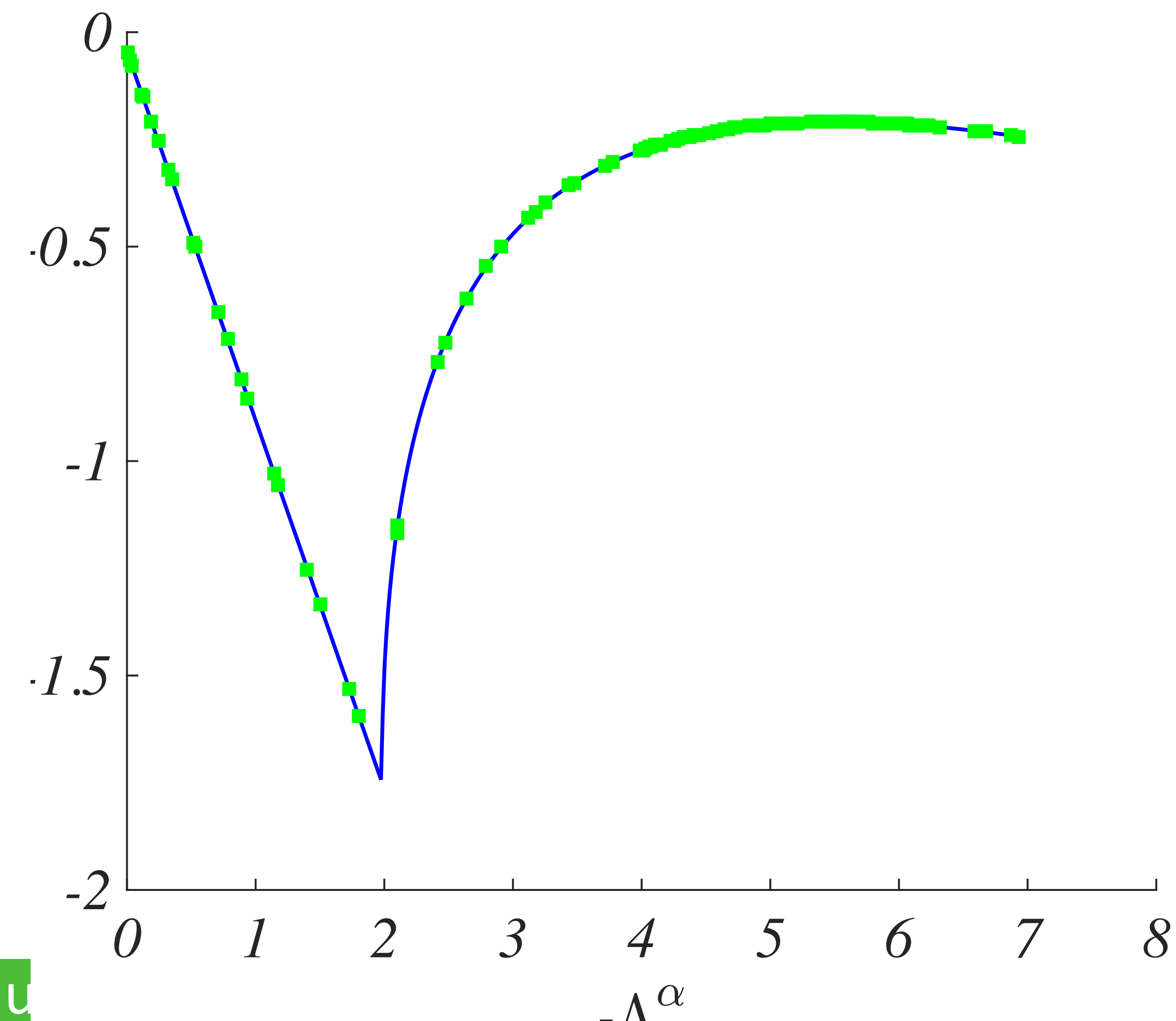
General strategy for the network case

iii) λ_α (called relation dispersion) is solution of

$$\det\left[\lambda_\alpha - \begin{pmatrix} f_u + D_u\Lambda^\alpha & f_v \\ g_u & g_v + D_v\Lambda^\alpha \end{pmatrix}\right] = 0$$

iv) if there exists Λ^{α_c} such that $\Re\lambda_{\alpha_c} > 0$ then Turing patterns do emerge.

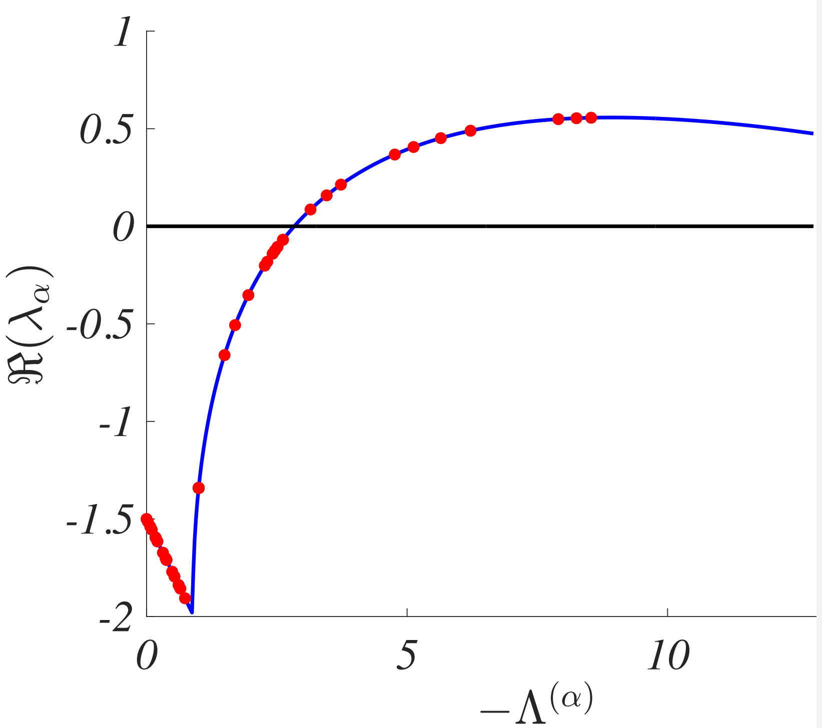


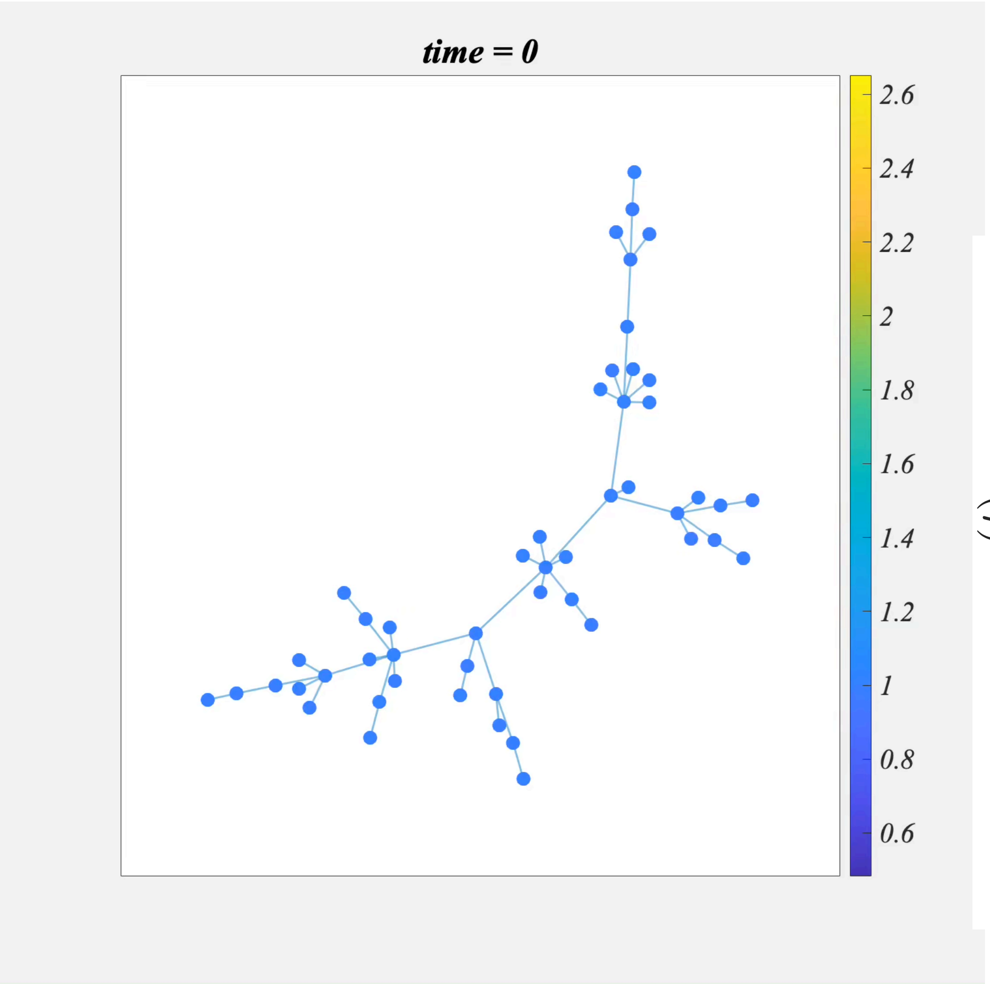


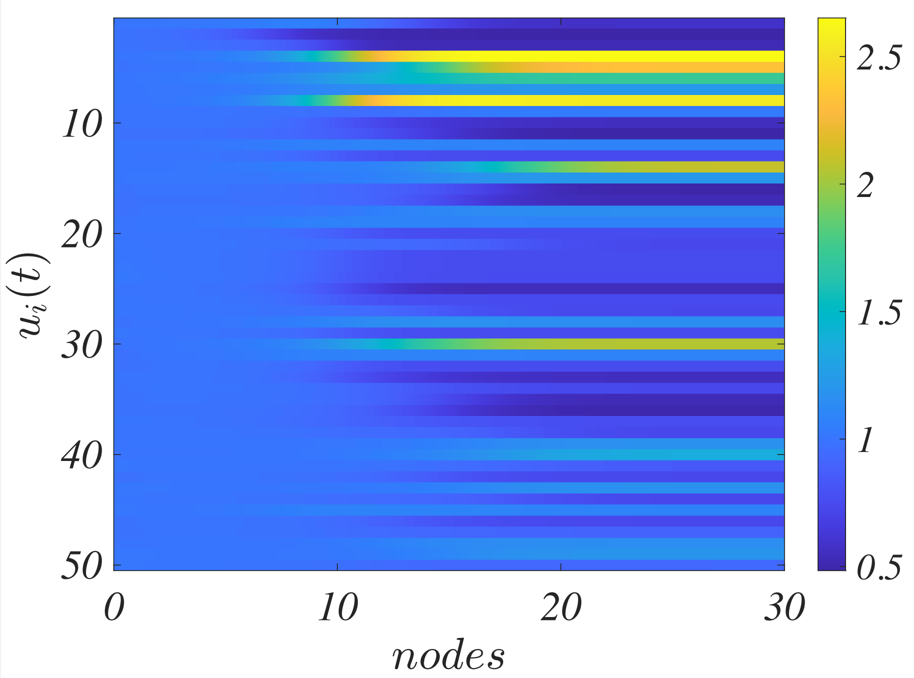
The Brusselator

$$\begin{cases} \dot{u}_i &= 1-(b+1)u_i + c u_i^2 v_i + D_u \sum_j L_{ij} u_j \\ \dot{v}_i &= b u_i - c u_i^2 v_i + D_v \sum_j L_{ij} v_j \end{cases}$$

$(u^*, v^*) = (1, b/c)$ equilibrium isolated system (no diffusion)







Turing mechanism: diffusion driven instability

Elegant and simple, but unable to describe patterns onset in many real scenarios.

- At least two diffusing species are needed;
- Activator and inhibitor are both necessary :

$$f_u g_v < 0$$

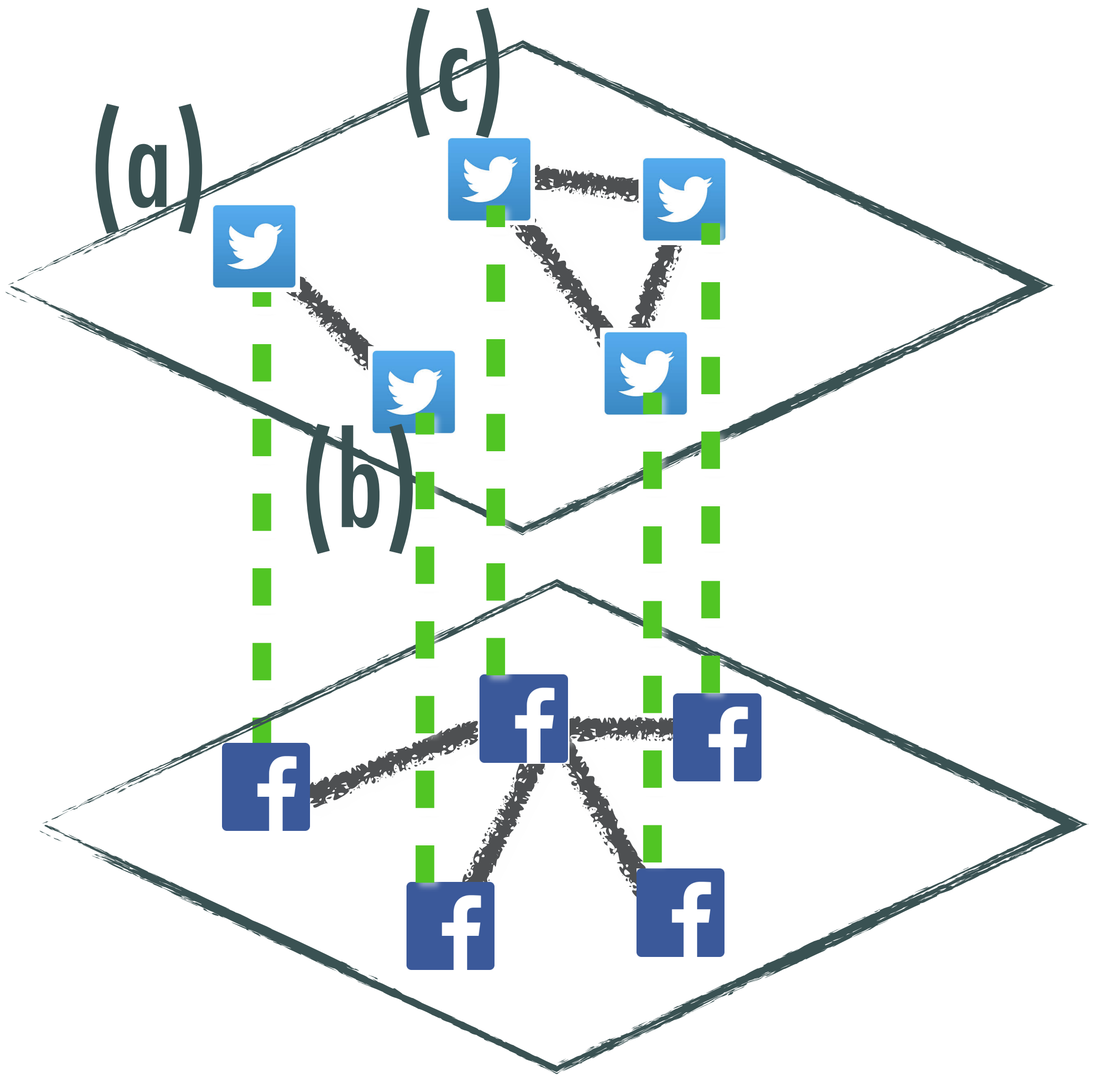
- The inhibitor must diffuse much faster than the activator;

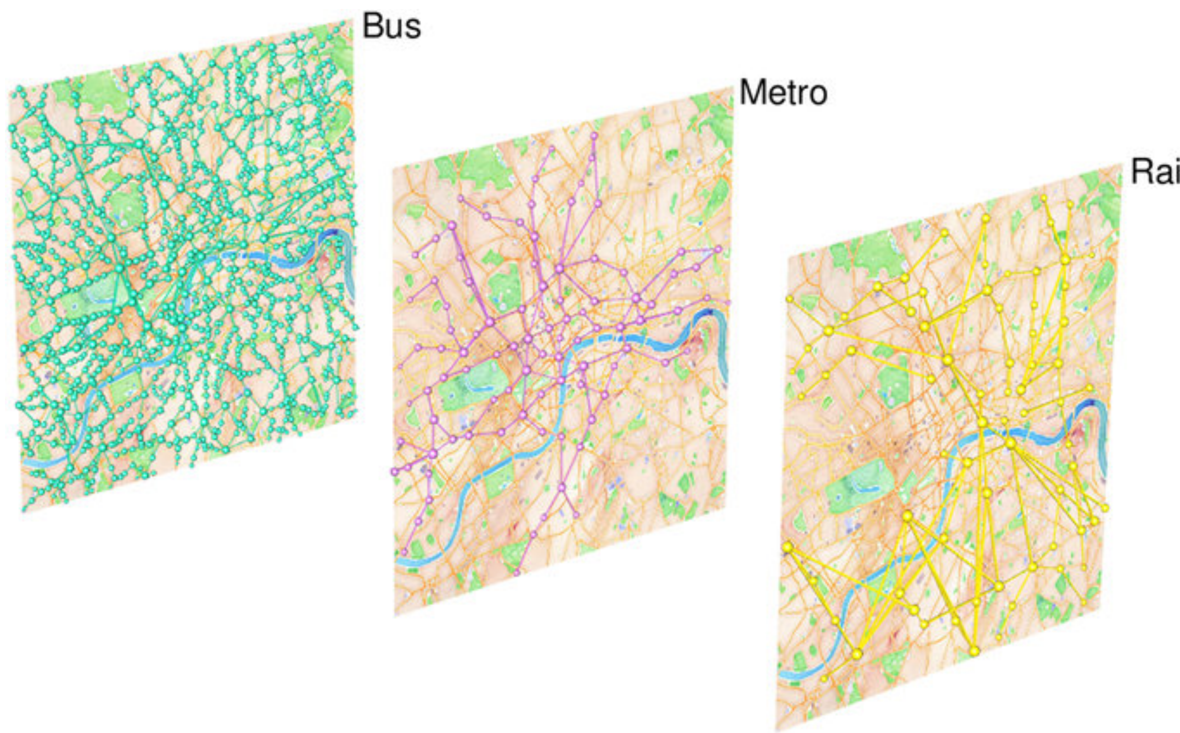
$$D_v \gg D_u$$

- Based on parabolic PDE (heat equation), hence infinite propagation of signals

Systems composed by layers of networks: **Multiplexes**

<u>Social networks</u>
layers=different social networks
nodes=same agent in each SN





<u>Transportation networks</u>
layers=different modalities
nodes=same spatial location

timoteo.carletti@unamur.be

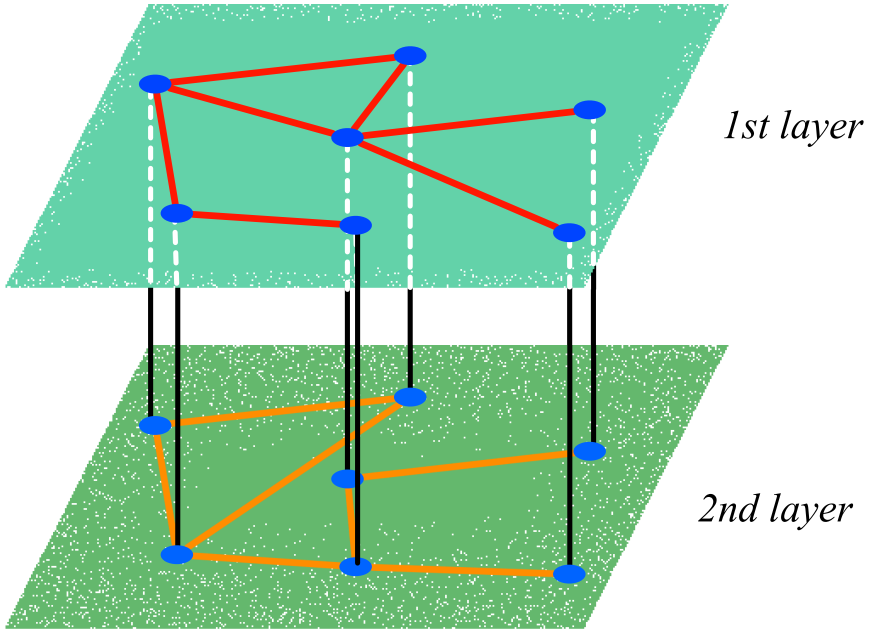
Turing instabilities on multiplex networks

adjacency matrix of layer K

degree of i-th node in layer K

$$L_{ij}^K = A_{ij}^K - \delta_{ij} k_i^K$$

Laplacian matrix of layer K



The same Ω nodes are present in each layer

$D_{u,v}^K$ **inter**-layer diffusion coefficient

$D_{u,v}^{12}$ **intra**-layer diffusion coefficient

$$\begin{cases} \dot{u}_i^K &= f(u_i^K, v_i^K) + D_u^K \sum_{j=1}^{\Omega} L_{ij}^K u_j^K + D_u^{12}\left(u_i^{K+1} - u_i^K\right) \\ \dot{v}_i^K &= g(u_i^K, v_i^K) + D_v^K \sum_{j=1}^{\Omega} L_{ij}^K v_j^K + D_v^{12}\left(v_i^{K+1} - v_i^K\right) \end{cases}$$

Small intra-layer diffusion case

Assume $D_v^{12} = \epsilon << 1 \qquad D_u^{12} = \mathcal{O}(\epsilon)$

$$\tilde{\mathcal{J}} = \begin{pmatrix} f_u \mathbf{I}_{2\Omega} + \boldsymbol{\mathcal{L}}_u & f_v \mathbf{I}_{2\Omega} \\ g_u \mathbf{I}_{2\Omega} & g_v \mathbf{I}_{2\Omega} + \boldsymbol{\mathcal{L}}_v \end{pmatrix} + \epsilon \begin{pmatrix} \frac{D_u^{12}}{D_v^{12}} \boldsymbol{L}^1 & \mathbf{0} \\ \mathbf{0} & \boldsymbol{L}^2 \end{pmatrix}$$

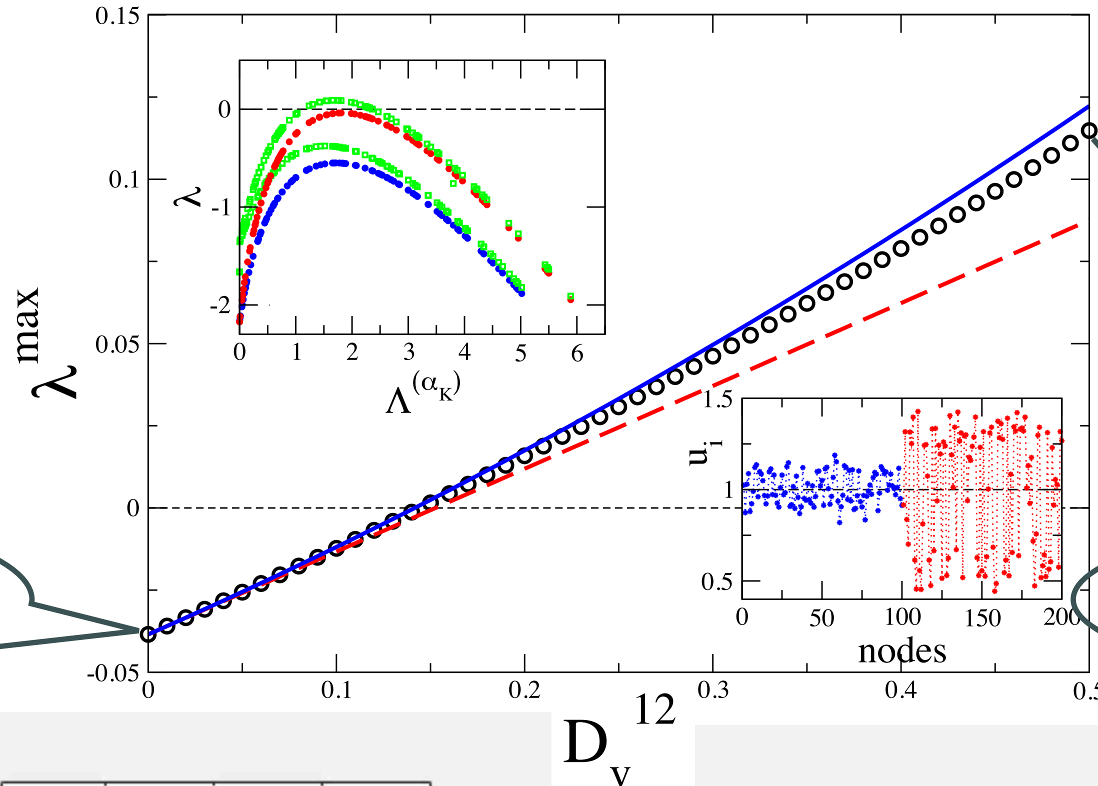
$$= \tilde{\mathcal{J}}_0 + \epsilon \boldsymbol{\mathcal{D}}_0$$

Perturbative approach to compute the spectrum

$$\lambda^{max}(\epsilon) = \lambda_0^{max} + \epsilon (U_0 \boldsymbol{\mathcal{D}}_0 V_0)_{k_{max} k_{max}} + \mathcal{O}(\epsilon^2)$$

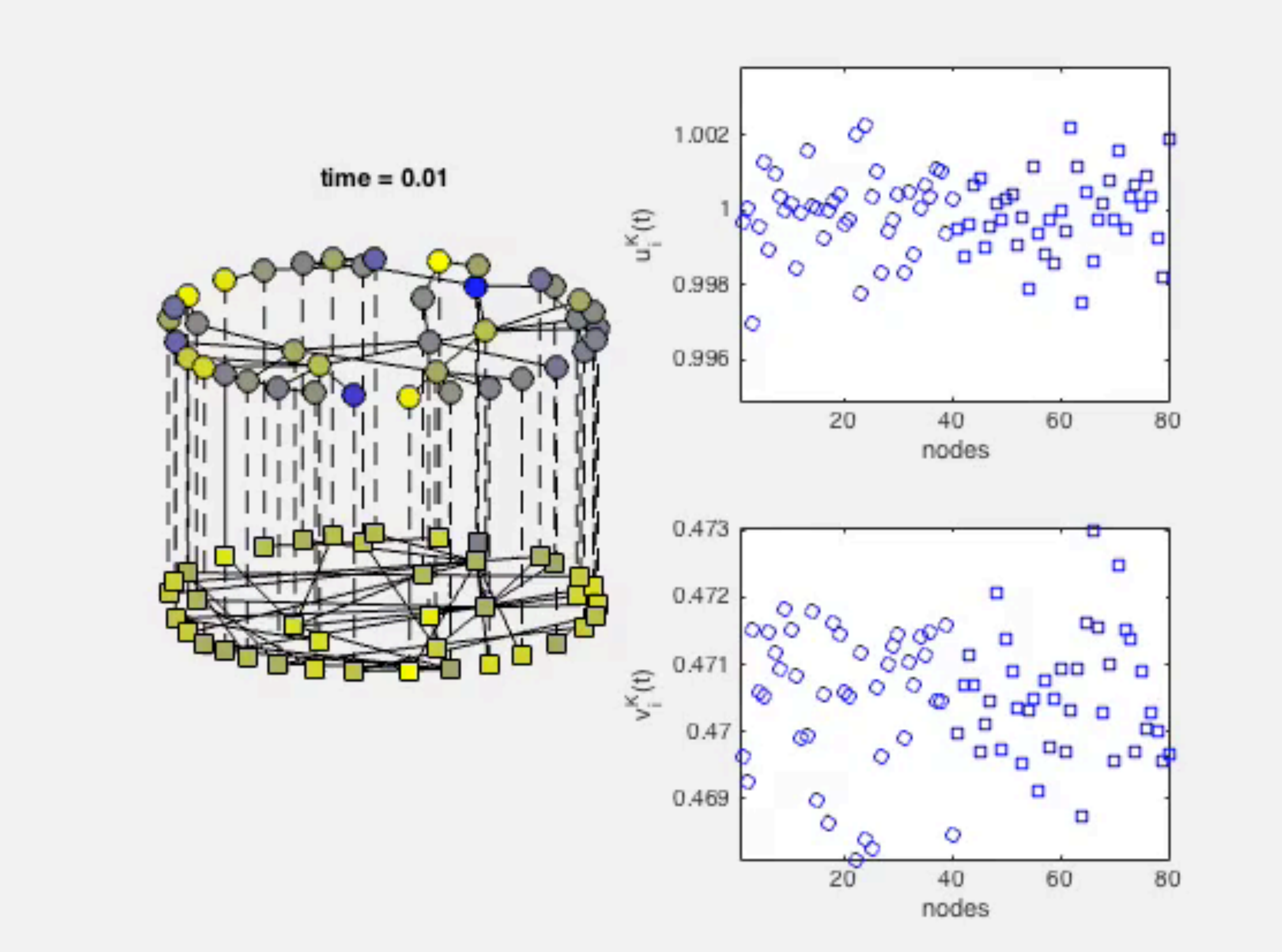
$$\lambda_0^{max} = \max \lambda_k(\epsilon = 0) \qquad k_{max} = \arg\max \lambda_k(\epsilon = 0)$$

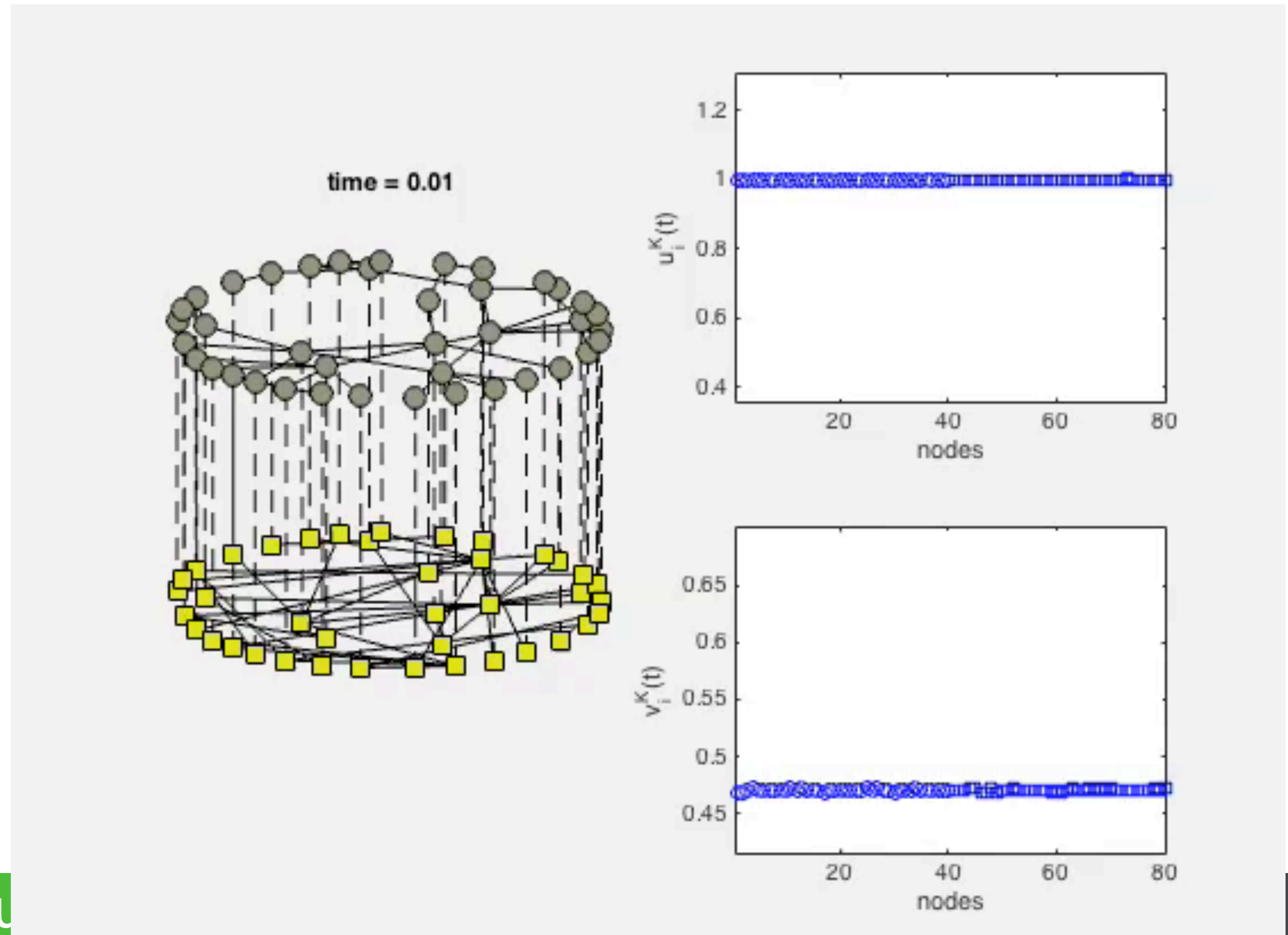
Small intra-layer diffusion case: onset of patterns



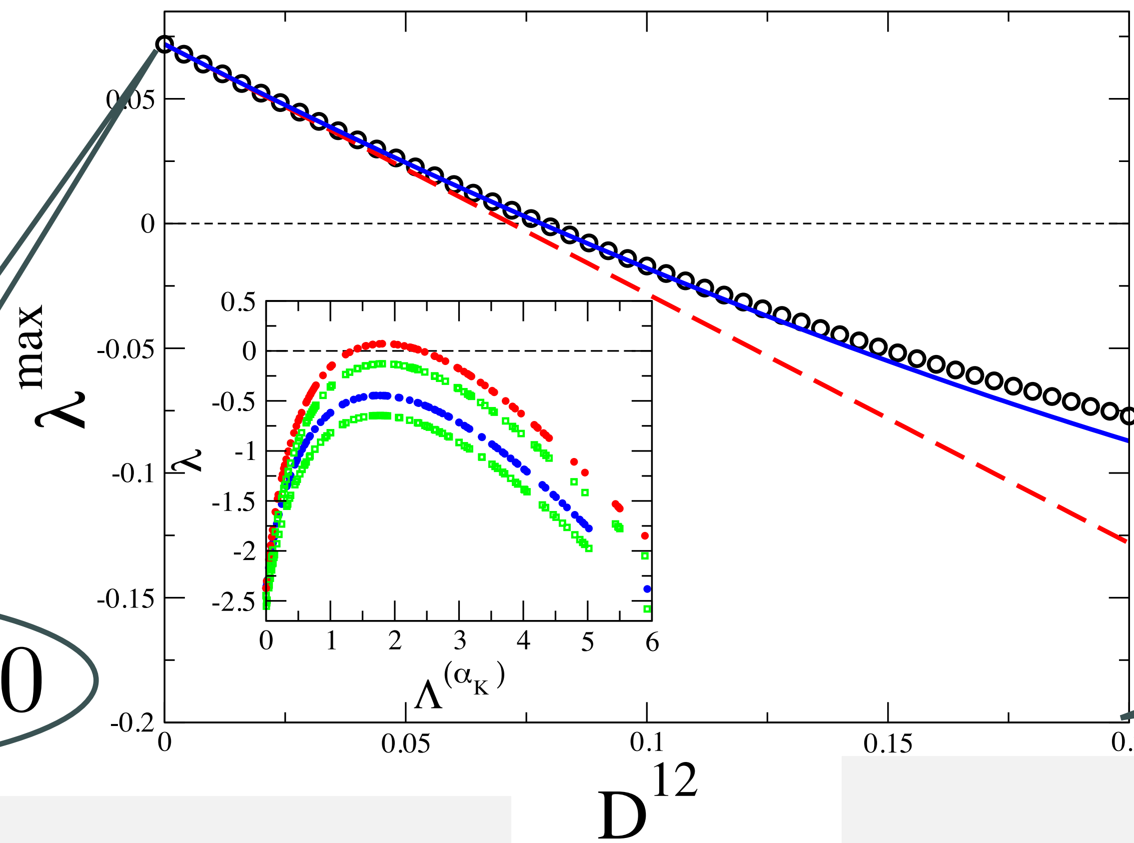
$$D_v^{12} = D_u^{12} = 0$$

$$D_u^{12} = 0 \quad D_v^{12} = 0.5$$



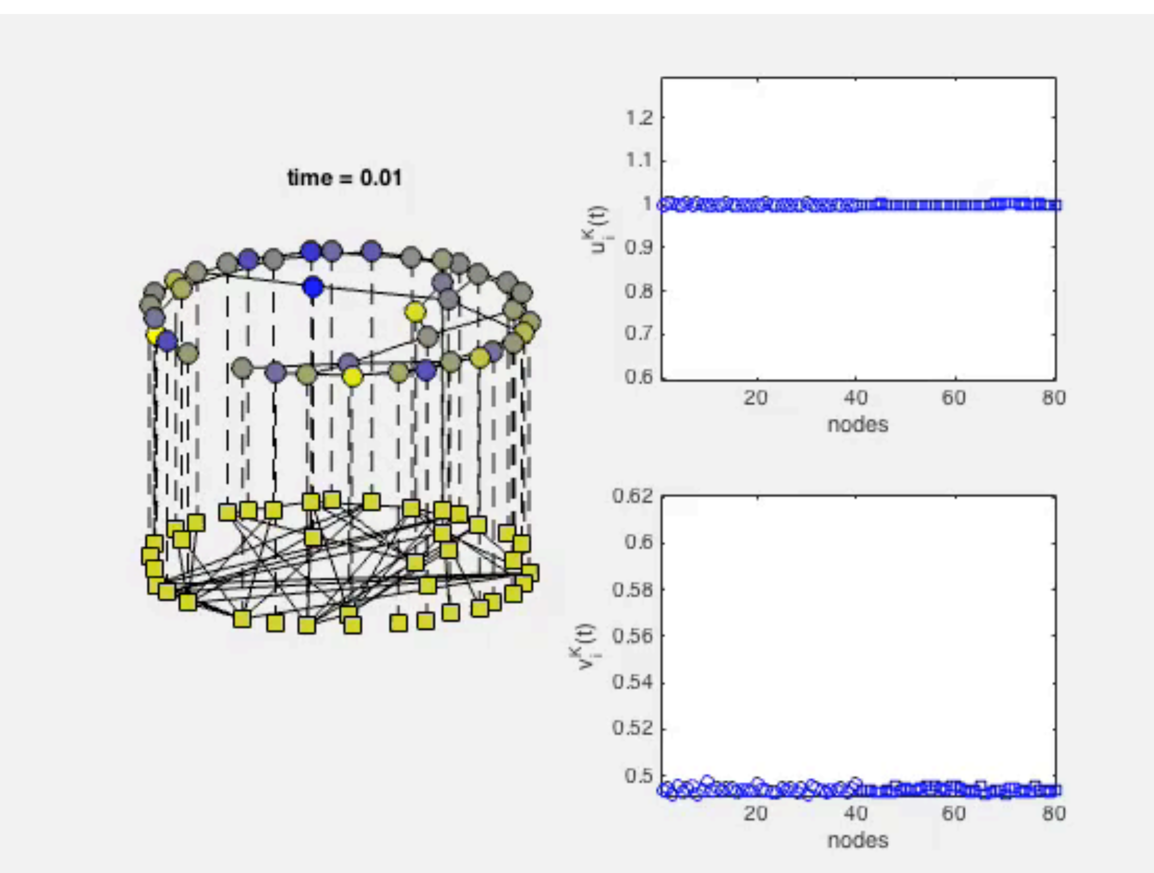


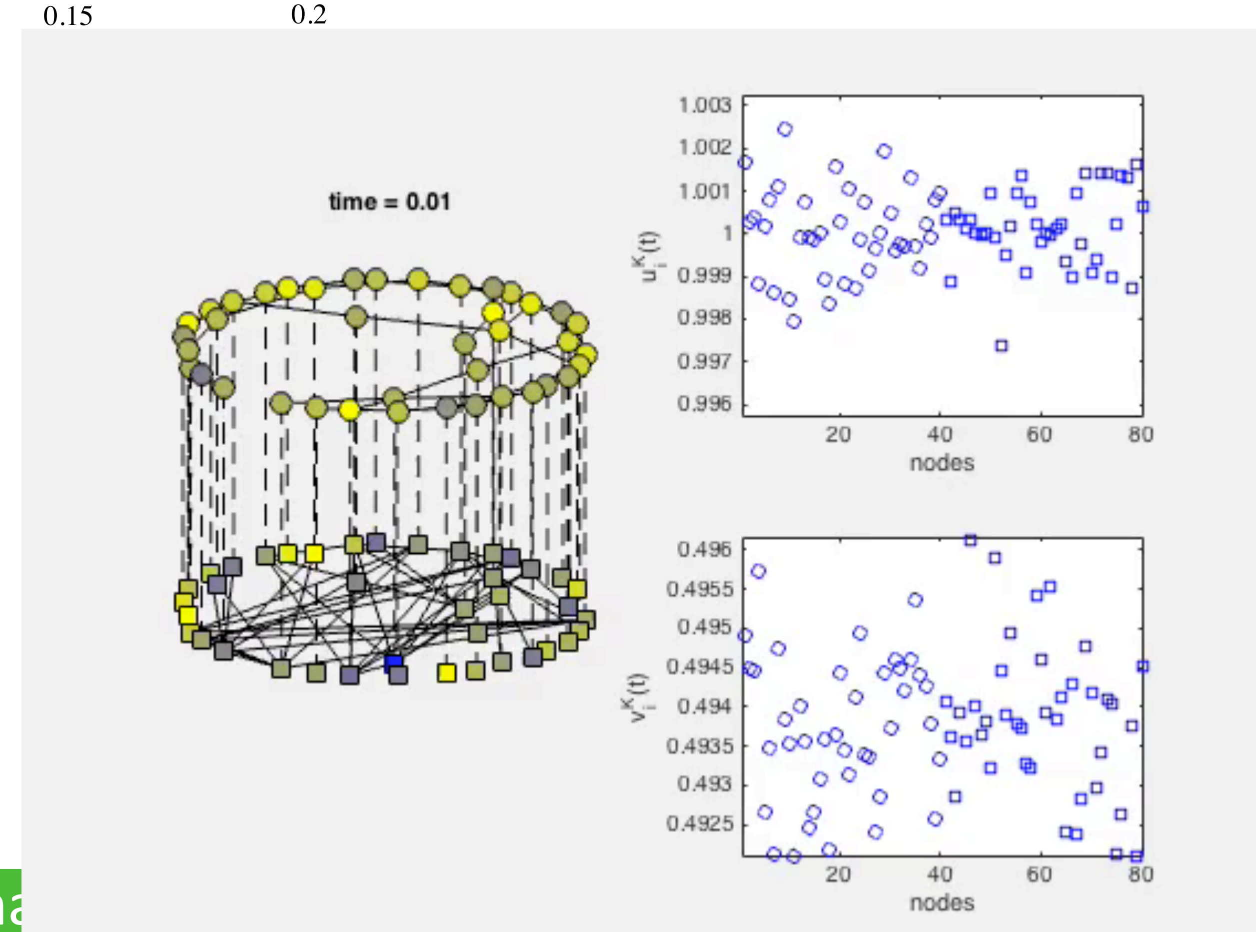
Small intra-layer diffusion case: destruction of patterns



$$D_v^{12} = D_u^{12} = 0$$

$$D_u^{12} = D_v^{12} = 0.2$$

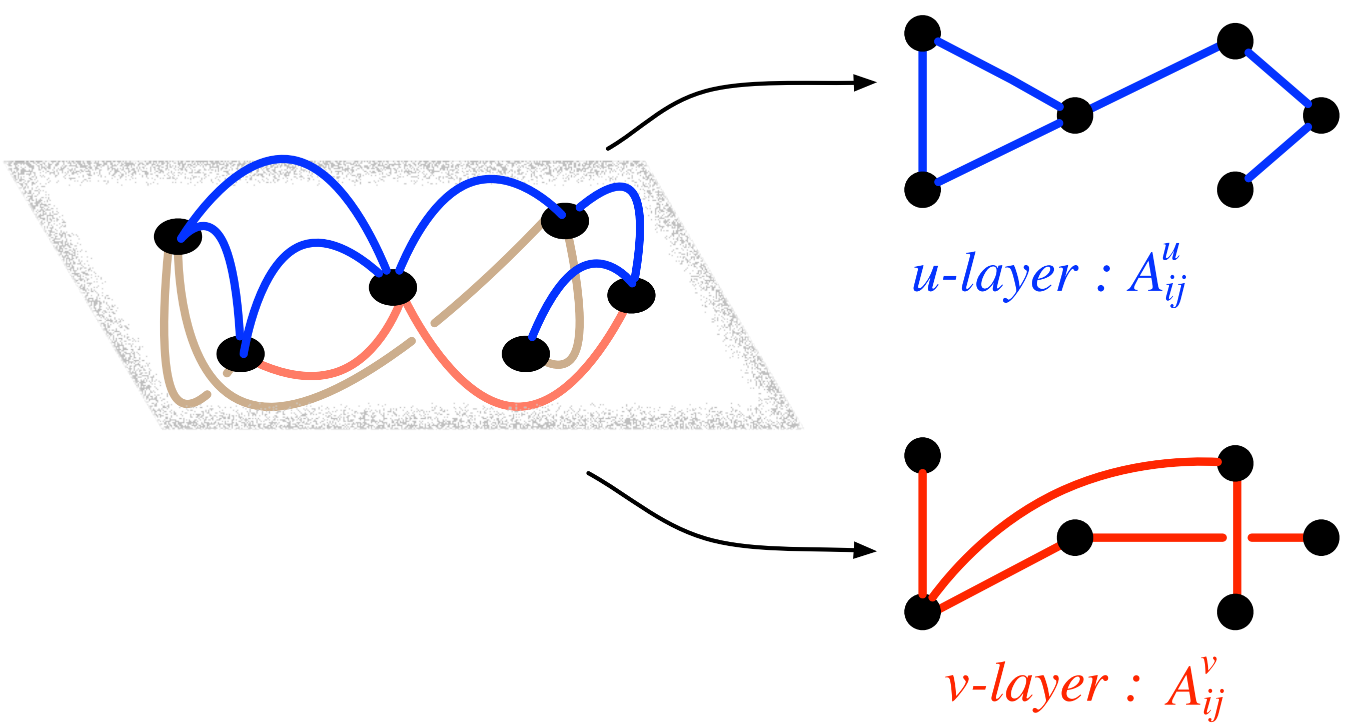




.carletti@una

Can we control the topology to create (destroy) patterns?

Let us consider a <u>multigraph</u>, e.g. two nodes can be connected through different edges



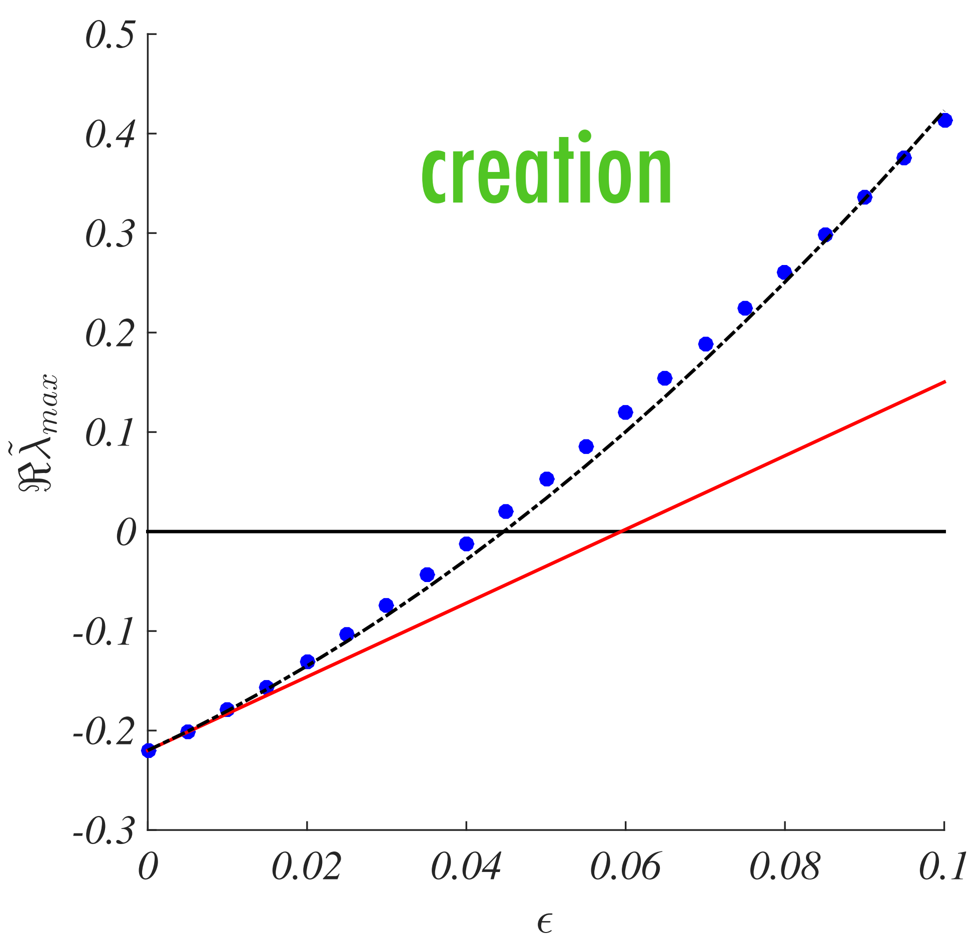
$\epsilon = 0$ $\qquad\qquad$ $\epsilon = 1$

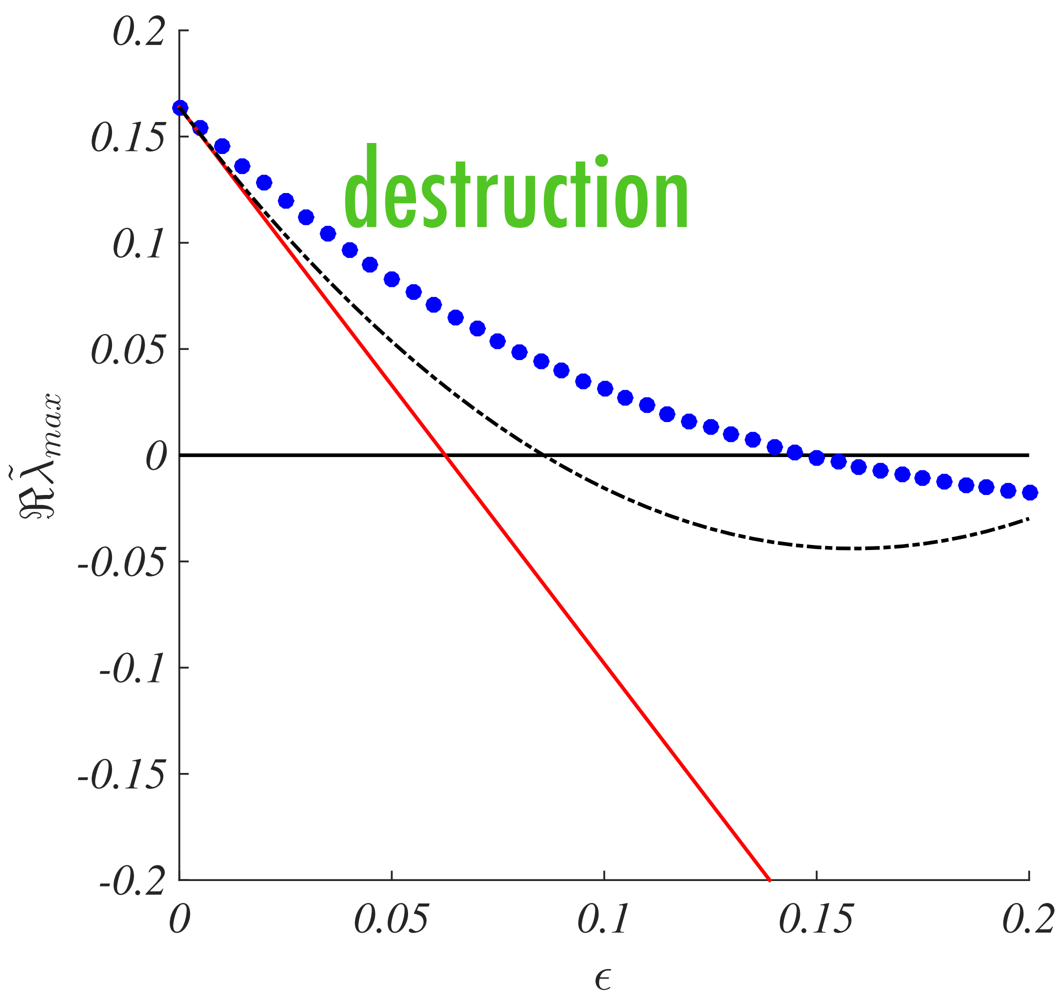
$$A^u(0) = A^0 \qquad A^u(\epsilon) = A^0 + \epsilon(A^1 - A^0) \qquad A^u(1) = A^1$$

$$A^v(0) = A^0 \qquad A^v(\epsilon) = A^0 + \epsilon(A^2 - A^0) \qquad A^v(1) = A^2$$

Can we control the topology to create (destroy) patterns?

theory vs simulations





$\epsilon = 0$ → $\epsilon = 1$

$$A^u(0) = A^0 \qquad A^u(\epsilon) = A^0 + \epsilon(A^1 - A^0) \qquad A^u(1) = A^1$$

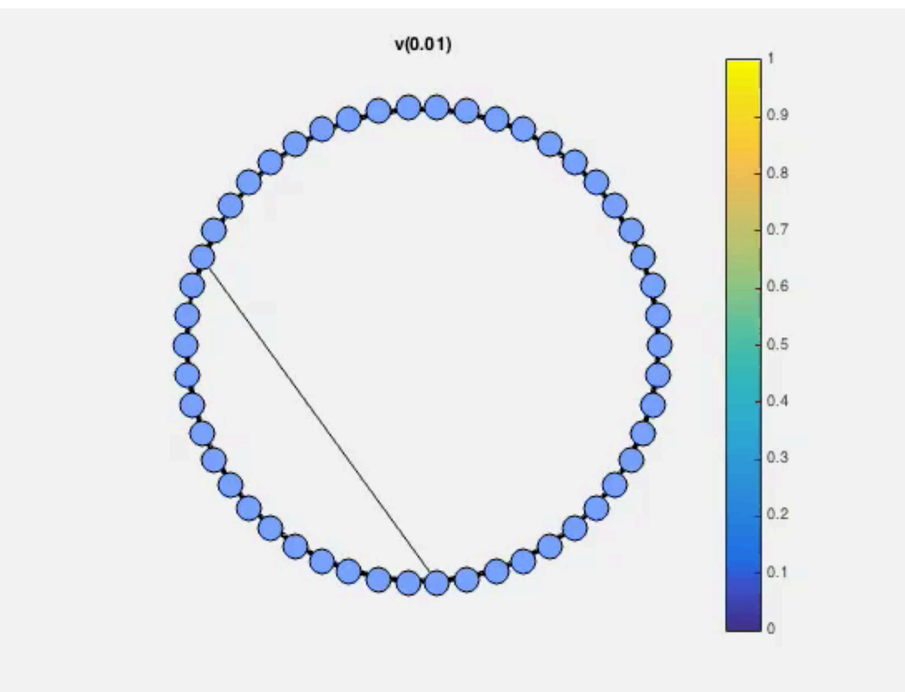
$$A^v(0) = A^0 \qquad A^v(\epsilon) = A^0 + \epsilon(A^2 - A^0) \qquad A^v(1) = A^2$$

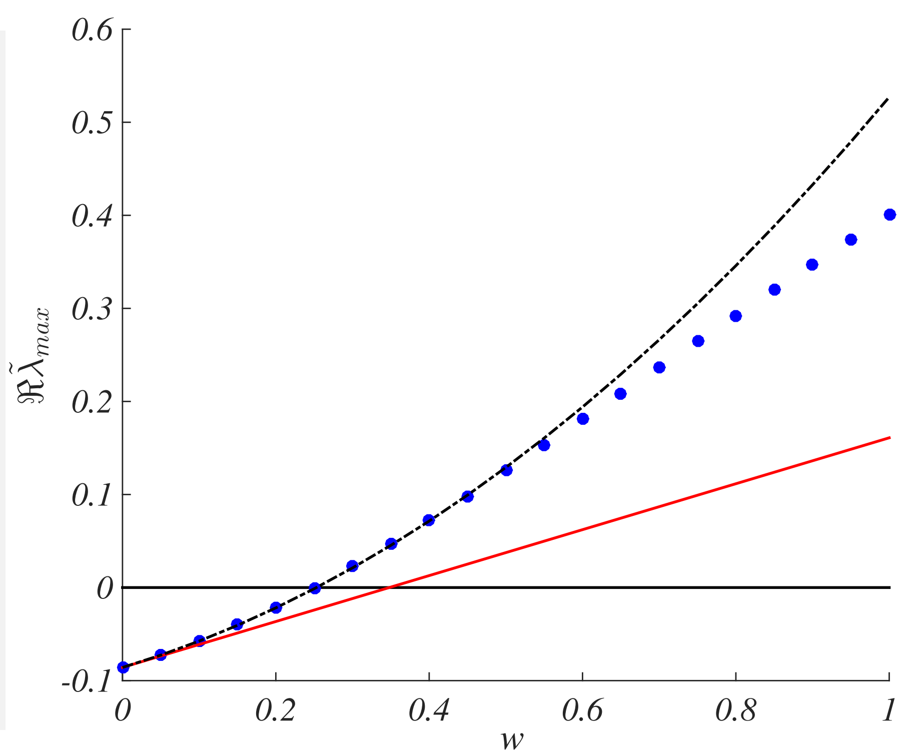
timoteo.carletti@unamur.be

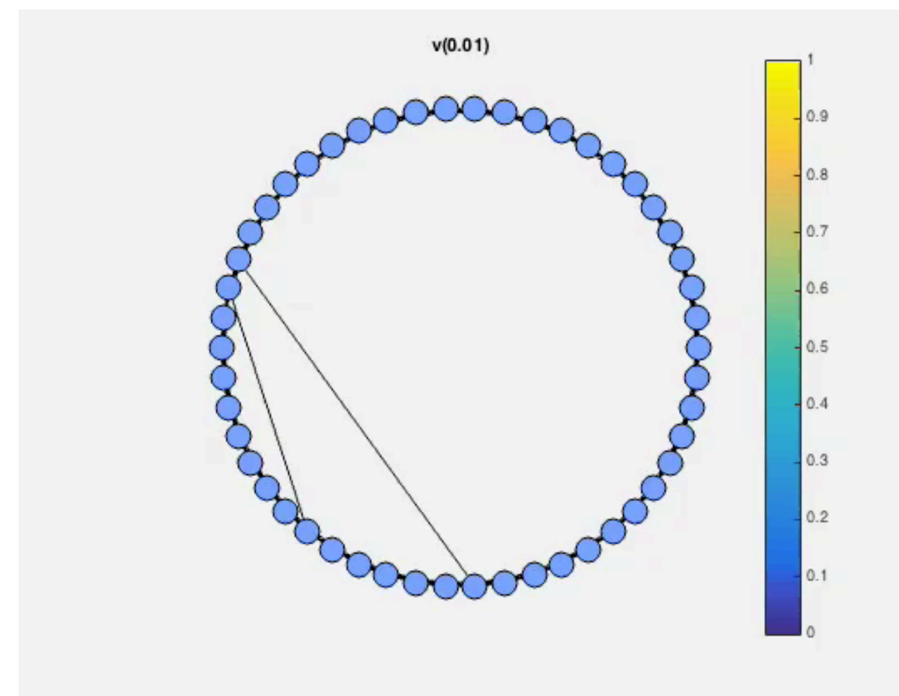
Can we control the topology to create (destroy) patterns?

Create patterns by adding a single (optimally chosen) link

$$A^u(w) = A^0 \qquad\qquad A^v(w) = A^0 + wT^{(ij)}$$



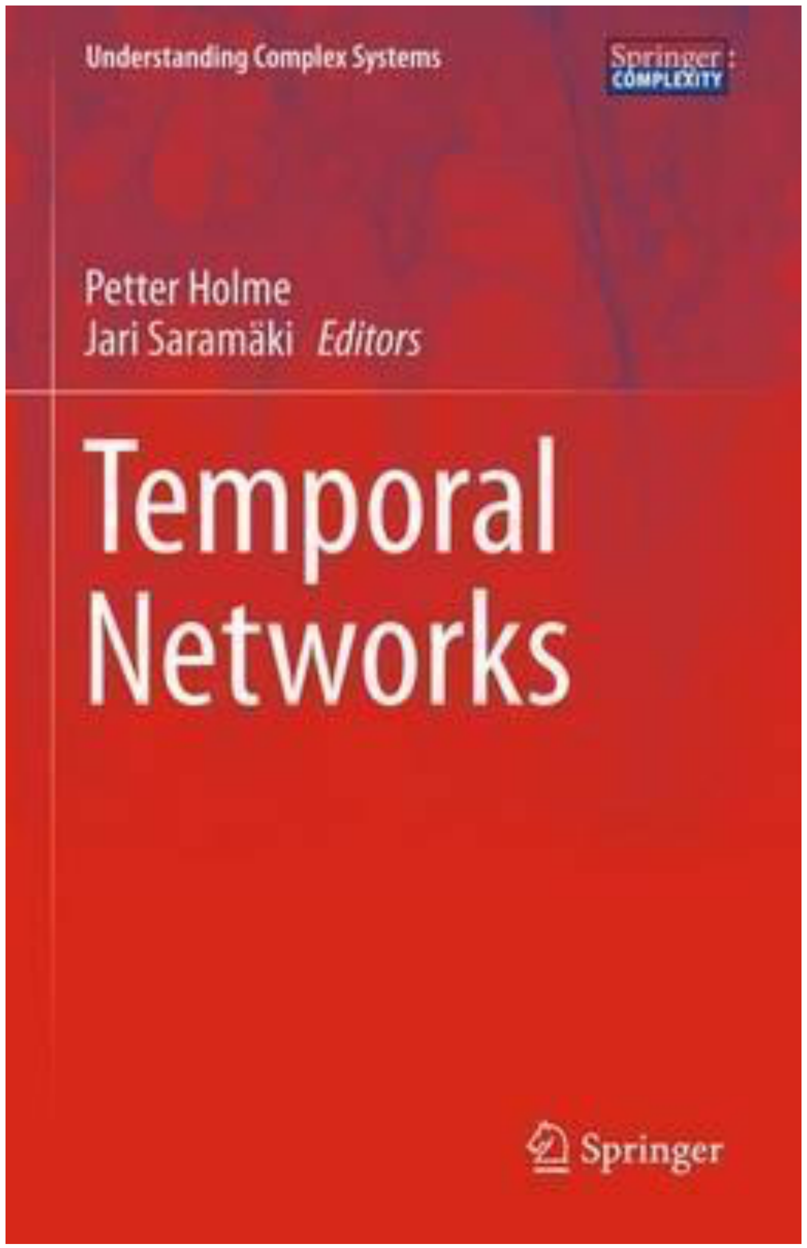


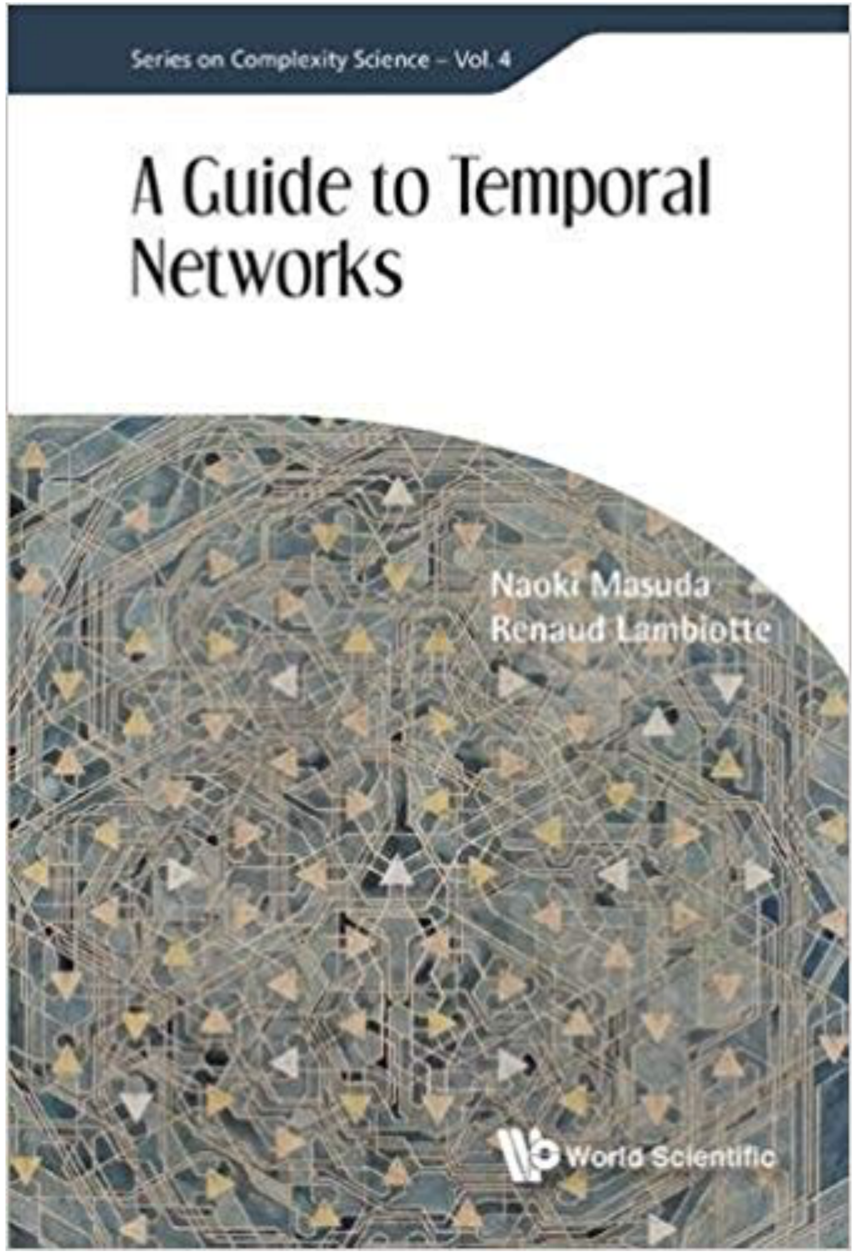


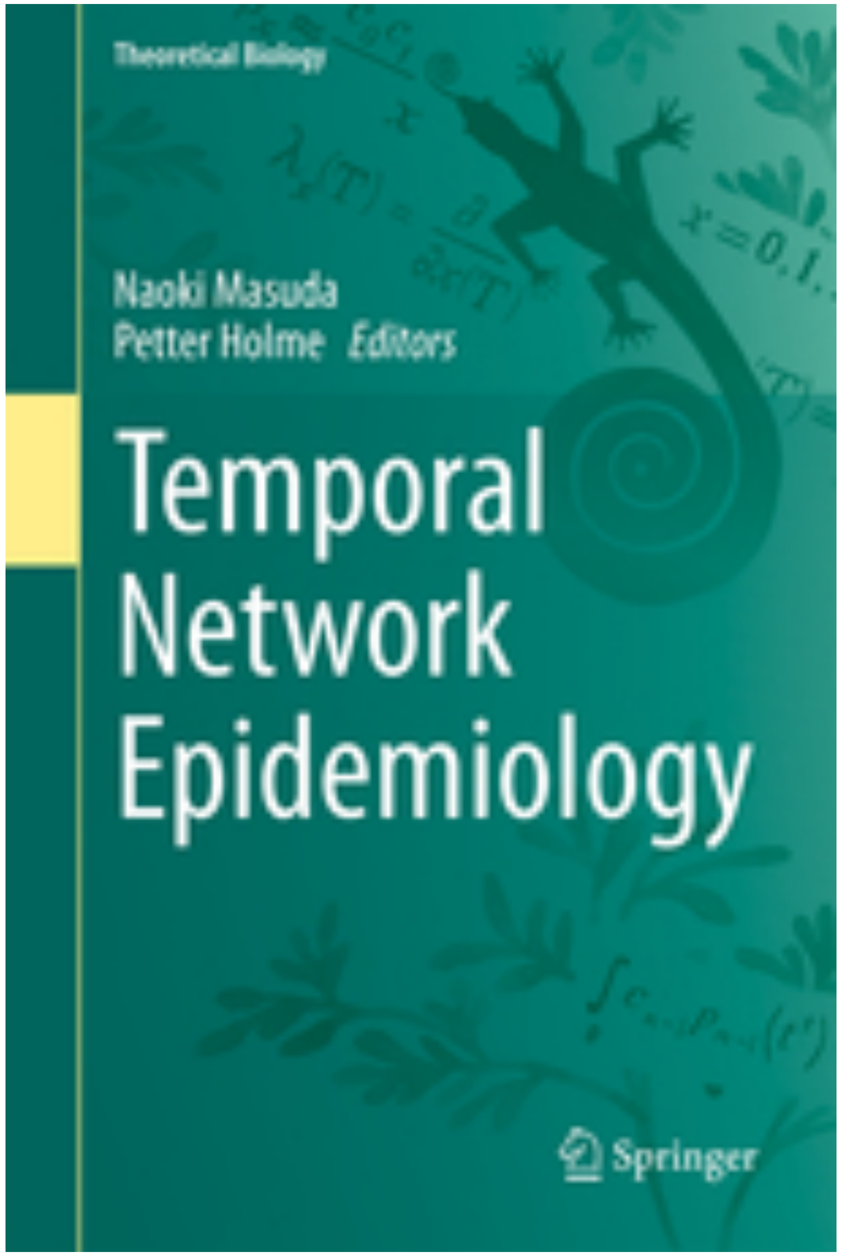
w=0

w=1

Temporal networks



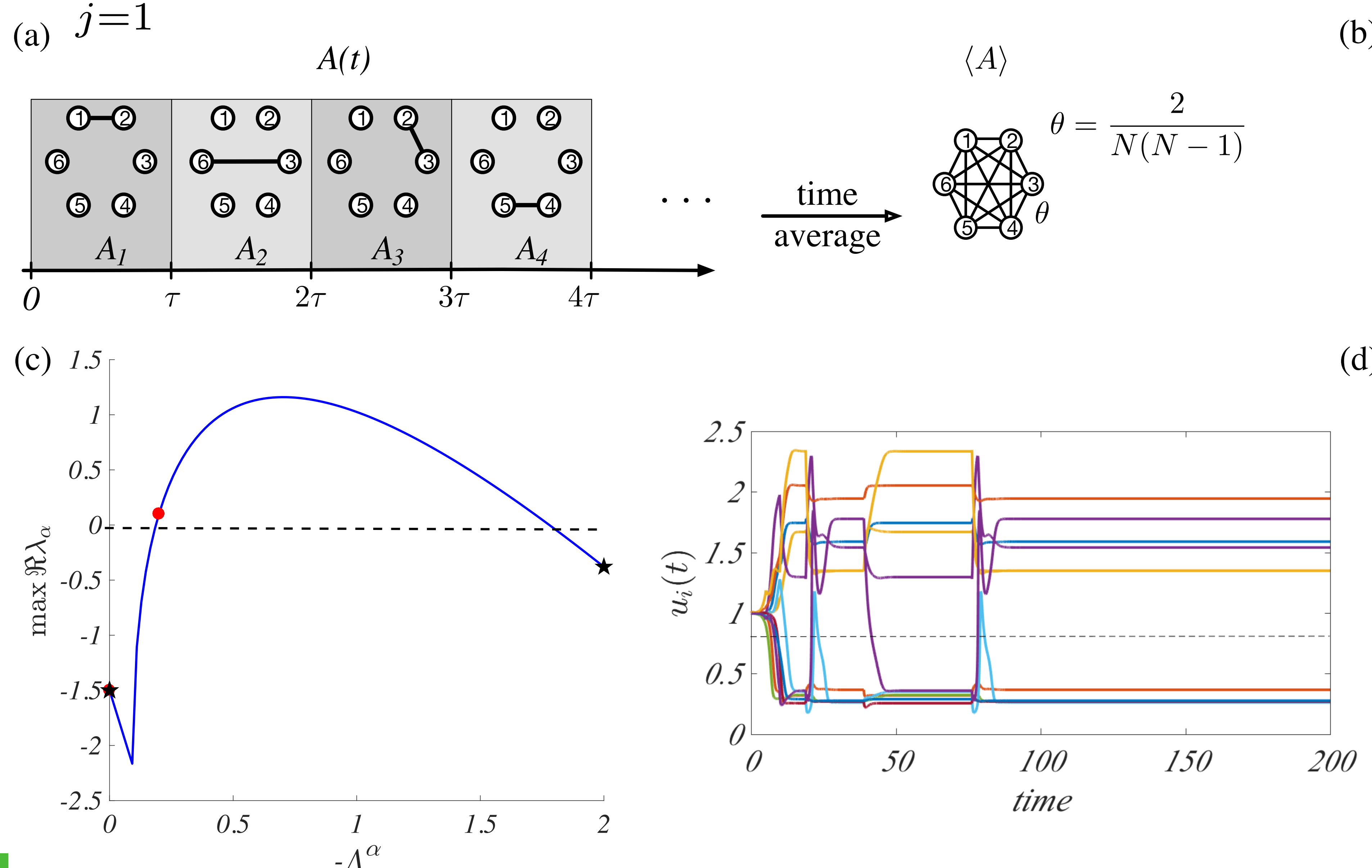




Temporal networks: fast switch

$$\dot{u}_i(t) = f(u_i, v_i) + D_u \sum_{j=1}^{N} L_{ij}(t/\epsilon) u_j(t)$$

$$\dot{v}_i(t) = g(u_i, v_i) + D_v \sum_{j=1}^{N} L_{ij}(t/\epsilon) v_j(t)$$



Temporal networks: generic case

$$\frac{d\mathbf{x}_i}{dt} = \mathbf{F}(\mathbf{x}_i) + \varepsilon \sum_j L_{ij}(t) \mathbf{H}(\mathbf{x}_j),$$

$$\mathbf{L}(t)\vec{\phi}^{(\alpha)}(t) = \Lambda^{(\alpha)}(t)\vec{\phi}^{(\alpha)}(t) \quad \forall \alpha = 1, \ldots, n \text{ and} \forall t \qquad \left(\vec{\phi}^{(\alpha)}(t)\right)^T \cdot \vec{\phi}^{(\beta)}(t) = \delta_{\alpha\beta}$$

$$\frac{d\vec{\phi}^{(\alpha)}}{dt}(t) = \sum_\beta c_{\alpha\beta}(t)\vec{\phi}^{(\beta)}(t) \quad \forall \alpha = 1, \ldots, n.$$

$c_{\alpha\beta} + c_{\beta\alpha} = 0$ and $c_{1\alpha} = 0$.

Temporal networks: generic case

$$\frac{d\mathbf{x}_i}{dt} = \mathbf{F}(\mathbf{x}_i) + \varepsilon \sum_j L_{ij}(t)\mathbf{H}(\mathbf{x}_j),$$

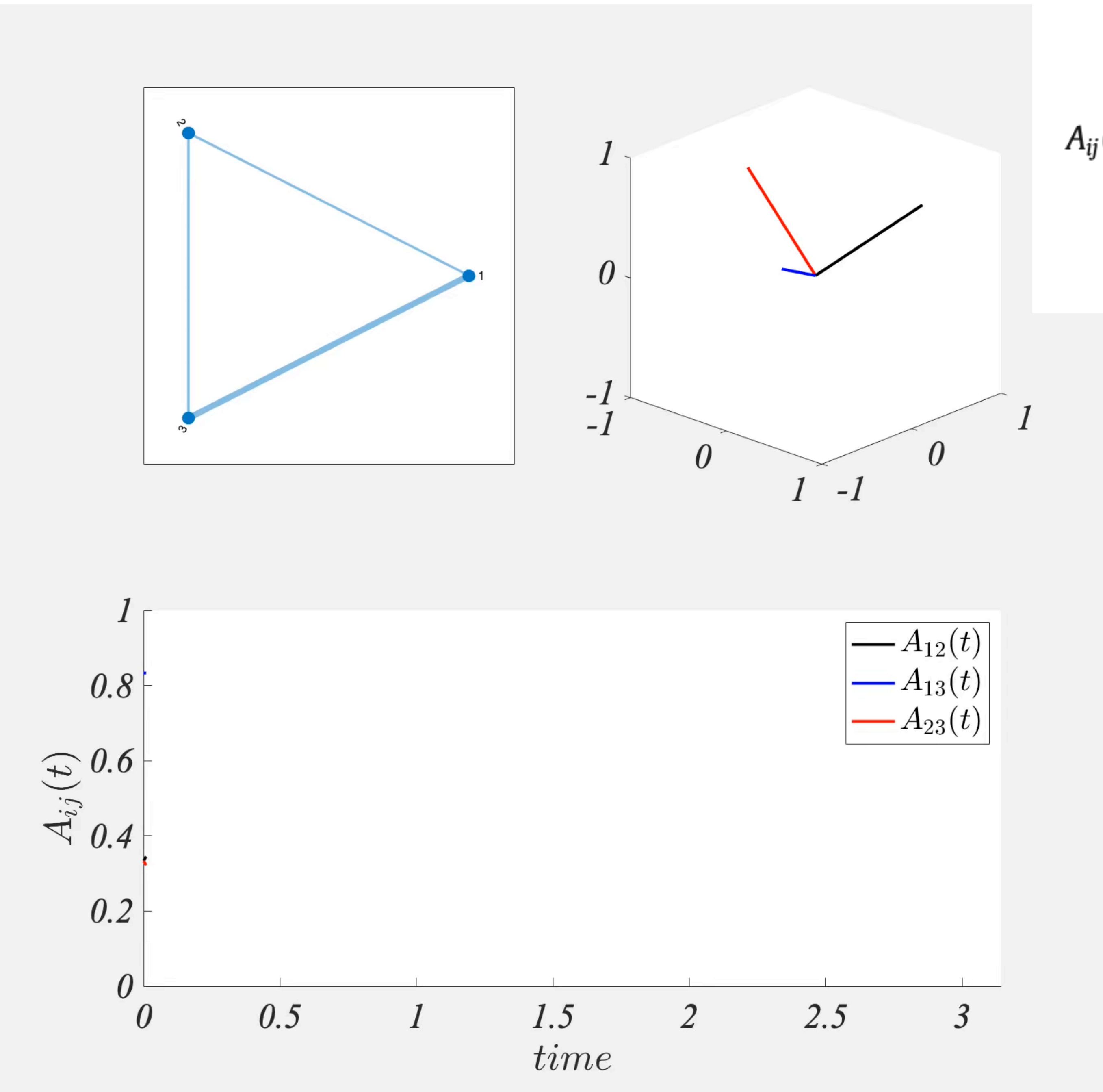
Linearisation $\delta\mathbf{x}_i = \mathbf{x}_i - \mathbf{s}$

$$\frac{d\delta\mathbf{x}_i}{dt} = \mathbf{J_F}(\mathbf{s}(t))\delta\mathbf{x}_i + \varepsilon \sum_j L_{ij}(t)\mathbf{J_H}(\mathbf{s}(t))\delta\mathbf{x}_j,$$

Projection eigenbasis

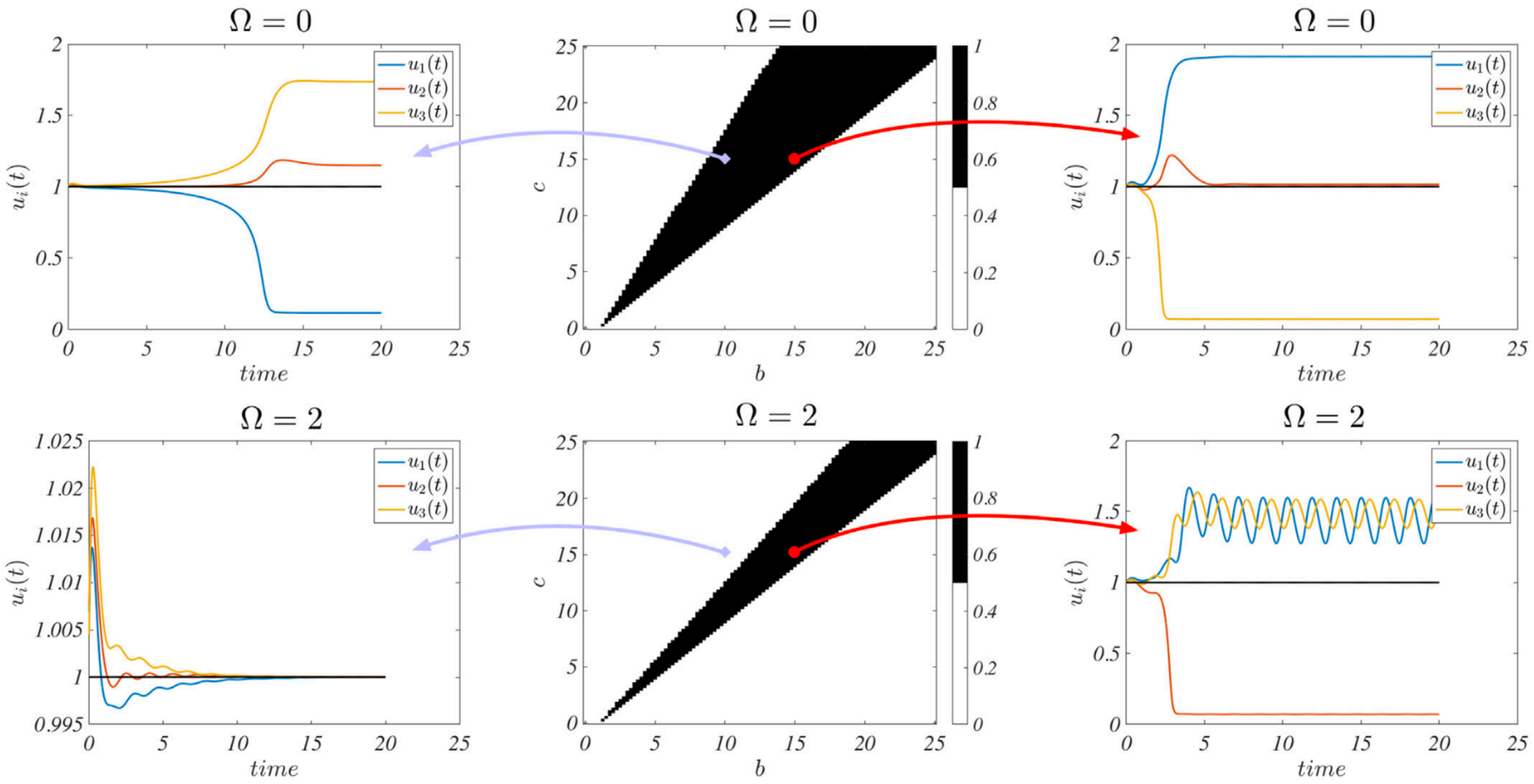
$$\frac{d\delta\widehat{\mathbf{x}}_\beta}{dt} = \sum_\alpha c_{\beta\alpha}(t)\delta\widehat{\mathbf{x}}_\alpha + \left[\mathbf{J_F}(\mathbf{s}(t)) + \varepsilon\Lambda^{(\beta)}(t)\mathbf{J_H}(\mathbf{s}(t))\right]\delta\widehat{\mathbf{x}}_\beta.$$

Temporal networks: generic case

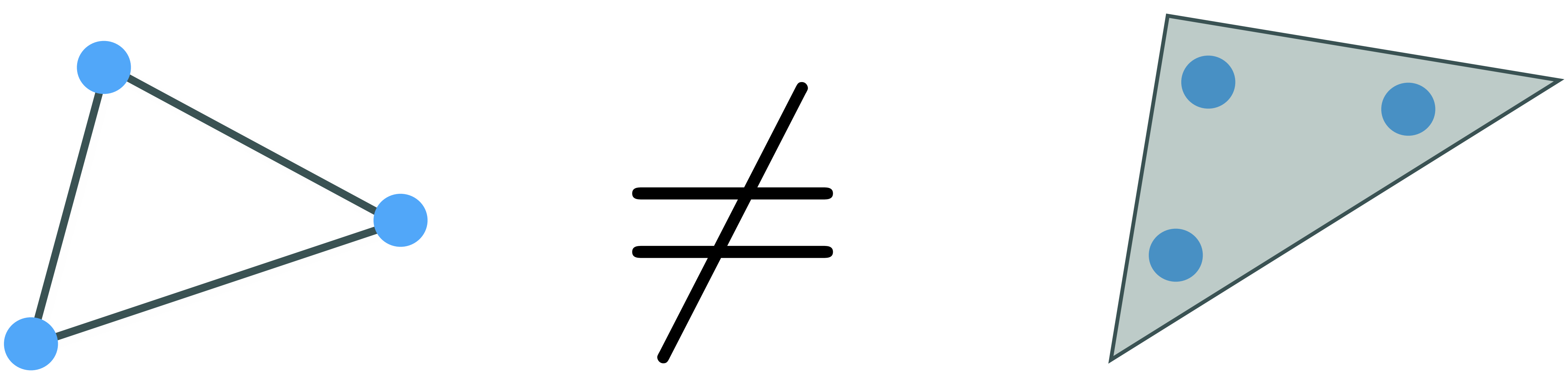


$$A_{ij}(t) = \begin{pmatrix} 0 & \frac{1}{2} - \frac{\cos\left(\frac{\pi}{3} + 2\Omega t\right)}{3} & \frac{\cos(2\Omega t)}{3} + \frac{1}{2} \\ \frac{1}{2} - \frac{\cos\left(\frac{\pi}{3} + 2\Omega t\right)}{3} & 0 & \frac{1}{2} - \frac{\cos\left(\frac{\pi}{3} - 2\Omega t\right)}{3} \\ \frac{\cos(2\Omega t)}{3} + \frac{1}{2} & \frac{1}{2} - \frac{\cos\left(\frac{\pi}{3} - 2\Omega t\right)}{3} & 0 \end{pmatrix}$$

Turing patterns on temporal networks: generic case

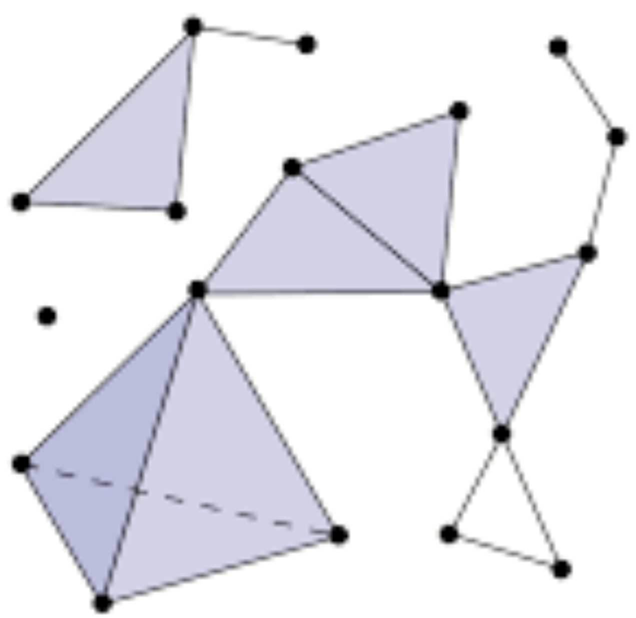


networks limitation



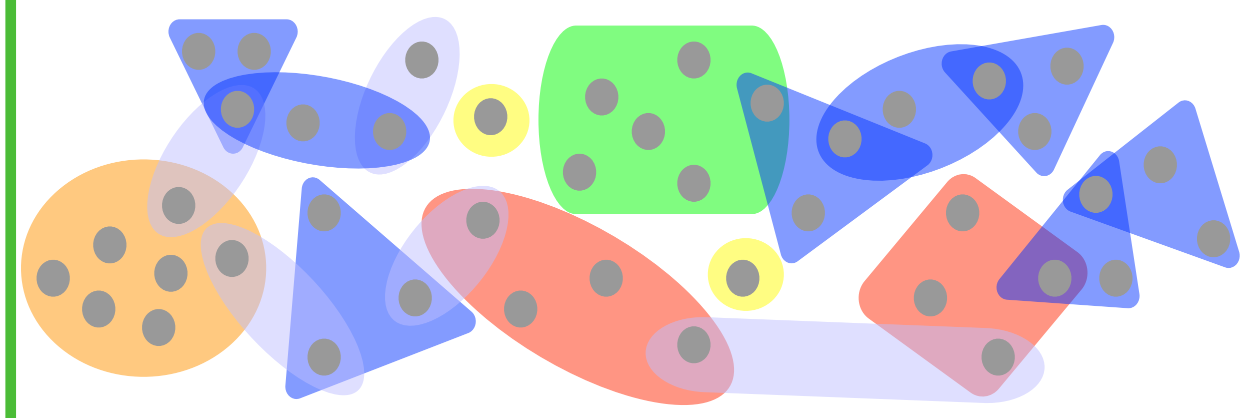
Simplicial complexes and Hypergraphs

Simplicial complexes



d-simplex = d+1 nodes
(all linked together)
0-simplex = node
1-simplex = link
2-simplex = triangle
3-simplex = tetrahedron

Hypergraphs

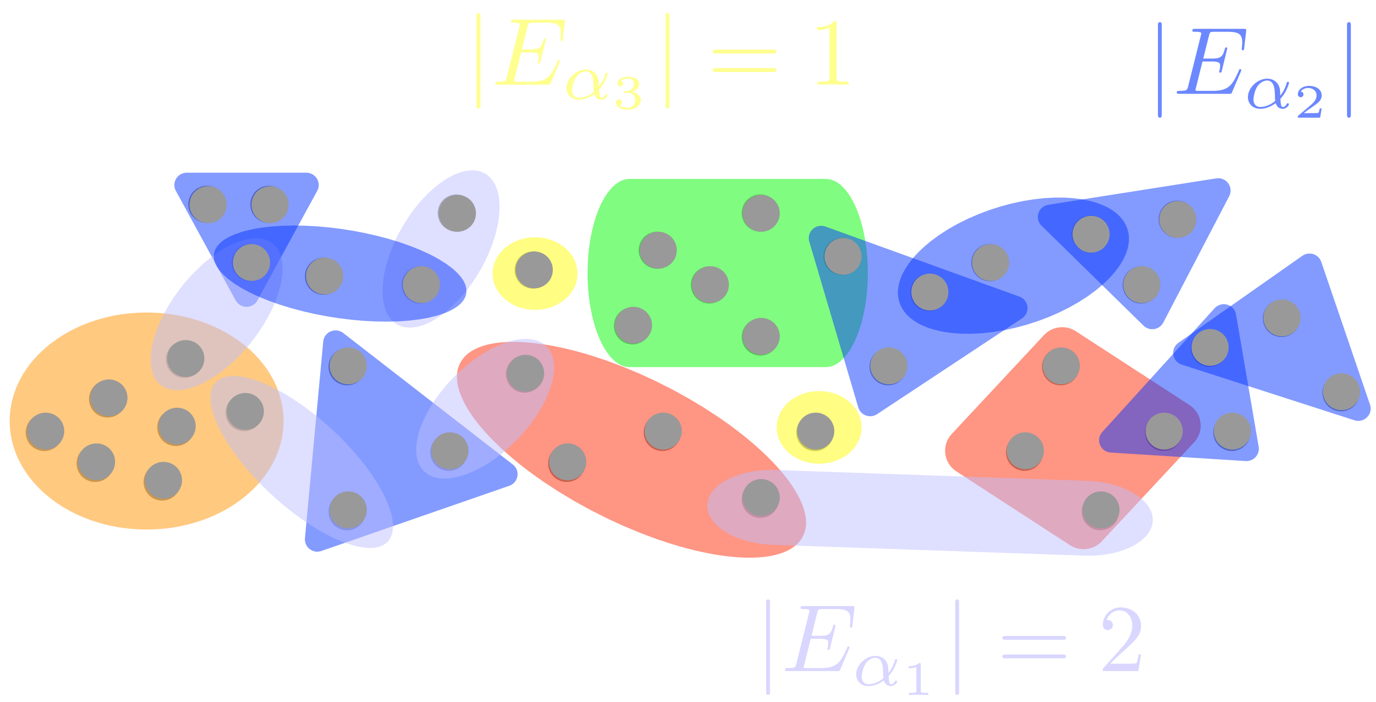


hyperedge = set of nodes

Hypergraphs

timoteo.carletti@unamur.be

Hypergraphs. Some definitions.



ensemble of nodes
=
hyperedges

Incidence matrix

$$e_{i\alpha} = 1 \quad \text{iff } i \in E_\alpha$$

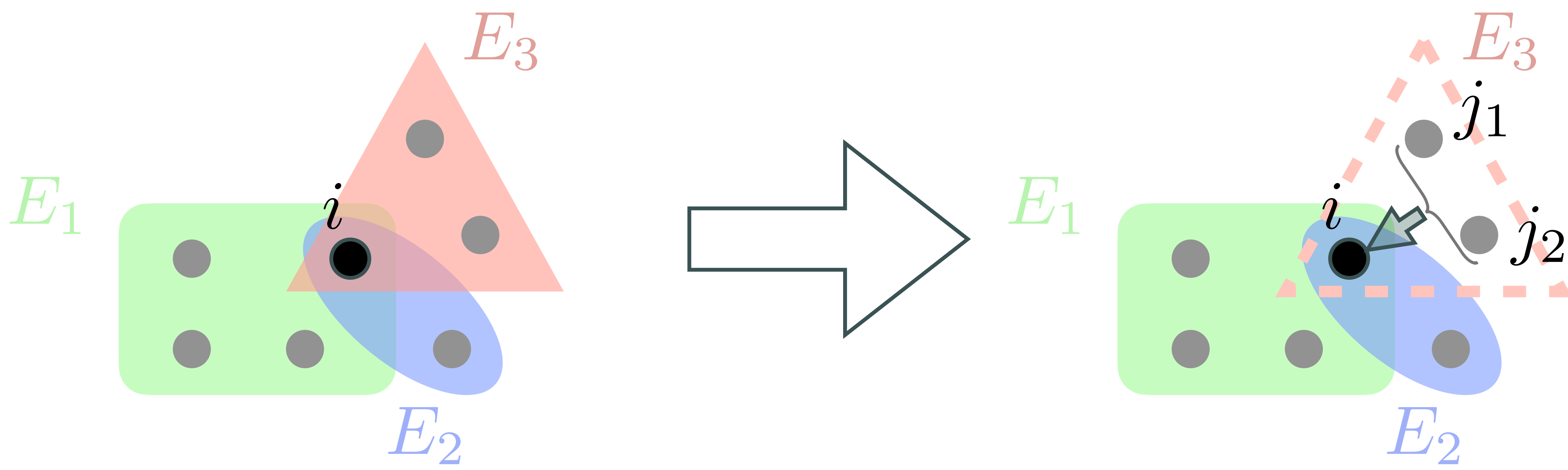
Hyperadjacency matrix

$$A = ee^{\mathrm{T}}$$

Hyperedge matrix

$$C = e^{\mathrm{T}}e$$

Hyperedge Mean Field



non-linearity

$$k_{ij}^H = \sum_\alpha (C_{\alpha\alpha} - 1)^\tau e_{i\alpha} e_{j\alpha}$$

hyperedge size

incidence matrices

Turing patterns on hypergraphs

$$\frac{\mathrm{d}\mathbf{x}_i}{\mathrm{d}t} = \mathbf{F}(\mathbf{x}_i) - \varepsilon \sum_{\alpha,j} e_{i\alpha} e_{j\alpha} (C_{\alpha\,\alpha} - 1) \left(\mathbf{G}(\mathbf{x}_i) - \mathbf{G}(\mathbf{x}_j)\right)$$

$$\tau = 1$$

$$= \mathbf{F}(\mathbf{x}_i) - \varepsilon \sum_{j} k_{ij}^{H} \left(\mathbf{G}(\mathbf{x}_i) - \mathbf{G}(\mathbf{x}_j)\right) = \mathbf{F}(\mathbf{x}_i) - \varepsilon \sum_{j} \left(\delta_{ij} k_i^{H} - k_{ij}^{H}\right) \mathbf{G}(\mathbf{x}_j)$$

$$= \mathbf{F}(\mathbf{x}_i) - \varepsilon \sum_{j} L_{ij}^{H} \mathbf{G}(\mathbf{x}_j),$$

L_{ij}^{H} Higher-order Laplace matrix

Turing patterns on hypergraphs

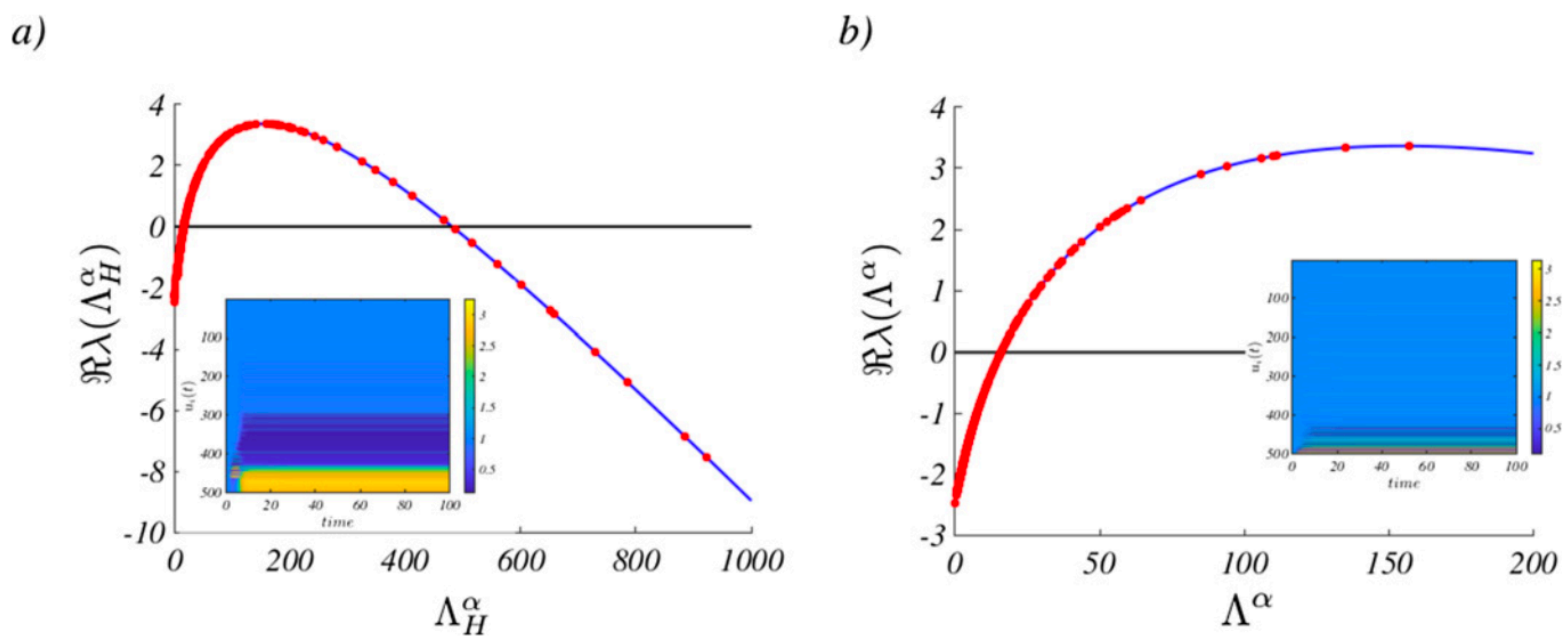
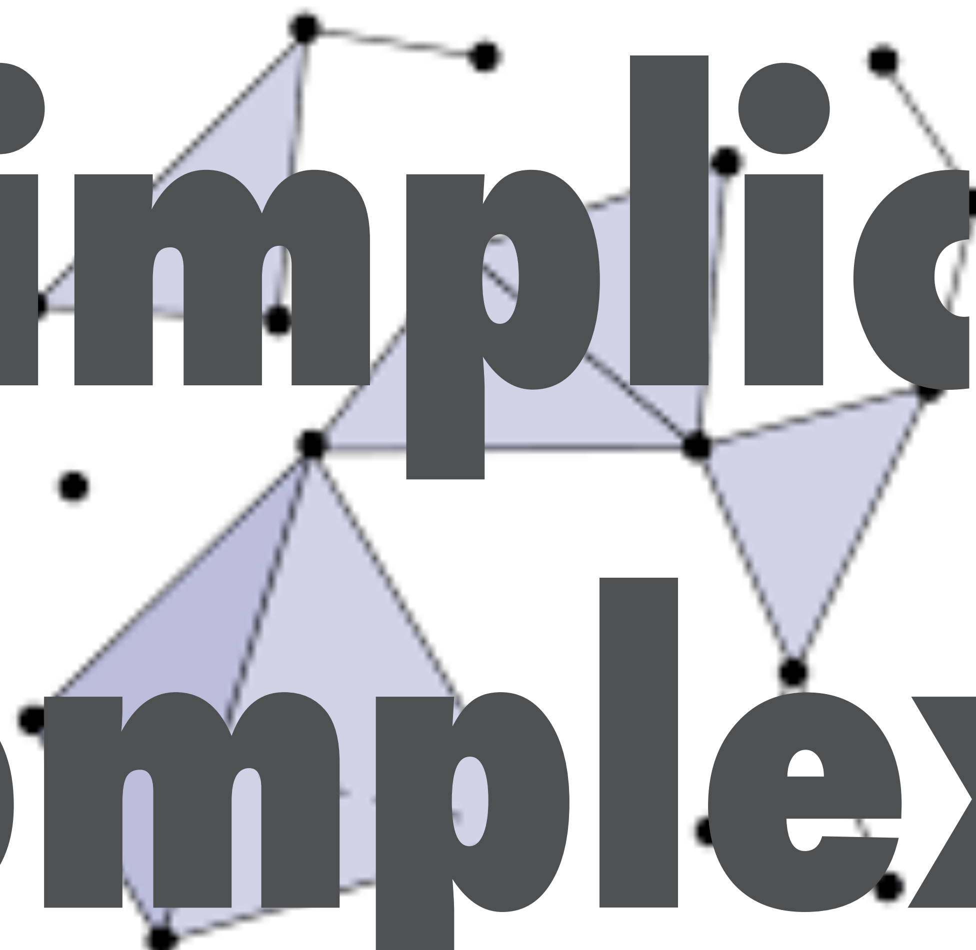


Figure 5. Turing patterns on hypergraphs. Main panels: the dispersion relation for the Brusselator model defined on the hypergraph—panel (a)—and the projected network—panel (b). One can observe that in both cases there are eigenvalues for which the dispersion relation is positive (red dots); the blue line represents the dispersion relation for the Brusselator model defined on a continuous regular support. Being both Laplace matrices symmetric, the dispersion relation computed for the discrete spectra lies on top of the one obtained for the continuous support. Insets: the Turing patterns on the hypergraph (panel (a)) and the projected network (panel (b)). We report the time evolution of the concentration of the species $u_i(t)$ in each node as a function of time, by using an appropriate colour code (yellow associated to large values, blue to small ones). In the former case, nodes are ordered for increasing hyper degree while in the second panel for increasing degree. One can hence conclude that nodes associated to large hyper degrees display a large concentration amount for species u_i. This yields a very localised pattern. The hypergraph and the projected network are the same used in figure 2.

Simplicial complexes



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Higher-order (many-body) interactions

$$\begin{cases} \dfrac{du_i}{dt} = f_1(u_i, v_i) + \displaystyle\sum_{d=1}^{P} \sigma_d \sum_{j_1=1}^{N} \cdots \sum_{j_d=1}^{N} A^{(d)}_{i,j_1,\dots,j_d} \Big[h_1^{(d)}(u_{j_1}, \dots, u_{j_d}, v_{j_1}, \dots, v_{j_d}) \\ \qquad\qquad - h_1^{(d)}(u_i, \dots, u_i, v_i, \dots, v_i) \Big] \\ \dfrac{dv_i}{dt} = f_2(u_i, v_i) + \displaystyle\sum_{d=1}^{P} \sigma_d \sum_{j_1=1}^{N} \cdots \sum_{j_d=1}^{N} A^{(d)}_{i,j_1,\dots,j_d} \Big[h_2^{(d)}(u_{j_1}, \dots, u_{j_d}, v_{j_1}, \dots, v_{j_d}) \\ \qquad\qquad - h_2^{(d)}(u_i, \dots, u_i, v_i, \dots, v_i) \Big] \end{cases}$$

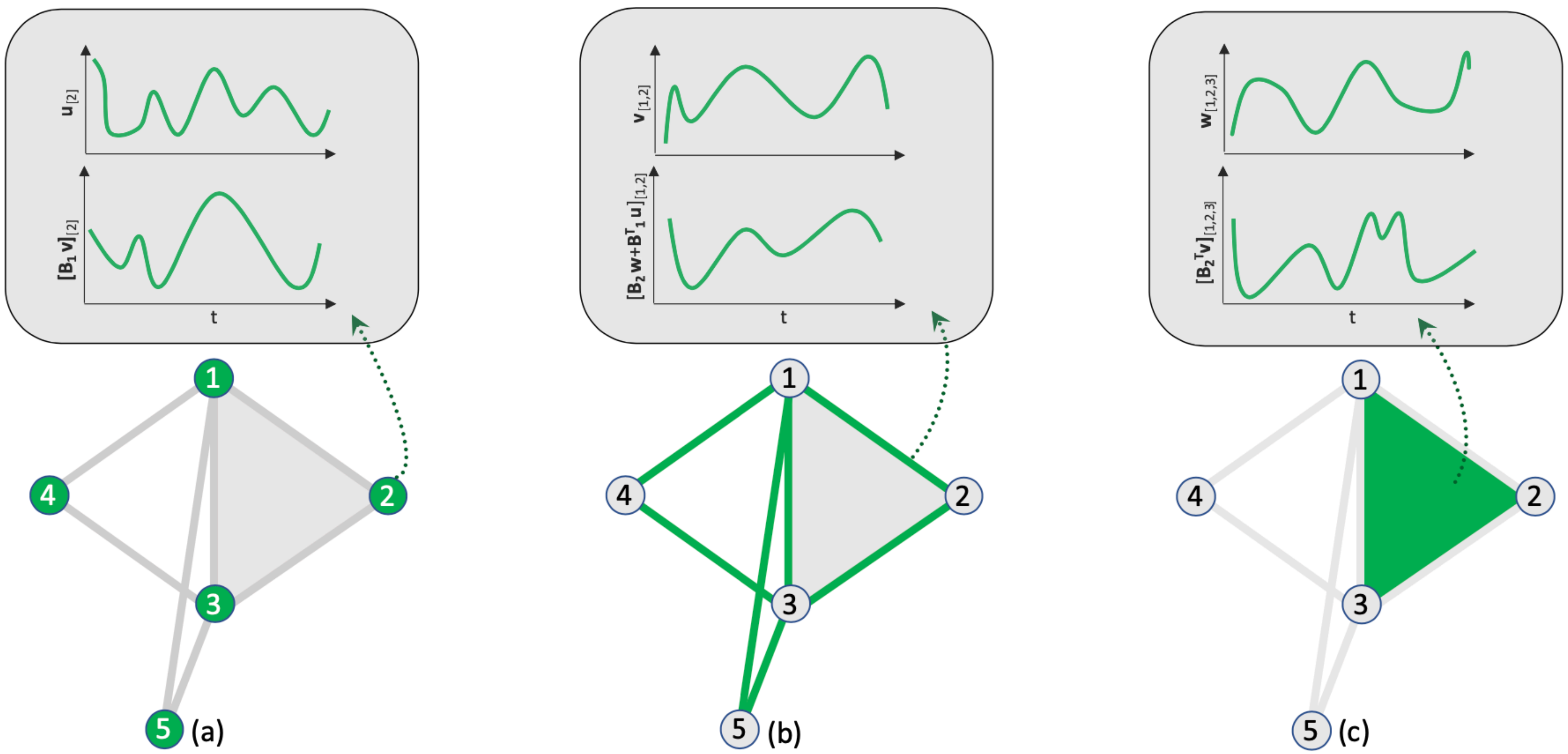
Equilibrium $f_1(u^*, v^*) = f_2(u^*, v^*) = 0$

Linearize $\delta u_i = u_i - u^*, \ \delta v_i = v_i - v^*$

d = 2 $\dfrac{d\vec{\xi}}{dt} = \left(\mathbb{I}_N \otimes \mathbf{J}_0 + \sigma_1 \mathbf{L}^{(1)} \otimes \mathbf{J}_{H^{(1)}} + \sigma_2 \mathbf{L}^{(2)} \otimes \mathbf{J}_{H^{(2)}}\right) \vec{\xi}$

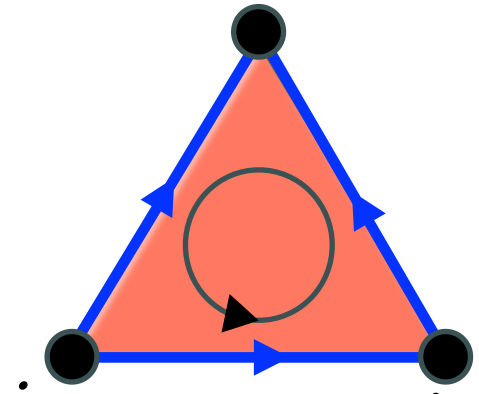
Note: need for assumptions on L or H

Topological signals



Simplicial complex: an example

$k = 2$ Three nodes, hence a triangle $\sigma^{(2)} = [i_0, i_1, i_2]$



$\sigma_1^{(1)} = [i_0, i_1]$ $\sigma_2^{(1)} = [i_1, i_2]$ $\sigma_3^{(1)} = [i_0, i_2]$

Incidence matrices

$$\mathbf{B}_1 \in M^{N_0 \times N_1}$$

$$\mathbf{B}_2 \in M^{N_1 \times N_2}$$

$$\mathbf{B}_1(\sigma_i^{(0)}, \sigma_j^{(1)}) = \begin{array}{c} \\ i_0 \\ i_1 \\ i_2 \end{array} \begin{array}{c} \begin{array}{ccc} [i_0, i_1] & [i_1, i_2] & [i_0, i_2] \end{array} \\ \begin{pmatrix} -1 & 0 & -1 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \end{array}$$

$$\mathbf{B}_2(\sigma_i^{(1)}, \sigma_j^{(2)}) = \begin{array}{c} \\ [i_0, i_1] \\ [i_1, i_2] \\ [i_0, i_2] \end{array} \begin{array}{c} [i_0, i_1, i_2] \\ \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \end{array}$$

Simplicial complex

$$\sigma_i^{(k)} = [i_0, \dots, i_k]$$

$$\mathbf{B}_k \in M^{N_{k-1} \times N_k}$$

Incidence matrix

$$B_k(\sigma_i^{(k-1)}, \sigma_j^{(k)}) = 1 \text{ if } \sigma_i^{(k-1)} \sim \sigma_j^{(k)}$$

$$B_k(\sigma_i^{(k-1)}, \sigma_j^{(k)}) = -1 \text{ if } \sigma_i^{(k-1)} \not\sim \sigma_j^{(k)}$$

$$B_k(\sigma_i^{(k-1)}, \sigma_j^{(k)}) = 0 \text{ otherwise}$$

$$\mathbf{L}_k = \mathbf{B}_k^\top \mathbf{B}_k + \mathbf{B}_{k+1} \mathbf{B}_{k+1}^\top$$

Hodge Laplace matrix

Turing patterns on simplicial complex

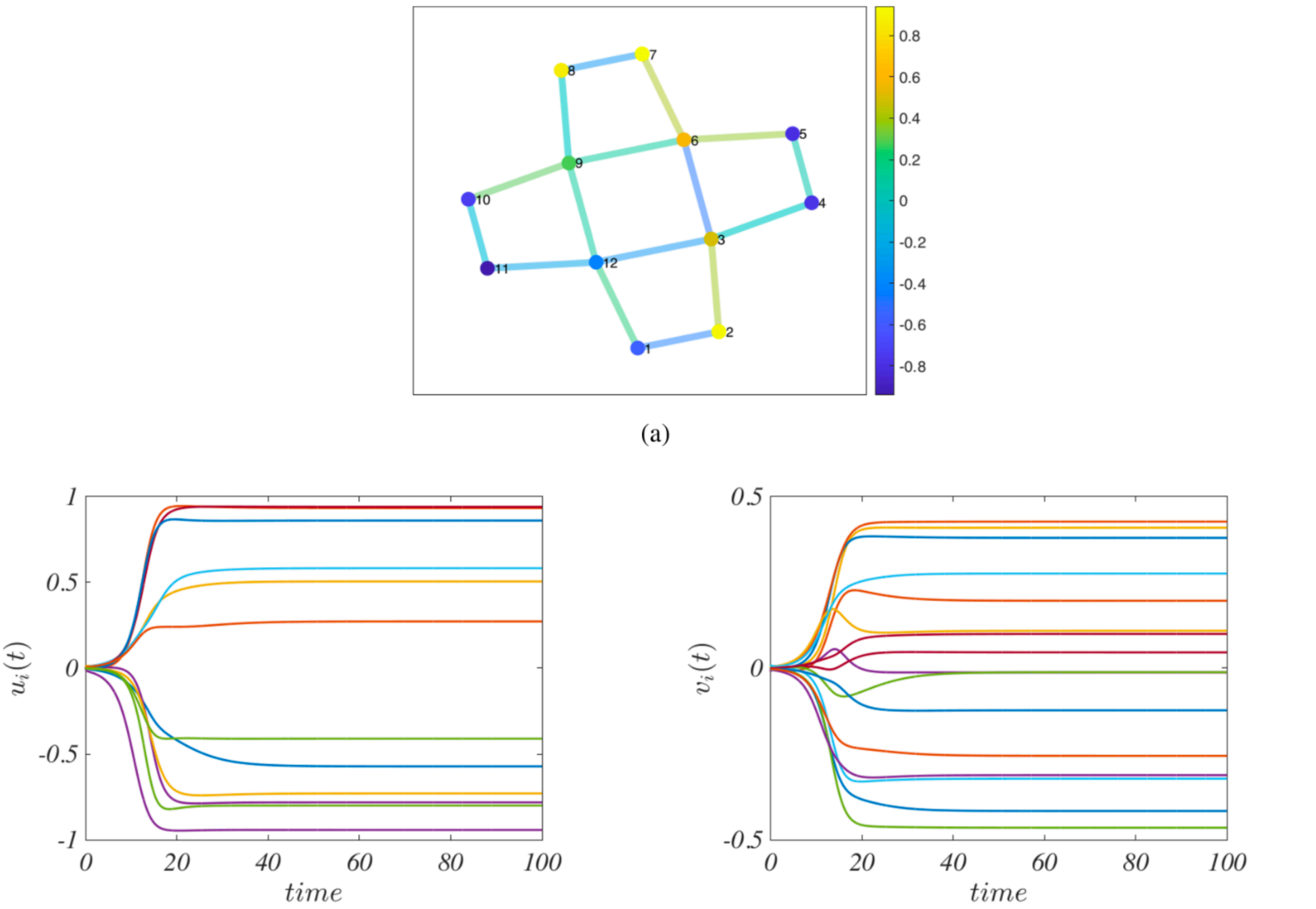
$$\frac{du}{dt} = f(u, \mathbf{B}_1 v) - D_0 \mathbf{L}_0 u, \qquad 0 = f(u^*, 0) \quad \text{and} \quad 0 = g(v^*, 0),$$

$$\frac{dv}{dt} = g\left(v, \mathbf{B}_1^\top u\right) - D_1 \mathbf{L}_1 v,$$

Note. We need extra conditions for the homogenous solutions to be a solution for the whole system.

$$\mathbf{B}_1 v^* = \mathbf{0} \qquad \mathbf{B}_1^\top u^* = \mathbf{0}.$$

Turing patterns on simplicial complex



(a)

(b)

(c)

timoteo.carletti@unamur.be

Some papers where results can be found

Global Topological Synchronization on Simplicial and Cell Complexes, T. Carletti, L. Giambagli, G. Bianconi, Physical Review Letters, **130**, pp. 187401, (2023)

Turing patterns in systems with high-order interactions, R. Muolo, L.Gallo, V. Latora, M. Frasca, T. Carletti, Chaos Solitons & Fractals. **106**, pp. 112912 (2023)

Finite propagation enhances turing patterns in reaction–diffusion networked systems, T. Carletti , R. Muolo. J Phys: Complexity **2**(4), pp. 045004 (2021)

Diffusion-driven instability of topological signals coupled by the Dirac operator, L. Giambagli et al, Physical Review E , **106** pp. 064314 , (2022)

Dynamical systems on hypergraphs, T. Carletti, D. Fanelli, S. Nicoletti, J Phys: Complexity **1**(3):035006 (2020)

Patterns of non-normality in networked systems., R. Muolo, M. Asllani, D. Fanelli, PK. Maini, T. Carletti, J Theoret Biol **480**:81, (2019)

Some papers where results can be found.

Theory of Turing Patterns on Time Varying Networks, J. Petit, B. Lauwens, D. Fanelli, T. Carletti, Physical Review Letters, **119**, pp. 148301-1–5, (2017)

Tune the topology to create or destroy patterns, M. Asllani, T. Carletti, D. Fanelli, European Physical Journal B. **89**, pp. 260 (2016)

Pattern formation in a two-component reaction-diffusion system with delayed processes on a network, J. Petit, M. Asllani, D. Fanelli, B. Lauwens, T. Carletti, Physica A, **462**, pp.230, (2016)

Delay induced Turing-like waves for one species reaction–diffusion model on a network, J. Petit, T. Carletti, M. Asllani, D. Fanelli, Europhysics Letters. **111**, 5, pp. 58002, (2015)

Turing instabilities on Cartesian product networks, M. Asllani, D.M. Busiello, T. Carletti, D. Fanelli, G. Planchon, Scientific Reports. **5**, pp. 12927, (2015)

Turing patterns in multiplex networks, M. Asllani, D.M. Busiello, T. Carletti, D. Fanelli, G. Planchon, Physical Review E ,**90**, 4, pp. 042814, (2014)

January the 23rd, 2024, ISI Kolkata, India

Timoteo Carletti

Thank you

Any questions?

A personal journey
in the zoo of Turing patterns

Namur Institute for Complex Systems

UNIVERSITÉ DE NAMUR

Delayed models

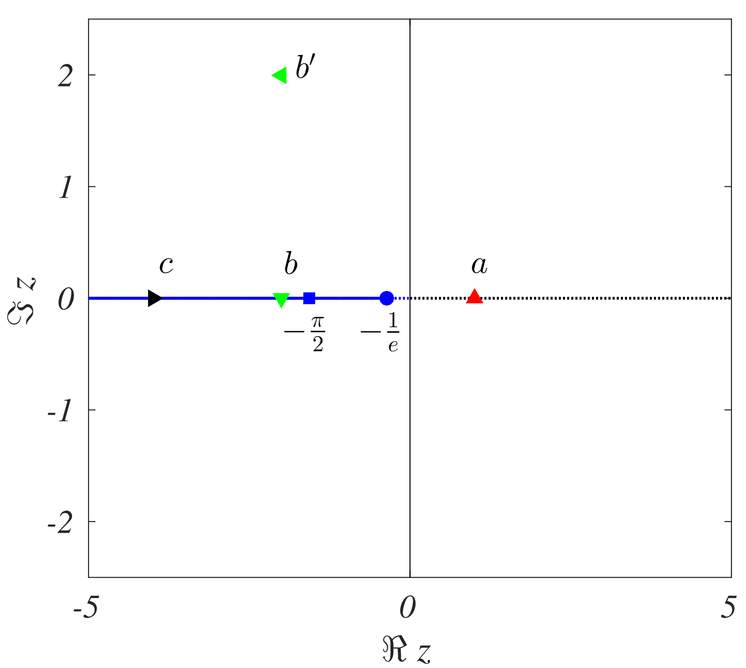
Movement across links takes time, so the diffusion part should contain a delay term.

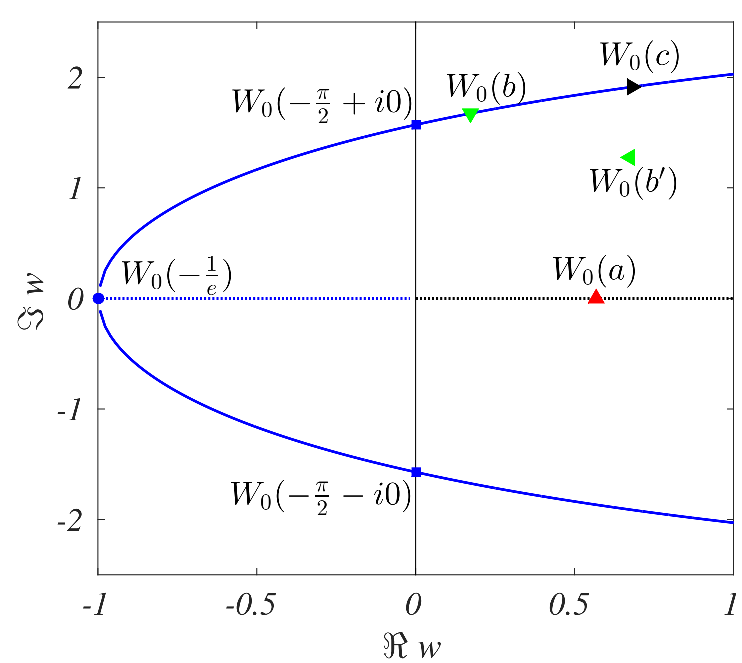
Also reactions can take time, so the reaction part should contain a delay term.

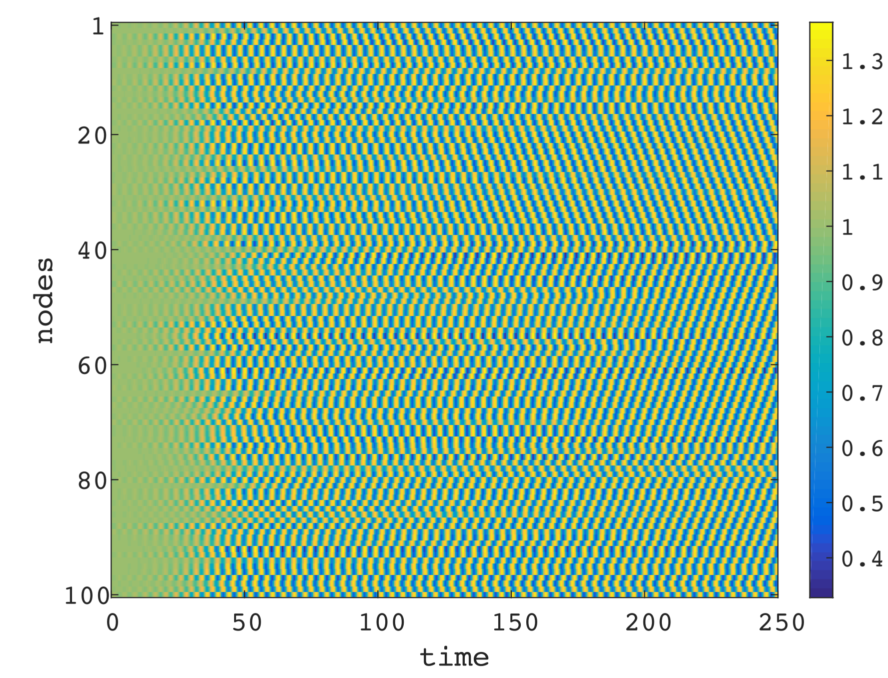
$$\dot{x}_i(t) = f(x_i(t-\tau_r)) + D\sum_j L_{ij}x_j(t-\tau_d)$$

Observe that one single species is enough to have Turing patterns

The relation dispersion can be analytically computed using the Lambert W-function







Finite propagation on complex networks

$\mathbf{M}$ Incidence matrix

$\mathbf{L} = -\mathbf{M}^\top \mathbf{M}$ Laplace matrix

Fick's law

$$\chi_e(t) = -D_u\left[u_j(t) - u_i(t)\right] \equiv D_u\left[\mathbf{M}\vec{u}(t)\right]_e$$

$$\frac{du_i}{dt}(t) = -\left[\mathbf{M}^\top \vec{\chi}(t)\right]_i$$

$$\frac{d\vec{u}}{dt}(t) = -\mathbf{M}^\top \vec{\chi} = -D_u \mathbf{M}^\top \mathbf{M}\vec{u} = D_u \mathbf{L}\vec{u}$$

Finite propagation on complex networks

$\mathbf{M}$ Incidence matrix $\mathbf{L} = -\mathbf{M}^\top \mathbf{M}$ Laplace matrix

Fick's law

$$\chi_e(t) = -D_u \left[u_j(t) - u_i(t)\right] \equiv D_u \left[\mathbf{M}\vec{u}(t)\right]_e \qquad \frac{du_i}{dt}(t) = -\left[\mathbf{M}^\top \vec{\chi}(t)\right]_i$$

$$\frac{d\vec{u}}{dt}(t) = -\mathbf{M}^\top \vec{\chi} = -D_u \mathbf{M}^\top \mathbf{M}\vec{u} = D_u \mathbf{L}\vec{u}$$

Cattaneo's theory

τ_u relaxation / inertial time

$$\chi_e(t) + \tau_u \frac{d\chi_e}{dt}(t) = D_u \left[\mathbf{M}\vec{u}(t)\right]_e \qquad \frac{du_i}{dt}(t) = -\left[\mathbf{M}^\top \vec{\chi}(t)\right]_i$$

$$\frac{d\vec{u}}{dt}(t) = -\tau_u \frac{d^2\vec{u}}{dt^2} + D_u \mathbf{L}\vec{u}(t)$$

Relativistic Turing mechanism on complex networks

$$\begin{cases} \dot{u}_i &= f(u_i, v_i) + D_u \sum_j L_{ij} u_j \\ \dot{v}_i &= g(u_i, v_i) + D_v \sum_j L_{ij} v_j \end{cases}$$

Parabolic RD
(Heat equation)

$$\begin{cases} \dfrac{du_i}{dt} + \tau_u \dfrac{d^2 u_i}{dt^2} &= f(u_i, v_i) + D_u \sum_{j=1}^{n} L_{ij} u_j \\ \dfrac{dv_i}{dt} + \tau_v \dfrac{d^2 v_i}{dt^2} &= g(u_i, v_i) + D_v \sum_{j=1}^{n} L_{ij} v_j \end{cases}$$

Hyperbolic RD
(Relativistic Heat equation)

(Cattaneo equation, telegraph equation, damped nonlinear Klein-Gordon equations)

Some results

- Existence of Turing pattern in activator - inhibitor systems with inhibitor diffusing faster than the activator

$$D_v \gg D_u$$

- Existence of Turing pattern in activator - inhibitor systems with inhibitor diffusing slower than the activator

$$D_v \leq D_u$$

- Existence of Turing pattern in inhibitor - inhibitor systems

$$f_u < 0 \text{ and } g_v < 0$$

Some results

CLASSICAL CASE

- Existence of Turing pattern in activator - inhibitor systems with <u>inhibitor diffusing</u> <u>faster</u> than the <u>activator</u>

$$D_v \gg D_u$$

NEW CASE

- Existence of Turing pattern in activator - inhibitor systems with <u>inhibitor diffusing</u> <u>slower</u> than the <u>activator</u>

$$D_v \leq D_u$$

NEW CASE

- Existence of Turing pattern in <u>inhibitor - inhibitor</u> systems

$$f_u < 0 \text{ and } g_v < 0$$

Inertia driven Turing instability

Turing instability in relativistic reaction-diffusion

$$\begin{cases} \frac{d\delta u_i}{dt} + \tau_u \frac{d^2\delta u_i}{dt^2} & = \partial_u f \delta u_i + \partial_v f \delta v_i + D_u \sum_{j=1}^{n} L_{ij}\delta u_j \\ \frac{d\delta v_i}{dt} + \tau_v \frac{d^2\delta v_i}{dt^2} & = \partial_u g \delta u_i + \partial_v g \delta v_i + D_v \sum_{j=1}^{n} L_{ij}\delta v_j \end{cases}$$

$$u_i(t) = \sum_\alpha u^\alpha e^{\lambda_\alpha t}\phi_i^{(\alpha)}$$

$$\sum_j L_{ij}\phi_j^{(\alpha)} = \Lambda_\alpha \phi_i^{(\alpha)} \quad \forall i, \alpha$$

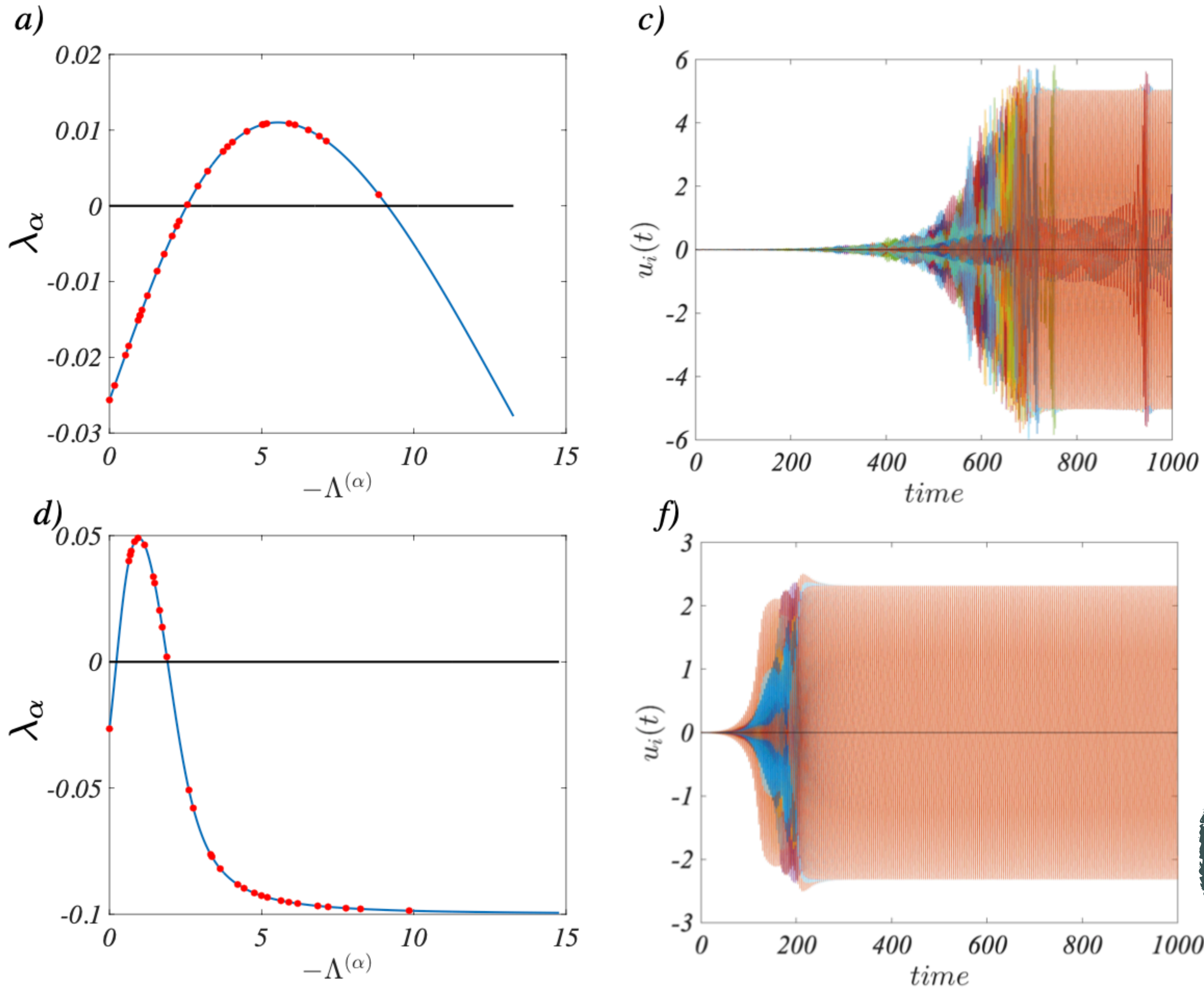
$$\begin{cases} \frac{d\hat{u}_\alpha}{dt}(t) + \tau_u \frac{d^2\hat{u}_\alpha}{dt^2}(t) & = \partial_u f \hat{u}_\alpha(t) + \partial_v f \hat{v}_\alpha(t) + D_u \Lambda^{(\alpha)}\hat{u}_\alpha(t) \\ \frac{d\hat{v}_\alpha}{dt}(t) + \tau_v \frac{d^2\hat{v}_\alpha}{dt^2}(t) & = \partial_u g \hat{u}_\alpha(t) + \partial_v g \hat{v}_\alpha(t) + D_v \Lambda^{(\alpha)}\hat{v}_\alpha(t) \end{cases}$$

$$\det\begin{pmatrix} \lambda_\alpha + \tau_u\lambda_\alpha^2 - \partial_u f - \Lambda^{(\alpha)}D_u & -\partial_v f \\ -\partial_u g & \lambda_\alpha + \tau_v\lambda_\alpha^2 - \partial_v g - \Lambda^{(\alpha)}D_v \end{pmatrix} = p_\alpha(\lambda_\alpha) = 0$$

Fourth order polynomial ⟺ Routh - Hurwitz criterium

FitzHugh - Nagumo model : inertia driven patterns

$$\begin{cases} \dfrac{du_i}{dt} + \tau_u \dfrac{d^2u_i}{dt^2} = \mu u_i - u_i^3 - v_i + D_u \displaystyle\sum_{j=1}^{n} L_{ij} u_j \\ \dfrac{dv_i}{dt} + \tau_v \dfrac{d^2v_i}{dt^2} = \gamma(u_i - \beta v_i) + D_v \displaystyle\sum_{j=1}^{n} L_{ij} v_j \end{cases}$$



$\beta = 0.6\,, \mu = 1.0\,, \gamma = 4.0$

$\tau_u = 5.0\,, \tau_v = 1.0$

$D_u = 2.2 > D_v = 0.2$

$\beta = 2.5\,, \mu = 0.18\,, \gamma = 4.0$

$\tau_u = 1.0\,, \tau_v = 5.0$

$D_u = 2.2 = D_v = 2.2$