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Published in:
SIAM Journal on Control and Optimization

Publication date:
2005

Document Version
Peer reviewed version

SIAM J. Control Optim.
Vol. 43, No. 4, pp. 1435–1466

SPECTRAL FACTORIZATION BY SYMMETRIC EXTRACTION FOR DISTRIBUTED PARAMETER SYSTEMS*

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Abstract. The spectral factorization problem of a scalar coercive spectral density is considered in the framework of the Callier–Desoer algebra of distributed parameter system transfer functions. Criteria are given for the infinite product representation of a meromorphic coercive spectral density of finite order and for the convergence of infinite product representations of spectral factors, i.e., for the convergence of the symmetric extraction method for solving the spectral factorization problem of such spectral density. These convergence criteria are applied to the solution of the linear-quadratic optimal control problem by spectral factorization for a specific class of semigroup Hilbert state-space systems with a Riesz-spectral generator. The speed of convergence of the symmetric extraction method is also considered. As an example a damped vibrating string model is handled.

Key words. distributed parameter systems, spectral factorization, coercivity, meromorphic function, entire function, finite order, infinite product, symmetric extraction, convergence analysis

AMS subject classifications. Primary, 47A68, 47A70, 49R20; Secondary, 93B52, 93C05

DOI. 10.1137/S0363012902416456

1. Introduction.

The spectral factorization problem plays a central role in the framework of the fractional representation approach (which is also known in the literature as the "factorization approach") for feedback control system design; see, e.g., [8], [35]. In addition, spectral factorization constitutes an essential step in the solution of the linear-quadratic (LQ) optimal control problem for infinite-dimensional state-space systems; see, e.g., [9], [10], [18], [33], [36] and the references therein. The spectral factorization problem is also used as a main tool for solving linear operator inequalities (Lur'e equations); see, e.g., [16], [19]. As far as the LQ-optimal control problem is concerned, it is shown in [9] and [10] that the latter is solvable by spectral factorization for C_0-semigroup Hilbert state-space systems with bounded observation and control operators and with finite-dimensional output and input spaces. The philosophy developed in those papers has been extended, e.g., in [33] and [36] to C_0-semigroup Hilbert state-space systems with unbounded observation and control operators. In those references, the authors analyze spectral factorization problems of operator-valued Popov functions, giving an H^∞ spectral factor and showing notably the existence of solutions to related operator Riccati equations. However, they do not develop any method to perform the spectral factorization iteratively, which is done here similarly as on the heat diffusion model dealt with in [10].

Fundamental questions concerning the spectral factorization problem have been studied in the literature: in particular the existence and multiplicity of spectral factors and the continuity of the spectral factorization mapping have been analyzed; see, e.g., [12], [23], [22] and the references therein. As far as computational questions are

*Received by the editors October 22, 2002; accepted for publication (in revised form) April 7, 2004; published electronically January 27, 2005.
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concerned, several methods have been developed and analyzed for the approximate computation of spectral factors or solutions of related operator Riccati equations for distributed parameter systems; see, e.g., the references cited in [9], [10], and [36]. The LQ-control based normalized coprime fraction spectral factorization problem considered in [10] was solved by an ad hoc iterative symmetric extraction method for a one-dimensional heat diffusion model, involving only elementary rational factors with real poles and zeros (first order case). The symmetric extraction method of spectral factorization was also studied in detail for multivariable finite-dimensional state-space systems in [4].

The major aim of this paper is to extend this method to a class of single-variable distributed parameter (i.e., in particular infinite-dimensional state-space) systems, including standard models like heat diffusion or wave propagation, and thereby involving the symmetric extraction of elementary rational factors with complex conjugate poles and zeros (second order case) as well. More specifically, this paper is devoted to the description and the convergence analysis of the symmetric extraction method for the spectral factorization of a scalar coercive spectral density, which is assumed to be a meromorphic function of finite order (see, e.g., [27]), in the framework of the Callier–Desoer algebra of distributed parameter system transfer functions (see, e.g., [5], [6], [11], [17]). Criteria for the infinite product representation of a meromorphic coercive spectral density of finite order and for the convergence of the symmetric extraction method of spectral factorization are developed, which extend some previous partial results (see [8], [10]). These criteria are mainly based on the knowledge of the comparative asymptotic behavior of the spectral density poles and zeros. Some comments concerning the speed of convergence of such a method are also given. Moreover, the symmetric extraction method is shown to work for the spectral factorization of a coprime fraction (coercive) spectral density. The analysis is performed in the framework of C_0-semigroup Hilbert state-space systems, whose infinitesimal generator is a Riesz-spectral operator, with eigenvalues satisfying some asymptotic conditions, and with transfer function in the Callier–Desoer algebra. The results are illustrated by an example, namely, the LQ-control based normalized coprime fraction spectral factorization problem for a vibrating string model. Some of these results were reported in [13], [14].

The methodology for the convergence analysis of the symmetric extraction method of spectral factorization which is used here is based on entire function theory; see, e.g., [37], [3], [26]. In particular, Hadamard's theorem on the infinite product representation of entire functions of finite order plays a central role here. Basically the analysis developed in this paper uses the material which is contained in [37] and which is paramount for the proof of a related important result, viz., Akhiezer's theorem, concerning the spectral factorization of entire functions of exponential type; see, e.g., [25, p. 567], [3, Theorem 7.5.1, p. 125].

Related results dealing with matrix-valued functions can also be found, e.g., in [29] and [30, Theorem 2.1]. These contributions do not deal with the symmetric case, whereas the present paper does. In addition, e.g., in [29], the starting point of the analysis is the function to be factorized (requiring a realization step in the analysis). Here the starting point of the analysis is basically a system transfer function, the main objective being to apply the methodology in a system theoretic framework.

The paper is organized as follows. Some preliminaries concerning the frequency domain framework, and properties of coercive spectral densities and invertible spectral factors are given in section 2. Fundamental results concerning the representation of a meromorphic coercive spectral density of finite order as an infinite product of elementary rational spectral densities are developed in section 3. These results are used in section 4 in order to establish spectral criteria for the convergence of the symmetric extraction method for the computation of a spectral factor. Section 5 is devoted to the implementation of this method for solving the LQ-optimal control based spectral factorization problem for a vibrating string model. Finally, section 6 contains some concluding remarks and perspectives.

2. Preliminary concepts and results. The analysis of the symmetric extraction method of spectral factorization is performed in the framework of the Callier–Desoer transfer function algebra (see, e.g., [5], [6], [11], [17, section 7.1]). The latter is briefly described below.

Let $\sigma \leq 0$. An impulse response f is said to be in $\mathcal{A}(\sigma)$ if for $t < 0, f(t) = 0$, and for $t \geq 0, f(t) = f_a(t) + f_{sa}(t)$, where the regular functional part $f_a \in L^1_\sigma$, i.e., $\exp(-\sigma\cdot)\ f_a(\cdot)$ is in $L^1(0,\infty)$, and the singular atomic part $f_{sa} := \sum_{i=0}^{\infty} f_i \delta(.-t_i)$, where $t_0 = 0, t_i > 0$ for $i = 1, 2, \ldots$, and $f_i \in \mathbb{C}$ for $i = 0, 1, \ldots$ with $\sum_{i=0}^{\infty} |f_i| \exp(-\sigma t_i) < \infty$. The norm of a distribution f in $\mathcal{A}(\sigma)$ is defined by

$$||f||_{\mathcal{A}(\sigma)} := \int_0^\infty |f_a(t)|\ e^{-\sigma t}\ dt + \sum_{i=0}^{\infty} |f_i|\ e^{-\sigma t_i}\ .$$

The Laplace transform of a distribution f is denoted by $\hat{f}$, and the class of Laplace transforms of elements in $\mathcal{A}(\sigma)$ is denoted by $\hat{\mathcal{A}}(\sigma)$. The norm of $\hat{f}$ in $\hat{\mathcal{A}}(\sigma)$ is defined by

$$||\hat{f}||_{\hat{\mathcal{A}}(\sigma)} := ||f||_{\mathcal{A}(\sigma)}\ .$$

An impulse response f is said to be in $\mathcal{A}_-(\sigma)$ if $f \in \mathcal{A}(\sigma_1)$ for some $\sigma_1 < \sigma$. We write $\mathcal{A}_-$ for $\mathcal{A}_-(0)$. $\mathcal{A}(\sigma)$ and $\mathcal{A}_-$ are convolution algebras. By $\hat{\mathcal{A}}_-(\sigma)$ and $\hat{\mathcal{A}}_-$ we denote the classes of Laplace transforms of elements in $\mathcal{A}_-(\sigma)$ and $\mathcal{A}_-$, respectively. Then $\hat{\mathcal{A}}_-$ is our selected class of distributed proper-stable transfer functions. It contains the multiplicative subset $\hat{\mathcal{A}}_-^\infty$ of transfer functions that are bounded away from zero at infinity in $\mathbb{C}_+$, i.e., that are biproper-stable. The Callier–Desoer algebra $\hat{\mathcal{B}}$ of possibly unstable transfer functions consists of those $\hat{f}$ such that $\hat{f} = \hat{n}\hat{d}^{-1}$ with $\hat{n} \in \hat{\mathcal{A}}_-$ and $\hat{d} \in \hat{\mathcal{A}}_-^\infty$. A transfer function is in $\hat{\mathcal{B}}$ if and only if it is the sum of a completely unstable strictly proper rational function and a stable transfer function in $\hat{\mathcal{A}}_-$; hence $\hat{d}$ above can always be chosen biproper-stable *rational*; see [11], [17].

DEFINITION 2.1. *A complex–valued function f is said to be* (1) parahermitian *if* $f(s) \equiv f_*(s) := \overline{f(-\overline{s})}$, (2) real *if* $\overline{f(s)} \equiv f(\overline{s})$, *and* (3) real parahermitian *if* $\overline{f(s)} \equiv f(\overline{s})$ *and* $f(s) \equiv f_*(s)$.

A function $\hat{F}$ is said to be a (real) spectral density *if $\hat{F}$ is (real) parahermitian such that $\hat{F} = \hat{F}_* = \hat{G}_* + \hat{G}$, where $\hat{G}$ is in $\hat{\mathcal{A}}_-$, and $\hat{F}$ is nonnegative on the imaginary axis, i.e., $\hat{F}(j\omega) \geq 0$ for all $\omega \in \mathbb{R}$. A spectral density $\hat{F}$ is said to be* coercive *if there exists $\eta > 0$ such that $\hat{F}(j\omega) \geq \eta$ for all $\omega \in \mathbb{R}$. A transfer function $\hat{R}$ in $\hat{\mathcal{A}}_-$ is said to be a* spectral factor *of a spectral density $\hat{F}$ if $\hat{F}(j\omega) = \hat{R}_*(j\omega)\hat{R}(j\omega)$ for all $\omega \in \mathbb{R}$. A spectral factor $\hat{R}$ is said to be* invertible *if $\hat{R}^{-1}$ is in $\hat{\mathcal{A}}_-$.*

A spectral density is also called a *Popov function* in the literature; see, e.g., [36] and the references therein. It is known that a spectral density has an invertible spectral factor if and only if it is coercive, and that spectral factors are unique up to multiplication by a constant of modulus one (see, e.g., [8], [9], [12]); furthermore, to coercive real spectral densities correspond real spectral factors unique up to the $\pm$ sign. Moreover, a coercive spectral density $\hat{F}$ such that $\hat{F}(\infty) = 1$, i.e., $\hat{F} = \hat{G}_* + \hat{G}$ with $\hat{G} = G_0 + \hat{G}_a \in \hat{\mathcal{A}}_-$ and $\mathrm{Re}\, G_0 = 2^{-1}$, has a unique invertible *standard* spectral factor $\hat{R} = 1 + \hat{R}_a \in \hat{\mathcal{A}}_-$, i.e., such that $\hat{R}(\infty) = 1$. The following properties will be needed for the analysis of the following sections, especially in the proof of Theorem 3.4. The proof of the following lemma can be found in [8].

LEMMA 2.2 (algebraic properties of coercive spectral densities).

(a) *If $\hat{F}$ is a coercive (real) spectral density, then so is its inverse $\hat{F}^{-1}$.*

(b) *If $\hat{F}$ and $\hat{G}$ are coercive (real) spectral densities, then so is their product $\hat{F}\cdot\hat{G}$.*

Remark 2.1. All impulse responses f considered below have no delayed impulses (delays); i.e., their singular atomic part is of the form $f_{sa} = f_0\delta(\cdot)$.

In this paper we are essentially interested in meromorphic spectral densities. Recall that a function f is said to be *meromorphic* (in $\mathbb{C}$) if there exists a countable set $S \subset \mathbb{C}$ such that S has no limit point, f is holomorphic in S^c, and f has a pole at each point of S; see, e.g., [31, p. 241]. In particular a meromorphic coercive real spectral density $\hat{F}$ with a meromorphic inverse has a countable set of zeros $\mathcal{Z} \subset \mathbb{C}$ and a countable set of poles $\mathcal{P} \subset \mathbb{C}$ such that $\mathcal{Z}$ and $\mathcal{P}$ have no limit points, there exists a vertical strip S_δ, $\delta > 0$, such that $\mathcal{Z} \cap S_\delta = \emptyset$ and $\mathcal{P} \cap S_\delta = \emptyset$, and for any $z \in \mathcal{Z}$ and any $p \in \mathcal{P}$, $-z$, $\overline{z}$, $-\overline{z}$ are in $\mathcal{Z}$ and $-p$, $\overline{p}$, $-\overline{p}$ are in $\mathcal{P}$.

In addition, the order of a meromorphic function is defined as follows (see, e.g., [27, Chapters VI and VIII]): For a meromorphic function f, with no poles at $s = 0$, define the counting function $N(r, f)$ and the proximity function $m(r, f)$, where $r \geq 0$, respectively, by

$$N(r, f) \;:=\; \int_0^r \frac{n(t, f)}{t}\, dt,$$

where $n(t, f)$ denotes the number of poles of f (counting their multiplicities) in the closed disk $\{s \in \mathbb{C} :\ |s| \leq t\}$, and

$$m(r, f) \;:=\; \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{j\theta})|\, d\theta,$$

where $\log^+ \ x := \max\{0, \log\ x\}$ and f is assumed to have no poles on the circle $\{s \in \mathbb{C} :\ |s| = r\}$. The function T which is given by

$$T(r) := T(r, f) := m(r, f) + N(r, f)$$

is called the *characteristic function* of f. Observe that T is positive and monotonically increasing for $r > 0$. The *order* of f is defined to be the order of its characteristic function T, viz.,

$$\rho \;:=\; \limsup_{r\to\infty} \frac{\log\ T(r)}{\log\ r}.$$

In particular, for an entire function f, let $M(r)$ be its maximum modulus defined by

$$M(r) \;:=\; \max\{|f(s)| :\ |s| = r\};$$

then the functions T and $\log\ M$ are of the same order, viz.,

$$\rho \;=\; \limsup_{r\to\infty} \frac{\log\ \log\ M(r)}{\log\ r}$$

(see, e.g., [27, p. 216] and [37]). Thus an entire function f is of finite order if and only if there exists a constant $k > 0$ such that $\max\{|f(s)| : |s| = r\} \leq \exp(r^k)$ for r large (see, e.g., [37]). Moreover, the sum, the product, and the quotient of two meromorphic functions of finite order are meromorphic functions of finite order as well; see, e.g., [27, p. 216]. The following concept will be very useful for the analysis of meromorphic spectral densities of finite order.

DEFINITION 2.3. *A* paraconjugate symmetric (ps-)family $\mathbf{M}$ with countably many defining parameters $\mu_n \in \mathbf{M}$ *is a countable family of complex numbers* (ρ_l), *containing* (μ_n) *as a subfamily, such that* (a) $\mathbf{M}$ *is paraconjugate symmetric, i.e., the* ρ_l*'s are either real such that* $\rho_l = \mu_n$ *and* $-\mu_n \in \mathbb{R}$, *with* $\mu_n < 0$, *or complex nonreal such that* $\rho_l = \mu_n, \overline{\mu_n}, -\mu_n$ *and* $-\overline{\mu_n} \in \mathbb{C}$, *with* $\text{Re}\ \mu_n < 0$, $\text{Im}\ \mu_n > 0$, $n \in \mathbb{N}$, (b) *the defining parameters may be finitely repeated in* $\mathbf{M}$, *i.e.,* (μ_n) *(or, equivalently,* (ρ_l)*) does not contain any constant subfamily, and* (c) *all the points* ρ_l *of* $\mathbf{M}$ *are located outside a vertical strip containing the imaginary axis in its interior, i.e., there exists some* $\kappa > 0$ *such that, for all* n, $|\text{Re}\ \mu_n| \geq \kappa$.

It turns out that there is a strong connection between meromorphic real spectral densities and real parahermitian entire functions; see Lemma 2.5 below. This result is based on the following additional lemma concerning the infinite product representation of real parahermitian entire functions, which also will be needed in the proof of Theorem 3.4.

LEMMA 2.4 (infinite product representation of real parahermitian entire functions).

(1) *Consider a ps-family* $\mathbf{M}$ *with defining parameters* μ_n. *Assume that*

$$\sum_{n=1}^{\infty} \frac{1}{|\mu_n|^2} < \infty. \tag{2.1}$$

Then there exists an entire function P *of finite order such that* P *has a zero at each point of* $\mathbf{M}$ *and no other zero in* $\mathbb{C}$, *and such that* (a) P *has a product factorization of the form*

$$P(s) = \prod_{n=1}^{\infty} P_n(s), \tag{2.2}$$

where

$$P_n(s) = 1 - \left(\frac{s}{\mu_n}\right)^2, \quad \textit{with } \mu_n \in \mathbb{R} \textit{ such that } \mu_n < 0, \tag{2.3}$$

and

(2.4)
$$P_n(s) = \left(1 - \left(\frac{s}{\mu_n}\right)^2\right)\left(1 - \left(\frac{s}{\overline{\mu_n}}\right)^2\right), \quad \textit{with } \mu_n \in \mathbb{C}\backslash\mathbb{R} \textit{ such that } \text{Re}\ \mu_n < 0,$$

whence, for all $\omega \in \mathbb{R}$, $P(j\omega) \geq 0$. *Moreover, the convergence of the infinite product in* (2.2) *is uniform and absolute on closed discs* $D(r) := \{s \in \mathbb{C} : |s| \leq r\}$, *and*
(b) P *is real parahermitian, whence for all* $s \in \mathbb{C}$, $P(s) = P(-s)$, *and there exists* $\delta > 0$ *such that, for all* $s \in S_\delta$, $P(s) \neq 0$.

(2) *Let f be a real parahermitian entire function having zeros in $\mathbb{C}$ as described in part* (1)*; then $f(s)$ has the product representation*

$$f(s) = e^{g(s)}\ P(s), \tag{2.5}$$

where $P(s)$ is as in part (1) *and $g(s)$ is a real parahermitian entire function. If in addition f is of finite order ρ, then $g(s)$ is a polynomial of degree $\delta[g] \leq \rho$.*

Proof. See, e.g., [37, pp. 54–57] for more detail.

Assume that $\mathbf{M} =: (\rho_l)_{l=1}^{\infty}$. Let p be the least nonnegative integer such that $\sum_{l=1}^{\infty} \frac{1}{|\rho_l|^{p+1}} < \infty$. It follows from (2.1) that p is 0 or 1. Then

$$P(s) := \prod_{l=1}^{\infty} E\left(\frac{s}{\rho_l}, p\right), \tag{2.6}$$

where

$$E\left(\frac{s}{\rho_l}, p\right) = \left(1 - \frac{s}{\rho_l}\right)\ e^{\frac{ps}{\rho_l}} \quad \text{for } p = 0, 1$$

is the *canonical product of genus p* associated with the sequence $(\rho_l)_{l=1}^{\infty}$. By the reasoning of [37, pp. 55–56] it is an entire function, which has a zero at each point ρ_l and no other zeros in $\mathbb{C}$. Moreover, the convergence of its infinite product is uniform and absolute on closed discs $D(r)$, whence it can be reordered arbitrarily. Now as the factors $E(\frac{s}{\rho_l}, p)$ can be grouped to form factors of the form (2.3) or (2.4) that have real coefficients and are invariant when s is exchanged for $-s$, there holds that in either case (i.e., $p = 0$ or $p = 1$), $P(s)$ is real parahermitian and can be rewritten as

$$P(s) = \prod_{n=1}^{\infty} P_n(s) = P(-s), \tag{2.7}$$

where the P_n's are the polynomial functions given by (2.3)–(2.4). In addition, the entire function P is of *finite order* $\rho \leq 2$ (i.e., $|P(s)| \leq e^{|s|^{\rho+\epsilon}}$ for $|s|$ large [37, pp. 63–64], for any $\epsilon > 0$). Indeed by [37, pp. 65–66], *the exponent of convergence of the zeros ρ_l* of the canonical product in (2.2)–(2.4) is the greatest lower bound of the nonnegative numbers α such that $\sum_{l=1}^{\infty} |\rho_l|^{-\alpha} < \infty$. By (2.1) it is less than or equal to 2. Then, by [37, Theorem 6, p. 69], the entire function P is of order $\rho \leq 2$. Finally, the function $P(s)$ satisfies all conclusions of part (1) of the lemma.

Part (2) follows by the reasoning around Weierstrass's factorization theorem [37, pp. 55–57] and the reasoning of the proof of part (1). The last statement concerning an entire function f of finite order follows by Hadamard's theorem; see, e.g., [37, Theorem 9, p. 74]. □

Remark 2.2. Let f be a real parahermitian entire function of finite order $\rho < 2$ with zeros as in part (1) of Lemma 2.4, whence for some $\delta > 0$, for all $s \in S_\delta$, $f(s) \neq 0$, and for all $\omega \in \mathbb{R}$, $f(j\omega) \geq 0$. Then f has a product factorization of the form

$$f(s) = kP(s), \tag{2.8}$$

where k is a positive constant and $P(s)$ is of the form (2.2)–(2.4).

Indeed, f has the product factorization (2.5), where the entire function g is a polynomial of degree $\delta[g] \leq \rho < 2$. Since g is a real coefficient polynomial function of $(-s^2)$, it should be a constant c, such that (2.8) holds with $k = e^c$.

It follows from the lemma above that a coercive real spectral density that is a meromorphic function of finite order can be written as a ratio of two real parahermitian entire functions of finite order, provided that its poles tend to infinity sufficiently fast: see the following lemma.

LEMMA 2.5. *Consider a coercive real spectral density $\hat{F}$ given by $\hat{F} = \hat{F}_* = \hat{G}_* + \hat{G}$, where $\hat{G} \in \hat{\mathcal{A}}_-$ is such that $G_{sa} = G_0\delta(\cdot)$ for some $G_0 \in \mathbb{C}$, whence $\hat{F}$ is holomorphic in some open vertical strip containing the imaginary axis $S_\delta := \{s \in \mathbb{C} : \text{Re } s \in (-\delta, \delta)\}$, where $\delta > 0$. Assume that $\hat{F}$ is a meromorphic function of finite order. Let the poles of $\hat{F}$ form a ps-family $\mathbf{P}$ with defining parameters p_n and let*

$$\sum_{n=1}^{\infty} \frac{1}{|p_n|^2} < \infty.$$

Then $\hat{F}$ can be written as a fraction

$$\hat{F}(s) = \frac{N(s)}{D(s)},$$

where the denominator D and numerator N are real parahermitian entire functions of finite order, such that $D(s) = D(-s)$, $N(s) = N(-s)$, and the zeros and poles of $\hat{F}$ are those of N and D, respectively.

Proof. Let the ps-family $\mathbf{P}$ be given by $\mathbf{P} =: (\pi_l)$. By Lemma 2.4 part (1), there exists a real parahermitian entire function D of finite order such that D has a zero at each pole π_l of $\hat{F}$ and no other zeros in $\mathbb{C}$. Now, as in the proof of [31, Theorem 15.12, p. 327], consider the function $N := \hat{F}\ D$, which is obviously real parahermitian and such that $\hat{F} = N\ D^{-1}$. Moreover, the singularities of N at the points π_l are removable, whence N can be extended such that it is holomorphic in $\mathbb{C}$, i.e., entire, and N and D have no common zeros in $\mathbb{C}$. Finally, $N = \hat{F}\ D$ is of finite order, since so are $\hat{F}$ and D. ∎

Remark 2.3. (α) The converse of Lemma 2.5 obviously holds: any coercive real spectral density whose poles satisfy the condition above and which can be written as a fraction as in the statement of Lemma 2.5 is a meromorphic function of finite order.

(β) The proof above contains the essential arguments of the proofs of [31, Theorems 15.11 and 15.12, pp. 326–327]). Lemma 2.5 will be used in the proof of Lemma 4.7.

(γ) Related results concerning the canonical representation of any meromorphic function of finite order are given in [27, pp. 218–221].

Later on, after having transformed the spectral density under study, we shall also need the following auxiliary technical result.

LEMMA 2.6 (entire coercive real spectral density of finite order without zeros). *Let $\hat{F}$ be a coercive real spectral density given by $\hat{F} = \hat{F}_* = \hat{G}_* + \hat{G}$, where $\hat{G} \in \hat{\mathcal{A}}_-$ is such that $G_{sa} = G_0\delta(\cdot)$ for some $G_0 \in \mathbb{C}$. Assume that*

(2.9) $\hat{F}$ *is an entire function of finite order without zeros.*

In addition, assume that the limit of $\hat{F}$ at infinity exists on the imaginary axis such that

$$\hat{F}(\pm j\infty) \ := \lim_{|\omega|\to\infty} \hat{F}(j\omega) \ = \ 1, \tag{2.10}$$

(or, equivalently, Re $G_0 = 2^{-1}$*). Then $\hat{F}$ is a constant function, i.e., $\hat{F}(s) = 1$, for all $s \in \mathbb{C}$.*

Proof. By Hadamard's theorem [37, Theorem 9, p. 74], $\hat{F}$ has the form

$$\hat{F}(s) = e^{g(s)}, \tag{2.11}$$

where $g(s)$ is a polynomial, i.e., $g(s) = \sum_{k=0}^{n} g_k s^k$. One has $\hat{F}(0) = e^{g(0)} = e^{g_0}$ is real and positive. Hence $g_0 = \log(\hat{F}(0))$ is real. Moreover, as $\hat{F}$ is real, i.e., $\hat{F}(\overline{s}) = \overline{\hat{F}(s)}$, there holds by (2.11) and the continuity of $g(s)$ that there exists a unique integer l such that $g(\overline{s}) = \overline{g(s)} + jl2\pi$, which for $s = 0$ reads $\mathrm{Im}(g_0) = l\pi$, whence $l = 0$ as g_0 is real. Thus $g(\overline{s}) = \overline{g(s)}$, i.e., g is a real polynomial or, equivalently, g has real coefficients.

In addition there holds that $\hat{F}$ is real parahermitian, whence $\hat{F}(s) = \hat{F}(-s)$. A similar reasoning using (2.11) shows then that g is a real polynomial in s^2, whence it can be rewritten as a real polynomial h in $-s^2$, i.e.,

$$g(s) = \sum_{k=0}^{m} g_{2k} s^{2k} = \sum_{k=0}^{m} h_{2k}(-s^2)^k =: h(-s^2) \in \mathbb{R}[-s^2] \quad \text{with} \quad h_{2k} = (-1)^k g_{2k}.$$

Observe now that, by the structure of $\hat{F}$, its coercivity, and (2.10), $\hat{F}(j\omega)$ is a real positive uniformly continuous function on $\omega \in \overline{\mathbb{R}}$ (i.e., the extended real line) and bounded as well as bounded away from zero on $\overline{\mathbb{R}}$. Hence $\log(\hat{F}(j\omega))$ is a real uniformly continuous function on $\omega \in \overline{\mathbb{R}}$ and bounded above and below on $\overline{\mathbb{R}}$; moreover, by (2.10), $\lim_{|\omega|\to\infty} \log(\hat{F}(j\omega)) = 0$. Furthermore, there holds that $\log(\hat{F}(j\omega)) = h(\omega^2)$ for all $\omega \in \mathbb{R}$, whence $\lim_{|\omega|\to\infty} h(\omega^2) = 0$ with $h(\omega^2) \in \mathbb{R}[\omega^2]$. As a consequence, the polynomial $h(\omega^2)$ is identical to $\log(\hat{F}(j\omega))$ on $\omega \in \overline{\mathbb{R}}$, bounded there above and below, and zero at infinity. Hence $h(\omega^2)$ must reduce to a constant polynomial which is identically zero, i.e., $g(s) \equiv 0$. Thus $\hat{F}(s) \equiv 1$. ∎

Remark 2.4. (α) Often (2.11) reads $\hat{F}(s) = ke^{g(s)}$, where k is a positive constant. Then with $k = e^c$, where $c \in \mathbb{R}$, one gets $\hat{F}(s) = e^{g(s)+c}$, where $g(s)+c$ is a polynomial. Hence (2.11) holds without loss of generality.

(β) Assumption (2.9) is realized for a meromorphic coercive real spectral density of finite order for which, "after the removal of the poles and zeros," there remains an entire function of finite order without zeros, or, equivalently, "after the removal of the poles" there remains an entire function of finite order (see Lemmas 2.5 and 2.4 (part 2)).

(γ) In Lemma 2.6, the assumption that the order of the spectral density $\hat{F}$ (as an entire function) is finite cannot be omitted. This fact is illustrated by the following simple example. Consider the function $\hat{F}$ given by

$$\hat{F}(s) := \exp\left(2 \cdot \frac{\sinh\ s}{s}\right).$$

Observe that $\hat{F}$ is a coercive real spectral density of the form $\hat{F} = \hat{G}_* + \hat{G}$, where $\hat{G} = G_0 + \hat{G}_a \in \hat{\mathcal{A}}_-$ and Re $G_0 = 2^{-1}$. Indeed, let $\hat{R}$ be the function defined by

$$\hat{R}(s) := \exp\left(\hat{g}(s)\right), \quad \text{where} \quad \hat{g}(s) := \frac{1 - e^{-s}}{s}.$$

Then $\hat{g}$ belongs to $\hat{\mathcal{A}}_-$ as the Laplace transform of the function of finite support $g := \chi_{[0,1]}$; i.e., $g(t) = 1$ if $0 \le t \le 1$ and $g(t) = 0$ elsewhere. Moreover, $\hat{g}$ is strictly proper, i.e., $\hat{g}(\infty) = 0$, where $\hat{g}(\infty)$ should be interpreted as the limit of $\hat{g}(s)$ as $|s| \to \infty$ in any right half-plane strictly containing the closed right half-plane. Hence $\hat{R}$ is in $\hat{\mathcal{A}}_-$ together with its inverse $\hat{R}^{-1} = \exp(-\hat{g}(s))$, since they are exponentials of elements of a Banach algebra, viz., $\hat{\mathcal{A}}(\sigma) \subset \hat{\mathcal{A}}_-$, $\sigma < 0$; moreover, $\hat{R}(\pm j\infty) = 1$. Since $\hat{F}(s) = \hat{R}(-s)\cdot\hat{R}(s)$, it follows that $\hat{F}$ is a coercive real spectral density with invertible standard real spectral factor $\hat{R}$. In addition, $\hat{g}$ is an entire function, whence $\hat{R}$ is an entire function without zeros, and so is $\hat{F}$. However, $\hat{F}$ is not a constant function. Observe that there is no contradiction with Lemma 2.6, since $\hat{F}$ is of infinite order. Indeed, the function $2\cdot\frac{\sinh s}{s}$ is an entire function which is not a polynomial. Hence, in view of Hadamard's theorem (see, e.g., [37, p. 74]), $\hat{F}$ cannot be of finite order.

3. Meromorphic spectral densities of finite order. The main objective of this section is to show that, under certain conditions, a coercive real spectral density that is a meromorphic function of finite order can be written as an infinite product of coercive real rational spectral densities. First it is shown that, under certain technical conditions, such an infinite product is necessarily a coercive real spectral density.

3.1. Product of rational spectral densities.

THEOREM 3.1 (infinite product of coercive rational spectral densities). *Consider a function $\hat{F}$ given, for all s in some vertical strip symmetric with respect to the imaginary axis, by an infinite product of pole-zero pairs of the form*

$$\hat{F}(s) = \prod_{n=1}^{\infty} \hat{F}_n(s), \tag{3.1}$$

where the elementary factors $\hat{F}_n$ are coercive real rational spectral densities, which are given either by

$$\hat{F}_n(s) = \frac{z_n^2 - s^2}{p_n^2 - s^2}, \tag{3.2}$$

where z_n and $p_n \in \mathbb{R}$, with z_n and $p_n < 0$, or by

$$\hat{F}_n(s) = \frac{(z_n^2 - s^2)(\overline{z}_n^2 - s^2)}{(p_n^2 - s^2)(\overline{p}_n^2 - s^2)}, \tag{3.3}$$

where z_n and $p_n \in \mathbb{C}\backslash\mathbb{R}$, with $\operatorname{Re} z_n$ *and* $\operatorname{Re} p_n < 0$. *Consider the standard invertible (real) spectral factors $\hat{R}_n$ of the spectral densities $\hat{F}_n$, which are such that $\hat{R}_n(\infty) = 1$ and which are given by*

$$\hat{R}_n(s) = \frac{z_n - s}{p_n - s} \tag{3.4}$$

(first order factor) when $\hat{F}_n$ is defined by (3.2) *and by*

$$\hat{R}_n(s) = \frac{(z_n - s)(\overline{z}_n - s)}{(p_n - s)(\overline{p}_n - s)} \tag{3.5}$$

(second order factor) when $\hat{F}_n$ is defined by (3.3), *respectively. Assume that there exists a constant $\sigma < 0$ such that $\hat{R}_n$ and $\hat{R}_n^{-1}$ are in $\hat{\mathcal{A}}(\sigma)$, for all n, with*

$$\sum_{n=1}^{\infty} ||(R_n)_a||_{\mathcal{A}(\sigma)} < \infty \tag{3.6}$$

and

$$\sum_{i=1}^{\infty} ||(R_n^{-1})_a||_{\mathcal{A}(\sigma)} < \infty. \tag{3.7}$$

Then the following assertions hold:

(a) *The infinite product in* (3.1) *converges to* $\hat{F}$ *in the Banach algebra*

$$\widehat{L\Delta}(\sigma) := \{\hat{f} = \hat{f}_- + \hat{f}_+ \ : \ (f_-)_* \text{ and } f_+ \ \in \mathcal{A}(\sigma)\}$$

equipped with the norm

$$||\hat{f}||_\sigma := ||f||_\sigma := ||(f_-)_*||_{\mathcal{A}(\sigma)} + ||f_+||_{\mathcal{A}(\sigma)};$$

(b) *The function* $\hat{F}$ *is a coercive real spectral density such that* $\hat{F} = \hat{F}_* = \hat{G}_* + \hat{G}$, *where* $\hat{G}$ *is in* $\hat{\mathcal{A}}(\sigma) \subset \hat{\mathcal{A}}_-$.

Proof. (a) The proof goes along the lines of [10, proof of Theorem 5].

(b) It follows from assertion (a) and from the fact that every elementary factor $\hat{F}_n$ is real parahermitian and positive semidefinite on the imaginary axis, that $\hat{F}$ is a real spectral density of the form $\hat{F} = \hat{G}_* + \hat{G}$ for some $\hat{G}$ in $\hat{\mathcal{A}}(\sigma)$. Finally observe that, by the fact that there exists a constant $\sigma < 0$ such that $\hat{R}_n$ and $\hat{R}_n^{-1}$ are in $\hat{\mathcal{A}}(\sigma)$ for all n, every elementary factor spectral density $\hat{F}_n$ is coercive such that, for some $\eta > 0$, $\prod_{n=1}^{N} \hat{F}_n(j\omega) \geq \eta$ for all $\omega \in \mathbb{R}$ and for all $N \geq 1$. Hence the spectral density $\hat{F}$ is coercive. □

Remark 3.1. Convergence in the $\widehat{L\Delta}(\sigma)$-norm implies convergence in the sup-norm on a vertical strip without singularities containing the $j\omega$-axis in its interior.

This result leads to a criterion for the convergence of an infinite product of coercive rational spectral densities, which is based on the knowledge of the spectrum, i.e., more precisely, on the knowledge of the comparative asymptotic behavior of the spectral density poles and zeros, p_n and z_n, as n tends to infinity; see Corollary 3.3 below. The proof of this spectral criterion is based on the following preliminary result.

LEMMA 3.2 (estimates of $||(R_n)_a||_{\mathcal{A}(\sigma)}$ and $||(R_n^{-1})_a||_{\mathcal{A}(\sigma)}$). *Consider the rational elementary factors* $\hat{R}_n$, $n = 1, 2, \ldots,$ *given by* (3.4)–(3.5), *and assume that there exists a constant* $\sigma < 0$ *such that* $2 \cdot |\sigma| \leq \min(|\text{Re } p_n|, |\text{Re } z_n|)$ *for all* n. *Then* $\hat{R}_n$ *and* $\hat{R}_n^{-1}$ *are in* $\hat{\mathcal{A}}(\sigma)$ *for all* n *and the following inequalities hold for all* n*: when* $\hat{R}_n$ *is given by* (3.4), *then*

$$||(R_n)_a||_{\mathcal{A}(\sigma)} \leq 2 \, \frac{|z_n - p_n|}{|p_n|},$$

and

$$||(R_n^{-1})_a||_{\mathcal{A}(\sigma)} \leq 2 \, \frac{|z_n - p_n|}{|z_n|},$$

and when $\hat{R}_n$ *is given by* (3.5), *then*

$$||(R_n)_a||_{\mathcal{A}(\sigma)} \leq 4 \, \frac{|z_n - p_n|}{|\text{Re } p_n|} \left(1 + \frac{|z_n - p_n|}{|\text{Re } p_n|}\right),$$

and

$$||(R_n^{-1})_a||_{\mathcal{A}(\sigma)} \leq 4 \, \frac{|z_n - p_n|}{|\text{Re } z_n|} \left(1 + \frac{|z_n - p_n|}{|\text{Re } z_n|}\right).$$

Proof. For the case of a first order factor, see [10, Fact 1]. Concerning the case of a second order factor, we derive only the inequality for $(R_n)_a$. The other one can be proved similarly. Now, for all $t \geq 0$,

$$(R_n)_a(t) \ = (p_n - z_n) \cdot e^{p_n t} + (\overline{p_n - z_n}) \cdot e^{\overline{p_n} t} + \ \frac{|p_n - z_n|^2}{(\overline{p_n} - p_n)} \cdot (e^{\overline{p_n} t} - e^{p_n t}),$$

whence, upon noting that $|\overline{p_n} - p_n| = 2|\text{Im } p_n|$ and $|e^{\overline{p_n} t} - e^{p_n t}| = 2 \cdot e^{\text{Re } p_n t} \cdot |\sin(\text{Im } p_n t)|$, one obtains

$$|(R_n)_a(t)| \leq 2 \ |z_n - p_n| \cdot e^{\text{Re } p_n t} + |z_n - p_n|^2 \cdot t \cdot e^{\text{Re } p_n t}.$$

By using the assumption that $|\text{Re } p_n| \geq 2 \cdot |\sigma|$, one gets

$$||e^{\text{Re } p_n t}||_{\mathcal{A}(\sigma)} = \int_0^\infty e^{(\text{Re } p_n - \sigma)t} dt = |\text{Re } p_n - \sigma|^{-1} \leq \frac{2}{|\text{Re } p_n|},$$

and

$$||t \cdot e^{\text{Re } p_n t}||_{\mathcal{A}(\sigma)} = \int_0^\infty t \cdot e^{(\text{Re } p_n - \sigma)t} dt = |\text{Re } p_n - \sigma|^{-2} \leq \frac{4}{|\text{Re } p_n|^2}.$$

As a result one gets

$$||(R_n)_a||_{\mathcal{A}(\sigma)} \leq 4 \ \frac{|z_n - p_n|}{|\text{Re } p_n|} \left(1 + \frac{|z_n - p_n|}{|\text{Re } p_n|}\right). \qquad \square$$

COROLLARY 3.3 (spectral criterion for the convergence of an infinite product of coercive rational spectral densities). *Consider a function $\hat{F}$ given by* (3.1)–(3.3) *for all s in some vertical strip symmetric with respect to the imaginary axis. Let $\hat{R}_n$, $n = 1, 2, \ldots$, be the rational elementary factors defined by* (3.4)–(3.5), *with $2 \cdot |\sigma| \leq \min(|\text{Re } p_n|, |\text{Re } z_n|)$, for all n, for some $\sigma < 0$. Now assume that*

$$\sum_{n=1}^{\infty} \frac{|z_n - p_n|}{|\text{Re } p_n|} < \infty, \tag{3.8}$$

and

$$\sum_{n=1}^{\infty} \frac{|z_n - p_n|}{|\text{Re } z_n|} < \infty. \tag{3.9}$$

Then the conclusions of Theorem 3.1 *hold.*

Proof. Consider the series

$$\sum_{n=1}^{\infty} \frac{|z_n - p_n|}{|\text{Re } p_n|} \left(1 + \frac{|z_n - p_n|}{|\text{Re } p_n|}\right).$$

By (3.8) the sequence

$$\left(\frac{|z_n - p_n|}{|\text{Re } p_n|}\right)_{n=1}^{\infty}$$

is in l^1, hence also in l^∞, and thus also in l^2. Therefore the series above converges, and in view of Lemma 3.2, (3.6) holds. Moreover, by similar arguments (3.7) holds. Hence the conclusion follows by Theorem 3.1. ∎

Remark 3.2. The conclusions of Corollary 3.3 still hold if conditions (3.8) and (3.9) are replaced, respectively, by

$$\frac{|z_n - p_n|}{|\text{Re } p_n|} = O\left(\frac{1}{n^\alpha}\right) \tag{3.10}$$

and

$$\frac{|z_n - p_n|}{|\text{Re } z_n|} = O\left(\frac{1}{n^\alpha}\right) \tag{3.11}$$

for some exponent $\alpha > 1$.

3.2. Product representation of meromorphic spectral densities. The fact that a coercive real spectral density $\hat{F}$ has an infinite product representation of the form (3.1)–(3.3) is not automatically satisfied in applications. Typically one should check this by using the Weierstrass factorization theorem for entire functions and related results; see, e.g., [37, Chapter 2, part 1], [26, section 7.1, p. 343], and the references therein. This was done in [10, p. 765] for a heat diffusion example. In the following theorem, conditions on the poles and zeros of a meromorphic spectral density of finite order are derived, under which this methodology can be used.

THEOREM 3.4 (criterion for the infinite product representation of a meromorphic coercive spectral density of finite order). *Let $\hat{F}$ be a coercive real spectral density given by $\hat{F} = \hat{F}_* = \hat{G}_* + \hat{G}$, where $\hat{G} \in \hat{\mathcal{A}}_-$ is such that $G_{sa} = G_0\delta(\cdot)$ for some $G_0 \in \mathbb{C}$; whence $\hat{F}$ is holomorphic in some open vertical strip containing the imaginary axis, namely, S_δ, where $\delta > 0$. Assume that the limit of $\hat{F}$ at infinity exists in this vertical strip such that*

$$\hat{F}(\infty) := \lim_{|s|\to\infty;\ s\in S_\delta} \hat{F}(s) = \lim_{|\omega|\to\infty;\ -\delta<\sigma<\delta} \hat{F}(\sigma + j\omega) = 1, \tag{3.12}$$

(or, equivalently, Re $G_0 = 2^{-1}$*). In addition, assume that $\hat{F}$ is a meromorphic function of finite order such that*

(1) *the poles of $\hat{F}$ form a ps-family $\mathbf{P}$ with defining parameters p_n,*
(2) *the zeros of $\hat{F}$ form a ps-family $\mathbf{Z}$ with defining parameters z_n, and*
(3)

$$\sum_{n=1}^{\infty} \frac{1}{|p_n|^2} < \infty \quad \textit{and} \quad \sum_{n=1}^{\infty} \frac{1}{|z_n|^2} < \infty. \tag{3.13}$$

Assume that the set of zeros (poles, respectively) of $\hat{F}$ consists of countably many real zeros (poles, respectively) and countably many complex zeros (poles, respectively) such that its zeros and poles can be associated by a one-to-one relationship, leading to elementary factors of the form (3.17)–(3.18)*. Finally, assume that conditions* (3.8)–(3.9) *hold.*

Then (a) *$\hat{F}$ can be written as a fraction*

$$\hat{F}(s) = \frac{N(s)}{D(s)}, \tag{3.14}$$

where the denominator D and numerator N are real parahermitian entire functions of finite order such that $D(s) = D(-s)$ and $N(s) = N(-s)$ and the zeros and poles of $\hat{F}$ are those of N and D, respectively;

(b) *the spectral density $\hat{F}$ admits an infinite product representation of pole-zero pairs that is of the form*

$$\hat{F}(s) = \prod_{n=1}^{\infty} \hat{F}_n(s), \tag{3.15}$$

and the inverse spectral density $\hat{F}^{-1}$ admits the infinite product representation

$$\hat{F}(s)^{-1} = \prod_{n=1}^{\infty} \hat{F}_n(s)^{-1}, \tag{3.16}$$

where the elementary factors $\hat{F}_n$ are coercive real rational spectral densities that are given either by

$$\hat{F}_n(s) = \frac{z_n^2 - s^2}{p_n^2 - s^2} \tag{3.17}$$

if z_n and $p_n \in \mathbb{R}$, with z_n and $p_n < 0$, or by

$$\hat{F}_n(s) = \frac{(z_n^2 - s^2)(\overline{z}_n^2 - s^2)}{(p_n^2 - s^2)(\overline{p}_n^2 - s^2)} \tag{3.18}$$

if z_n and $p_n \in \mathbb{C}\backslash\mathbb{R}$, with Re z_n *and* Re $p_n < 0$.

Remark 3.3. (α) Conditions (3.13) are not the most general ones encountered in the theory of infinite product representation of entire functions; see, e.g., [26, p. 358] and [37, Theorem 1, pp. 55–56]. However, according to [26], these conditions are applicable to many problems. In addition these assumptions together with the fact that a spectral density is parahermitian lead to a simpler structure for the corresponding infinite product elementary factors.

(β) Concerning the definitions of the elementary factors (3.17)–(3.18), it is natural and usual to take the upper half-plane zeros and poles in increasing order of real part. This is done in this way in the application dealt with in this paper; see section 5.

Proof of Theorem 3.4. (a) follows from Lemma 2.5.

(b) *Step* 1. D and N have the infinite product representations

$$D(s) = e^{G_D(s)} \cdot \prod_{n=1}^{\infty} D_n(s), \tag{3.19}$$

and

$$N(s) = e^{G_N(s)} \cdot \prod_{n=1}^{\infty} N_n(s), \tag{3.20}$$

respectively, where G_D and G_N are polynomial functions and where D_n and N_n are the polynomial functions given by

$$D_n(s) = \begin{cases} 1 - (\frac{s}{p_n})^2 & \text{if } p_n \in \mathbb{R} \text{ with } p_n < 0, \\ (1 - (\frac{s}{p_n})^2) \cdot (1 - (\frac{s}{\overline{p}_n})^2) & \text{if } p_n \in \mathbb{C} \text{ with Re } p_n < 0 \end{cases} \tag{3.21}$$

and by

$$(3.22) \qquad N_n(s) = \begin{cases} 1-(\frac{s}{z_n})^2 & \text{if } z_n \in \mathbb{R} \text{ with } z_n < 0, \\ (1-(\frac{s}{z_n})^2)\cdot(1-(\frac{s}{\overline{z_n}})^2) & \text{if } z_n \in \mathbb{C} \text{ with } \operatorname{Re} z_n < 0, \end{cases}$$

respectively. Moreover, the convergence of the infinite products is uniform and absolute on any closed disc $D(r)$.

Indeed this follows from the data concerning D and N, where in particular (3.13) holds, and from Lemma 2.4, part (2).

Step 2. On closed discs $D(r)$ where small neighborhoods of the poles are omitted, the spectral density $\hat{F}$ has the infinite product representations

$$(3.23) \qquad \hat{F}(s) = e^{H(s)} \cdot \prod_{n=1}^{\infty} \frac{N_n(s)}{D_n(s)} = e^{H(s)} \cdot \prod_{n=1}^{\infty} \phi_n \cdot \prod_{n=1}^{\infty} \hat{F}_n(s),$$

where $H := G_N - G_D$ is a polynomial function, the $\hat{F}_n(s)$ are given by (3.17)–(3.18), and where the constants ϕ_n are given by

$$(3.24) \qquad \phi_n = \begin{cases} \left|\frac{p_n}{z_n}\right|^2 & \text{if } p_n,\ z_n \in \mathbb{R} \text{ with } p_n \text{ and } z_n < 0, \\ \\ \left|\frac{p_n}{z_n}\right|^4 & \text{if } p_n,\ z_n \in \mathbb{C}\backslash\mathbb{R} \text{ with } \operatorname{Re} p_n \text{ and } \operatorname{Re} z_n < 0. \end{cases}$$

Moreover, the convergence of the s-dependent products is uniform and absolute on the aforementioned punctured discs.

Indeed this follows from identity (3.14), from (3.19)–(3.22), and from assumptions (3.8)–(3.9). First it can be shown that for n sufficiently large, for the case that $p_n,\ z_n \in \mathbb{R}$ with p_n and $z_n < 0$,

$$\left|\hat{F}_n(s) - 1\right| = O\left\{\left|1 - \left(\frac{z_n}{p_n}\right)^2\right|\right\},$$

and for the case that $p_n,\ z_n \in \mathbb{C}$ with $\operatorname{Re} p_n$ and $\operatorname{Re} z_n < 0$,

$$\left|\hat{F}_n(s) - 1\right| = O\left\{\left|1 - \left|\frac{z_n}{p_n}\right|^4\right| + 2\left|1 - \left(\frac{z_n}{p_n}\right)^2\right|\right\}.$$

In addition, observe that $\hat{F}(\infty) = 2 \cdot \operatorname{Re} G_0$; whence, by assumption (3.12), $\operatorname{Re} G_0 = 2^{-1}$. Moreover, it follows from (3.13) that $|p_n| \to \infty$ and $|z_n| \to \infty$ as $n \to \infty$. Finally by assumptions (3.8)–(3.9) and the inequalities

$$\left|1 - \left(\frac{z_n}{p_n}\right)\right| \le \frac{|z_n - p_n|}{|\operatorname{Re} p_n|}$$

and

$$\left|1 - \left(\frac{p_n}{z_n}\right)\right| \le \frac{|z_n - p_n|}{|\operatorname{Re} z_n|},$$

there holds that the spectral density poles and zeros will be asymptotically close (as $n \to \infty$), i.e.,

$$\lim_{n\to\infty}\left(\frac{z_n}{p_n}\right) = 1 \quad \text{and} \quad \lim_{n\to\infty}\left(\frac{p_n}{z_n}\right) = 1.$$

It then follows easily that

$$\sum_{n=1}^{\infty}\left|1-\left(\frac{z_n}{p_n}\right)^2\right|<\infty \qquad \text{and} \qquad \sum_{n=1}^{\infty}\left|1-\left(\frac{p_n}{z_n}\right)^2\right|<\infty. \tag{3.25}$$

Hence by (3.25), the infinite product $\prod_{n=1}^{\infty} \hat{F}_n(s)$ converges uniformly and absolutely in any closed disk $D(r)$ where small neighborhoods of the poles are omitted. Next by (3.25) it can also be shown that the infinite product $\prod_{n=1}^{\infty} \phi_n$ is well defined. Now, on closed discs $D(r)$ where small neighborhoods of the poles are omitted, there holds for any $M \in \mathbb{N}$

$$\frac{\prod_{n=1}^{M} N_n}{\prod_{n=1}^{M} D_n} = \prod_{n=1}^{M} \phi_n \cdot \prod_{n=1}^{M} F_n.$$

Hence, as the limits exist for $M \to \infty$ uniformly and absolutely on such discs, Step 2 follows.

Step 3. There holds that

$$e^{H(s)} \cdot \prod_{n=1}^{\infty} \phi_n \;\equiv\; 1, \tag{3.26}$$

whence, in view of (3.23), (3.15) and (3.17)–(3.18) hold.

Indeed observe that the infinite product $\prod_{n=1}^{\infty} \hat{F}_n(s)$ converges uniformly and absolutely in any closed disk not containing any of its poles and that $\prod_{n=1}^{\infty} \hat{F}_n(\infty) = 1$, such that $\prod_{n=1}^{\infty} \hat{F}_n(s)$ exists at infinity and converges uniformly in the vertical strip S_δ by Corollary 3.3 and (3.8)–(3.9). Thus

$$\prod_{n=1}^{\infty} \hat{F}_n(s) \;\to\; 1 \quad \text{as} \quad |s| \to \infty \quad \text{in} \quad S_\delta. \tag{3.27}$$

Since, by assumption, $\hat{F}(\infty) = 1$, it follows from (3.23) and (3.27) that

$$e^{H(\infty)} \cdot \prod_{n=1}^{\infty} \phi_n \;:=\; \lim_{|s|\to\infty;\; s\in S_\delta} e^{H(s)} \cdot \prod_{n=1}^{\infty} \phi_n \;=\; 1. \tag{3.28}$$

Observe that, by Corollary 3.3, the function $\prod_{n=1}^{\infty} \hat{F}_n$ is a coercive real spectral density; whence, in view of Lemma 2.2, so is the function

$$e^{H(s)} \cdot \prod_{n=1}^{\infty} \phi_n \;= \hat{F}(s) \;\cdot \left(\prod_{n=1}^{\infty} \hat{F}_n(s)\right)^{-1},$$

which in addition, as H is a polynomial, is an entire function of finite order without zeros. Hence, by Lemma 2.6 (see also Remark 2.4 (α)), it follows from (3.28) that Step 3 holds.

Step 4. Observe that $\hat{F}^{-1}$ is a meromorphic function of finite order and mutatis mutandis satisfies the same conditions as $\hat{F}$. Hence, by reasoning similar to that above, (3.16) and (3.17)–(3.18) hold. ∎

4. Spectral factorization by symmetric extraction.

4.1. Main results. By the proof of [10, Theorem 5], the following result holds.

THEOREM 4.1 (criterion for infinite product representation of invertible spectral factors). *Consider a coercive real spectral density* $\hat{F}$ *given by* (3.1)–(3.3) *for all* s *in some vertical strip symmetric with respect to the imaginary axis. Let* $\hat{R}_n$, $n = 1, 2, \dots$, *be the rational elementary factors defined by* (3.4)–(3.5). *Assume that there exists a constant* $\sigma < 0$ *such that* $\hat{R}_n$ *and* $\hat{R}_n^{-1}$ *are in* $\hat{\mathcal{A}}(\sigma)$, *for all* n, *with*

$$\sum_{n=1}^{\infty} ||(R_n)_a||_{\mathcal{A}(\sigma)} < \infty \tag{4.1}$$

and

$$\sum_{i=1}^{\infty} ||(R_n^{-1})_a||_{\mathcal{A}(\sigma)} < \infty. \tag{4.2}$$

Then the invertible standard spectral factor $\hat{R}$ *in* $\hat{\mathcal{A}}_-$ *of* $\hat{F}$ *is given by the infinite product representation*

$$\hat{R}(s) = \prod_{n=1}^{\infty} \hat{R}_n(s) = \lim_{N\to\infty} \prod_{n=1}^{N} \hat{R}_n(s), \tag{4.3}$$

where the limit is taken in the framework of the topology induced by the norm $||\cdot||_{\hat{\mathcal{A}}(\sigma)}$ *on the Banach algebra* $\hat{\mathcal{A}}(\sigma)$.

We are now ready to get a spectral criterion for the convergence of the symmetric extraction method of spectral factorization: it is based on the knowledge of the comparative asymptotic behavior of the spectral density poles and zeros.

THEOREM 4.2 (spectral criterion for the convergence of the symmetric extraction method). *Consider a coercive real spectral density* $\hat{F}$ *given by* (3.1)–(3.3) *for all* s *in some vertical strip symmetric with respect to the imaginary axis. Let* $\hat{R}_n$, $n = 1, 2, \dots,$ *be the rational elementary factors defined by* (3.4)–(3.5), *with* $2 \cdot |\sigma| \le \min(|\mathrm{Re}\ p_n|, |\mathrm{Re}\ z_n|)$, *for all* n, *for some* $\sigma < 0$. *Now assume that* $\hat{R}_n$ *is a first order factor given by* (3.4) *or a second order factor given by* (3.5) *satisfying* (3.8) *and* (3.9). *Then* (a) *the conclusions of Theorem* 4.1 *hold. In particular, the sequence*

$$\left(\prod_{n=1}^{N} \hat{R}_n \right)_{N\ge 1} \tag{4.4}$$

of invertible approximate (rational) spectral factors converges to the invertible standard spectral factor $\hat{R} \in \hat{\mathcal{A}}_-$ *of* $\hat{F}$ *in the* $\hat{\mathcal{A}}(\sigma)$ *norm, and the sequence*

$$\left(\prod_{n=1}^{N} \hat{R}_n^{-1} \right)_{N\ge 1} \tag{4.5}$$

converges to the corresponding inverse spectral factor $\hat{R}^{-1} \in \hat{\mathcal{A}}_-$;

(b) *with* $\hat{W}_N \in \widehat{L\Delta}^{+}(\sigma) \subset \hat{\mathcal{A}}_-$ *denoting the approximate spectral factor defined by*

$$W_N := \prod_{i=1}^{N} R_i,$$

the spectral factor, inverse spectral factor, and spectral factorization relative errors *can be estimated for all $N = 1, 2, \ldots,$ respectively, by the following inequalities:*

$$\|(W_N - R) * R^{-1}\|_\sigma \;\leq\; \exp S_{N+1} - 1 \;\leq\; 2S_{N+1}, \tag{4.6}$$

where the last inequality holds if

$$S_{N+1} \;:=\; 4 \sum_{i=N+1}^{\infty} \frac{|z_n - p_n|}{|\text{Re } z_n|} \left(1 + \frac{|z_n - p_n|}{|\text{Re } z_n|}\right) \;\leq\; 1;$$

$$\|(W_N^{-1} - R^{-1}) * R\|_\sigma \;\leq\; \exp T_{N+1} - 1 \;\leq\; 2T_{N+1}, \tag{4.7}$$

where the last inequality holds if

$$T_{N+1} \;:=\; 4 \sum_{i=N+1}^{\infty} \frac{|z_n - p_n|}{|\text{Re } p_n|} \left(1 + \frac{|z_n - p_n|}{|\text{Re } p_n|}\right) \leq 1;$$

and finally

$$\|((W_N)_* \cdot W_N - F) * F^{-1}\|_\sigma \;\leq\; \exp(2S_{N+1}) - 1 \;\leq\; 4S_{N+1}, \tag{4.8}$$

where the last inequality holds if $S_{N+1} \;\leq\; 2^{-1}$.

Proof. (a) The conclusion follows directly from Theorem 4.1, by the proof of Corollary 3.3.

(b) The relative error estimates (4.6)–(4.8) can be derived by following the lines of the proof of [10, Theorem 5, pp. 766–767] and by using Lemma 3.2. ☐

Remark 4.1. (α) The conclusions of Theorem 4.2 still hold if conditions (3.8) and (3.9) are replaced by conditions (3.10) and (3.11). Note that this remark is also applicable to any other result that holds here under these conditions, e.g., Theorem 3.4.

(β) Typically, in applications, e.g., LQ-optimal control or spectral factorization of a normalized coprime fraction spectral density (see, e.g., [9], [10]) for an infinite-dimensional (without loss of generality) stable system, the p_n's and $\overline{p}_n$'s are the poles of the open-loop transfer function, and the z_n's and $\overline{z}_n$'s are the poles of the closed-loop transfer function.

(γ) The symmetric extraction method works very well for the heat diffusion equation; see [10], [7]. Indeed, in that case, the spectral density zeros and poles are all real, and the corresponding relative spectral errors $|z_n - p_n| \cdot |p_n|^{-1}$ and $|z_n - p_n| \cdot |z_n|^{-1}$ tend to zero exponentially fast as n tends to infinity, whence (3.10) and (3.11) obviously hold for $\alpha = \infty$, i.e., for any $\alpha > 1$.

(δ) The speed of convergence of the sequences (4.4) and (4.5) towards an invertible spectral factor $\hat{R}$ and its inverse $\hat{R}^{-1}$, respectively, is dictated by the magnitude of the parameter α of conditions (3.10) and (3.11). The larger it is, the better is the speed of convergence of the symmetric extraction method. However, this speed of convergence might not be as good as in the heat equation example mentioned above; see Example 4.1.

(ϵ) It is possible to compute absolute and relative error estimates, in the $\hat{\mathcal{A}}(\sigma)$-norm, for the spectral factor as well as for its inverse, especially when $||(R_n)_a||_{\mathcal{A}(\sigma)}$ and $||(R_n^{-1})_a||_{\mathcal{A}(\sigma)}$ are of the order of the general term of a converging power series; see [10].

Example 4.1. Consider the following coercive spectral density $\hat{F}(s)$ given by (3.1) with a countable number of elementary factors of the form (3.3) with complex conjugate poles and zeros such that

(1) for some $\sigma < 0$

$$2 \cdot |\sigma| \leq \min(|\text{Re } p_n|, |\text{Re } z_n|), \quad n = 1, 2, \ldots ,$$

(2) for some constants $a > 0$ and $b > 0$

$$p_n = -a + j \cdot b \cdot n, \quad n = 1, 2, \ldots ,$$

(3) for n sufficiently large,

$$|z_n - p_n| = O(\frac{1}{n^2}).$$

Then by Theorem 4.2 any invertible spectral factor of the spectral density $\hat{F}$ can be approximated arbitrarily precisely in the $\hat{\mathcal{A}}(\sigma)$-norm, by an invertible approximate (rational) spectral factor of the form (4.4). Here convergence is achieved but at a much slower speed ($\alpha = 2$) than in the heat equation example mentioned above ($\alpha = \infty$). Note that, in section 5, a physical example (vibrating string) is chosen to illustrate the possibility of slow convergence.

COROLLARY 4.3. *Let $\hat{F}$ be a coercive real spectral density satisfying the assumptions of Theorem* 3.4. *Then $\hat{F}$ admits an infinite product representation of pole-zero pairs that is of the form* (3.1)–(3.3) *for all s in some vertical strip symmetric with respect to the imaginary axis. Let $\hat{R}_n$, $n = 1, 2, \ldots$, be the rational elementary factors defined by* (3.4)–(3.5), *with $2 \cdot |\sigma| \leq \min(|\text{Re } p_n|, |\text{Re } z_n|)$, for all n, for some $\sigma < 0$.*

Then the sequence

$$\left(\prod_{n=1}^{N} \hat{R}_n \right)_{N \geq 1} \tag{4.9}$$

of invertible approximate (rational) spectral factors converges to the invertible standard spectral factor $\hat{R} \in \hat{\mathcal{A}}_-$ of $\hat{F}$ in the $\hat{\mathcal{A}}(\sigma)$-norm, and the sequence

$$\left(\prod_{n=1}^{N} \hat{R}_n^{-1} \right)_{N \geq 1} \tag{4.10}$$

converges to the corresponding inverse spectral factor $\hat{R}^{-1} \in \hat{\mathcal{A}}_-$.

Proof. The conclusion follows from Theorems 3.4 and 4.2. ∎

We conclude this subsection by yet another spectral criterion for the convergence of the symmetric extraction spectral factorization method. Its sufficient conditions are stronger than those established in the previous results. However, they fit specific classes of systems quite well, as shown in the next subsection.

THEOREM 4.4 (spectral criterion for the convergence of the symmetric extraction method). *Let $\hat{F}$ be a coercive real spectral density given by $\hat{F} = \hat{F}_* = \hat{G}_* + \hat{G}$, where $\hat{G} \in \hat{\mathcal{A}}_-$ is such that $G_{sa} = G_0 \delta(\cdot)$ for some $G_0 \in \mathbb{C}$; whence $\hat{F}$ is holomorphic in some open vertical strip containing the imaginary axis, namely, $S_\delta := \{s \in \mathbb{C} : \text{Re } s \in (-\delta, \delta)\}$, where $\delta > 0$. Assume that the limit of $\hat{F}$ at infinity exists in this vertical strip such that*

$$\hat{F}(\infty) \ := \lim_{|s| \to \infty;\, s \in S_\delta} \hat{F}(s) \ = \lim_{|\omega| \to \infty;\, -\delta < \sigma < \delta} \hat{F}(\sigma + j\omega) \ = \ 1, \tag{4.11}$$

(or, equivalently, $\operatorname{Re} G_0 = 2^{-1}$*). In addition, assume that* $\hat{F}$ *is a meromorphic function of finite order and is given by the fraction*

$$\hat{F}(s) = \frac{N(s)}{D(s)}, \tag{4.12}$$

where the denominator D *and numerator* N *are real parahermitian entire functions of finite order such that* $D(s) = D(-s)$ *and* $N(s) = N(-s)$*.*

Moreover, (1) *the zeros of* D *form a ps-family* $\mathbf{P}$ *with defining parameters* p_n*,*

(2) *the zeros of* N *form a ps-family* $\mathbf{Z}$ *with defining parameters* z_n*,*

(3) *one assumes that*

$$\sum_{n=1}^{\infty} \frac{1}{|p_n|^2} < \infty, \tag{4.13}$$

and

(4) *the pole-zero absolute error sequence is absolutely summable; i.e.,*

$$\sum_{n=1}^{\infty} |z_n - p_n| < \infty. \tag{4.14}$$

Then (a) $\hat{F}$ *admits the infinite product representation of pole-zero pairs* (3.1)–(3.3).

(b) *In addition, let* $\hat{R}_n$*,* $n = 1, 2, \dots$*, be the rational elementary factors defined by* (3.4)–(3.5)*, with* $2 \cdot |\sigma| \le \min(|\operatorname{Re} p_n|, |\operatorname{Re} z_n|)$*, for all* n*, for some* $\sigma < 0$*. Then the sequence* $(\prod_{n=1}^{N} \hat{R}_n)_{N \ge 1}$ *of invertible approximate (rational) spectral factors converges to the exact invertible standard spectral factor* $\hat{R} \in \hat{\mathcal{A}}_-$ *of* $\hat{F}$ *in the* $\hat{\mathcal{A}}(\sigma)$*-norm, and the sequence* $(\prod_{n=1}^{N} \hat{R}_n^{-1})_{N \ge 1}$ *converges to the corresponding inverse spectral factor* $\hat{R}^{-1} \in \hat{\mathcal{A}}_-$*.*

Proof. In view of Corollary 4.3, it suffices to check that the assumptions of Theorem 3.4 hold. Since the spectral density $\hat{F}$ is coercive, there exists some $\gamma > 0$ such that, for all $n \ge 1$,

$$|\operatorname{Re} z_n| \ \ge \ \gamma.$$

Therefore by condition (4.14) there holds

$$\sum_{n=1}^{\infty} \frac{|z_n - p_n|}{|\operatorname{Re} z_n|} < \infty;$$

i.e., assumption (3.9) holds.

Now observe that

$$\sum_{n=1}^{\infty} \left| \frac{1}{z_n} - \frac{1}{p_n} \right| < \infty. \tag{4.15}$$

Indeed, for all $n \ge 1$,

$$\left| \frac{1}{z_n} - \frac{1}{p_n} \right| \ \le \ \gamma^{-1} \cdot |z_n - p_n| \cdot |p_n|^{-1},$$

where, by (4.13) and (4.14), the sequences $(|z_n - p_n|)$ and $(|p_n|^{-1})$ are, respectively, in l^1 and in l^∞. Whence (4.15) holds. It follows by (4.15) and (4.13) that (3.13) holds. Finally observe that, by the holomorphicity of F in the strip S_δ, one has for all $n \geq 1$

$$|\text{Re } p_n| \geq \delta.$$

This together with (4.14) implies

$$\sum_{n=1}^{\infty} \frac{|z_n - p_n|}{|\text{Re } p_n|} < \infty,$$

i.e., assumption (3.8) holds. Hence all assumptions of Theorem 3.4 are valid and we are done. □

Remark 4.2. It can be shown that Theorem 4.4 can be applied to the heat diffusion model studied in [10], [7] (see Remark 4.1 (γ)). Actually this result is applicable to an important class of semigroup state-space systems in the framework of the LQ-optimal control problem for such systems. This question is addressed in the following subsection.

4.2. Semigroup state-space systems. Consider a single-input C_0-semigroup state-space system with bounded control and observation operators (see, e.g., [17], [28]), viz.,

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0, \quad y(t) = Cx(t), \quad t \geq 0, \tag{4.16}$$

where $x(t) \in \text{H}$, with H a separable Hilbert state-space with inner product $\langle \cdot\ ,\ \cdot \rangle$, $u(t) \in \mathbb{R}$, $y(t) \in \mathbb{R}^p$, and

(1) $A : D(A) \subset \text{H} \to \text{H}$ is the generator of a C_0-semigroup $(e^{At})_{t \geq 0} \subset \mathbf{L}(\text{H})$,

(2) $B \in \mathbf{L}(\mathbb{R}, \text{H})$ is a bounded linear control operator given by

$$Bu = bu \quad \text{for all} \;\; u \in \mathbb{R}\,, \quad b \in \text{H},$$

(3) $C \in \mathbf{L}(\text{H}, \mathbb{R}^p)$ is a bounded linear observation operator.

Furthermore assume that A is a Riesz-spectral operator [17] with discrete spectrum

$$\sigma(A) = \sigma_p(A) = \{\lambda_n \ : \ n \in \mathbb{N}\} \subset \mathbb{C} \tag{4.17}$$

consisting of simple eigenvalues such that

$$\delta := \inf \{ \, |\lambda_n - \lambda_m| \ : \ n, m \in \mathbb{N}\,,\ n \neq m\} > 0 \tag{4.18}$$

and

$$\mu := \sup \left\{ \sum_{\substack{l=1 \\ l \neq n}}^{\infty} \frac{1}{|\lambda_l - \lambda_n|^2} \ : \ n \in \mathbb{N} \right\} < \infty. \tag{4.19}$$

Remark 4.3. Since the operator A is the (infinitesimal) generator of a C_0-semigroup of bounded linear operators $(e^{At})_{t \geq 0}$ on H, it holds (see [17]) that $\sup \{\text{Re } \lambda_n \ : \ n \in \mathbb{N}\} < \infty$.

Finally assume that

(4.20) (A, B) is exponentially stabilizable and (C, A) is exponentially detectable.

Observe that, by (4.20), there exists some $\sigma < 0$ such that the spectrum of A can be decomposed according to

$$\sigma(A) = (\sigma(A) \cap \overset{o}{\mathbb{C}}_{\sigma-}) \ \dot{\cup} \ (\sigma(A) \cap \mathbb{C}_+), \tag{4.21}$$

where $\overset{o}{\mathbb{C}}_{\sigma-}$ denotes the open left half-plane $\{s \in \mathbb{C} \ : \ \mathrm{Re}\ s < \sigma\}$ and $\mathbb{C}_+$ denotes the closed right half-plane $\{s \in \mathbb{C} \ : \ \mathrm{Re}\ s \geq 0\}$. The two sets on the right-hand side of identity (4.21) are disjoint and the unstable spectrum $\sigma(A) \cap \mathbb{C}_+$ is a finite point set. Moreover, the following holds.

LEMMA 4.5. *The* nonzero *eigenvalues* $\lambda_n, n \in \mathbb{N}$, *of* A *satisfy*

$$\sum_{n=1, \lambda_n \neq 0}^{\infty} \frac{1}{|\lambda_n|^2} < \infty. \tag{4.22}$$

Proof. This follows immediately from (4.19). ∎

Now for system (4.16)–(4.20) consider the LQ-optimal control problem: for any initial state $x_0 \in \mathrm{H}$, find a square-integrable control $u_0 \in L^2(0, \infty; \mathbb{R})$ which minimizes the cost functional

$$J(x_0, u) := \int_0^\infty \left(\|Cx(t)\|^2 + \|u(t)\|^2\right) \, dt.$$

It is well known (see, e.g., [17] and the references cited therein) that the optimal control $u_0(t)$ is given by

$$u_0(t) = K_0 \, x_0(t), \qquad x_0(t) = e^{(A+BK_0)t} x_0,$$

where the optimal feedback operator $K_0 \in \mathcal{L}(\mathrm{H}, \mathbb{R})$ is given by

$$K_0 = -B^* \, Q_0,$$

where the operator $Q_0 \in \mathbf{L}(H)$ is the unique nonnegative self-adjoint solution of the operator Riccati equation on the domain of the operator A:

$$A^* Q_0 + Q_0 A + C^* C - Q_0 B B^* Q_0 = 0 \quad \text{on } D(A),$$

where $Q_0(D(A)) \subset D(A^*)$. Moreover, the optimal feedback K_0 is stabilizing; i.e., the feedback semigroup $(e^{(A+BK_0)t})_{t \geq 0}$ is exponentially stable, and $K_0 \in \mathbf{L}(\mathrm{H}, \mathbb{R})$ is also given by

$$K_0 \, x = \langle k_0 \, , \, x \rangle \quad \text{for all } x \in \mathrm{H}, \quad k_0 \in \mathrm{H}. \tag{4.23}$$

We have then the following.

LEMMA 4.6. *Consider the C_0-semigroup state-space system given by* (4.16)–(4.20). *Let $(\phi_n)_{n \in \mathbb{N}}$ be a Riesz basis of eigenvectors of A (corresponding to the eigenvalues λ_n) and let $(\psi_n)_{n \in \mathbb{N}}$ be the corresponding biorthogonal dual Riesz basis of eigenvectors of the adjoint operator A^*. Consider the LQ-optimal feedback operator $K_0 \in \mathcal{L}(H, \mathbb{R})$ given by* (4.23). *Then the feedback semigroup generator*

$$A_c := A + BK_0 = A + b \, \langle k_0 \, , \cdot \rangle \tag{4.24}$$

has a discrete spectrum of eigenvalues $\lambda_{cn}, n \in \mathbb{N}$, *with*

$$\sigma(A_c) = \sigma_p(A_c) = \{\lambda_{cn} \;:\; n \in \mathbb{N}\}, \tag{4.25}$$

and corresponding eigenvectors forming a Riesz basis of H. Moreover,

$$|\lambda_{cn} - \lambda_n| \;=\; O\,(\,|\langle k_0, \phi_n\rangle \cdot \langle b, \psi_n\rangle|\,) \quad \textit{for } \; n \textit{ sufficiently large}, \tag{4.26}$$

whence

$$\sum_{n=1}^{\infty} |\lambda_{cn} - \lambda_n| < \infty. \tag{4.27}$$

Proof. Results (4.25)–(4.26), where $\{\lambda_{cn} \;:\; n \in \mathbb{N}\}$ is replaced by its closure, follow by (4.17)–(4.19), [34, Theorem 2.1], and [15, Appendix B, p. 66]. Now in (4.26) the sequences $(\langle k_0, \phi_n\rangle)$ and $(\langle b, \psi_n\rangle)$ are in l^2, whence the product sequence $(\langle k_0, \phi_n\rangle \cdot \langle b, \psi_n\rangle)$ is in l^1. Hence (4.27) holds. Furthermore, by Lemma 4.5, (4.22) holds.

Now as $(e^{(A+BK_0)t})_{t\geq 0}$ is exponentially stable and $\sigma(A)$ is as in (4.21), there exists some $\eta > 0$ and $M \in \mathbb{N}$ such that for all $n \geq M$, $\min(|\text{Re}\lambda_n|, |\text{Re}\lambda_{cn}|) \geq \eta$. This together with (4.27) and (4.22) and arguments similar to those in the proof of Theorem 4.4, gives

$$\sum_{n=M}^{\infty} \left| \frac{1}{\lambda_n} - \frac{1}{\lambda_{cn}} \right| < \infty.$$

Consequently

$$\sum_{n=1}^{\infty} \frac{1}{|\lambda_{cn}|^2} \;<\; \infty.$$

Hence $\sigma_p(A_c)$ cannot have limit points in $\mathbb{C}$, and the last equality of (4.25) holds. ◻

Remark 4.4. In view of [24, Corollary 4.6], the feedback semigroup generator A_c given by (4.24) is a *Riesz-spectral operator* [17, p. 41], whenever its eigenvalues $\lambda_{cn}, n \in \mathbb{N}$ are simple. *This will be tacitly assumed in what follows.*

It is also known (see [9], [10]) that the LQ-optimal control problem can be solved by the spectral factorization of a specific spectral density, whose spectral factor gives the state-feedback operator K_o via a Diophantine equation. In the sequel we shall concentrate upon that spectral factorization problem. More precisely, assuming that the operator pair (A, B) is exponentially stabilizable, there exists a stabilizing feedback operator $K \in \mathcal{L}(\text{H}, \mathbb{R})$, i.e., such that the C_0-semigroup $(e^{(A+BK)t})_{t\geq 0}$ is exponentially stable. Moreover, with the spectrum of A as in (4.21), the feedback $K = [0\; K_2]$ (K_2 = vector) can be chosen such that

$$\sigma(A + BK) \;=\; (\sigma(A) \cap \overset{o}{\mathbb{C}}_{\sigma-}) \;\dot{\cup}\; \Sigma, \tag{4.28}$$

where $\Sigma \subset \overset{o}{\mathbb{C}}_{\sigma-}$ is a finite set having the same number of elements as $\sigma(A) \cap \mathbb{C}_+$. Under these conditions the pair $(\hat{\mathcal{N}}, \hat{\mathcal{D}}) \in \hat{\mathcal{A}}_-^{p\times 1} \times \hat{\mathcal{A}}_-$ defined by

$$(\hat{\mathcal{N}}(s), \hat{\mathcal{D}}(s)) := (C(sI - A - BK)^{-1}B,\; 1 + K(sI - A - BK)^{-1}B) \tag{4.29}$$

generates a right fraction of the semigroup state-space system transfer function $\hat{G}(s) = C(sI - A)^{-1}B \in \hat{\mathcal{B}}^{p\times 1}$ with no common zeros in $\mathbb{C}_+$, where $\hat{\mathcal{D}}$ is a biproper stable rational function whose zeros are in $\sigma(A) \cap \mathbb{C}_+$ and whose poles are in Σ. Moreover, $\hat{\mathcal{D}}$ equals 1 at infinity.

Now consider the function $\hat{F}(s)$ defined by

$$\hat{F} := \hat{\mathcal{N}}_* \, \hat{\mathcal{N}} + \hat{\mathcal{D}}_* \, \hat{\mathcal{D}} = \hat{\mathcal{D}}_* \, (1 + \hat{G}_* \, \hat{G}) \, \hat{\mathcal{D}}. \tag{4.30}$$

It is shown in [9, Theorem 3, pp. 70–71] and [10, Theorem 3, pp. 761–762] that $\hat{F}$ is a coercive spectral density whose spectral factorization is the main step towards the solution of the LQ-optimal control problem, i.e., for the computation of the feedback operator K_0. Moreover for the specific case at hand one has the following.

LEMMA 4.7. *Under the assumptions of Lemma* 4.6, *let* $\hat{G}(s) = C(sI - A)^{-1}B$ *be the transfer function of the* C_0*-semigroup state-space system* (4.16)–(4.20). *Consider the real function* $\hat{F}$ *given by* (4.30), *where* $(\hat{\mathcal{N}}, \hat{\mathcal{D}})$ *is the right fraction* (4.29) *with no common zeros in* $\mathbb{C}_+$ *of* $\hat{G}(s)$, *where* $K \in \mathbf{L}(H, \mathbb{R})$ *is a stabilizing feedback operator such that* (4.28) *holds.*

Then $\hat{F}$ *is a coercive real spectral density such that* $\hat{F}$ *is holomorphic in a vertical strip* S_δ *for some* $\delta > 0$ *and such that* $\hat{F}(\infty) = 1$, *i.e.,* (4.11) *holds. Moreover,* $\hat{F}$ *is a meromorphic function of finite order* $\rho \leq 2$ *and can be described as a fraction of real parahermitian entire functions, i.e.,*

$$\hat{F}(s) = \frac{N(s)}{D(s)}, \tag{4.31}$$

where the functions $D = D_*$ *and* $N = N_*$ *are entire functions with countable zero sets* $\mathcal{Z}[D]$ *and* $\mathcal{Z}[N]$, *respectively, such that, with* $\mathcal{P}[\hat{F}]$ *denoting the set of poles of* $\hat{F}$ *and* $\sigma < 0$ *chosen such that* (4.21) *holds,*

$$\mathcal{Z}[D] = \mathcal{P}[\hat{F}] \subset \{p, -\overline{p} \;:\; p \in (\sigma(A) \cap \overset{o}{\mathbb{C}}_{\sigma-}) \;\dot{\cup}\; \Sigma\}, \tag{4.32}$$

and

$$\mathcal{Z}[N] = \mathcal{Z}[\hat{F}] \subset \{z, -\overline{z} \;:\; z \in \sigma(A_c) = \sigma(A + BK_0)\}. \tag{4.33}$$

Remark 4.5. (α) When the C_0-semigroup $(e^{At})_{t\geq 0}$ is exponentially stable, one can choose the feedback K to be zero. In this case, the right fraction $(\hat{\mathcal{N}}, \hat{\mathcal{D}})$ defined by (4.29) is given by $(\hat{\mathcal{N}}, \hat{\mathcal{D}}) = (\hat{G}\ ,\ 1)$, and the spectral density reads

$$\hat{F} = 1 + \hat{G}_* \, \hat{G}. \tag{4.34}$$

Furthermore, the denominator entire function D in (4.31) is such that

$$\mathcal{Z}[D] = \mathcal{P}[\hat{F}] \subset \{p, -\overline{p} \;:\; p \in \sigma(A)\}. \tag{4.35}$$

(β) In general the inclusions in (4.32) and (4.33) are not equalities. This is due to the fact that the system is not necessarily approximately controllable and/or observable, whence the numerator and denominator can be simplified by common zero cancellations.

In standard examples, like the heat diffusion (see [10]) and the vibrating string (see below), the spectral density above is obtained (by applying the Laplace transform to the PDE describing the system) as a fraction of entire functions where an infinite number of common zero cancellations may occur.

Proof of Lemma 4.7. Property (4.11) follows directly from (4.30) and (4.29), which ensure that $\hat{G}(s)$ is zero at infinity in $\mathbb{C}_{\sigma+}$ and that $\hat{\mathcal{D}}$ equals 1 at infinity. By [17, Lemma 4.3.10, p. 183], the transfer function $\hat{G}(s)$ is given by

$$\hat{G}(s) \;=\; \sum_{n=1}^{\infty} \;(C\phi_n)\; \overline{\langle b, \psi_n \rangle}\; (s-\lambda_n)^{-1}, \tag{4.36}$$

which is holomorphic in the resolvent set $\rho(A)$, whose complement $\sigma(A)$ is a pure point spectrum of isolated points by (4.18). It follows, using [31, Definition 10.41, p. 241]), that $\hat{G}(s)$ is meromorphic in $\mathbb{C}$ with poles contained in $\sigma(A)$. Thus, upon noting that in (4.30) $\hat{\mathcal{D}}$ is a biproper rational function, there holds that the spectral density $\hat{F}$ given by (4.30) is real parahermitian meromorphic in $\mathbb{C}$, with poles given by the inclusion in (4.32). Consequently, by Lemma 2.5, $\hat{F} = N/D$, where N and D are real parahermitian entire functions with no common zeros in $\mathbb{C}$; i.e., (4.31) holds with $\mathcal{Z}[D] = \mathcal{P}[\hat{F}]$ and $\mathcal{Z}[N] = \mathcal{Z}[\hat{F}]$. Hence (4.32) follows and we have to show that the inclusion of (4.33) holds. Now, by [9, Theorem 2, p. 67], the inverse spectral density $\hat{F}^{-1}$ can be written as

$$\hat{F}^{-1} \;=\; \hat{W}\,\hat{W}_*,$$

where

$$\hat{W}(s) \;=\; \hat{\mathcal{D}}(s)^{-1}\,[1 + K_0(sI - A - BK_0)^{-1}B] \;\in\; \hat{\mathcal{A}}_-.$$

Therefore the zero set of $\hat{F}$, i.e., the pole set of $\hat{F}^{-1}$, satisfies

$$\mathcal{Z}[\hat{F}] = \mathcal{P}[\hat{F}^{-1}] \;\subset\; \{z, -\overline{z} \;:\; z \in \sigma(A + BK_0)\},$$

and the inclusion in (4.33) holds. Finally $\hat{F}$ is holomorphic in a vertical strip S_δ for some $\delta > 0$ by its pole structure and because it is coercive.

It remains to be proved that the transfer function $\hat{G}(s) = C(sI - A)^{-1}B$ is a meromorphic function *of finite order*, whence so will be the spectral density $\hat{F}$ given by (4.30). Since C and B are bounded linear operators of finite-rank, and (ϕ_n) and (ψ_n) are Riesz bases, the sequences $((C\phi_n))$ and $(\overline{\langle b, \psi_n \rangle})$ are in l^2. It follows by (4.36) that

$$\hat{G}(s) = \sum_{n=1}^{\infty} d_n (s - \lambda_n)^{-1},$$

where the product sequence $(d_n) := ((C\phi_n))\cdot\overline{\langle b, \psi_n \rangle})$ is in l^1, i.e., absolutely summable, and convergence is pointwise. It follows that

$$|\hat{G}(s)| \leq \frac{K}{d(s, \sigma(A))}, \tag{4.37}$$

for some constant K, where $d(s, \sigma(A))$ denotes the distance between s and $\sigma(A)$, which is given by

$$d(s, \sigma(A)) \;:=\; \inf\,\{|s - \lambda| \;:\; \lambda \in \sigma(A)\}.$$

In view of Lemma 4.5, $|\lambda_n| \longrightarrow \infty$ as $n \to \infty$. Hence, for any positive real number R, there exists a nonnegative integer $n(R)$ such that

$$|\lambda_n| > R \text{ for all } n > n(R).$$

Because of assumption (4.18), the maximum number of poles of the function $\hat{G}$ in a disc $\{s \in \mathbb{C} : |s| \le R\}$ of arbitrarily large radius R, namely, $n(R)$, is such that

$$n(R) = O(R^2). \tag{4.38}$$

Moreover, in the annulus defined by the circles centered at the origin with radii R and $R+1$, respectively, the number of poles of $\hat{G}$ is $O(R)$, i.e., is equal to κR for some constant κ. These poles can be arranged in increasing order of modulus, say,

$$R_0 := R \le R_1 \le \cdots \le R_{\kappa R} \le R_{\kappa R+1} := R + 1.$$

This set is formed of at most $\kappa R + 2$ numbers which are contained in an interval of length one. So the maximum gap between two consecutive numbers among them is at least $(\kappa R + 1)^{-1}$. Now if a point $s \in \mathbb{C}$ is such that $|s|$ lies in the middle of this gap, then the distance from s to the nearest pole of $\hat{G}$ is at least $(2(\kappa R + 1))^{-1}$, i.e., $O(R^{-1})$. Hence

$$d(s, \sigma(A)) = O\left(\frac{1}{R}\right) \text{ on a circle } C(0, r_R) := \{s \in \mathbb{C} : |s| = r_R\}, \text{ where } R < r_R < R + 1.$$

Thus, by inequality (4.37),

$$|\hat{G}(s)| = O(R) \text{ on the circle } C(0, r_R). \tag{4.39}$$

It follows from (4.38) and (4.39) that the counting and proximity functions of $\hat{G}$ satisfy, respectively,

$$N(r_n, \hat{G}) = O(r_n^2) \text{ and } m(r_n, \hat{G}) = O(\log r_n) \text{ as } n \to \infty,$$

where the sequence of points (r_n) is such that, for all n, $n < r_n < n + 1$; whence

$$T(r_n) = O\left(r_n^2\right) \quad \text{as } n \to \infty.$$

Since the characteristic function $T(r) := T(r, \hat{G})$ of $\hat{G}$ is a monotonically increasing function of $r > 0$, it follows that

$$T(r) = O(r^2) \text{ as } r \to \infty. \tag{4.40}$$

Observe that the order ρ of the meromorphic function $\hat{G}$ is the lower bound of all positive numbers k such that $T(r) = O(r^k)$ as $r \to \infty$. Hence, in view of (4.40), the function $\hat{G}$ is of finite order $\rho \le 2$. □

We are now ready to show that the symmetric extraction method of spectral factorization works for such systems.

THEOREM 4.8. *Let the assumptions of Lemma* 4.6 *hold. Consider the coercive real spectral density* $\hat{F}$ *given by* (4.30), *where* $(\hat{\mathcal{N}}, \hat{\mathcal{D}})$ *is the right-coprime fraction of the transfer function* $\hat{G}(s) = C(sI - A)^{-1}B$, *which is given by* (4.29) *for some stabilizing feedback operator* $K \in \mathbf{L}(H, \mathbb{R})$ *such that* (4.28) *holds. Then the symmetric* *extraction method of spectral factorization of the spectral density* $\hat{F}$ *is convergent, i.e., the conclusions* (a) *and* (b) *of Theorem* 4.4 *hold.*

Proof. From the initial stabilization procedure described above, it is clear that one may assume without loss of generality that the open-loop C_0-semigroup $(e^{At})_{t\geq 0}$ is exponentially stable. Hence without loss of generality, poles and zeros of $\hat{F}$ are a subset of, respectively, the λ_n and the λ_{cn} mentioned above. The conclusions then follow directly from Lemmas 4.5–4.7, by using Theorem 4.4. ∎

5. Example: Vibrating string with low damping. The main result of the last section, viz., Theorem 4.8, is now used in order to apply the symmetric extraction method to a lowly damped vibrating string model, with the purpose of illustrating a case of slow convergence.

In what follows $z(t,x)$ denotes the vertical position of a damped vibrating string at the place $x \in [0,1]$ and time $t \geq 0$ that is described by the PDE

$$z_{tt}(t,x) = z_{xx}(t,x) - 2\beta z_t(t,x) + b(x)u(t), \tag{5.1}$$

where the damping parameter $\beta \in (0,\pi)$ (low damping) and for all $t \geq 0$, $z(t,0) = z(t,1) = 0$; moreover, $u(t) \in \mathbb{R}$ is a scalar input, and $b(x)$ is a window function given for $\nu_i > 0$ small and $[x_i - \nu_i, x_i + \nu_i] \subset [0,1]$ by

$$b(x) := (2\nu_i)^{-1}\chi_{[x_i-\nu_i,x_i+\nu_i]}(x), \quad x \in [0,1]. \tag{5.2}$$

The scalar output $y(t) \in \mathbb{R}$ is given by

$$y(t) := \int_0^1 c(x)z(t,x)dx, \tag{5.3}$$

where $c(x)$ is a window function which for $\nu_o > 0$ small and $[x_o - \nu_o, x_o + \nu_o] \subset [0,1]$ reads

$$c(x) := (2\nu_o)^{-1}\chi_{[x_o-\nu_o,x_o+\nu_o]}(x), \quad x \in [0,1]. \tag{5.4}$$

It is moreover assumed that

$$x_i - \nu_i > 0, \quad x_o - \nu_o > x_i + \nu_i, \quad \text{and} \quad x_o + \nu_o < 1. \tag{5.5}$$

In order to show that the theory of subsection 4.2 applies to this example, we first derive a semigroup state-space model of the form (4.16) for this system. The reader is referred to [17, Examples 2.2.5 and 2.3.8] and [1, Example 3.5.3] for more detail. Consider the Hilbert space $\mathrm{H} = \mathrm{L}^2(0,1)$ with standard scalar product $\langle\cdot,\cdot\rangle_2$, which is antilinear in its second argument. Let $A : (D(A) \subset \mathrm{H}) \to \mathrm{H}$ be the generator of a C_0-semigroup $(e^{At})_{t\geq 0}$ on H given by

$$Az = z'', \qquad D(A) = \{z \in \mathrm{H}^2(0,1) : \ z(1) = 0, \ z(0) = 0\} = \mathrm{H}^2(0,1) \cap \mathrm{H}_0^1(0,1). \tag{5.6}$$

Since $A = A^* < 0$, A generates on H an analytic semigroup that is exponentially stable. Moreover, $D[(-A)^{\frac{1}{2}}]$ equipped with the graph norm of $(-A)^{\frac{1}{2}}$ is a Hilbert space that can be identified with $\mathrm{H}_0^1(0,1)$ equipped with the norm $||z'||_2$ for any $z \in \mathrm{H}_0^1(0,1)$ [1, Example 3.5.3]. In this sense it is possible to consider the Hilbert space

$$\mathcal{H} := D[(-A)^{\frac{1}{2}}] \oplus \mathrm{H} = \mathrm{H}_0^1(0,1) \oplus \mathrm{H}, \tag{5.7}$$

with scalar product

$$\langle \zeta, \eta \rangle_{\mathcal{H}} := \langle \zeta_1', \eta_1' \rangle_2 + \langle \zeta_2, \eta_2 \rangle_2 \qquad \forall\, \zeta = \begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix},\ \eta = \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} \in \mathcal{H}. \tag{5.8}$$

Recall now $b(\cdot)$ and $c(\cdot)$ given by (5.2) and (5.4), and define $B \in \mathbf{L}(\mathbb{R}, \mathrm{H})$ and $C \in \mathbf{L}(\mathrm{H}, \mathbb{R})$ by, respectively,

$$Bu := b(\cdot)u \quad \forall\, u \in \mathbb{R} \quad \text{and} \quad Cz := \langle c(\cdot), z(\cdot) \rangle_2 = \int_0^1 c(x)z(x)dx \quad \forall\, z \in \mathrm{H}. \tag{5.9}$$

Consider now $\mathcal{A}\colon\ (D(\mathcal{A}) \subset \mathcal{H}) \to \mathcal{H}$ given by

$$\mathcal{A} := \begin{bmatrix} 0 & I \\ A & -2\beta I \end{bmatrix}, \qquad D(\mathcal{A}) = D(A) \oplus \mathrm{H}_0^1(0,1), \tag{5.10}$$

and $\mathcal{B} \in \mathbf{L}(\mathbb{R}, \mathcal{H})$ and $\mathcal{C} \in \mathbf{L}(\mathcal{H}, \mathbb{R})$ defined by

$$\mathcal{B}u := \begin{bmatrix} 0 \\ Bu \end{bmatrix} \quad \forall\, u \in \mathbb{R} \quad \text{and} \quad \mathcal{C}\zeta := \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix} \quad \forall\, \zeta \in \mathcal{H}. \tag{5.11}$$

Observe that $\mathcal{C} \in \mathbf{L}(\mathcal{H}, \mathbb{R})$ because $\langle c(\cdot), z(\cdot) \rangle_2 = \langle -\int_0^\cdot c(\xi)d\xi, z'(\cdot) \rangle_2$ for any $z \in \mathrm{H}_0^1(0,1)$. It turns out that $\mathcal{A}$ is the generator of an exponentially stable C_0-semigroup $(e^{\mathcal{A}t})_{t\geq 0}$ of contraction on $\mathcal{H}$ [17, Example 2.2.5] and $\mathcal{A}$ is a Riesz-spectral operator [17, Definition 2.3.4, Example 2.3.8]. More precisely, $\mathcal{A}$ is a Riesz-spectral operator that has for $k \in \mathbb{Z}_0$ countably many complex eigenvalues λ_k given by

$$\lambda_k = -\beta + j\,\mathrm{sign}(k)\sqrt{(k\pi)^2 - \beta^2}, \tag{5.12}$$

with primal Riesz basis of eigenvectors

$$\phi_k(x) = \begin{bmatrix} \phi_{k1}(x) \\ \phi_{k2}(x) \end{bmatrix} = \begin{bmatrix} 1 \\ \lambda_k \end{bmatrix} \frac{\mathrm{sign}(k)\sin(k\pi x)}{\lambda_k} \quad x \in [0,1] \quad \forall k \in \mathbb{Z}_0, \tag{5.13}$$

and dual Riesz basis of eigenvectors

$$\psi_k(x) = \begin{bmatrix} \psi_{k1}(x) \\ \psi_{k2}(x) \end{bmatrix} = \begin{bmatrix} 1 \\ -\overline{\lambda_k} \end{bmatrix} \frac{\mathrm{sign}(k)\sin(k\pi x)}{j\,\mathrm{Im}(\lambda_k)} \quad x \in [0,1] \quad \forall k \in \mathbb{Z}_0, \tag{5.14}$$

such that

$$||\phi_k||_{\mathcal{H}}^2 = 1 \quad \text{and} \quad ||\psi_k||_{\mathcal{H}}^2 = \frac{(k\pi)^2}{(k\pi)^2 - \beta^2}.$$

Hence by [17, Theorem 2.3.5] the spectrum of $\mathcal{A}$ satisfies

$$\sigma(\mathcal{A}) = \overline{\{\lambda_k : k \in \mathbb{Z}_0\}},$$

where the eigenvalues λ_k are given by (5.12) and the growth constant of the semigroup $(e^{\mathcal{A}t})_{t\geq 0}$ generated by $\mathcal{A}$ is given by

$$\omega_0 := \inf_{t>0} \left(\frac{1}{t} \log ||e^{\mathcal{A}t}|| \right) = \sup_{k\in\mathbb{Z}_0} \mathrm{Re}\ \lambda_k = -\beta.$$

Thus, for $\sigma \in (-\beta, 0]$, $||e^{\mathcal{A}t}|| \leq M \exp(\sigma t)$ for all $t \geq 0$, and for such σ, $(e^{\mathcal{A}t})_{t\geq 0}$ is σ-exponentially stable [17, Definition 5.1.1].

Upon identifying $\zeta_1(t)(\cdot) := z(t, \cdot)$ and $\zeta_2(t)(\cdot) := z_t(t, \cdot)$, the PDE model described by (5.1)–(5.5) can be given an infinite-dimensional state-space description of the form (4.16) on the state-space $\zeta = [\zeta_1, \zeta_2]^T \in \mathcal{H}$ given by

$$\dot{\zeta} = \mathcal{A}\zeta + \mathcal{B}u(t) \quad \text{and} \quad y(t) = \mathcal{C}\zeta(t), \tag{5.15}$$

where one uses the mild solution of the state differential equation, viz.,

$$\zeta(t) = e^{\mathcal{A}t}\zeta(0) + \int_0^t e^{\mathcal{A}(t-\tau)}\mathcal{B}u(\tau)d\tau, \quad t \geq 0, \quad \zeta(0) \in \mathcal{H}, \tag{5.16}$$

and where $\mathcal{A}$ is a Riesz-spectral operator satisfying condition (4.17).

In addition, it follows from the fact that the semigroup $(e^{\mathcal{A}t})_{t\geq 0}$ is exponentially stable that (4.20) holds. Thus it remains to be proved that (4.18) and (4.19) hold. Now observe that, for all $k, l \in \mathbb{Z}_0$ such that $k \neq l$, there holds

$$|\lambda_k - \lambda_l| \geq \sqrt{\pi^2 - \beta^2};$$

whence (4.18) holds with

$$\delta = \inf \{ |\lambda_k - \lambda_l| : k, l \in \mathbb{Z}_0 , k \neq l\} \geq \sqrt{\pi^2 - \beta^2} > 0.$$

Finally, for all k, l in $\mathbb{Z}_0$ such that $k \neq l$, there holds

$$|\lambda_l - \lambda_k|^2 > (l\pi)^2 - \beta^2 > 0,$$

when $\text{sign}(k) \neq \text{sign}(l)$, and

$$|\lambda_l - \lambda_k|^2 = \left(\frac{\pi^2 (k^2 - l^2)}{\sqrt{(k\pi)^2 - \beta^2} + \sqrt{(l\pi)^2 - \beta^2}} \right)^2 > \pi^2 (k - l)^2 > 0,$$

when $\text{sign}(k) = \text{sign}(l)$. It follows that, for all k in $\mathbb{Z}_0$,

$$\sum_{\substack{l \in \mathbb{Z}_0 \\ l \neq k}} \frac{1}{|\lambda_l - \lambda_k|^2} \leq \sum_{l \in \mathbb{N}} \frac{1}{(l\pi)^2 - \beta^2} + \frac{1}{\pi^2} \cdot \sum_{\substack{l \in \mathbb{Z} \\ l \neq k}} \frac{1}{(l - k)^2}.$$

Hence condition (4.19) is satisfied with

$$\mu = \sup \left\{ \sum_{\substack{l \in \mathbb{Z}_0 \\ l \neq k}} \frac{1}{|\lambda_l - \lambda_k|^2} : k \in \mathbb{Z}_0 \right\} \leq \sum_{l \in \mathbb{N}} \frac{1}{(l\pi)^2 - \beta^2} + \frac{1}{\pi^2} \cdot \sum_{l \in \mathbb{Z}_0} \frac{1}{l^2} < \infty.$$

In view of Remark 4.5 (α), it follows from the exponential stability of the semigroup $(e^{\mathcal{A}t})_{t\geq 0}$ that the corresponding LQ-optimal control based (coercive) spectral density $\hat{F}$ to be factorized can without loss of generality be chosen to be

$$\hat{F} = 1 + \hat{g}_* \, \hat{g}, \tag{5.17}$$

where $\hat{g} \in \hat{\mathcal{A}}_-$ is the vibrating string model transfer function, which is given by

$$\hat{g}(s) = \frac{\sin(\sqrt{\rho_s}(1-x_o))}{\sin(\sqrt{\rho_s})} \cdot \frac{\sin(\sqrt{\rho_s}\nu_o)}{\sqrt{\rho_s}\nu_o} \cdot \frac{\sin(\sqrt{\rho_s}x_i)}{\sqrt{\rho_s}} \cdot \frac{\sin(\sqrt{\rho_s}\nu_i)}{\sqrt{\rho_s}\nu_i} \tag{5.18}$$

or, equivalently,

$$\hat{g}(s) = \frac{\sinh(\sqrt{r_s}(1-x_o))}{\sinh(\sqrt{r_s})} \cdot \frac{\sinh(\sqrt{r_s}\nu_o)}{\sqrt{r_s}\nu_o} \cdot \frac{\sinh(\sqrt{r_s}x_i)}{\sqrt{r_s}} \cdot \frac{\sinh(\sqrt{r_s}\nu_i)}{\sqrt{r_s}\nu_i}, \tag{5.19}$$

where $\rho_s := -s(2\beta + s)$ and $r_s := -\rho_s$.

Remark 5.1. As the semigroup $(e^{\mathcal{A}t})_{t\geq 0}$ generated by $\mathcal{A}$ is σ-exponentially stable for $\sigma \in (-\beta, 0]$, and $\mathcal{B} \in \mathbf{L}(\mathbb{R}, \mathcal{H})$ and $\mathcal{C} \in \mathbf{L}(\mathcal{H}, \mathbb{R})$, there holds by [17, Lemma 7.3.1] that the transfer function given above belongs to the class $\hat{\mathcal{A}}_-(\sigma)$ and to the class $\hat{\mathcal{A}}(\sigma)$ for $\sigma \in (-\beta, 0]$. As the corresponding impulse response $g(t)$ has no impulses, one has $\exp(-\sigma\cdot)g(\cdot) \in \mathrm{L}^1(0,\infty)$, $\sup_{\mathrm{Re}\, s\geq\sigma} |\hat{g}(s)| < \infty$, and $\hat{g}(s)$ is zero at infinity in $\mathbb{C}_{\sigma+} := \{s \in \mathbb{C} : \mathrm{Re}\, s \geq \sigma\}$.

Now observe that the spectral density $\hat{F}$ can be written as

$$\hat{F} \;=\; \frac{N}{D}, \tag{5.20}$$

where N and D are the real parahermitian entire functions given, respectively, by

$$N(s) \;:=\; n(-s)\cdot n(s) + d(-s)\cdot d(s) \qquad \text{and} \qquad D(s) \;:=\; d(-s)\cdot d(s), \tag{5.21}$$

where d and n are, respectively, the denominator and numerator of the transfer function $\hat{g}$, which are the entire functions given, respectively, by

$$d(s) \;:=\; \frac{\sin(\sqrt{\rho_s})}{\sqrt{\rho_s}} \;=\; \frac{\sinh(\sqrt{r_s})}{\sqrt{r_s}}, \tag{5.22}$$

and

$$n(s) := \hat{g}(s)\; d(s), \tag{5.23}$$

where the zeros of d are exactly the open-loop eigenvalues λ_k, given by (5.12). Observe that the numerator N and denominator D of the spectral density $\hat{F}$ above may have infinitely many common zeros (see Remark 4.5 (β)). In addition observe that, in view of (5.21)–(5.22), the entire functions N and D are of finite order (see, e.g., [37, Example 1, p. 76]).

It follows from the analysis above that, by Theorem 4.8, the spectral factorization by symmetric extraction of the spectral density $\hat{F}$ given by (5.20)–(5.22) is convergent; i.e., the conclusions (a) and (b) of Theorem 4.4 hold.

Numerical results are presented in Table 5.1. These results were obtained for the following parameter values: $\beta = 2$, $x_i = 0.02$, $x_o = 1 - x_i = 0.98$, and $\nu_i = \nu_o = 0.01$. It is found that the closed-loop eigenvalues λ_{cn} have numerically a constant real part equal to -2 and hence are vertically distant from the open-loop ones by $|\lambda_n - \lambda_{cn}|$. One can observe that the convergence is slow. Moreover, the absolute and relative errors are overall decreasing in an oscillatory manner. Further numerical evidence leads us to conjecture that $|\lambda_n - \lambda_{cn}|$ is of order $n^{-\alpha}$, where α is slightly larger than one.

Table 5.1
Eigenvalues, eigenvalue errors $\delta_n := |\lambda_n - \lambda_{cn}|$ *(versus the sequences* $(1/n)$ *and* $(1/n^2)$*).*

n	λ_n	δ_n	$n \cdot \delta_n$	$n^2 \cdot \delta_n$
1	-2+2.42j	1.55e-7	1.55e-7	1.55e-7
2	-2+5.96j	3.26e-7	6.52e-7	1.30e-6
3	-2+9.21j	6.05e-7	1.81e-6	5.44e-6
4	-2+12.41j	9.79e-7	3.92e-6	1.57e-5
5	-2+15.58j	1.43e-6	7.16e-6	3.58e-5
16	-2+50.23j	6.28e-6	1.01e-4	1.62e-3
17	-2+53.37j	6.31e-6	1.07e-4	1.82e-3
18	-2+56.51j	6.25e-6	1.12e-4	2.02e-3
19	-2+59.66j	6.10e-6	1.16e-4	2.22e-3
20	-2+62.80j	5.86e-6	1.17e-4	2.34e-3
21	-2+65.94j	5.56e-6	1.17e-4	2.46e-3
22	-2+69.09j	5.21e-6	1.15e-4	2.53e-3
23	-2+72.23j	4.81e-6	1.11e-4	2.55e-3
24	-2+75.37j	4.38e-6	1.05e-4	2.52e-3
25	-2+78.51j	3.93e-6	9.83e-5	2.46e-3
36	-2+113.08j	4.19e-7	1.51e-5	5.44e-4
37	-2+116.22j	3.01e-7	1.12e-5	4.14e-4
38	-2+119.36j	2.11e-7	8.01e-6	3.04e-4
39	-2+122.51j	1.42e-7	5.56e-6	2.17e-4
40	-2+125.65j	9.26e-8	3.71e-6	1.48e-4

This is theoretically confirmed by the facts that (1) by Lemma 4.6, (5.11), and (5.14), $|\lambda_n - \lambda_{cn}| = O(\frac{x_n}{n})$, where (x_n) is a square-summable sequence, whence $\alpha > 1$, and (2) the linearized, i.e., Newton–Raphson, estimate of $|\lambda_n - \lambda_{cn}|$ is $O(\frac{1}{n^2})$. Thus as nonlinear perturbations do not improve the speed of convergence, one has $\alpha \in (1, 2]$: the situation is comparable with that of Example 4.1.

Notice further that the tail-sums used in (4.6)–(4.8) are here of order $\frac{1}{n^{\alpha-1}}$. Hence the error analysis of Theorem 4.2 reveals that approximate spectral factorization will be achieved very slowly, the main reason being the asymptotically linear distribution of the spectra along a vertical line in the open left half-plane. A better situation is to be expected when this is not the case, i.e., acceleration by the fact that the real parts of the closed-loop eigenvalues Re λ_{cn} tend to $-\infty$.

6. Conclusion. As we have seen, the symmetric extraction method may be applied to a wide class of distributed parameter systems, for which its convergence has been established theoretically.

Another example for which the symmetric extraction method is appropriate is the beam equation with structural damping (see, e.g., [32, pp. 131–133]), and in this case one would expect the convergence to be faster, since the real parts of the eigenvalues tend to $-\infty$. More generally, one would expect the convergence to be faster when the semigroup is analytic, since in that case the spectrum lies in a sector contained in some left half-plane; see, e.g., [2].

Other possible techniques for approaching the spectral factorization problem for distributed parameter systems include a direct approximation of the spectral density function, but this needs to be treated with caution, since the mapping from spectral density to spectral factor is discontinuous in the uniform norm (see, e.g., [22]). It would also be of interest to extend the present methods to multivariable systems (the finite-dimensional case was analyzed in [4]), but this introduces additional function-theoretic difficulties.

As a referee has observed, there may be connections between the factorization approach taken here and the invariant subspace approach. This could be an interesting topic for further research, in particular for the special class of Riesz-spectral systems; see [24].

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